TILTING OBJECTS IN STABLE CATEGORIES OF PREPROJECTIVE ALGEBRAS

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ABSTRACT. In this paper, we construct a tilting object in stable categories of factor algebras of preprojective algebras. In [4], for a finite acyclic quiver Q and its preprojective algebra Π, Buan-Iyama-Reiten-Scott introduced and studied the factor algebra Π_w associated with an element w in the Coxeter group of Q. The algebra Π_w has a natural $\mathbb{Z}$-grading, and we prove that $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w$ has a tilting object if w is a c-sortable element.

1. Introduction

The preprojective algebra Π of a finite acyclic quiver Q has an important role in representation theory of algebras. One of them is categorifications of cluster algebras introduced by Fomin-Zelevinsky [6]. In the study of categorifications of cluster algebras, 2-Calabi-Yau triangulated categories (2-CY for short) and their cluster tilting objects are important.

If Q is a Dynkin quiver, then the preprojective algebra Π of Q is a finite dimensional selfinjective algebra and Geiss-Leclerc-Schröer showed that the stable category $\underline{\mathsf{mod}}\,\Pi$ is a 2-CY category and $\underline{\mathsf{mod}}\,\Pi$ has cluster tilting objects [7]. If Q is finite acyclic non-Dynkin quiver, Buan-Iyama-Reiten-Scott introduced and studied the factor algebra Π_w associated with an element w in the Coxeter group of Q [4]. They showed that the stable category of $\mathsf{Sub}\,\Pi_w$ is a 2-CY category and has cluster tilting objects, where $\mathsf{Sub}\,\Pi_w$ is the full subcategory of $\mathsf{mod}\,\Pi_w$ of submodules of finitely generated free Π_w-modules.

There are other classes of 2-CY triangulated categories. For a finite dimensional algebra A of finite global dimension, the cluster category $\mathcal{C}_A$ were introduced [1, 5]. The category $\mathcal{C}_A$ is a 2-CY category and has cluster tilting objects. Amiot-Reiten-Todorov [3] showed that there are close connections between 2-CY categories $\underline{\mathsf{Sub}}\,\Pi_w$ and $\mathcal{C}_A$. That is, for any finite acyclic quiver Q and any element w of the Coxeter group, there is a triangle equivalence

$$\underline{\mathsf{Sub}}\,\Pi_w \simeq \mathcal{C}_{A_w}$$

for some finite dimensional algebra A_w of global dimension at most two.

The aim of this paper is to construct a derived category version of this equivalence. More precisely, we regard Π_w as a $\mathbb{Z}$-graded algebra and consider the stable category $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w$ of graded Π_w-submodules of graded free Π_w-modules. Then we construct a tilting object in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w$.

The detailed version of this paper will be submitted for publication elsewhere.

2. Preliminaries

Through out this paper, let k be an algebraically closed field. By a module, we mean a left module unless stated otherwise. In this section, we give definitions used in the next section.

Definition 1. Let Q be a finite acyclic quiver.

(1) The *double quiver* $\overline{Q} = (\overline{Q}_0, \overline{Q}_1, s, t)$ of Q is defined by $\overline{Q}_0 = Q_0$, $\overline{Q}_1 = Q_1 \sqcup \{\bar{\alpha} \mid \alpha \in Q_1\}$, where $s(\bar{\alpha}) = t(\alpha)$, $t(\bar{\alpha}) = s(\alpha)$ for all $\alpha \in Q_1$.

(2) Then we have the *preprojective algebra* Π of Q by
$$\Pi := k\overline{Q}/\langle \sum_{\beta \in Q_1} \beta\bar{\beta} - \bar{\beta}\beta \rangle.$$

In this paper, we assume Q is non-Dynkin quiver, that is, the underlying graph of Q is not a simply laced Dynkin diagram. Note that, if Q is non-Dynkin quiver, then the preprojective algebra of Q is not a finite dimensional algebra. Next we define the Coxeter group of Q.

Definition 2. The *Coxeter group* W of a quiver Q is the group generated by the set $\{s_i \mid i \in Q_0\}$ with relations

- $s_i^2 = 1$,
- $s_j s_i = s_i s_j$ if there are no arrows between i and j,
- $s_i s_j s_i = s_j s_i s_j$ if there is exactly one arrow between i and j.

An expression $w = s_{i_1} s_{i_2} \ldots s_{i_l}$ is *reduced* if for any other expression $w = s_{i_1} s_{i_2} \cdots s_{i_m}$, we have $l \leq m$.

Let i be a vertex of Q. We define the two-sided ideal I_i of Π by
$$I_i := \Pi(1 - e_i)\Pi,$$
where e_i is the idempotent associated to i. Let $w = s_{i_1} s_{i_2} \ldots s_{i_l}$ be a reduced expression of w. We define a two-sided ideal I_w of Π by
$$I_w := I_{i_1} I_{i_2} \cdots I_{i_l}.$$
Note that, an ideal I_w is independent of the choice of a reduced expression of w by [4, Theorem Ⅲ. 1.9]. In [4], the authors studied the algebra Π/I_w.

Let $\mathsf{mod}\,\Pi_w$ be the category of finitely generated Π_w-modules. We denote by $\mathsf{Sub}\,\Pi_w$ the full subcategory of $\mathsf{mod}\Pi_w$ of submodules of finitely generated free Π_w-modules.

Proposition 3. [4] *Let Q be a finite acyclic non-Dynkin quiver. For an element w of the Coxeter group of Q, we have the following results.*

(a) *The algebra Π_w is finite dimensional and $\mathsf{inj.dim}({}_{\Pi_w}\Pi_w) \leq 1$.*

(b) *The category $\mathsf{Sub}\,\Pi_w$ is a Frobenius category.*

(c) *The stable category $\underline{\mathsf{Sub}}\,\Pi_w$ is 2-Calabi-Yau triangulated category, that is, for any objects $X, Y \in \underline{\mathsf{Sub}}\,\Pi_w$, there is a functorial isomorphism $\underline{\mathrm{Hom}}_{\Pi_w}(X, Y) \simeq \mathrm{D}\,\underline{\mathrm{Hom}}_{\Pi_w}(Y, X[2])$, where $\mathrm{D} = \mathrm{Hom}_k(\ , k)$.*

(d) *For any reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_l}$, the object $T = \bigoplus_{j=1}^{l} \Pi_{s_{i_1} s_{i_2} \cdots s_{i_j}}$ is a cluster tilting object of $\underline{\mathsf{Sub}}\,\Pi_w$.*

Next we consider the grading of a preprojective algebra. We regard the path algebra $k\overline{Q}$ as a $\mathbb{Z}$-graded k-algebra by the following grading:

$$\deg\beta = \begin{cases} 1 & \beta = \bar{\alpha}, \alpha \in Q_1 \\ 0 & \beta = \alpha, \alpha \in Q_1. \end{cases}$$

Since the element $\sum\limits_{\beta \in Q_1} (\beta\bar{\beta} - \bar{\beta}\beta)$ in $k\overline{Q}$ is homogeneous of degree 1, the grading of $k\overline{Q}$ naturally gives a grading on the preprojective algebra $\Pi = \bigoplus\limits_{i \geq 0} \Pi_i$.

Remark 4. (a) We have $\Pi_0 = kQ$, since Π_0 is spanned by all paths of degree 0.
(b) For any $w \in W$ the ideal I_w of Π is a graded ideal of Π since so is each I_i.
(c) In particular, the quotient algebra Π_w is a graded algebra.

For a graded module $M = \bigoplus\limits_{i \in \mathbb{Z}} M_i$ and an integer j, we define a new graded module $M(j)$ by $(M(j))_i = M_{i+j}$. For any integer j, we define a graded submodule $M_{\geq j}$ of M by

$$(M_{\geq j})_i = \begin{cases} M_i & i \geq j \\ 0 & \text{else} \end{cases}$$

and a graded factor module of M by $M_{\leq j} = M/M_{\geq j+1}$.

Let $\mathsf{mod}^{\mathbb{Z}}\Pi_w$ be the category of finitely generated $\mathbb{Z}$-graded Π_w-modules with degree zero morphisms. We denote by $\mathsf{Sub}^{\mathbb{Z}}\Pi_w$ the full subcategory of $\mathsf{mod}^{\mathbb{Z}}\Pi_w$ of submodules of graded free Π_w-modules, that is,

$$\mathsf{Sub}^{\mathbb{Z}}\Pi_w = \left\{ X \in \mathsf{mod}^{\mathbb{Z}}\Pi_w \mid X \subset \bigoplus_{j=1}^{m} \Pi_w(i_j),\ i_j \in \mathbb{Z} \right\}.$$

By Proposition 3 (a), $\mathsf{Sub}^{\mathbb{Z}}\Pi_w$ is a Frobenius category. Then we have a triangulated category $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w$. In this paper, we get a tilting object in this category.

3. c-sortable words and grading

In this section, we define a c-sortable words of the Coxeter group of Q and calculate the graded structure of Π_w.

Definition 5. Let Q be a finite acyclic quiver with vertices $Q_0 = \{1, 2, \dots, n\}$ and W be the Coxeter group of Q.

(1) An element c in W is called a *Coxeter element* if c has an expression $c = s_{i_1}s_{i_2}\dots s_{i_n}$, where $i_1, \dots, i_n$ is a permutation of $1, \dots, n$.
(2) A Coxeter element $c = s_{i_1}s_{i_2}\dots s_{i_n}$ in W is said to be *admissible with respect to the orientation of* Q if c satisfies $e_{i_j}(kQ)e_{i_k} = 0$ for $k < j$.

Since Q is acyclic, W has a Coxeter element c admissible with respect to the orientation of Q. There are some expression of $c = s_{i_1}s_{i_2}\dots s_{i_n}$ satisfying $\{i_1, \dots, i_n\} = \{1, \dots, n\}$ and $e_{i_j}(kQ)e_{i_k} = 0$ for $k < j$. However, it is shown that c is uniquely determined as an element of W. From now on, we call a Coxeter element admissible with respect to the orientation of Q simply a Coxeter element.

Then we define a c-sortable words.

Definition 6. Let c be a Coxeter element of W. An element $w \in W$ is said to be *c-sortable* if there is a reduced expression $w = c^{(0)}c^{(1)} \cdots c^{(l)}$, where each $c^{(i)}$ is subsequence of c and

$$\mathsf{Supp}(c^{(l)}) \subset \mathsf{Supp}(c^{(l-1)}) \subset \cdots \subset \mathsf{Supp}(c^{(0)}) \subset Q_0,$$

where $\mathsf{Supp}(c^{(i)})$ is the set of i_j such that s_{i_j} appears in $c^{(i)}$.

Example 7. Let $Q = \begin{tikzcd}[row sep=small, column sep=small] & 1 \arrow[dl] \arrow[dr] & \\ 2 \arrow[rr] & & 3 \end{tikzcd}$. A Coxeter element is $c = s_3s_2s_1$. Then an element $w = s_3s_2s_1s_3s_2s_3$ is a c-sortable element. Actually, $c^{(0)} = s_3s_2s_1$, $c^{(1)} = s_3s_2$, and $c^{(2)} = s_3$.

If $w = c^{(0)}c^{(1)} \cdots c^{(l)}$ is a c-sortable element, then the grading of Π_w is calculated as follows.

Proposition 8. *Let $w = c^{(0)}c^{(1)} \cdots c^{(l)} \in W$ be a c-sortable element. If $i \leq l$, then we have $(\Pi_w)_{\leq i} = (\Pi_{c^{(0)}c^{(1)}\cdots c^{(i)}})_{\leq i} = \Pi_{c^{(0)}c^{(1)}\cdots c^{(i)}}$. If $i > l$, then we have $(\Pi_w)_{\geq i} = 0$.*

4. Main theorem

In this section, we state the main theorem of this paper. Let $\mathcal{T}$ be a triangulated category. Recall that, an object M in $\mathcal{T}$ is called a *tilting object* if following holds.

- $\mathsf{Hom}_{\mathcal{T}}(M, M[j]) = 0$ for any $j \neq 0$,
- $\mathsf{thick}M = \mathcal{T}$, where $\mathsf{thick}M$ is the smallest triangulated full subcategory of $\mathcal{T}$ containing M and closed under direct summands.

Let $\mathcal{T}$ be the stable category of a Frobenius category, and assume that $\mathcal{T}$ is Krull-Schmidt. If there is a tilting object M in $\mathcal{T}$, then it follows from [8, (4.3)] that we have a triangle equivalence

$$\mathcal{T} \simeq \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\underline{\mathsf{End}}_{\mathcal{T}}(M)),$$

where $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\mathsf{End}_{\mathcal{T}}(M))$ is the homotopy category of bounded complexes of projective $\mathsf{End}_{\mathcal{T}}(M)$-modules.

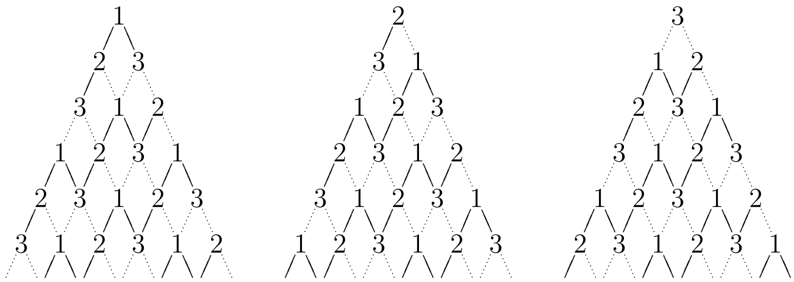
Theorem 9. *Let $w = s_{i_1}s_{i_2} \cdots s_{i_l}$ be a c-sortable element. For an integer $1 \leq j \leq l$, let m_j be the number of integers $1 \leq k \leq j-1$ satisfying $i_j = i_k$. Then $M = \bigoplus_{j=1}^{l} \Pi/I_{s_{i_1}\cdots s_{i_j}} e_{i_j}(m_j)$ is a tilting object in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w$.*

Actually, the module $M = \bigoplus_{j=1}^{l} \Pi/I_{s_{i_1}\cdots s_{i_j}} e_{i_j}(m_j)$ belongs to $\mathsf{Sub}^{\mathbb{Z}}\Pi_w$, since M corresponds to the cluster tilting object of $\underline{\mathsf{Sub}}\,\Pi_w$ of Proposition 3 (d) by forgetting the grading.

The first condition of tilting objects follows from Proposition 8 and calculating a projective resolution of M. The second condition of tilting objects follows from the following Theorem which is shown in [2]. For a c-sortable element $w = s_{i_1}s_{i_2} \cdots s_{i_l}$ and $i \in \mathsf{Supp}(w)$, let t_i be the number of integers $1 \leq k \leq l$ satisfying $i_k = i$.

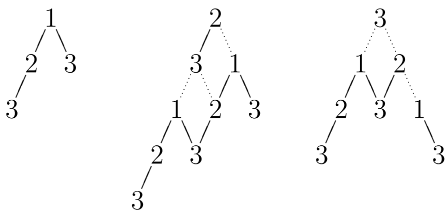
Theorem 10. [2, Theorem 3.11] *Let $w = s_{i_1}s_{i_2} \cdots s_{i_l}$ be a c-sortable element. Then $\bigoplus_{i \in Q'_0} (\Pi_w e_i)_{t_i - 1}$ is a tilting kQ'-module, where Q' is the full subquiver of Q such that $Q'_0 = \mathsf{Supp}(w)$.*

Example 11. Let Q be a quiver $\begin{tikzcd}[column sep=small] & 1 \arrow[dl] \arrow[dr] & \\ 2 \arrow[rr] & & 3 \end{tikzcd}$. Then we have a graded algebra $\Pi = \Pi e_1 \oplus \Pi e_2 \oplus \Pi e_3$, and these are represented by their radical filtrations

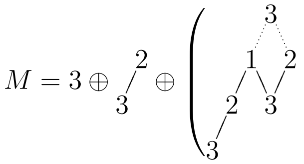


where numbers connected by solid lines are in the same degree, and the tops of the Πe_i are concentrated in degree 0.

Let $w = s_3 s_2 s_1 s_3 s_2 s_3$, then we have a graded algebra $\Pi_w = \Pi_w e_1 \oplus \Pi_w e_2 \oplus \Pi_w e_3$,



and a tilting module



(1)

in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w$, where graded projective Π_w-modules are removed. The endomorphism algebra $\underline{\mathsf{End}}^{\mathbb{Z}}_{\Pi_w}(M)$ of M is given by the following quiver with relations

$$\Delta = \bullet \xleftarrow{b} \bullet \xleftarrow{a} \bullet \qquad ba = 0.$$

Since the algebra $k\Delta/\langle ba \rangle$ has global dimension two, we have a triangle equivalence

$$\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w \simeq \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, k\Delta/\langle ba \rangle) \simeq \mathsf{D}^{\mathrm{b}}(k\Delta/\langle ba \rangle).$$

References

[1] C. Amiot, Cluster categories for algebras of global dimension 2 and quivers with potential, Ann. Inst. Fourier (Grenoble) 59 (2009), no. 6, 2525-2590.

[2] C. Amiot, O. Iyama, I. Reiten, G. Todorov, Preprojective algebras and c-sortable words, Proc. Lond. Math. Soc. (3) 104 (2012), no. 3, 513-539.

[3] C. Amiot, I. Reiten, G. Todorov, The ubiquity of the generalized cluster categories, Adv. Math. 226 (2011), no. 4, 3813-3849.

[4] A. Buan, O. Iyama, I. Reiten, J. Scott, Cluster structures for 2-Calabi-Yau categories and unipotent groups, Compos. Math. 145 (2009), no. 4, 1035-1079.

[5] A. Buan, R. Marsh, I. Reiten, M. Reineke, G. Todorov, Tilting theory and cluster combinatorics, Adv. Math. 204 (2006), no. 2, 572-618.
[6] S. Fomin, A. Zelevinsky, Cluster Algebras I: Foundations, J. Amer. Math. Soc. 15, no. 2 (2002), 497-529.
[7] C. Geiss, B. Leclerc, J. Schröer, Rigid modules over preprojective algebras, Invent. Math. 165 (2006), no. 3, 589-632.
[8] B. Keller, Deriving DG categories, Ann. Sci. Ècole Norm. Sup. (4) 27 (1994), no. 1, 63-102.

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