MUTATION AND MUTATION QUIVERS OF SYMMETRIC SPECIAL BISERIAL ALGEBRAS

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Abstract. The notion of mutation plays crucial roles in representation theory of algebras. Two kinds of mutation are well-known: tilting/silting mutation and quiver-mutation. In this paper, we focus on tilting mutation for symmetric algebras. Introducing mutation of SB quivers, we explicitly give a combinatorial description of tilting mutation of symmetric special biserial algebras. As an application, we generalize Rickard's star theorem. We also introduce flip of Brauer graphs and apply our results to Brauer graph algebras. Moreover we study tilting quivers of symmetric algebras and show that a Brauer graph algebra is tilting-discrete if and only if its Brauer graph is of type odd.

Key Words: special biserial algebra, mutation of SB quivers, tilting mutation, Brauer graph algebra, flip of Brauer graphs.

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1. Introduction

In representation theory of algebras, the notion of mutation plays an important role. We refer to two kinds of mutation: quiver-mutation and tilting/silting mutation. Quiver-mutation was introduced by Fomin-Zelevinsky [FZ] to develop a combinatorial approach to canonical bases of quantum groups, and yields the notion of Fomin-Zelevinsky cluster algebras which has spectacular growth thanks to the many links with a wide range of subjects of mathematics.

Tilting mutation, which is a special case of silting mutation [AI], was introduced by Riedtmann-Schofield [RS] and Happel-Unger [HU] to investigate the structure of the derived category. For example, Bernstein-Gelfand-Ponomarev reflection functors [BGP], Auslander-Platzeck-Reiten tilting modules [APR] and Okuyama-Rickard tilting complexes [O, R2] are special cases of tilting mutation. In the case that a given algebra is symmetric, tilting mutation yields infinitely many tilting complexes, which are extremely important complexes from Morita theoretic viewpoint of derived categories [R1]. It is because they give rise to derived equivalences which preserve many homological properties.

The aim of this paper is to find some similarities between the effects of tilting mutation and Fomin-Zelevinsky quiver-mutations.

The following problem is naturally asked:

Problem 1. Give an explicit description of the endomorphism algebra of a tilting complex given by tilting mutation.

The detailed version of this paper will be submitted for publication elsewhere.

In this paper we give a complete answer to this problem for symmetric special biserial algebras, which is one of the important classes of algebras in representation theory. Some of special biserial algebras were first studied by Gelfand-Ponomarev [GP], and also naturally appear in modular representation theory of finite groups [Al, E]. Moreover such an algebra is always representation-tame and the classification of all indecomposable modules of such an algebra was provided in [WW, BR]. The derived equivalence classes of special biserial algebras were also discussed in [BHS, K, KR].

To realize our goal, we start with describing symmetric special biserial algebras in terms of combinatorial data, which we call *SB quivers*. Moreover we will study symmetric special biserial algebras from graph theoretic viewpoint, which is described by *Brauer graphs*. Indeed, we have the result below (see Lemma 8 and Proposition 26):

Proposition 2. *There exist one-to-one correspondences among the following three classes:*

(1) *Symmetric special biserial algebras;*
(2) *Special quivers with cycle-decomposition (*SB quivers*);*
(3) *Brauer graphs.*

We introduce *mutation of SB quivers* (see Definition 15, 18 and 21), which is similar to Fomin-Zelevinsky quiver-mutation. Moreover we will show that mutation of SB quivers corresponds to a certain operation on Brauer graphs, which we call *flip* and is a generalization of mutation/flip of Brauer trees introduced in [A].

The main theorem of this paper is the following:

Theorem 3 (Theorem 14 and Theorem 29)**.** *The following three operations are compatible each other:*

(1) *Tilting mutation of symmetric special biserial algebras;*
(2) *Mutation of SB quivers;*
(3) *Flip of Brauer graphs.*

We note that certain special cases of the compatibility of (1) and (3) in Theorem 3 were given by [K, An] (see Remark 30).

As an application of Theorem 3, we generalize "Rickard's star theorem" for Brauer tree algebras, which gives nice representatives of Brauer tree algebras up to derived equivalence [R2, M]. We introduce *Brauer double-star algebras*, as the corresponding class for Brauer tree algebras, and prove the following (see Section 4.3 for the details):

Theorem 4 (Theorem 31)**.** *Any Brauer graph algebra is derived equivalent to a Brauer double-star algebra whose Brauer graph has the same number of the edges and the same multiplicities of the vertices.*

As an application of Theorem 4, we deduce Rickard's star theorem (Corollary 33).

Finally we study tilting quivers which were introduced in [AI] to observe the behavior of tilting mutation. We are interested in the connectedness of tilting quivers. A symmetric algebra is said to be *tilting-connected* if its tilting quiver is connected. It was proved in [AI, A1] that a symmetric algebra is tilting-connected if it is either local or representation-finite. On the other hand, an example of symmetric algebras which are not tilting-connected was found by Grant, Iyama and the author. In this paper, we discuss the tilting-connectedness of Brauer graph algebras and aim to understand when a Brauer graph algebra is tilting-connected. We introduce Brauer graphs of *type odd*, and have the main theorem (see Section 5 for the details).

Theorem 5 (Theorem 39)**.** *Any Brauer graph algebra with a Brauer graph of type odd is tilting-connected.*

2. Symmetric special biserial algebras

This section is devoted to introducing the notion of SB quivers. We will give a relationship between symmetric special biserial algebras and SB quivers. Moreover we study tilting mutation, which is a special case of silting mutation introduced by [AI].

Throughout this paper, we use the following notation.

Notation. Let A be a finite dimensional algebra over an algebraically closed field k.

(1) We always assume that A is basic and indecomposable.
(2) We often write $A = kQ/I$ where Q is a finite quiver with relations I. The sets of vertices and arrows of Q are denoted by Q_0 and Q_1, respectively.
(3) We denote by $\operatorname{mod} A$ the category of finitely generated right A-modules. A simple (respectively, indecomposable projective) A-module corresponding to a vertex i of Q is denoted by S_i (respectively, by P_i). We always mean that a module is finitely generated.

A quiver of the form $\bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet$ with n arrows is called an *n-cycle* (for simplicity, *cycle*). We mean 1-cycle by *loop*.

Let us start with introducing SB quivers.

Definition 6. We say that a finite connected quiver Q is *special* if any vertex i of Q is the starting point of at most two arrows and also the end point of at most two arrows. For a special quiver Q with at least one arrow, a set $\mathsf{C} = \{C_1, C_2, \cdots, C_v\}$ of cycles in Q with a function $\mathsf{mult} : \mathsf{C} \to \mathbb{N}$ is said to be a *cycle-decomposition* if it satisfies the following conditions:

(1) Each C_ℓ is a subquiver of Q with at least one arrow such that $Q_0 = (C_1)_0 \cup \cdots \cup (C_v)_0$ and $Q_1 = (C_1)_1 \amalg \cdots \amalg (C_v)_1$: For any $\alpha \in Q_1$, we denote by C_α a unique cycle in C which contains α.
(2) Any vertex of Q belongs to at most two cycles.
(3) $\mathsf{mult}(C_\ell) > 1$ if C_ℓ is a loop.

A *SB quiver* is a pair (Q, C) of a special quiver Q and its cycle-decomposition C.

Let (Q, C) be a SB quiver. For each cycle C in C, we call $\mathsf{mult}(C)$ the *multiplicity* of C.

For any arrow α of Q, we denote by $\mathsf{na}(\alpha)$ a unique arrow β such that $\bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet$ appears in C_α.

We construct a finite dimensional algebra from a SB quiver.

Definition 7. Let (Q, C) be a SB quiver. An ideal $I_{(Q,\mathsf{C})}$ of kQ is generated by the following three kinds of elements:

(1) $(\alpha_t\alpha_{t+1}\cdots\alpha_{t+s-1})^m\alpha_t$ for each cycle C in C of the form

$$\begin{tikzcd} i_1 \arrow[r, "\alpha_1"] & i_2 \arrow[r, "\alpha_2"] & \cdots \arrow[r, "\alpha_{s-1}"] & i_s \arrow[lll, bend left, "\alpha_s"] \end{tikzcd}$$

and $t = 1, 2, \cdots, s$, where $m = \mathsf{mult}(C)$ and the indices are considered in modulo s.

(2) $\alpha\beta$ if $\beta \neq \mathsf{na}(\alpha)$.

(3) $(\alpha_1\alpha_2\cdots\alpha_s)^m - (\beta_1\beta_2\cdots\beta_t)^{m'}$ whenever we have a diagram

$$\begin{tikzcd} \cdots \arrow[r] & i_s \arrow[dr, "\alpha_s"] & & i'_t \arrow[dl, "\beta_t"'] & \cdots \arrow[l] \\ & & i \arrow[dl, "\alpha_1"'] \arrow[dr, "\beta_1"] & & \\ \cdots & i_2 \arrow[l, "\alpha_2"'] & & i'_2 \arrow[r, "\beta_2"] & \cdots \end{tikzcd}$$

where $C_{\alpha_\ell} = C_{\alpha_1}, C_{\beta_{\ell'}} = C_{\beta_1}$ for any $1 \leq \ell \leq s, 1 \leq \ell' \leq t$ and $m = \mathsf{mult}(C_{\alpha_1}), m' = \mathsf{mult}(C_{\beta_1})$.

We define a k-algebra $A := A_{(Q,\mathsf{C})}$ associated with (Q, C) by $A = kQ/I_{(Q,\mathsf{C})}$. Then the algebra $A_{(Q,\mathsf{C})}$ is finite dimensional and symmetric. The cycle-decomposition C is also said to be the *cycle-decomposition* of $A_{(Q,\mathsf{C})}$.

An algebra $A := kQ/I$ is said to be *special biserial* if Q is special and for any arrow β of Q, there is at most one arrow α with $\alpha\beta \notin I$ and at most one arrow γ with $\beta\gamma \notin I$.

Thanks to [Ro] (see Proposition 26), we have the following result.

Lemma 8. *The assignment $(Q, \mathsf{C}) \mapsto A_{(Q,\mathsf{C})}$ gives rise to a bijection between the isoclasses of SB quivers and those of symmetric special biserial algebras.*

Example 9. (1) Let Q be the quiver

$$\begin{tikzcd} & 1 \arrow[dl, shift left, "\alpha"] \arrow[dr, shift left, "\gamma'"] & \\ 2 \arrow[ur, shift left, "\alpha'"] \arrow[rr, shift left, "\beta"] & & 3 \arrow[ul, shift left, "\gamma"] \arrow[ll, shift left, "\beta'"] \end{tikzcd}$$

with the relations $I := \langle \alpha\beta, \beta\gamma, \gamma\alpha, \alpha'\gamma', \gamma'\beta', \beta'\alpha', \alpha'\alpha - \beta\beta', \beta'\beta - \gamma\gamma', \gamma'\gamma - \alpha\alpha' \rangle$. Then the algebra $A := kQ/I$ is symmetric special biserial associated with the SB quiver (Q, C) where the cycle-decomposition is

$$\mathsf{C} = \left\{ \left(\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] & 2 \arrow[l, shift left, "\alpha'"] \end{tikzcd} \right), \left(\begin{tikzcd} 2 \arrow[r, shift left, "\beta"] & 3 \arrow[l, shift left, "\beta'"] \end{tikzcd} \right), \left(\begin{tikzcd} 3 \arrow[r, shift left, "\gamma"] & 1 \arrow[l, shift left, "\gamma'"] \end{tikzcd} \right) \right\}$$

such that the multiplicity of every cycle is 1.

(2) Let Q be the quiver

$$
\begin{tikzcd}
 & 1 \arrow[loop above, "\delta"] \arrow[dl, "\alpha"'] & \\
2 \arrow[rr, "\beta"'] & & 3 \arrow[ul, "\gamma"']
\end{tikzcd}
$$

with the relations $I := \langle \gamma\alpha, (abcd)^2 a \mid \{a, b, c, d\} = \{\alpha, \beta, \gamma, \delta\} \rangle$. Then $A := kQ/I$ is a symmetric special biserial algebra which is isomorphic to $A_{(Q,\mathsf{C})}$, where C is the cycle-decomposition

$$
\mathsf{C} = \left\{ \left(\begin{tikzcd}
1 \arrow[r, "\alpha"] & 2 \arrow[d, "\beta"] \\
1 \arrow[u, "\delta"] & 3 \arrow[l, "\gamma"]
\end{tikzcd} \right) \right\}
$$

with the multiplicity 2.

(3) Let Q be the quiver

$$
\begin{tikzcd}
 & 1 \arrow[d, shift right, "\alpha"'] \arrow[d, shift left, "\alpha'"] & \\
3 \arrow[ur, "\gamma"] & 2 \arrow[l, "\beta"] \arrow[r, "\beta'"'] & 4 \arrow[ul, "\gamma'"']
\end{tikzcd}
$$

with cycle-decomposition $\mathsf{C} = \{C_1, C_2\}$ where

$$
C_1 = \left(\begin{tikzcd}
 & 1 \arrow[d, "\alpha"] \\
3 \arrow[ur, "\gamma"] & 2 \arrow[l, "\beta"]
\end{tikzcd} \right), C_2 = \left(\begin{tikzcd}
1 \arrow[d, "\alpha'"'] & \\
2 \arrow[r, "\beta'"'] & 4 \arrow[ul, "\gamma'"']
\end{tikzcd} \right)
$$

and $\mathsf{mult}(C_1) = \mathsf{mult}(C_2) = 1$. Then we have an isomorphism $A_{(Q,\mathsf{C})} \simeq kQ/I$ where $I = \langle \alpha\beta', \alpha'\beta, \gamma\alpha', \gamma'\alpha, (abc)a \mid \{a, b, c\} = \{\alpha, \beta, \gamma\}, \{\alpha', \beta', \gamma'\} \rangle$.

We know that the property of being symmetric special biserial is derived invariant.

Proposition 10. *Let A and B be finite dimensional algebras. Suppose that A and B are derived equivalent. If A is a symmetric special biserial algebra, then so is B.*

Proof. Combine [R1] and [P]. □

Next, we recall the notion of tilting mutation. We refer to [AI] for details.
The bounded derived category of $\mathrm{mod}\, A$ is denoted by $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\, A)$.
We give the definition of tilting complexes.

Definition 11. Let A be a finite dimensional algebra. We say that a bounded complex T of finitely generated projective A-modules is *tilting* if it satisfies $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\, A)}(T, T[n]) = 0$ for any integer $n \neq 0$ and produces the complex A concerned in degree 0 by taking direct summands, mapping cones and shifts.

The following result shows the importance of tilting complexes.

Theorem 12. [R1] *Let A and B be finite dimensional algebras. Then A and B are derived equivalent if and only if there exists a tilting complex T of A such that B is Morita equivalent to the endomorphism algebra* $\mathrm{End}_{\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\, A)}(T)$.

For each vertex i of Q, we denote by e_i the corresponding primitive idempotent of A.

We recall a complex given by Okuyama and Rickard [O, R2], which is a special case of tilting mutation (see [AI]).

Definition-Theorem 13. [O] Fix a vertex i of Q. We define a complex by

$$T_j := \begin{cases} \begin{array}{ccc} (0\mathrm{th}) & & (1\mathrm{st}) \\ P_j & \longrightarrow & 0 \\ P & \xrightarrow{\pi_i} & P_i \end{array} & \begin{array}{c} \\ (j \neq i) \\ (j = i) \end{array} \end{cases}$$

where $P \xrightarrow{\pi_i} P_i$ is a minimal projective presentation of $e_iA/e_iA(1-e_i)A$. Now we call $T(i) := \bigoplus_{j \in Q_0} T_j$ an *Okuyama-Rickard complex* with respect to i and put $\mu_i^+(A) := \mathrm{End}_{\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\, A)}(T(i))$. If A is symmetric, then T is tilting. In particular, $\mu_i^+(A)$ is derived equivalent to A.

3. Mutation of SB quivers

The aim of this paper is to give a purely combinatorial description of tilting mutation of symmetric special biserial algebras.

To do this, we introduce *mutation of SB quivers* by dividing to three cases, which is a new SB quiver $\mu_i^+(Q, \mathsf{C})$ made from a given one (Q, C).

Now, the main theorem in this paper is stated, which gives the compatibility between tilting mutation and mutation of SB quivers.

Theorem 14. *Let A be a symmetric special biserial algebra and take a SB quiver (Q, C) satisfying $A \simeq A_{(Q,\mathsf{C})}$. Let i be a vertex of Q. Then we have an isomorphism $A_{\mu_i^+(Q,\mathsf{C})} \simeq \mu_i^+(A)$. In particular, $A_{\mu_i^+(Q,\mathsf{C})}$ is derived equivalent to A.*

Let (Q, C) be a SB quiver and i be a vertex of Q. We say that Q is *multiplex* at i if there exists arrows $i \underset{\beta}{\overset{\alpha}{\rightleftarrows}} j$ with $\beta \neq \mathsf{na}(\alpha)$ and $\alpha \neq \mathsf{na}(\beta)$.

3.1. **Non-multiplex case.**

We introduce mutation of SB quivers at non-multiplex vertices.

Let (Q, C) be a SB quiver and fix a vertex i of Q. We define a new SB quiver $\mu_i^+(Q, \mathsf{C}) = (Q', \mathsf{C}')$ as follows.

3.1.1. *Mutation rules.*

Definition 15. Suppose that Q is non-multiplex at i. We define a quiver Q' as the following three steps:

(QM1) Consider any path

$$h \xrightarrow{\alpha} i \xrightarrow{\beta} j \text{ with } \beta = \mathsf{na}(\alpha) \qquad \text{or} \qquad \begin{tikzcd} h \arrow[r, "\alpha"] & i \arrow[loop above, "\gamma"] \arrow[r, "\beta"] & j \end{tikzcd} \text{ with } \gamma = \mathsf{na}(\alpha), \beta = \mathsf{na}(\gamma)$$

for $h \neq i \neq j$. Then draw a new arrow $h \xrightarrow{x} j$

(QM1-1) if $h \neq j$ or

(QM1-2) if $h = j, \alpha = \mathsf{na}(\beta)$ and $\mathsf{mult}(C_\alpha) > 1$.

(QM2) Remove all arrows $h \longrightarrow i$ for $h \neq i$.

(QM3) Consider any arrow $i \xrightarrow{\alpha} h$ for $h \neq i$.

(QM3-1) If there exists a path $i \xrightarrow{\alpha} h \xrightarrow{\beta} j$ with $\beta \neq \mathsf{na}(\alpha)$, then replace it by a new path $h \xrightarrow{x} i \xrightarrow{y} j$.

(QM3-2) Otherwise, add a new arrow $h \xrightarrow{x} i$.

It is easy to see that the new quiver Q' is again special.

3.1.2. *Cycle-decompositions.* We give a cycle-decomposition C' of Q'.

Definition 16. We use the notation of Definition 15.

(1) We define a cycle containing a new arrow x in (QM1) as follows:
   (a) In the case (QM1-1), C_x is obtained by replacing $\alpha\beta$ or $\alpha\gamma\beta$ in C_α by x.
   (b) In the case (QM1-2), C_x is a new cycle $\begin{tikzcd} h \arrow[loop right, "x"] \end{tikzcd}$ with multiplicity $\mathsf{mult}(C_\alpha)$.

(2) We define a cycle containing a new arrow x and y in (QM3) as follows:
   (a) In the case (QM3-1), $C_x = C_y$ and replace β in C_β by xy.
   (b) In the case (QM3-2),
      (i) if there exists an arrow $h \xrightarrow{\beta} i$ of Q, then C_x is defined by replacing β in C_β by x.
      (ii) Otherwise, C_x is a new cycle

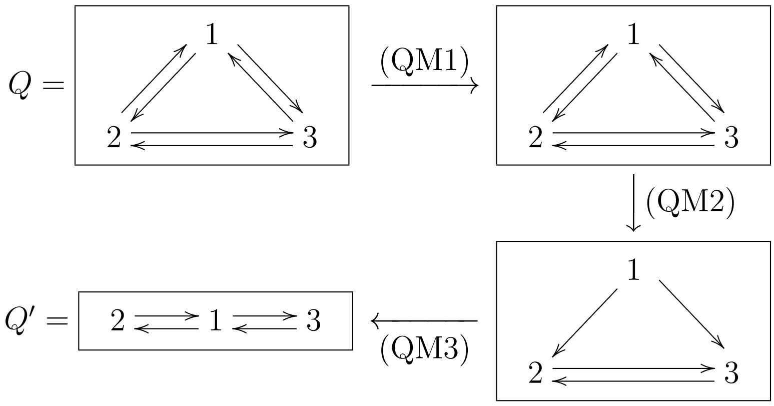
$$\begin{cases} \begin{tikzcd} i \arrow[r, shift left, "\alpha"] & h \arrow[l, shift left, "x"] \end{tikzcd} & \text{if there is no loop at } i \text{ belonging to } C_\alpha \\ \begin{tikzcd} & i \arrow[dr, "\alpha"] & \\ i \arrow[ur, "\beta"] & & h \arrow[ll, "x"] \end{tikzcd} & \text{if there is a loop } \beta \text{ at } i \text{ belonging to } C_\alpha \end{cases}$$

with multiplicity 1.

Then we obtain a cycle-decomposition C' of Q'.

Thus, we get a new SB quiver $\mu_i^+(Q, \mathsf{C}) := (Q', \mathsf{C}')$, called *right mutation* of (Q, C) at i. Dually, we define the *left mutation* $\mu_i^-(Q, \mathsf{C})$ of (Q, C) at i by $\mu_i^-(Q, \mathsf{C}) := \mu_i^+(Q^{\mathrm{op}}, \mathsf{C}^{\mathrm{op}})^{\mathrm{op}}$, where Q^{op} is the opposite quiver of Q and C^{op} is the cycle-decomposition of Q^{op} corresponding to C.

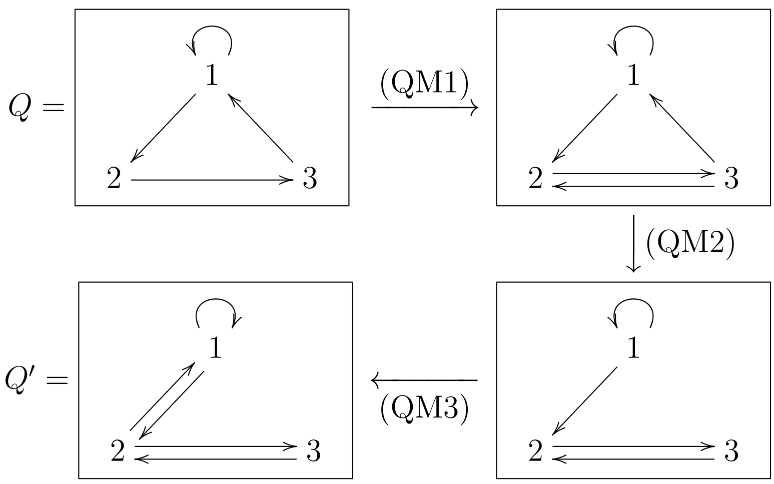
Example 17. (1) Let (Q, C) be the SB quiver as in Example 9 (1). Then we have the right mutation $\mu_1^+(Q, \mathsf{C}) = (Q', \mathsf{C}')$ of Q at 1 as follows:



and

$$\mathsf{C}' = \left\{ \left(\begin{tikzcd} 1 \arrow[r] & 2 \arrow[d] \\ 3 \arrow[u] & 1 \arrow[l] \end{tikzcd} \right) \right\}$$

(2) Let (Q, C) be the SB quiver of Example 9 (2). Then the right mutation $\mu_1^+(Q, \mathsf{C}) = (Q', \mathsf{C}')$ of Q at 1 is obtained as follows:

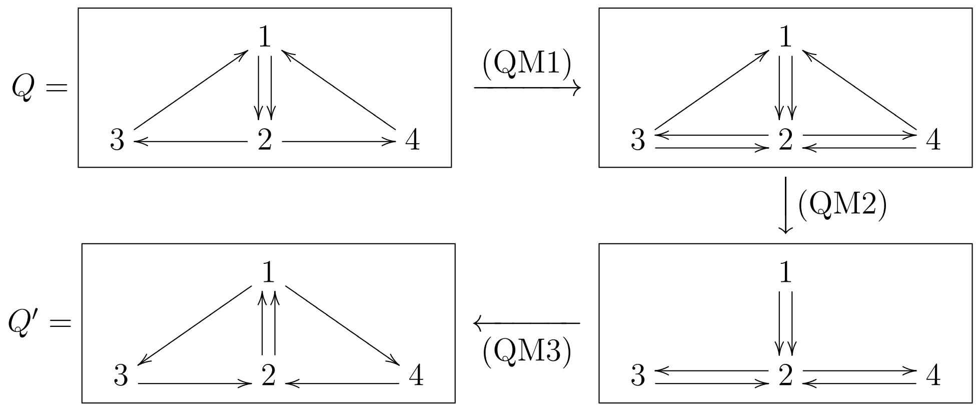


and

$$\mathsf{C}' = \left\{ \left(\begin{tikzcd} & 1 \arrow[dr] & \\ 1 \arrow[ur] & & 2 \arrow[ll] \end{tikzcd} \right), \left(\begin{tikzcd} 2 \arrow[r, shift left] & 3 \arrow[l, shift left] \end{tikzcd} \right) \right\}$$

where the first and the second cycles have multiplicity 1 and 2, respectively.

(3) Let (Q, C) be the SB quiver as in Example 9 (3). Then we get the right mutation $\mu_1^+(Q, \mathsf{C}) = (Q', \mathsf{C}')$ of Q at 1 as follows:



and

$$\mathsf{C}' = \left\{ \left(\begin{tikzcd} & 1 \arrow[dl] \\ 3 \arrow[r] & 2 \arrow[u] \end{tikzcd} \right), \left(\begin{tikzcd} 1 \arrow[dr] & \\ 2 \arrow[u] & 4 \arrow[l] \end{tikzcd} \right) \right\}$$

3.2. **Multiplex case (1).** Next, we introduce mutation at multiplex vertices and its cycle-decomposition. They are defined by making minor alterations to mutation at non-multiplex vertices.

Let (Q, C) be a SB quiver and fix a vertex i of Q. We consider the following situation:

$$\begin{tikzcd} j' \arrow[dr, "\alpha'"] & & \\ & i \arrow[dl, "\beta'"] \arrow[r, shift left, "\alpha"] & j \arrow[l, shift left, "\beta"] \\ h & & \end{tikzcd}$$

with $\beta \neq \mathsf{na}(\alpha)$ and $\alpha \neq \mathsf{na}(\beta)$: in this case, it is observed that $\alpha = \mathsf{na}(\alpha')$ and $\beta' = \mathsf{na}(\beta)$.

We define a new SB quiver $\mu_i^+(Q, \mathsf{C}) = (Q', \mathsf{C}')$ as follows.

3.2.1. *Mutation rules.*

Definition 18. We assume that $j' \neq h$. A quiver Q' of Q at i is defined by the following three steps:

(QM1)' Draw a new arrow $j' \xrightarrow{x} j$

(QM1-1)' if $j' \neq j$ or

(QM1-2)' if $j' = j$ and $\mathsf{mult}(C_\alpha) > 1$.

(QM2)' Remove two arrows α and α'.

(QM3)' Add new arrows in the following way:

(QM3-1)' If there is an arrow $\gamma : h \longrightarrow h'$ with $\gamma \neq \mathsf{na}(\beta')$, then remove γ and add new arrows $h \xrightarrow{x} i \xrightarrow{y} h'$.

(QM3-2)' Otherwise, add new arrows $\begin{tikzcd} h \arrow[r, shift left, "x"] & i \arrow[l, shift left, "y"] \end{tikzcd}$.

We can easily check that the new quiver Q' is again special.

3.2.2. *Cycle-decompositions.* We give a cycle-decomposition C' of Q'.

Definition 19. Assume that $j' \neq h$. We use the notation of Definition 18.

(1) We define a cycle containing a new arrow x in (QM1)' as follows:
   (a) In the case (QM1-1)', C_x is obtained by replacing $\alpha'\alpha$ in C_α by x.
   (b) In the case (QM1-2)', C_x is a new cycle $\begin{tikzcd} j \arrow[loop right, "x"] \end{tikzcd}$ with multiplicity $\mathsf{mult}(C_\alpha)$.

(2) We define a cycle containing a new arrow x and y in (QM3)' as follows:
   (a) In the case (QM3-1)', $C_x = C_y$ and replace γ in C_γ by xy.
   (b) In the case (QM3-2)', C_x and C_y are new cycles satisfying $C_x = C_y = \left(\begin{tikzcd} h \arrow[r, shift left, "x"] & i \arrow[l, shift left, "y"] \end{tikzcd} \right)$ with multiplicity 1.

Then we have a cycle-decomposition C' of Q'.

Thus, we get a new SB quiver $\mu_i^+(Q, \mathsf{C}) = (Q', \mathsf{C}')$, called *right mutation* of (Q, C) at i. Dually, we define the *left mutation* $\mu_i^-(Q, \mathsf{C})$ of (Q, C) at i by $\mu_i^-(Q, \mathsf{C}) := \mu_i^+(Q^{\mathrm{op}}, \mathsf{C}^{\mathrm{op}})^{\mathrm{op}}$.

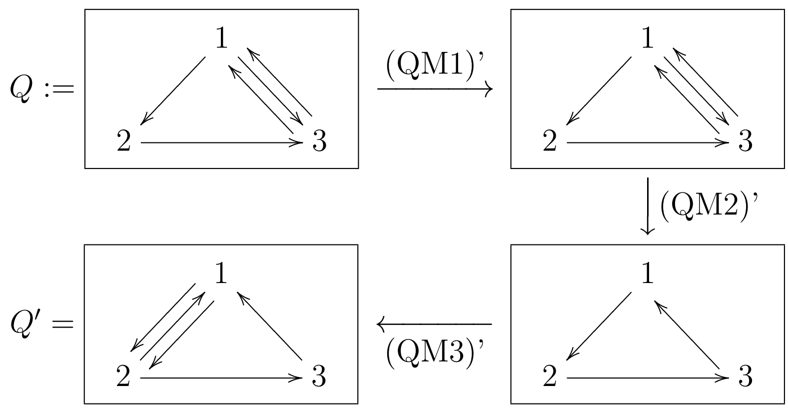
Example 20. Let Q be the quiver

$$\begin{tikzcd} & 1 \arrow[dl, "\alpha_1"'] \arrow[dr, shift left, "\beta"] & \\ 2 \arrow[rr, "\alpha_2"'] & & 3 \arrow[ul, "\alpha_3"] \arrow[ul, shift right=3, "\beta'"'] \end{tikzcd}$$

with cycle-decomposition C:

$$Q = \left(\begin{tikzcd} & 1 \arrow[dl, "\alpha_1"'] & \\ 2 \arrow[rr, "\alpha_2"'] & & 3 \arrow[ul, "\alpha_3"'] \end{tikzcd} \right) \cup \left(\begin{tikzcd} 1 \arrow[dr, shift right, "\beta"'] & \\ & 3 \arrow[ul, shift right, "\beta'"'] \end{tikzcd} \right)$$

such that the multiplicity of each cycle is 1. Then we see that the right mutation $\mu_1^+(Q, \mathsf{C}) = (Q', \mathsf{C}')$ of Q at 1 is



and

$$\mathsf{C}' = \left\{ \left(\begin{tikzcd} & 1 \arrow[dl] & \\ 2 \arrow[rr] & & 3 \arrow[ul] \end{tikzcd} \right), \left(\begin{tikzcd} & 1 \arrow[dl, shift left] \\ 2 \arrow[ur, shift left] & \end{tikzcd} \right) \right\}$$

3.3. **Multiplex case (2).** Finally, we introduce the last case of mutation of SB quivers.

Let (Q, C) be a SB quiver and fix a vertex i of Q. Suppose that the subquiver of Q around the vertex i is

$$\begin{tikzcd} j' \arrow[r, shift left, "\alpha'"] & i \arrow[l, shift left, "\beta'"] \arrow[r, shift left, "\alpha"] & j \arrow[l, shift left, "\beta"] \end{tikzcd}$$

with $\beta \neq \mathsf{na}(\alpha)$ and $\alpha \neq \mathsf{na}(\beta)$: i.e., the case of $j' = h$ in Multiplex case (1).

Definition 21. We define the *right mutation* $\mu_i^+(Q, \mathsf{C})$ of (Q, C) at i by $\mu_i^+(Q, \mathsf{C}) = (Q, \mathsf{C})$.

Dually, the *left mutation* $\mu_i^-(Q, \mathsf{C})$ of (Q, C) at i is also defined by $\mu_i^-(Q, \mathsf{C}) = (Q, \mathsf{C})$.

4. Brauer graph algebras

In this section, we introduce *flip* of Brauer graphs and show that it is compatible with tilting mutation of Brauer graph algebras. Note that any Brauer graph algebra is symmetric special biserial (see Proposition 26).

We recall the definition of Brauer graphs.

Definition 22. A *Brauer graph* G is a graph with the following data:

(1) There exists a cyclic ordering of the edges adjacent to each vertex, usually described by the clockwise ordering.
(2) For every vertex v, there exists a positive integer m_v assigned to v, called the *multiplicity* of v. We say that a vertex v is *exceptional* if $m_v > 1$

4.1. **Flip of Brauer graphs.** Let G be a Brauer graph. For a cyclic ordering $(\cdots, i, j, \cdots)$ adjacent to a vertex v with $j \neq i$, we write j by $e_v(i)$ and denote by $v_j(i)$ (simply, $v(i)$) the vertex of j distinct from v if it exists, otherwise $v(i) := v$.

We say that an edge i of G is *external* if it has a vertex with cyclic ordering which consists of only i, otherwise it is said to be *internal*.

We now introduce flip of Brauer graphs.

Definition 23. Let G be a Brauer graph and fix an edge i of G. We define the *flip* $\mu_i^+(G)$ of G as follows:

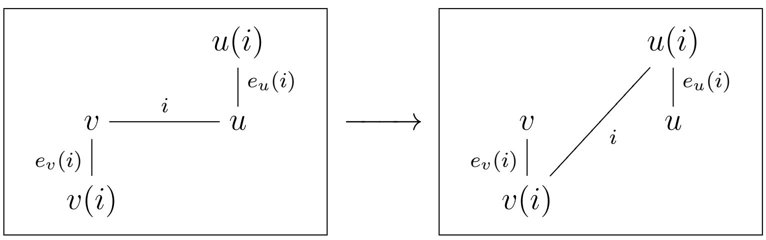
Case (1) The edge i has the distinct two vertices v and u:

- If i is internal, then

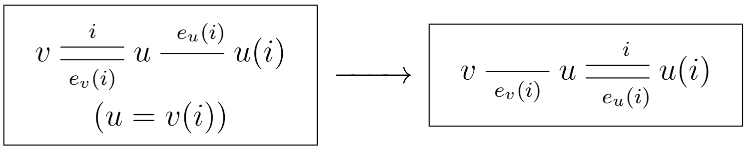
(Step 1) detach i from v and u;
(Step 2) attach it to $v(i)$ and $u(i)$ by $e_{v(i)}(e_v(i)) = i$ and $e_{u(i)}(e_u(i)) = i$, respectively.

Locally there are the following three cases:

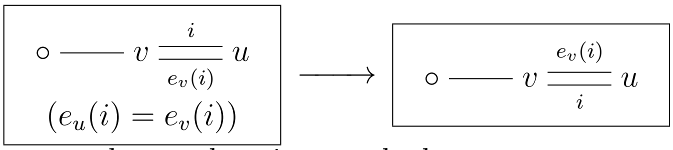
(i)



(ii)



(iii)

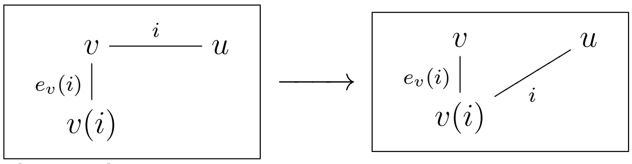


- If i is external, namely u is at end, then

(Step 1) detach i from v;
(Step 2) attach it to $v(i)$ by $e_{v(i)}(e_v(i)) = i$.

The local picture is the following:

(iv)



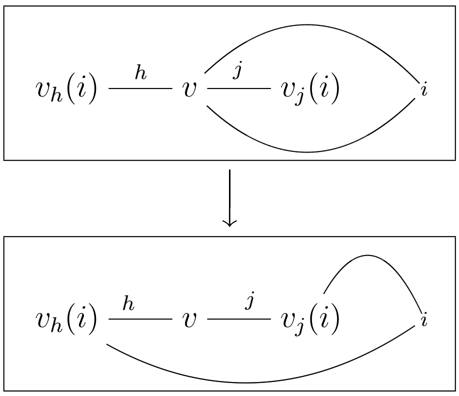
Case (2) The edge i has only one vertex v:

- If there exists the distinct two edges h and j written by $e_v(i)$, then

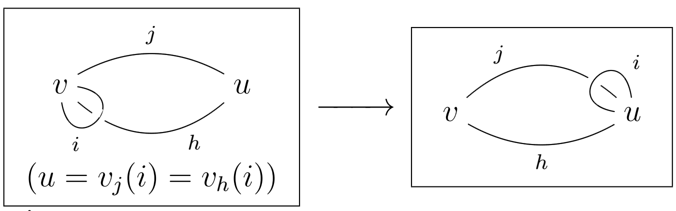
(Step 1) detach i from v;
(Step 2) attach it to $v_h(i)$ and $v_j(i)$ by $e_{v_h(i)}(h) = i$ and $e_{v_j(i)}(j) = i$.

Locally there are the following two cases:

(v)



(vi)

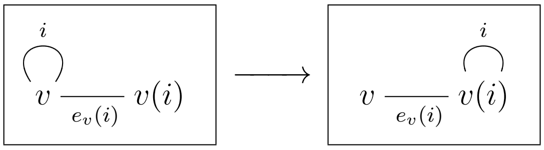


- Otherwise,

(Step 1) detach i from v;
(Step 2) attach it to the only one vertex $v(i)$ by $e_{v(i)}(e_v(i)) = i$.

The local picture is the following:

(vii)



In all cases, the multiplicity of any vertex does not change.

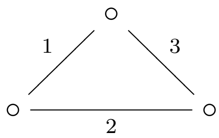
Dually, we define $\mu_i^-(G)$ by $\mu_i^-(G) := \left(\mu_i^+(G^{\rm op})\right)^{\rm op}$ where the opposite Brauer graph, namely its cyclic ordering is described by counter-clockwise, is denoted by $G^{\rm op}$.

Every case of flip of Brauer graphs is covered in Definition 23.

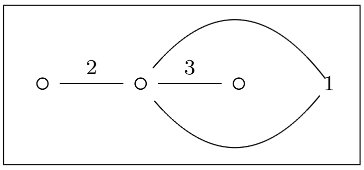
We also point out that our flip of Brauer graphs can be regarded as a generalization of flip of triangulations of surfaces [FST].

Example 24. For a Brauer graph, we denote by $\bullet$ an exceptional vertex and by $\circ$ a non-exceptional vertex.

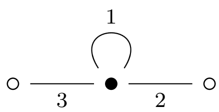
(1) Let G be the Brauer graph



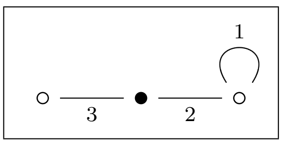
Then the flip of G at 1 is

$\mu_1^+(G) =$ 

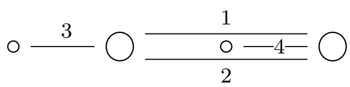
(2) Let G be the Brauer graph



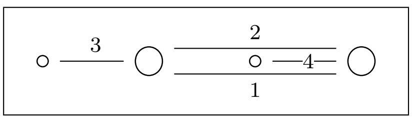
such that the multiplicity of the exceptional vertex $\bullet$ is 2. Then we have the flip of G at 1:

$\mu_1^+(G) =$ 

(3) Let G be the Brauer graph



Then the flip of G at 1 is observed by

$\mu_1^+(G) =$ 

(4) Let G be the Brauer graph

$$\begin{tikzcd} \circ \arrow[r, dash, shift left, "1"] \arrow[r, dash, shift right, "3"'] & \circ \arrow[r, dash, "2"] & \circ \end{tikzcd}$$

Then the flip of G at 1 is:

$$\mu_1^+(G) = \boxed{\begin{tikzcd} \circ \arrow[r, dash, "3"] & \circ \arrow[r, dash, shift left, "1"] \arrow[r, dash, shift right, "2"'] & \circ \end{tikzcd}}$$

4.2. **Compatibility of flip and tilting mutation.** For a Brauer graph G, we denote by $\mathsf{vx}(G)$ the set of the vertices of G.

We construct a SB quiver from a Brauer graph.

Definition 25. Let G be a Brauer graph. A *Brauer quiver* $Q = Q_G$ is a finite quiver given by a Brauer graph G as follows:

(1) There exists a one-to-one correspondence between vertices of Q and edges of G.
(2) For two distinct edges i and j, an arrow $i \longrightarrow j$ of Q is drawn if there exists a cyclic ordering of the form $(\cdots, i, j, \cdots)$.
(3) For an edge i of G, we draw a loop at i if it has an exceptional vertex which is at end.

Then Q is special.

For each vertex v of G, let $(i_1, i_2, \cdots, i_s, i_1)$ be a cyclic ordering at v. Then we define a cycle C_v by

$$\begin{tikzcd} i_1 \arrow[r] & i_2 \arrow[r] & \cdots \arrow[r] & i_s \arrow[lll, bend right] \end{tikzcd}$$

with multiplicity m_v if $s \neq 1$, otherwise by an empty set. We have a cycle-decomposition $\mathsf{C} = \mathsf{C}_G = \{C_v \mid v \in \mathsf{vx}(G)\}$.

Thus we obtain a SB quiver (Q, C).

For a Brauer graph G, a *Brauer graph algebra* $A = A_G$ is a symmetric special biserial algebra associated with the SB quiver (Q_G, C_G).

It is known that the notion of Brauer graph algebras is nothing but that of symmetric special biserial algebras. The following result is obtained.

Proposition 26. (1) [Ro] *An algebra is a Brauer graph algebra if and only if it is symmetric special biserial.*
(2) *The property of being a Brauer graph algebra is derived invariant.*

Proof. The second assertion follows from the first assertion and Proposition 10. $\square$

For an edge i of a Brauer graph G, we say that G *has multi-edges* at i if there exists a subgraph $\overset{v}{\circ} \overset{i}{\underset{j}{=\!=}} \overset{u}{\circ}$ of G such that the cyclic orderings at v and u are $(\cdots, i, j, \cdots)$ and $(\cdots, j, i, \cdots)$, respectively; it is allowed that $u = v$.

We have the following easy observation.

Proposition 27. *Let G be a Brauer graph and i be an edge of G. Then Q_G is multiplex at i if and only if G has multi-edges at i.*

It is not difficult to see that flip of each Brauer graph G coincides with right mutation of the corresponding SB quiver (Q_G, C_G), that is, we have:

Proposition 28. *Let G be a Brauer graph and i be an edge of G. Then one has $(Q_{\mu_i^+(G)}, \mathsf{C}_{\mu_i^+(G)}) = \mu_i^+(Q_G, \mathsf{C}_G)$.*

We observe that Example 17 (1)–(3) and Example 20 coincide with Example 24 (1)–(3) and (4), respectively.

Applying Theorem 14 to Brauer graph algebras, we figure out that flip of Brauer graph is compatible with tilting mutation of Brauer graph algebras.

Theorem 29. *Let G be a Brauer graph and i be an edge of G.*

(1) *We have an isomorphism $A_{\mu_i^+(G)} \simeq \mu_i^+(A_G)$.*
(2) *The algebra $\mu_i^+(A_G)$ has the Brauer graph $\mu_i^+(G)$.*
(3) *The algebra $A_{\mu_i^+(G)}$ is derived equivalent to A_G.*

Remark 30. Special cases of this theorem were given in [K], where he considered the cases (i)(iv) and (vii) in Definition 23.

4.3. **(Double-) Star theorem.** In this subsection, we generalize Rickard's star theorem.

Let G be a Brauer graph. We denote by m_G the sequence $(m_{v_1}, \cdots, m_{v_\ell})$ of the multiplicities of all vertices satisfying $m_{v_1} \geq \cdots \geq m_{v_\ell}$.

For a Brauer graph algebra A, the Brauer graph of A is denoted by G_A.

A Brauer graph G is said to be *double-star* if there exist two vertices v and u of G such that any edge is either of the following:

- It is external having the vertex v:
- It has both the vertices v and u:
- It has only the vertex v, that is, it is of the form v.

We call v and u *center* and *vice-center*, respectively.

We say that a Brauer double-star G satisfies *multiplicity condition* if the multiplicities of the center and the vice-center are the first and the second greatest among them of all vertices of G, respectively.

The following theorem is obtained.

Theorem 31. *Any Brauer graph algebra A is derived equivalent to a Brauer graph algebra with Brauer double-star G satisfying multiplicity condition such that*

(1) *the number of the edges of G coincides with that of G_A and*
(2) *$\mathsf{m}_G = \mathsf{m}_{G_A}$, in particular G and G_A have the same number of exceptional vertices.*

We raise a question on classification of derived equivalence classes of Brauer graph algebras.

Question 32. *For a given Brauer graph G, is a Brauer double-star algebra satisfying multiplicity condition which is derived equivalent to A_G unique, up to isomorphism and opposite isomorphism?*

It is well-known that this question has a positive answer if G is a tree as a graph. Such a Brauer graph is said to be a *generalized Brauer tree*. It is called *Brauer tree* if it has at most one exceptional vertex. A *(generalized) Brauer star* is a (generalized) Brauer tree and a Brauer double-star. Note that any edge of a generalized Brauer star is external and every vertex can be a vice-center.

From Theorem 31, we deduce star theorem for generalized Brauer tree algebras.

Corollary 33. [R2, M]

(1) *Any generalized Brauer tree algebra A is derived equivalent to a generalized Brauer star algebra B with $\mathsf{m}_{G_B} = \mathsf{m}_{G_A}$ such that the multiplicity of the center is maximal.*
(2) *Derived equivalence classes of generalized Brauer tree algebras are determined by the number of the edges and the multiplicities of the vertices.*

5. Tilting quivers of Brauer graph algebras

This section is based on joint work with Adachi and Chan [AAC].

In this section, we discuss tilting quivers which were introduced in [AI] to observe the behavior of tilting mutation.

We denote by $\mathsf{tilt}\,\Lambda$ the set of non-isomorphic basic tilting complexes of a finite dimensional algebra Λ.

Let us recall the definition of tilting quivers (see [AI] for the details).

Definition 34. Let Λ be a symmetric algebra. The *tilting quiver* of Λ is defined as follows:

- The set of vertices is $\mathsf{tilt}\,\Lambda$.
- An arrow $T \to U$ is drawn if T corresponds to an Okuyama-Rickard complex of the endomorphism algebra $\mathrm{End}_{\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)}(U)$ under the derived equivalence induced by U.

We naturally ask whether the tilting quiver of Λ is always connected or not. The answer of this question is No. An example of symmetric algebras whose tilting quivers are not connected was found by Grant, Iyama and the author.

A symmetric algebra is said to be *tilting-connected* if its tilting quiver is connected.

We raise the following question.

Question 35. *When is a symmetric algebra tilting-connected?*

We introduce a partial order on $\mathsf{tilt}\,\Lambda$.

Definition-Theorem 36. [AI] Let Λ be a symmetric algebra. For $T, U \in \mathsf{tilt}\,\Lambda$, we write $T \geq U$ if it satisfies $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)}(T, U[i]) = 0$ for any $i > 0$. Then $\geq$ gives a partial order on $\mathsf{tilt}\,\Lambda$.

We say that a symmetric algebra Λ is *tilting-discrete* if for any $\ell > 0$, there exist only finitely many tilting complexes T in $\mathsf{tilt}\,\Lambda$ satisfying $\Lambda \geq T \geq \Lambda[\ell]$. It is seen that a tilting-discrete symmetric algebra is tilting-connected [A1].

Two examples of tilting-discrete symmetric algebras are well-known.

Example 37. [AI, A1] A symmetric algebra is tilting-discrete if it is either local or representation-finite.

We refer to [AI] for more general examples of tilting/silting-connected algebras.

The aim of this section is to give a partial answer to Question 35 for Brauer graph algebras.

We say that a Brauer graph is of *type odd* if it has at most one cycle of odd length and none of even length. For example, any (generalized) Brauer tree is of type odd.

We easily observe the following result, which says that flip of Brauer graphs preserves the property to be of type odd.

Lemma 38. *Let G be a Brauer graph and i be an edge of G. If G is of type odd, then so is $\mu_i^+(G)$.*

Now we state the main theorem of this section.

Theorem 39. *Let G be a Brauer graph. Then G is of type odd if and only if A_G is tilting-discrete.*

Example 40. Any generalized Brauer graph algebra is tilting-discrete, and hence is tilting-connected.

The following corollary is an immediate consequence of Theorem 39.

Corollary 41. *Let G be a Brauer graph of type odd and B a derived equivalent algebra to A_G. Then the following hold:*

(1) *The algebra B is a Brauer graph algebra whose Brauer graph can be obtained from G by iterated flip.*
(2) *The Brauer graph of B is of type odd.*

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