CLASSICAL DERIVED FUNCTORS AS FULLY FAITHFUL EMBEDDINGS

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ABSTRACT. Given associative unital algebras A and B and a complex $T^\bullet$ of $B-A-$bimodules, we give necessary and sufficient conditions for the derived functors $\mathbf{R}\mathrm{Hom}_A(T^\cdot, ?)$: $\mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ and $? \otimes_B^{\mathbf{L}} T^\bullet : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ to be fully faithful. We also give criteria for these functors to be one of the fully faithful functors appearing in a recollement of derived categories. In the case when $T^\bullet$ is just a $B-A-$bimodule, we connect the results with (infinite dimensional) tilting theory and show that some open question on the fully faithfulness of $\mathbf{R}\mathrm{Hom}_A(T, ?)$ is related to the classical Wakamatsu tilting problem.

1. INTRODUCTION

In October 2013, the second named author was invited to the *46th Japan Symposium on Ring and Representation Theory* and the title of one of his talks was exactly the title of this paper, which tries to be a much expanded version of that talk. Several results that we will present here are particular cases of results given in [32] in the language of dg categories and will be published elsewhere. For different reasons, the language of dg categories tends to be difficult to understand by people working both in Ring Theory and Representation Theory, and it is specially so for beginners in the field. The main motivation of the present work is to isolate the material of [32] which applies to ordinary (always associative unital) algebras and rings, and use it to go further in its applications in terms of recollement situations that were only indirectly considered in [32]. We hope in this way that the results that we present interest ring and representation theorists. Only minor references to dg algebras will be needed, but the bulk of the contents stays within the scope of ordinary algebras and rings.

Apart from the extraordinary hospitality of the organizers, the most captivating thing for the mentioned author was the very active, lively and enthusiastic Japanese youth community in the field, who presented their recent work, sometimes impressive. This paper is written thinking mainly on them. Aimed at beginners, in the initial sections of the paper we have tried to be as self-contained as possible, referring to the written literature only for technical definitions, some proofs and specific details.

All throughout the paper the term 'algebra' will denote an associative unital algebra over a ground commutative ring k, fixed in the sequel. Unless otherwise stated, 'module' will mean 'right module' and the corresponding category of modules over an algebra A will be denoted by $\mathrm{Mod} - A$. Left A-modules will be looked at as right modules over

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the opposite algebra A^{op}. Then $\mathcal{D}(A)$ and $\mathcal{D}(A^{op})$ will denote the derived categories of the categories of right and left A-modules, respectively. On what concerns set-theoretical matters, unlike [32], in this paper we will avoid the universe axiom and, instead, we will distinguish between 'sets' and '(proper) classes'. All families will be set-indexed families and an expression of the sort 'it has (co)products' will always mean 'it has set-indexed (co)products'.

By now, the following is a classical result due to successive contributions by Happel , Rickard and Keller (see [15], [34], [35] and [19]). We refer the reader to sections 2 and 3 for the pertinent definitions.

Theorem 1.1. *Let A and B be ordinary algebras and let $T^\bullet$ be a complex of $B - A-bimodules$. The following assertions are equivalent:*

(1) *The functor $? \otimes_B^{\mathbf{L}} T^\bullet : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ is an equivalence of categories;*
(2) *The functor $\mathbf{R}Hom_A(T^\bullet, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ is an equivalence of categories;*
(3) *$T_A^\bullet$ is a classical tilting object of $\mathcal{D}(A)$ such that the canonical algebra morphism $B \longrightarrow End_{\mathcal{D}(A)}(T^\bullet)$ is an isomorphism.*

It is natural to ask what should be the substitute of assertion (3) in this theorem when, in assertion (1) (resp. assertion (2)), we only require that $? \otimes_B^{\mathbf{L}} T^\bullet$ (resp. $\mathbf{R}Hom_A(T^\bullet, ?)$) be fully faithful. That is the first goal of this paper. Namely, given a complex $T^\bullet$ of $B - A-bimodules$, we want to give necessary and sufficient conditions for $? \otimes_B^{\mathbf{L}} T^\bullet$ and $\mathbf{R}Hom_A(T^\bullet, ?)$ to be fully faithful functors.

On the other hand, a weaker condition than the one in the theorem appears when the algebras A and B admit a recollement situation

$$\mathcal{D}' \underset{i^!}{\overset{i^*}{\underset{\overline{}}{\rightleftarrows}}} \mathcal{D} \underset{j_*}{\overset{j_!}{\underset{\overline{}}{\rightleftarrows}}} \mathcal{D}''$$

as defined by Beilinson, Bernstein and Deligne ([7]), where either $\{\mathcal{D}, \mathcal{D}'\} = \{\mathcal{D}(A), \mathcal{D}(B)\}$ or $\{\mathcal{D}, \mathcal{D}''\} = \{\mathcal{D}(A), \mathcal{D}(B)\}$. In these cases, the functors $i_* = i_!$, $j_!$ and j_* are also fully faithful. This motivates the second goal of the paper. We want to give necessary and sufficient conditions for those recollements to exist, but imposing the condition that some of the functors in the picture be either $? \otimes_B^{\mathbf{L}} T^\bullet$ or $\mathbf{R}Hom_A(T^\bullet, ?)$.

Finally, the following are natural questions for which we want to have an answer.

Questions 1.2. *Let $T^\bullet$ be a complex of $B - A-bimodules$.*

(1) *Suppose that $\mathbf{R}Hom_A(T^\bullet, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ is fully faithful.*
   (a) *Is there a recollement*

$$\mathcal{D}(A) \underset{i^!}{\overset{i^*}{\underset{\overline{}}{\rightleftarrows}}} \mathcal{D}(B) \underset{j_*}{\overset{j_!}{\underset{\overline{}}{\rightleftarrows}}} \mathcal{D}'' \ ,$$

   with $i_* = \mathbf{R}Hom_A(T^\bullet, ?)$, for some triangulated category $\mathcal{D}''$?
   (b) *Is there a recollement*

$$\mathcal{D}' \underset{i^!}{\overset{i^*}{\underset{\overline{}}{\rightleftarrows}}} \mathcal{D}(B) \underset{j_*}{\overset{j_!}{\underset{\overline{}}{\rightleftarrows}}} \mathcal{D}(A) \ ,$$

whith $j_* = \mathbf{R}Hom_A(T^\bullet, ?)$, for some triangulated category $\mathcal{D}'$?

(2) *Suppose that* $? \otimes_B^{\mathbf{L}} T^\bullet : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ *is fully faithful.*

    (a) *Is there a recollement*

$$\mathcal{D}(B) \underset{i^!}{\overset{i^*}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xleftarrow{\ i_*=i_!\ }}}}} \mathcal{D}(A) \underset{j_*}{\overset{j_!}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xleftarrow{\ j^!=j^*\ }}}}} \mathcal{D}'' \ ,$$

    with $i_* = ? \otimes_B^{\mathbf{L}} T^\bullet$, *for some triangulated category* $\mathcal{D}''$?

    (b) *Is there a recollement*

$$\mathcal{D}' \underset{i^!}{\overset{i^*}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xleftarrow{\ i_*=i_!\ }}}}} \mathcal{D}(A) \underset{j_*}{\overset{j_!}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xleftarrow{\ j^!=j^*\ }}}}} \mathcal{D}(B) \ ,$$

whith $j_! = ? \otimes_B^{\mathbf{L}} T^\bullet$, *for some triangulated category* $\mathcal{D}'$?

The organization of the paper goes as follows. In section 2 we give the preliminary results on triangulated categories and the corresponding terminology used in the paper. This part has been prepared as an introductory material for beginners and, hence, tends to be as self-contained as possible.

Section 3 is specifically dedicated to the derived functors of Hom and the tensor product, but, due to the requirements of some later proofs in the paper, the development is made for derived categories of bimodules. In this context the material seems to be unavailable in the literature. Special care is put on describing the behavior of these derived functors when passing from the derived categoy of bimodules to derived category of modules on one side. In the final part of the section, we give a brief introduction to dg algebras and give a generalization of Rickard theorem, in the case of the derived category of a k-flat dg algebra (Theorem 3.15).

Section 4 contains the main results in the paper. We first show that the compact objects in the derived category of an algebra are precisely those for which the associated derived tensor product preserves products (proposition 4.2). We then go towards the mentioned goals of the paper. Proposition 4.4 gives a criterion for the fully faithfulness of $\mathbf{R}Hom_A(T^\bullet, ?)$, while proposition 4.10 gives criteria for the fully faithfulness of $? \otimes_B^{\mathbf{L}} T^\bullet$. In a parallel way, in corollary 4.5, theorem 4.6, theorem 4.13 and corollary 4.14 we give criteria for the existence of the recollements mentioned in questions 1.2 (1.a, 2.a, 2.b and 1.b, respectively). As a confluent point, when $T_A^\bullet$ is exceptional in $\mathcal{D}(A)$ and the algebra morphism $B \longrightarrow \mathrm{End}_{\mathcal{D}(A)}(T^\bullet)$ is an isomorphism, we show in theorem 4.18 that $T^\bullet$ defines a recollement as in question 1.2(1.b) if, only if, it defines a recollement as in question 1.2(2.b) on the derived categories of left modules, and this is turn equivalent to saying that A is in the thick subcategory of $\mathcal{D}(A)$ generated by $T_A^\bullet$. We end the section by giving counterexamples to all questions 1.2 and by proposing some other questions which remain open.

In the final section 5, we explicitly re-state some of the results of section 4 in the particular case that $T^\bullet = T$ is just a $B - A-$bimodule. One of the questions asked in section 4 asks whether $\mathbf{R}Hom_A(T^\bullet, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ preserves compact objects, when it is fully faithful, $T_A^\bullet$ is exceptional in $\mathcal{D}(A)$ and B is isomorphic to $\mathrm{End}_{\mathcal{D}(A)}(T^\bullet)$. We

end the paper by showing that, when $T^\bullet = T$ is a bimodule, this question is related to the classical Wakamatsu tilting problem.

Some of our results are connected to recent results of Bazzoni-Mantese-Tonolo [5], Bazzoni-Pavarin [6], Chen-Xi ([10], [11]), Han [14] and Yang [41]. All throughout sections 4 and 5, we give remarks showing these connections.

2. Preliminaries on triangulated categories and derived functors

The results of this section are well-known, but they sometimes appear scattered in the literature and with different notation. We give them here for the convenience of the reader and, also, as a way of unifying the terminology that we shall use throughout the paper. Most of the material is an adaptation of Verdier's work (see [38]), but we will refer also to several texts like [29], [40], [16], [18], [25]..., for specific results and proofs. As mentioned before, we will work over a fixed ground commutative ring k. Then the term 'category' will mean always 'k-category'. If $\mathcal{C}$ is such a category, then the set of morphisms $\mathcal{C}(C, D)$ has a structure of k-module, for all $C, D \in \mathcal{D}$, and compositions of morphisms are k-bilinear. Unless otherwise stated, subcategories will be always full and closed under taking isomorphic objects.

The reader is referred to [40, Chapter 10] for the explicit definition *triangulated category*, although some of its notation is changed. If $\mathcal{D}$ is such a category, then the *shifting*, also called *suspension (or translation) functor* $\mathcal{D} \longrightarrow \mathcal{D}$ will be denoted here by ?[1] and a triangle in $\mathcal{D}$ will be denoted by $X \longrightarrow Y \longrightarrow Z \xrightarrow{+}$, although we will also write $X \longrightarrow Y \longrightarrow Z \xrightarrow{h} X[1]$ when the connecting morphism h need be emphasized. Recall that Z is determined by the morphism $f : X \longrightarrow Y$ up to non-unique isomorphism. We will call Z the *cone of f*. A functor $F : \mathcal{D} \longrightarrow \mathcal{D}'$ between triangulated categories will be called a *triangulated or triangle-preserving functor* when it takes triangles to triangules.

2.1. **The triangulated structure of the stable category of a Frobenius exact category.** An exact category (in the sense of Quillen) is an additive category $\mathcal{C}$, together with a class of short exact sequences, called *conflations* or *admissible short exact sequences*. If $0 \to C \xrightarrow{f} D \xrightarrow{g} E \to 0$ is a conflation, then we say that f is an *inflation* or *admissible monomorphism* and that g is a *deflation* or *admissible epimorphism*. The class of conflations must satisfy the following axioms and their duals (see [18] and [8]):

Ex0 The identity morphism 1_X is a deflation, for each $X \in \mathrm{Ob}(C)$;
Ex1 The composition of two deflations is a deflation;
Ex2 Pullbacks of deflations along any morphism exist and are deflations.

Obviously the dual of an exact category is an exact category with the 'same' class of conflations. An object P of an exact category is called *projective* when the functor $\mathcal{C}(P, ?) : \mathcal{C} \longrightarrow k - \mathrm{Mod}$ takes deflations to epimorphisms. The dual notion is that of *injective* object. We say that $\mathcal{C}$ *has enough projectives (resp. injectives)* when, for each object $C \in \mathcal{C}$, there is a deflation $P \longrightarrow C$ (resp. inflation $C \longrightarrow I$), where P (resp. I) is a projective (resp. injective) object. The exact category $\mathcal{C}$ is said to be *Frobenius exact* when it has enough projectives and enough injectives and the classes of projective and injective objects coincide.

Given any exact Frobenius category $\mathcal{C}$, we can form its *stable category*, denoted $\underline{\mathcal{C}}$. Its objects are those of $\mathcal{C}$, but we have $\underline{\mathcal{C}}(C, D) = \frac{\mathcal{C}(C,D)}{\mathcal{P}(C,D)}$, where $\mathcal{P}(C, D)$ is the k-submodule of $\mathcal{C}(C, D)$ consisting of those morphisms which factor through some projective(=injective) object. The new category comes with a *projection functor* $p_{\mathcal{C}} : \mathcal{C} \longrightarrow \underline{\mathcal{C}}$, which has the property that each functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ which vanishes on the projective (=injective) objects factors through $p_{\mathcal{C}}$ in a unique way.

The so-called *(first) syzygy functor* $\Omega : \underline{C} \longrightarrow \underline{C}$ assigns to each object C the kernel of any deflation (=admissible epimorphism) $\epsilon : P \twoheadrightarrow C$ in $\mathcal{C}$, where P is a projective object. Up to isomorphism in $\underline{\mathcal{C}}$, the object $\Omega(C)$ does not depend on the projective object P or the deflation ϵ. Moreover, Ω is an equivalence of categories and its quasi-inverse is called the *(first) cosyzygy functor* $\Omega^{-1} : \underline{C} \longrightarrow \underline{C}$.

If $0 \to C \xrightarrow{f} D \xrightarrow{g} E \to 0$ a conflation in the Frobenius exact category $\mathcal{C}$, then we have the following commutative diagram, where the rows are conflations and I is a projective(=injective) object:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C & \xrightarrow{f} & D & \xrightarrow{g} & E & \longrightarrow & 0 \\
& & \| & & \Big\downarrow & & \Big\downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & C & \longrightarrow & I & \longrightarrow & \Omega^{-1}C & \longrightarrow & 0
\end{array}
$$

The following result is fundamental (see [16, Section I.2]):

Proposition 2.1 (Happel). *If $\mathcal{C}$ is a Frobenius exact category, then its stable category admits a structure of triangulated category, where:*

(1) *the suspension functor is the first cosyzygy functor;*
(2) *the distinguished triangles are those isomorphic in $\underline{\mathcal{C}}$ to a sequence of morphisms*

$$
C \xrightarrow{f} D \xrightarrow{g} E \xrightarrow{h} \Omega^{-1}(C)
$$

coming from a commutative diagram in $\mathcal{C}$ as above.

2.2. **The Frobenius exact structure on the category of chain complexes. The homotopy category.** In the rest of this section $\mathcal{A}$ will be an abelian category. The *graded category* associated to $\mathcal{A}$, denoted $\mathcal{A}^{\mathbb{Z}}$ has as objects the $\mathbb{Z}$-indexed families $X^{\bullet} := (X^n)_{n \in \mathbb{Z}}$, where $X_n \in \mathcal{A}$ for each $n \in \mathbb{Z}$. If $X^{\bullet}, Y^{\bullet} \in \mathcal{A}^{\mathbb{Z}}$ then $\mathcal{A}^{\mathbb{Z}}$ consists of the families $f = (f^n)_{n \in \mathbb{Z}}$, where $f^n \in \mathcal{A}(X^n, Y^n)$ for each $n \in \mathbb{Z}$. This category is abelian and comes with a canonical self-equivalence $?[1] : \mathcal{A}^{\mathbb{Z}} \longrightarrow \mathcal{A}^{\mathbb{Z}}$, called the *suspension* or *(1-)shift*, given by the rule $X^{\bullet}[1]^n = X^{n+1}$, for all $n \in \mathbb{Z}$. Keeping the same class of objects, we can increase the class of morphisms and form a new category $\mathrm{GR}\mathcal{A}$, where $\mathrm{GR}\mathcal{A}(X^{\bullet}, Y^{\bullet}) = \oplus_{n \in \mathbb{Z}} \mathcal{A}^{\mathbb{Z}}(X^{\bullet}, Y^{\bullet}[n])$. This is obviously a graded category where a *morphisms of degree n* is just a morphism $X^{\bullet} \longrightarrow Y^{\bullet}[n]$ in $\mathcal{A}^{\mathbb{Z}}$, and where $g \circ f$ is the composition $X^{\bullet} \xrightarrow{f} Y^{\bullet}[m] \xrightarrow{g[m]} Z[m + n]$ in $\mathcal{A}^{\mathbb{Z}}$, whenever f and g are morphisms in $GR\mathcal{A}$ of degrees m and n.

A *chain complex* of objects of $\mathcal{A}$ is a pair $(X^{\bullet}, d)$ consisting of an object $X^{\bullet}$ of $\mathrm{GR}\mathcal{A}$ together with a morphism $d : X^{\bullet} \longrightarrow X^{\bullet}$ in $GR\mathcal{A}$ of degree $+1$ such that $d \circ d = 0$. The *category of chain complexes* of objects of $\mathcal{A}$ will be denoted by $\mathcal{C}(\mathcal{A})$. It has as

objects the chain complexes of objects of $\mathcal{A}$ and a morphism $f : X^\bullet \longrightarrow Y^\bullet$, usually called a *chain map*, will be just a morphism of degree 0 in GR$\mathcal{A}$ such that $f \circ d = d \circ f$. The category $\mathcal{C}(\mathcal{A})$ is abelian, with pointwise calculation of limits and colimits, and we have an obvious forgetful functor $\mathcal{C}(\mathcal{A}) \longrightarrow \mathcal{A}^\mathbb{Z}$ which is faithful and dense, but not full. What is even more important for us is that $\mathcal{C}(\mathcal{A})$ admits a structure of Quillen exact category, usually called the *semi-split exact structure*. Recall that a short exact sequence $0 \to X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \to 0$ in $\mathcal{C}(\mathcal{A})$ is called *semi-split* when its image by the forgetful functor $\mathcal{C}(\mathcal{A}) \longrightarrow \mathcal{A}^\mathbb{Z}$ is a split exact sequence of $\mathcal{A}^\mathbb{Z}$. That is, when the sequence $0 \to X^n \xrightarrow{f^n} Y^n \xrightarrow{g^n} Z^n \to 0$ splits in $\mathcal{A}$, for each $n \in \mathbf{Z}$. For the semi-split exact structure in $\mathcal{C}(\mathcal{A})$ the conflations are precisely the semi-split exact sequences. The suspension functor of $\mathcal{A}^\mathbb{Z}$ induces a corresponding suspension functor $?[1] : \mathcal{C}(\mathcal{A}) \xrightarrow{\cong} \mathcal{C}(\mathcal{A})$, which is a self-equivalence and take $(X^\bullet, d_X) \rightsquigarrow (X^\bullet[1], d_{X[1]}$, where $d^n_{X^\bullet[1]} = -d^{n+1}_X$ for each $n \in \mathbb{Z}$.

It is well-known (see, e.g., [16, Section I.3.2]) that $\mathcal{C}(\mathcal{A})$ is a *Frobenius exact category* when considered with this exact structure. Its projective (=injective) objects with respect to the conflations are precisely the *contractible complexes*. These are those complexes isomorphic in $\mathcal{C}(\mathcal{A})$ to a coproduct of complexes of the form $C_n(X) : ...0 \longrightarrow X \xrightarrow{1_X} X \longrightarrow 0....$, where $C_n(X)$ is concentrated in degrees $n-1, n$ for all $n \in \mathbb{Z}$ and all $X \in \mathcal{A}$. For each $X^\bullet \in \mathcal{C}(\mathcal{A})$, we have a conflation

$$0 \to X^\bullet[-1] \longrightarrow \coprod_{n\in\mathbb{Z}} C_n(X_{n-1}) \longrightarrow X^\bullet \to 0.$$

Then the syzygy functor of $\mathcal{C}(\mathcal{A})$ with respect to the semi-split exact structure is identified with the inverse $?[-1] : \mathcal{C}(\mathcal{A}) \longrightarrow \mathcal{C}(\mathcal{A})$ of the suspension functor. The stable category of $\mathcal{C}(\mathcal{A})$ has a very familiar description due to the following result (see [16, p. 28]):

Lemma 2.2. *Let $(X^\bullet, d_X)$ and $(Y^\bullet, d_Y)$ be chain complexes of objects of $\mathcal{A}$ and let $f : X^\bullet \longrightarrow Y^\bullet$ be a chain map. The following assertions are equivalent:*

    (1) *f factors through a contractible complex;*
    (2) *f is null-homotopic, i.e., there exists a morphism $\sigma : X^\bullet \longrightarrow Y^\bullet$ of degree -1 in GR$\mathcal{A}$ such that $f = \sigma \circ d_X + d_Y \circ \sigma$.*

As a consequence of the previous result, the morphisms in $\mathcal{C}(\mathcal{A})$ which factor through a projective (=injective) object, with respect to the semi-split exact structure, are precisely the null-homotopic chain maps. As a consequence the associated stable category $\underline{\mathcal{C}(\mathcal{A})}$ is the *homotopy category* of $\mathcal{A}$, denoted $\mathcal{H}(\mathcal{A})$ in the sequel. We then get:

Corollary 2.3. *The homotopy category $\mathcal{H}(\mathcal{A})$ has a structure of triangulated category such that*

    (1) *the suspension functor $?[1] : \mathcal{H}(\mathcal{A}) \longrightarrow \mathcal{H}(\mathcal{A})$ is induced from the suspension functor of $\mathcal{C}(\mathcal{A})$;*
    (2) *each distinguished triangle in $\mathcal{H}(\mathcal{A})$ comes from a semi-split exact sequence $0 \to X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \to 0$ in $\mathcal{C}(\mathcal{A})$ in the form described in proposition 2.1.*

Recall that if $(\mathcal{D}, ?[1])$ is a triangulated category and $\mathcal{A}$ is an abelian category, then a *cohomological functor* $H : \mathcal{D} \longrightarrow \mathcal{A}$ is an additive functor such that if

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is any triangle in $\mathcal{D}$ and one puts $H^k = H \circ (?[k])$, for each $k \in \mathbb{Z}$, then we get a long exact sequence

$$...H^{k-1}(Z) \xrightarrow{H^{k-1}(h)} H^k(X) \xrightarrow{H^k(f)} H^k(Y) \xrightarrow{H^k(q)} H^k(Z) \xrightarrow{H^k(h)} H^{k+1}(X) \xrightarrow{H^{k+1}(f)} H^{k+1}(Y)...$$

Recall that if $X^\bullet$ is a chain complex and $k \in \mathbb{Z}$, then the k-th object of homology of X is $H^k(X^\bullet) = \frac{\text{Ker}(d^k)}{\text{Im}(d^{k-1})}$, where $d : X^\bullet \longrightarrow X^\bullet$ is the differential. The assignment $X^\bullet \rightsquigarrow H^k(X^\bullet)$ is the definition on objects of a functor $H^k : \mathcal{C}(\mathcal{A}) \longrightarrow \mathcal{A}$. The following is well-known:

Corollary 2.4. *Let $\mathcal{A}$ be any abelian category, $p : \mathcal{C}(\mathcal{A}) \longrightarrow \mathcal{H}(\mathcal{A})$ the projection functor and $k \in \mathbb{Z}$ be an integer. The functor $H^k : \mathcal{C}(\mathcal{A}) \longrightarrow \mathcal{A}$ vanishes on null-homotopic chain maps and there is a unique k-linear functor $\bar{H}^k : \mathcal{H}(\mathcal{A}) \longrightarrow \mathcal{A}$ such that $\bar{H}^k \circ p = H^k$. Moreover, $\bar{H}^k$ is a cohomological functor.*

Remark 2.5. *We will forget the overlining of H and will still denote by H^k the functor $\bar{H}^k : \mathcal{H}(\mathcal{A}) \longrightarrow \mathcal{A}$.*

2.3. **Localization of triangulated categories.** As in the previous sections, the symbol $\mathcal{A}$ will denote a fixed abelian category. We will use the term *big category* to denote a concept defined as an usual category, but where we do not require that the morphsims between two objects form a set. The following is well-known (see [13, Chapter 1]):

Proposition 2.6. *Given a category $\mathcal{C}$ and a class $\mathcal{S}$ of morphisms in $\mathcal{C}$, there is a big category $\mathcal{C}[\mathcal{S}^{-1}]$, together with a dense functor $q : \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{S}^{-1}]$ satisfying the following properties:*

(1) *$q(s)$ is an isomorphism, for each $s \in \mathcal{S}$;*
(2) *if $F : \mathcal{C} \longrightarrow \mathcal{D}$ is any functor between categories such that $F(s)$ is an isomorphism for each $s \in \mathcal{S}$, then there is a unique functor $\bar{F} : \mathcal{C}[\mathcal{S}^{-1}] \longrightarrow \mathcal{D}$ such that $\bar{F} \circ q = F$.*

Definition 1. *$\mathcal{C}[\mathcal{S}^{-1}]$ is called the* localization of $\mathcal{C}$ with respect to $\mathcal{S}$.

Remarks 2.7. (1) *The pair $(\mathcal{C}[\mathcal{S}^{-1}], q)$ is uniquely determined up to equivalence.*
(2) *A sufficient condition to guarantee that $\mathcal{C}[\mathcal{S}^{-1}]$ is an usual category is that the functor $q : \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{S}^{-1}]$ has a left or a right adjoint. If, say, $R : \mathcal{C}[\mathcal{S}^{-1}] \longrightarrow \mathcal{C}$ is right adjoint to q, then we have a bijection $\mathcal{C}[\mathcal{S}^{-1}](q(X), q(Y)) \cong \mathcal{C}(X, Rq(Y))$, for all $X, Y \in \mathcal{C}$. This proves that the morphisms between two objects of $\mathcal{C}[\mathcal{S}^{-1}]$ form a set since q is dense. A dual argument works in case q has a left adjoint.*

The explicit definition of $\mathcal{C}[\mathcal{S}^{-1}]$ was given in [13, Chapter 1], but the morphisms in this category are intractable in general. Then Gabriel and Zisman introduced some condition on $\mathcal{S}$ which makes much more tractable the morphisms in $\mathcal{C}[\mathcal{S}^{-1}]$.

Definition 2. *We shall say that $\mathcal{S}$ admits a calculus of left fractions when it satisfies the following conditions:*

(1) *$1_X \in \mathcal{S}$, for all $X \in \mathcal{C}$;*
(2) *for each diagram $X' \xrightarrow{f'} Y' \xleftarrow{s'} Y$ in $\mathcal{C}$, with $s' \in \mathcal{S}$, there exists a diagram $X' \xleftarrow{s} X \xrightarrow{f} Y$ such that $s \in \mathcal{S}$ and $f' \circ s = s' \circ f$.*

(3) *if $f, g : X \longrightarrow Y$ are morphisms in $\mathcal{C}$ and there exists $t \in \mathcal{S}$ such that $t \circ f = t \circ g$, then there exists $s \in \mathcal{S}$ such that $f \circ s = g \circ s$.*

We say that $\mathcal{S}$ admits a calculus of right fractions when it satisfies the duals of properties 1-3 above. Finally, we will say that $\mathcal{S}$ admits a calculus of fractions, or that $\mathcal{S}$ is a multiplicative system of morphisms, when it admits both a calculus of left fractions and a calculus of right fractions.

When $\mathcal{S}$ admits a calculus of left fractions, the morphisms in $\mathcal{C}[\mathcal{S}^{-1}]$ have a more tractable form. Indeed, if $X, Y \in \mathrm{Ob}(C) = \mathrm{Ob}(\mathcal{C}[\mathcal{S}^{-1}])$, then $\mathcal{C}[\mathcal{S}^{-1}](X, Y)$ consists of the formal left fractions $s^{-1}f$. Such a formal left fraction is the equivalence class of the pair (s, f) with respect to some equivalence relation defined in the class of diagrams $X \xleftarrow{s} X' \xrightarrow{f} Y$, with $s \in \mathcal{S}$. We refer the reader to [13, Chapter 1] for the precise definition of the equivalence relation and the composition of morphisms in $\mathcal{C}[\mathcal{S}^{-1}]$.

The process of localizing a category with respect to a class of morphism was developed in the context of triangulated categories by Verdier (see [38, Section II.2]).

Definition 3. *Let $\mathcal{D}$ be a triangulated category. A multiplicative system $\mathcal{S}$ in $\mathcal{D}$ is said to be* compatible with the triangulation *when the following properties hold:*

(1) *if $s : X \longrightarrow Y$ is a morphism in $\mathcal{S}$ and $n \in \mathbb{Z}$, then $s[n] : X[n] \longrightarrow Y[n]$ is in $\mathcal{S}$;*
(2) *each commutative diagram in $\mathcal{D}$*

$$
\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\
\downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s'} & & \downarrow{\scriptstyle s''} & & \downarrow{\scriptstyle s[1]} \\
X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & X'[1]
\end{array}
$$

where the rows are triangles and where $s, s' \in \mathcal{S}$, can be completed commutatively by an arrow s'' which is in $\mathcal{S}$.

As we can expect, one gets:

Proposition 2.8 (Verdier, Théorème 2.2.6)**.** *Let $\mathcal{D}$ be a triangulated category and $\mathcal{S}$ be a multiplicative system compatible with the triangulation in $\mathcal{D}$. There exists a unique structure of triangulated category in $\mathcal{D}[\mathcal{S}^{-1}]$ such that the canonical functor $q : \mathcal{D} \longrightarrow \mathcal{D}[\mathcal{S}^{-1}]$ is triangulated. Moreover, if $\mathcal{A}$ is an abelian category and $H : \mathcal{D} \longrightarrow \mathcal{A}$ is a cohomological functor such that $H(s)$ is an isomorphism, for each $s \in \mathcal{S}$, then there is a unique cohomological funtor $\tilde{H} : \mathcal{D}[\mathcal{S}^{-1}] \longrightarrow \mathcal{A}$ such that $\tilde{H} \circ q = H$.*

2.4. **Semi-orthogonal decompositions. Brown representability theorem.** Let $\mathcal{D}$ be a triangulated category. A *t-structure* in $\mathcal{D}$ (see [7]) is a pair $(\mathcal{U}, \mathcal{W})$ of full subcategories which satisfy the following properties:

i) $\mathcal{D}(U, W[-1]) = 0$, for all $U \in \mathcal{U}$ and $W \in \mathcal{W}$;
ii) $\mathcal{U}[1] \subseteq \mathcal{U}$;
iii) For each $X \in Ob(\mathcal{D})$, there is a triangle $U \longrightarrow X \longrightarrow V \xrightarrow{+}$ in $\mathcal{D}$, where $U \in \mathcal{U}$ and $V \in \mathcal{W}[-1]$.

It is easy to see that in such case $\mathcal{W} = \mathcal{U}^{\perp}[1]$ and $\mathcal{U} = {}^{\perp}(\mathcal{W}[-1]) = {}^{\perp}(\mathcal{U}^{\perp})$. For this reason, we will write a t-structure as $(\mathcal{U}, \mathcal{U}^{\perp}[1])$. We will call $\mathcal{U}$ and $\mathcal{U}^{\perp}$ the *aisle* and the *co-aisle* of the t-structure, respectively. The objects U and V in the above triangle are uniquely determined by X, up to isomorphism, and define functors $\tau_{\mathcal{U}} : \mathcal{D} \longrightarrow \mathcal{U}$ and $\tau^{\mathcal{U}^{\perp}} : \mathcal{D} \longrightarrow \mathcal{U}^{\perp}$ which are right and left adjoints to the respective inclusion functors. We call them the *left and right truncation functors* with respect to the given t-structure.

Recall that a subcategory $\mathcal{X}$ of a triangulated category $\mathcal{D}$ is *closed under extensions* when, given any triangle $X' \longrightarrow Y \longrightarrow X \overset{+}{\longrightarrow}$ in $\mathcal{D}$, with $X, X' \in \mathcal{X}$, the object Y is also in $\mathcal{X}$.

Definition 4. *Let $\mathcal{D}$ be a triangulated category and $\mathcal{U} \subseteq \mathcal{D}$ a full subcategory. We say that $\mathcal{U}$ is*

(1) suspended *when it is closed under extensions in $\mathcal{D}$ and $\mathcal{U}[1] \subseteq \mathcal{U}$;*
(2) triangulated *when it is closed under extensions and $\mathcal{U}[1] = \mathcal{U}$*
(3) thick *when it is triangulated and closed under taking direct summands.*

Notation and terminology.- Given a class $\mathcal{S}$ of objects of $\mathcal{D}$, we shall denote by $\text{susp}_{\mathcal{D}}(\mathcal{S})$ (resp. $\text{tria}_{\mathcal{D}}(\mathcal{S})$, resp. $\text{thick}_{\mathcal{D}}(\mathcal{S})$) the smallest suspended (resp. triangulated, resp. thick) subcategory of $\mathcal{D}$ containing $\mathcal{S}$. If $\mathcal{U}$ is any suspended (resp. triangulated, resp. thick) subcategory of $\mathcal{D}$ and $\mathcal{U} = \text{susp}_{\mathcal{D}}(\mathcal{S})$ (resp. $\mathcal{U} = \text{tria}_{\mathcal{D}}(\mathcal{S})$, resp. $\text{thick}_{\mathcal{D}}(\mathcal{S})$), we will say that $\mathcal{U}$ is the *suspended (resp. triangulated, resp. thick) subcategory of $\mathcal{D}$ generated by $\mathcal{S}$.* When $\mathcal{D}$ has coproducts, we will denote by $\text{Susp}_{\mathcal{D}}(\mathcal{S})$ (resp. $\text{Tria}_{\mathcal{D}}(\mathcal{S})$) the smallest suspended (resp. triangulated) subcategory of D containing $\mathcal{S}$ which is closed under taking coproducts in $\mathcal{D}$.

Remark 2.9. *If $\mathcal{D}$ has coproducts and $\mathcal{T}$ is a full subcategory closed under taking coproducts, then it is a triangulated subcategory if, and only if, it is thick (see [29, Prop.1.6.8 and its proof]).*

It is an easy exercise to prove now the following useful result.

Lemma 2.10. *Let $F : \mathcal{D} \longrightarrow \mathcal{D}'$ be a triangulated functor between triangulated categories. If $\mathcal{S} \subseteq \mathcal{D}$ is any class of objects, then $F(\text{susp}_{\mathcal{D}}(\mathcal{S})) \subseteq \text{susp}_{\mathcal{D}'}(F(\mathcal{S}))$ (resp. $F(\text{tria}_{\mathcal{D}}(\mathcal{S})) \subseteq \text{tria}_{\mathcal{D}'}(F(\mathcal{S}))$, resp. $F(\text{thick}_{\mathcal{D}}(\mathcal{S})) \subseteq \text{thick}_{\mathcal{D}'}(F(\mathcal{S}))$). When $\mathcal{D}$ and $\mathcal{D}'$ have coproducts and F preserves coproducts, we also have that $F(\text{Susp}_{\mathcal{D}}(\mathcal{S})) \subseteq \text{Susp}_{\mathcal{D}'}(F(\mathcal{S}))$ (resp. $F(\text{Tria}_{\mathcal{D}}(\mathcal{S})) \subseteq \text{Tria}_{\mathcal{D}'}(F(\mathcal{S}))$).*

The following result of Keller and Vossieck [23, Proposition 1.1] is fundamental to deal with t-structures and semi-orthogonal decompositions.

Proposition 2.11. *A full subcategory $\mathcal{U}$ of $\mathcal{D}$ is the aisle of a t-structure if, and only if, it is a suspended subcategory such that the inclusion functor $\mathcal{U} \hookrightarrow \mathcal{D}$ has a right adjoint.*

The type of t-structure which is most useful to us in this paper is the following.

Definition 5. *A semi-orthogonal decomposition or Bousfield localization pair in $\mathcal{D}$ is a t-structure $(\mathcal{U}, \mathcal{U}^{\perp}[1])$ such that $\mathcal{U}[1] = \mathcal{U}$ (equivalently, such that $\mathcal{U}^{\perp} = \mathcal{U}^{\perp}[1]$). That is, a semi-orthogonal decomposition is a t-structure such that $\mathcal{U}$ (resp. $\mathcal{U}^{\perp}$) is a triangulated*

subcategory of $\mathcal{D}$. In such case we will use $(\mathcal{U}, \mathcal{U}^\perp)$ instead of $(\mathcal{U}, \mathcal{U}^\perp[1])$ to denote the semi-orthogonal decomposition.

Certain adjunctions of triangulated functors provide semi-orthogonal decompositions.

Proposition 2.12. *Let* $F : \mathcal{D} \longrightarrow \mathcal{D}'$ *and* $G : \mathcal{D}' \longrightarrow \mathcal{D}$ *be triangulated functors between triangulated categories such that* (F, G) *is an adjoint pair. The following assertions hold:*

(1) *If* F *is fully faithful, then* $(Im(F), Ker(G))$ *is a semi-orthogonal decomposition of* $\mathcal{D}'$;

(2) *If* G *is fully faitfhful, then* $(Ker(F), Im(G))$ *is a semi-orthogonal decomposition of* $\mathcal{D}$.

Proof. Assertion (2) follows from assertion (1) by duality. To prove (1), note that the unit $\lambda : 1_\mathcal{D} \longrightarrow G \circ F$ is an isomorphism (see [17, Proposition II.7.5]) and, by the adjunction equations, we then get that $G(\delta)$ is also an isomorphism, where $\delta : F \circ G \longrightarrow 1_{\mathcal{D}'}$ is the counit. This implies that if $M \in \mathcal{D}'$ is any object and we complete δ_M to a triangle

$$(F \circ G)(M) \xrightarrow{\delta_M} M \longrightarrow Y_M \xrightarrow{+},$$

then $Y_M \in Ker(G)$.

But, by the adjunction, we have that $\mathcal{D}'(F(D), Y) = \mathcal{D}(D, G(Y)) = 0$, for each $Y \in Ker(G)$. It follows that $(Im(F), Ker(G))$ is a semi-orthogonal decomposition of $\mathcal{D}'$. $\square$

Given a triangulated subcategory $\mathcal{T}$ of $\mathcal{D}$, we shall denote by $\Sigma_\mathcal{T}$ the class of morphisms $s : X \longrightarrow Y$ in $\mathcal{D}$ whose cone is an object of $\mathcal{T}$. The following is a fundamental result of Verdier:

Proposition 2.13. *Let* $\mathcal{D}$ *be a triangulated category and* $\mathcal{T}$ *a thick subcategory. The following assertions hold:*

(1) $\Sigma_\mathcal{T}$ *is a multiplicative system of* $\mathcal{D}$ *compatible with the triangulation. The category* $\mathcal{D}[\Sigma_\mathcal{T}^{-1}]$ *is denoted by* $\mathcal{D}/\mathcal{T}$ *and called the* quotient category *of* $\mathcal{D}$ *by* $\mathcal{T}$.

(2) *The canonical functor* $q : \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{T}$ *satisfies the following universal property:*
    (*) *For each triangulated category* $\mathcal{D}'$ *and each triangulated functor* $F : \mathcal{D} \longrightarrow \mathcal{D}'$ *such that* $F(\mathcal{T}) = 0$, *there is a triangulated functor* $\bar{F} : \mathcal{D}/\mathcal{T} \longrightarrow \mathcal{D}'$, *unique up to natural isomorphism, such that* $\bar{F} \circ q \cong F$.

(3) *The functor* $q : \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{T}$ *has a right adjoint if, and only if,* $(\mathcal{T}, \mathcal{T}^\perp)$ *is a semi-orthogonal decomposition in* $\mathcal{D}$. *In this case, the functor* $\tau^{\mathcal{T}^\perp} : D \longrightarrow \mathcal{T}^\perp$ *induces an equivalence of triangulated categories* $\mathcal{D}/\mathcal{T} \xrightarrow{\cong} \mathcal{T}^\perp$.

(4) *The functor* $q : \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{T}$ *has a left adjoint if, and only if,* $(^\perp\mathcal{T}, \mathcal{T})$ *is a semi-orthogonal decomposition in* $\mathcal{D}$. *In this case, the functor* $\tau_{\perp\mathcal{T}} : D \longrightarrow {}^\perp\mathcal{T}$ *induces an equivalence of triangulated categories* $\mathcal{D}/\mathcal{T} \xrightarrow{\cong} {}^\perp\mathcal{T}$.

Proof. Assertions (1) is [38, Proposition II.2.1.8] while assertion (2) is included in [38, Corollaire II.2.2.11]. Assertions (3) and (4) are dual to each other. Assertion (3) is implicit in [38, Proposition II.2.3.3]. For an explicit proof, see [25, Proposition 4.9.1] and use proposition 2.11. $\square$

In many situations, we will need criteria for a given triangulated functor to have adjoints. The main tool is the following.

Definition 6. *Let $\mathcal{D}$ be a triangulated category with coproducts. We shall say that $\mathcal{D}$ satisfies Brown representability theorem when any cohomological contravariant functor $H : \mathcal{D} \longrightarrow Mod - k$ which takes coproducts to products is representable. That is, there exists an object Y of $\mathcal{D}$ such that H is naturally isomorphic to $\mathcal{D}(?, Y)$.*

The key point is the following.

Proposition 2.14. *Let $\mathcal{D}$ be a triangulated subcategory which satisfies Brown representability theorem and let $\mathcal{D}'$ be any triangulated category. Each triangulated functor $F : \mathcal{D} \longrightarrow \mathcal{D}'$ which preserves coproducts has a right adjoint.*

Proof. See [29, Theorem 8.8.4]. □

Definition 7. *Let $\mathcal{D}$ have coproducts. An object X of $\mathcal{D}$ is called* compact *when the functor $\mathcal{D}(X, ?) : \mathcal{D} \longrightarrow Mod - k$ preserves coproducts. The category $\mathcal{D}$ is called* compactly generated *when there is a* <u>set</u> *$\mathcal{S}$ of compact objects such that $Tria_{\mathcal{D}}(\mathcal{S}) = \mathcal{D}$. We then say that $\mathcal{S}$ is a* set of compact generators *of $\mathcal{D}$.*

Corollary 2.15. *The following assertions hold, for any compactly generated triangulated category $\mathcal{D}$ and any covariant triangulated functor $F : \mathcal{D} \longrightarrow \mathcal{D}'$:*

    (1) *F preserves coproducts if, and only if, it has a right adjoint;*
    (2) *F preserves products if, and only if, it has a left adjoint.*

Proof. See [25, Proposition 5.3.1]. □

The following lemma, whose proof can be found in [32], is rather useful.

Lemma 2.16. *Let $F : \mathcal{D} \longrightarrow \mathcal{D}'$ be a triangulated functor between triangulated categories and suppose that it has a left adjoint L and a right adjoint R. Then L is fully faithful if, and only if, so is R.*

2.5. **Recollements and TTF triples.** The following concept, introduced in [7], is fundamental in the theory of triangulated categories.

Definition 8. *A* recollement *of triangulated categories consists of a triple $(\mathcal{D}', \mathcal{D}, \mathcal{D}'')$ of triangulated categories and of six triangulated functors between them, assembled as follows*

$$\mathcal{D}' \underset{i^!}{\overset{i^*}{\underset{\longleftarrow}{\underset{i_* = i_!}{\overset{\longleftarrow}{\longrightarrow}}}}} \mathcal{D} \underset{j_*}{\overset{j_!}{\underset{\longleftarrow}{\underset{j^! = j^*}{\overset{\longleftarrow}{\longrightarrow}}}}} \mathcal{D}'' \ ,$$

which satisfy the following conditons:

    (1) *Each functor in the picture is left adjoint to the one immediately below, when it exists;*
    (2) *The composition $i^! \circ j_*$ (and hence also $j^! \circ i_! = j^* \circ i_*$ and $i^* \circ j_!$) is the zero functor;*
    (3) *The functors i_*, j_* and $j_!$ are fully faithful (and hence the unit maps $1_{\mathcal{D}'} \longrightarrow i^! \circ i_* = i^! \circ i_!$, $1_{\mathcal{D}''} \longrightarrow j^* \circ j_! = j^! \circ j_!$ and the counit maps $i^* \circ i_! = i^* \circ i_* \longrightarrow 1_{\mathcal{D}'}$, $j^! \circ j_* = j^* \circ j_* \longrightarrow 1_{\mathcal{D}''}$ are all isomorphisms);*
    (4) *The remaining unit and counit maps of the different adjunctions give rise, for each object $X \in \mathcal{D}$, to triangules*

$$(i_! \circ i^!)(X) \longrightarrow X \longrightarrow (j_* \circ j^*)(X) \overset{+}{\longrightarrow}$$
$$and$$
$$(j_! \circ j^!)(X) \longrightarrow X \longrightarrow (i_* \circ i^*)(X) \overset{+}{\longrightarrow}.$$

In such situation, we will say that $\mathcal{D}$ is a recollement of $\mathcal{D}'$ and $\mathcal{D}''$.

A less familiar concept, coming from torsion theory in module categories, is the following (see [31]):

Definition 9. *Given a triangulated category $\mathcal{D}$, a triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ of full subcategories is called a TTF triple in $\mathcal{D}$ when the pairs $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{Y}, \mathcal{Z})$ are both semi-orthogonal decompositions of $\mathcal{D}$.*

As shown in [31, Section 2.1], it turns out that recollements and TTF triples are equivalent concepts in the following sense:

Proposition 2.17. *Let $\mathcal{D}$ be a triangulated category. The following assertions hold:*
   (1) *If*

$$\mathcal{D}' \overset{\overset{i^*}{\longleftarrow}}{\underset{\underset{i^!}{\longleftarrow}}{\overset{i_*=i_!}{\longrightarrow}}} \mathcal{D} \overset{\overset{j_!}{\longleftarrow}}{\underset{\underset{j_*}{\longleftarrow}}{\overset{j^!=j^*}{\longrightarrow}}} \mathcal{D}''$$

   is a recollement of the triangulated category $\mathcal{D}$, then $(Im(j_!), Im(i_*), Im(j_*))$ is a TTF triple in $\mathcal{D}$;
   (2) *If $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is a TTF triple in $\mathcal{D}$, then*

$$\mathcal{Y} \overset{\overset{i^*}{\longleftarrow}}{\underset{\underset{i^!}{\longleftarrow}}{\overset{i_*=i_!}{\longrightarrow}}} \mathcal{D} \overset{\overset{j_!}{\longleftarrow}}{\underset{\underset{j_*}{\longleftarrow}}{\overset{j^!=j^*}{\longrightarrow}}} \mathcal{X}$$

   is a recollement, where:
   (a) *$i_* = i_! : \mathcal{Y} \longrightarrow \mathcal{D}$ and $j_! : \mathcal{X} \longrightarrow \mathcal{D}$ are the inclusion functors;*
   (b) *$i^* = \tau^{\mathcal{Y}} : \mathcal{D} \longrightarrow \mathcal{Y}$ is the right truncation with respect to the semi-orthogonal decomposition $(\mathcal{X}, \mathcal{Y})$ and $i^! = \tau_{\mathcal{Y}} : \mathcal{D} \longrightarrow \mathcal{Y}$ is the left truncation with respect to the semi-orthogonal decomposition $(\mathcal{Y}, \mathcal{Z})$;*
   (c) *$j^! = j^* = \tau_{\mathcal{X}} : \mathcal{D} \longrightarrow \mathcal{X}$ is the left truncation with respect to the semi-orthogonal decomposition $(\mathcal{X}, \mathcal{Y})$;*
   (d) *j_* is the composition $\mathcal{X} \hookrightarrow \mathcal{D} \overset{\tau^{\mathcal{Z}}}{\longrightarrow} \mathcal{Z} \hookrightarrow \mathcal{D}$, where the hooked arrows are the inclusions and $\tau^{\mathcal{Z}}$ is the right truncation with respect to the semi-orthogonal decomposition $(\mathcal{Y}, \mathcal{Z})$.*

Remark 2.18. *In the rest of the paper, whenever*

$$\mathcal{D}' \overset{\overset{i^*}{\longleftarrow}}{\underset{\underset{i^!}{\longleftarrow}}{\overset{i_*=i_!}{\longrightarrow}}} \mathcal{D} \overset{\overset{j_!}{\longleftarrow}}{\underset{\underset{j_*}{\longleftarrow}}{\overset{j^!=j^*}{\longrightarrow}}} \mathcal{D}''$$

is a recollement of triangulated categories, we will simply write $\mathcal{D}' \equiv \mathcal{D} \equiv \mathcal{D}''$ and the six functors of the recollement will be understood.

We now give a criterion for a triangulated functor to be one of the two central arrows of a recollement.

Proposition 2.19. *The following assertions hold:*

(1) *Let $F : \mathcal{D}' \longrightarrow \mathcal{D}$ be a triangulated functor between triangulated categories. There is a recollement $\mathcal{D}' \equiv \mathcal{D} \equiv \mathcal{D}''$, with $F = i_*$, for some triangulated category $\mathcal{D}''$, if and only if F is fully faithful and has both a left and a right adjoint;*

(2) *Let $G : \mathcal{D} \longrightarrow \mathcal{D}''$ be a triangulated functor between triangulated categories. There is a recollement $\mathcal{D}' \equiv \mathcal{D} \equiv \mathcal{D}''$, with $G = j^*$, for some triangulated category $\mathcal{D}'$, if and only if G has both a left and a right adjoint, and one of these adjoints is fully faithful.*

Proof. (1) We just need to prove the 'if' part of the assertion. If F is fully faithful, then it induces an equivalence of categories $\mathcal{D}' \xrightarrow{\cong} \mathrm{Im}(F) =: \mathcal{Y}$. The fact that F has both a left and a right adjoint implies that also the inclusion functor $i_{\mathcal{Y}} : \mathcal{Y} \hookrightarrow \mathcal{D}$ has both a left and a right adjoint. By proposition 2.11 and its dual, we get that $(\mathcal{Y}, \mathcal{Y}^{\perp})$ and $(^{\perp}\mathcal{Y}, \mathcal{Y})$ are semi-orthogonal decompositions of $\mathcal{D}$. Therefore $(^{\perp}\mathcal{Y}, \mathcal{Y}, \mathcal{Y}^{\perp})$ is a TTF triple in $\mathcal{D}$ and, by proposition 2.17, we have a recollement $\mathcal{Y} \equiv \mathcal{D} \equiv {^{\perp}\mathcal{Y}}$, with $i_* = i_{\mathcal{Y}} : \mathcal{Y} \hookrightarrow \mathcal{D}$ the inclusion functor. Using now the equivalence $\mathcal{D}' \xrightarrow{\cong} \mathcal{Y}$ given by F, we immediately get a recollement $\mathcal{D}' \equiv \mathcal{D} \equiv {^{\perp}\mathcal{Y}}$, with $i_* = F$.

(2) Again, we just need to prove the 'if' part. If $G : \mathcal{D} \longrightarrow \mathcal{D}''$ is a triangulated functor as stated and we denote by L and R its left and right adjoint, respectively, then lemma 2.16 tells us that L and R are both fully faithful. Then, by proposition 2.12, we get that $(\mathrm{Im}(L), \mathrm{Ker}(G), \mathrm{Im}(R))$ is a TTF triple in $\mathcal{D}$.

Since L gives an equivalence of triangulated categories $\tilde{L} : \mathcal{D}'' \xrightarrow{\cong} \mathrm{Im}(L) = \mathcal{X}$, we easily get that the left truncation functor $\tau_{\mathcal{X}}$ with respect to the semi-orthogonal decomposition $(\mathcal{X}, \mathcal{X}^{\perp}) = (\mathrm{Im}(L), \mathrm{Ker}(G))$ is naturally isomorphic to $\tilde{L} \circ G$. Using proposition 2.17, we then get a recollement $\mathrm{Ker}(G) \equiv \mathcal{D} \equiv D''$, where $j^* = G$.

$\square$

2.6. **The derived category of an abelian category.** In this subsection $\mathcal{A}$ will be an abelian category. A morphism $f : X^{\bullet} \longrightarrow Y^{\bullet}$ in $\mathcal{C}(\mathcal{A})$ is called a *quasi-isomorphism* when the morphism $H^k(f) : H^k(X^{\bullet}) \longrightarrow H^k(Y^{\bullet})$ is an isomorphism in $\mathcal{A}$, for each $k \in \mathbb{Z}$. Our main object of interest is the following category.

Definition 10. *The* derived category of $\mathcal{A}$, *denoted $\mathcal{D}(\mathcal{A})$, is the localization of $\mathcal{C}(\mathcal{A})$ with respect to the class of quasi-isomorphisms.*

Note that, in general, $\mathcal{D}(\mathcal{A})$ is a big category. Moreover, defined as above, we have the problem of the intractability of its morphisms. But, fortunately, this latter obstacle is overcome:

Proposition 2.20 (Verdier)**.** *Let $\mathcal{Q}$ be the class of quasi-isomoprhisms in $\mathcal{C}(\mathcal{A})$. The following assertions hold:*

(1) *The canonical functor $q : \mathcal{C}(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{A})$ factors through the projection functor $p : \mathcal{C}(\mathcal{A}) \longrightarrow \mathcal{H}(\mathcal{A})$. More concretely, there is a unique functor, up to natural isomorphism, $\bar{q} : \mathcal{H}(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{A})$ such that $\bar{q} \circ p = q$.*

(2) *$\overline{\mathcal{Q}} := p(\mathcal{Q})$ is a multiplicative system in $\mathcal{H}(\mathcal{A})$ compatible with the triangulation and the functor $\bar{q}$ induces an equivalence of categories $\mathcal{H}(\mathcal{A})[\overline{Q}^{-1}] \xrightarrow{\cong} \mathcal{D}(\mathcal{A})$.*

Proof. See Définition 1.2.2, Proposition 1.3.5 and Remarque 1.3.7 in [38, Chapitre III]. □

Corollary 2.21. $\mathcal{D}(\mathcal{A})$ *admits a unique structure of triangulated category such that the functor* $\bar{q} : \mathcal{H}(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{A})$ *is triangulated. Moreover, for each* $k \in \mathbf{Z}$*, the cohomological functor* $H^k : \mathcal{H}(\mathcal{A}) \longrightarrow \mathcal{A}$ *factors thorough* $\bar{q}$ *in a unique way.*

It remains to settle the set-theoretical problem that $\mathcal{D}(\mathcal{A})$ is a big category. Led by remark 2.7(2), we will characterize when the functor $q : \mathcal{H}(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{A})$ has an adjoint. Recall that an object $X^\bullet \in \mathcal{C}(\mathcal{A})$ is called an *acyclic complex* when it has zero homology, i.e., when $H^k(X^\bullet) = 0$, for all $k \in \mathbb{Z}$. Note that, when $\mathcal{A}$ is AB4, the k-th homology functor $H^k : \mathcal{C}(\mathcal{A}) \longrightarrow \mathcal{A}$ preserves coproducts. In this case, if $(X_i^\bullet)_{i \in I}$ is a family of acyclic complexes which has a coproduct in $\mathcal{C}(\mathcal{A})$ (equivalently, in $\mathcal{H}(\mathcal{A})$), then $\coprod_{i \in I} X_i^\bullet$ is also an acyclic complex. Viewed as a full subcategory of $\mathcal{H}(\mathcal{A})$, it follows that the class $\mathcal{Z}$ of acyclic complexes is a triangulated subcategory closed under taking coproducts. The dual fact applies to products when $\mathcal{A}$ is AB4*. The following result (resp. its dual), which is a direct consequence of proposition 2.13, gives a criterion for the canonical functor $q : \mathcal{H}(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{A})$ to have a right (resp. left) adjoint.

Proposition 2.22. *Let* $\mathcal{A}$ *be an AB4 abelian category and denote by* $\mathcal{Z}$ *the full subcategory of* $\mathcal{H}(\mathcal{A})$ *whose objects are the (images by the quotient functor* $p : \mathcal{C}(\mathcal{A}) \longrightarrow \mathcal{H}(\mathcal{A})$ *of) acyclic complexes. The following assertions are equivalent:*

    (1) *The pair* $(\mathcal{Z}, \mathcal{Z}^\perp)$ *is a semi-orthogonal decomposition in* $\mathcal{H}(\mathcal{A})$*.*
    (2) *The canonical functor* $q : \mathcal{H}(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{A})$ *has a right adjoint.*

In such case, there is an equivalence of categories $\mathcal{D}(\mathcal{A}) \cong \mathcal{Z}^\perp$*, so that* $\mathcal{D}(\mathcal{A})$ *is a real category and not just a big one.*

Definition 11. *A chain complex* $Q^\bullet \in \mathcal{C}(\mathcal{A})$ *is called* homotopically injective *(resp.* homotopically projective) *when* $p(Q^\bullet) \in \mathcal{Z}^\perp$ *(resp.* $p(Q^\bullet) \in {}^\perp\mathcal{Z}$*), where* $\mathcal{Z}$ *is as in the previous proposition and* $p : \mathcal{C}(\mathcal{A}) \longrightarrow \mathcal{H}(\mathcal{A})$ *is the projection functor. A chain complex* $X^\bullet$ *is said to have a* homotopically injective resolution *(resp.* homotopically projective resolution) *in* $\mathcal{H}(\mathcal{A})$ *when there is quasi-isomorphism* $\iota : X^\bullet \longrightarrow I_X^\bullet$ *(resp.* $\pi : P_X^\bullet \longrightarrow X^\bullet$*), where* $I_X^\bullet$ *(resp.* $P_X^\bullet$*) is a homotopically injective (resp. homotopically projective) complex.*

Note that $X^\bullet$ has a homotopically injective resolution if, and only if, there is a triangle $Z^\bullet \longrightarrow X^\bullet \stackrel{\iota}{\longrightarrow} I^\bullet \stackrel{+}{\longrightarrow}$ in $\mathcal{H}(\mathcal{A})$ such that $Z^\bullet \in \mathcal{Z}$ and $I^\bullet \in \mathcal{Z}^\perp$ (i.e. $Z^\bullet$ is acyclic and $I^\bullet$ is homotopically injective). A dual fact is true about the existence of a homotopically projective resolution.

As a direct consequence of the definition of semi-orthogonal decomposition and of proposition 2.22, we get the following result. The statement of the dual result is left to the reader.

Corollary 2.23. *Let* $\mathcal{A}$ *be an AB4 abelian category. The canonical functor* $q : \mathcal{H}(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{A})$ *has a right adjoint if, and only if, each chain complex* $X^\bullet$ *has a homotopically injective resolution* $\iota_X : X^\bullet \longrightarrow I_X^\bullet$*. In such case* $I_X^\bullet$ *is uniquely determined, up to isomorphism in* $\mathcal{H}(\mathcal{A})$*, and the assignment* $X \rightsquigarrow I_X^\bullet$ *is the definition on objects of a triangulated functor* $\mathbf{i}_\mathcal{A} : \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{H}(\mathcal{A})$ *which is right adjoint to* q*.*

Definition 12. *In the situation of the previous corollary, $\mathbf{i}_{\mathcal{A}} : \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{H}(\mathcal{A})$ is called the homotopically injective resolution functor. When the dual result holds, one defines the homotopically projective resolution functor $\mathbf{p}_{\mathcal{A}} : \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{H}(\mathcal{A})$, which is left adjoint to $q : \mathcal{H}(\mathcal{A}) \longrightarrow \mathcal{D}(A)$.*

Remark 2.24. *When the homotopically projective (resp. homotopically injective) resolution functor is defined, the counit $q \circ \mathbf{i}_{\mathcal{A}} \longrightarrow 1_{\mathcal{D}(A)}$ (resp. the unit $1_{\mathcal{D}(A)} \longrightarrow q \circ \mathbf{p}_{\mathcal{A}}$) of the adjunction pair $(q, \mathbf{i}_{\mathcal{A}})$ (resp. $(\mathbf{p}_{\mathcal{A}}, q)$) is a natural isomorphism. We will denote by $\iota : 1_{\mathcal{H}(\mathcal{A})} \longrightarrow \mathbf{i}_{\mathcal{A}} \circ q$ (resp. $\pi : \mathbf{p}_{\mathcal{A}} \circ q \longrightarrow 1_{\mathcal{H}(A)}$) the unit (resp. counit) of the first (resp. second) adjoint pair. But since q 'is' the identity on objects, we will simply write $\iota = \iota_{X^{\bullet}} : X^{\bullet} \longrightarrow \mathbf{i}_{\mathcal{A}} X^{\bullet}$ and $\pi = \pi_{X^{\bullet}} : \mathbf{p}_{\mathcal{A}} X^{\bullet} \longrightarrow X^{\bullet}$. Both of them are quasi-isomorphisms.*

The canonical examples that we should keep in mind are the following:

Examples 2.25. (1) *If $\mathcal{A} = \mathcal{G}$ is a Grothendieck category, then each chain complex of objects of $\mathcal{G}$ admits a homotopically injective resolution ([2, Theorem 5.4], see also [37, Theorem D]).*

(2) *If $\mathcal{A} = Mod - A$ is the category of (right) modules over a k-algebra A, then each chain complex of A-modules admits both a homotopically projective and a homotopically injective resolution (see [37, Theorem C]). Note that if A and B are k-algebras, then a $B - A-$bimodule is the same as a right $B^{op} \otimes A$-module, by the rule $m(b^o \otimes a) = bma$, for all $a \in A$, $b \in B$ and $m \in M$. Therefore the existence of homotopically injective and homotopically projective resolutions also applies when taking $\mathcal{A} = Mod - (B^{op} \otimes A)$ to be the category of $B - A-$bimodules.*

(3) *Since we have canonical equivalences $\mathcal{C}(\mathcal{A}^{op}) \cong \mathcal{C}(\mathcal{A})^{op}$ and $\mathcal{H}(\mathcal{A}^{op}) \cong \mathcal{H}(\mathcal{A})^{op}$, the opposite category of a (bi)module category is another example of abelian category $\mathcal{A}$ over which every complex admits both a homotopically projective resolution and a homotopically injective one.*

A consequence of Brown representability theorem is now the following result.

Proposition 2.26. *Let A be an algebra and $\mathcal{S}$ be any <u>set</u> of objects of $\mathcal{D}(A)$. The following assertions hold:*

(1) *$\mathcal{U} := Susp_{\mathcal{D}(A)}(\mathcal{S})$ is the aisle of a t-structure in $\mathcal{D}(A)$. The co-aisle $\mathcal{U}^{\perp}$ consists of the complexes $Y^{\bullet}$ such that $\mathcal{D}(A)(S^{\bullet}[k], Y^{\bullet}) = 0$, for all $S^{\bullet} \in \mathcal{S}$ and and all integers $k \geq 0$;*

(2) *$\mathcal{T} := Tria_{\mathcal{D}(A)}(\mathcal{S})$ is the aisle of a semi-orthogonal decomposition in $\mathcal{D}(A)$. In this case $\mathcal{T}^{\perp}$ consists of the complexes $Y^{\bullet}$ such that $\mathcal{D}(A)(S^{\bullet}[k], Y^{\bullet}) = 0$, for all $S^{\bullet} \in \mathcal{S}$ and $k \in \mathbb{Z}$.*

Proof. Assertion 1 is proved in [3, Proposition 3.2] (see also [36] and [22, Theorem 12.1] for more general versions). Assertion 2 follows from 1 since $Tria_{\mathcal{D}(A)}(\mathcal{S}) = Susp_{\mathcal{D}(A)}(\bigcup_{k \in \mathbb{Z}} \mathcal{S}[k])$. $\square$

2.7. Derived functors and adjunctions.

Lemma 2.27. *Let $\mathcal{C}$ and $\mathcal{D}$ be Frobenius exact categories and let $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{C}$ be functors which take conflations to conflations. Then the following statements hold true:*

(1) *If F takes projective objects to projective objects, then there is a triangulated functor $\underline{F} : \underline{C} \longrightarrow \underline{D}$, unique up to natural isomorphism, such that $p_{\mathcal{D}} \circ F = \underline{F} \circ p_{\mathcal{C}}$.*

(2) *If (F, G) is an adjoint pair, then the following assertions hold:*

   (a) *Both F and G preserve projective objects.*

   (b) *The induced triangulated functors $\underline{F} : \underline{C} \longrightarrow \underline{D}$ and $\underline{G} : \underline{D} \longrightarrow \underline{C}$ form an adjoint pair $(\underline{F}, \underline{G})$.*

Proof. (1) The existence of the functor is immediate and the fact that it is triangulated is due to the fact that all triangles in $\underline{C}$ and $\underline{D}$ are 'image' of conflations in $\mathcal{C}$ and $\mathcal{D}$ by the respective projection functors.

(2) (a) The proof is identical to the one which proves, for arbitrary categories, that each left (resp. right) adjoint of a functor which preserves epimorphisms (resp. monomorphisms) preserves projective (resp. injective) objects. Then use the fact that the injective and projective objects coincide.

(b) Let us fix an isomorphism $\eta : \mathcal{D}(F(?), ?) \overset{\cong}{\longrightarrow} \mathcal{C}(?, G(?))$ natural on both variables and let $C \in \mathcal{C}$ and $D \in \mathcal{D}$ be any objects. If $\alpha \in \mathcal{D}(F(C), D)$ is a morphism which factors through a projective object Q of $\mathcal{D}$, then we have a decomposition $F(C) \overset{\beta}{\longrightarrow} Q \overset{\gamma}{\longrightarrow} D$. Due to the naturality of η, we then have a commutative diagram:

$$
\begin{array}{ccc}
\mathcal{D}(F(C), Q) & \overset{\eta_{C,Q}}{\longrightarrow} & \mathcal{C}(C, G(Q)) \\
\downarrow{\scriptstyle \gamma_*} & & \downarrow{\scriptstyle G(\gamma)_*} \\
\mathcal{D}(F(C), D) & \overset{\eta_{C,D}}{\longrightarrow} & \mathcal{C}(C, G(D))
\end{array}
$$

with the obvious meaning of the vertical arrows. It follows that

$$\eta_{C,D}(\alpha) = \eta_{C,D}(\gamma \circ \beta) = (\eta_{C,D} \circ \gamma_*)(\beta) = G(\gamma)_* \circ \eta_{C,Q}(\beta) = G(\gamma) \circ \eta_{C,Q}(\beta),$$

which proves that $\eta_{C,D}(\alpha)$ factors through the projective object $G(Q)$ of $\mathcal{C}$. We then get and induced map $\bar{\eta}_{C,D} : \underline{D}(\underline{F}(C), D) \longrightarrow \underline{C}(C, \underline{G}(D))$, which is natural on both variables. Applying the symmetric argument to η^{-1}, we obtain a map $\overline{\eta^{-1}}_{C,D} : \underline{C}(C, \underline{G}(D)) \longrightarrow \underline{D}(\underline{F}(C), D)$, natural on both variables, which is clearly inverse to $\bar{\eta}$. Then $(\underline{F}, \underline{G})$ is an adjoint pair, as desired. $\qquad \square$

Bearing in mind that $\underline{\mathcal{C}(\mathcal{A})} = \mathcal{H}(\mathcal{A})$, for any abelian category category $\mathcal{A}$, the following definition makes sense.

Definition 13. *Let $\mathcal{A}$ be an AB4 abelian category such that every chain complex of objects of $\mathcal{A}$ admits a homotopically injective resolution, and let $\mathbf{i}_{\mathcal{A}} : \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{H}(\mathcal{A})$ be the homotopically injective resolution functor. If $\mathcal{B}$ is another abelian category and $F : \mathcal{C}(\mathcal{A}) \longrightarrow \mathcal{C}(\mathcal{B})$ is a k-linear functor which takes conflations to conflations and contractible complexes to contractible complexes, then the composition of triangulated funtors*

$$\mathcal{D}(\mathcal{A}) \overset{\mathbf{i}_{\mathcal{A}}}{\longrightarrow} \mathcal{H}(\mathcal{A}) \overset{F}{\longrightarrow} \mathcal{H}(\mathcal{B}) \overset{q_{\mathcal{B}}}{\longrightarrow} \mathcal{D}(\mathcal{B})$$

is called the right derived functor of F, *usually denoted $\mathbf{R}F$.*

Dually, one defines the the left derived functor of F, denoted $\mathbf{L}F$, whenever $\mathcal{A}$ is an AB4* abelian category on which every chain complex admits a homotopically projective resolution.

We now get:

Proposition 2.28. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Suppose that $\mathcal{A}$ is AB4* on which each chain complex has a homotopically projective resolution and $\mathcal{B}$ is AB4 on which each chain complex has a homotopically injective resolution. If $F : \mathcal{C}(\mathcal{A}) \longrightarrow \mathcal{C}(\mathcal{B})$ and $G : \mathcal{C}(\mathcal{B}) \longrightarrow \mathcal{C}(\mathcal{A})$ are k-linear functors which take conflations to conflations and form an adjoint pair (F, G), then the derived functors $\mathbf{L}F : \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{B})$ and $\mathbf{R}F : \mathcal{D}(\mathcal{B}) \longrightarrow \mathcal{D}(\mathcal{A})$ form an adjoint pair $(\mathbf{L}F, \mathbf{R}G)$.*

Proof. By definition, we have

$$\mathbf{L}F : \mathcal{D}(\mathcal{A}) \xrightarrow{\mathbf{p}_{\mathcal{A}}} \mathcal{H}(\mathcal{A}) \xrightarrow{\underline{F}} \mathcal{H}(\mathcal{B}) \xrightarrow{q_{\mathcal{B}}} \mathcal{D}(\mathcal{B})$$
$$\text{and}$$
$$\mathbf{R}G : \mathcal{D}(\mathcal{B}) \xrightarrow{\mathbf{i}_{\mathcal{B}}} \mathcal{H}(\mathcal{A}) \xrightarrow{\underline{G}} \mathcal{H}(\mathcal{A}) \xrightarrow{q_{\mathcal{A}}} \mathcal{D}(\mathcal{A}).$$

The result is an immediate consequence of the fact that $(\mathbf{p}_{\mathcal{A}}, q_{\mathcal{A}})$, $(\underline{F}, \underline{G})$ and $(q_{\mathcal{B}}, \mathbf{i}_{\mathcal{B}})$ are adjoint pairs. □

3. Classical derived functors defined by complexes of bimodules

3.1. **Definition and main adjunction properties.** If A is a k-algebra, we will write $\mathcal{C}(A)$, $\mathcal{H}(A)$ and $\mathcal{D}(A)$ instead of $\mathcal{C}(\mathrm{Mod}A)$, $\mathcal{H}(\mathrm{Mod}A)$ and $\mathcal{D}(\mathrm{Mod}A)$, respectively. This rule applies also to the algebra $B^{op} \otimes A$, for any algebras A and B.

Let A, B and C be k-algebras. Given complexes $T^{\bullet}$ and $M^{\bullet}$, of $B - A-$bimodules and $A - C-$bimodules respectively, we shall associate to them several functors between categories of complexes of bimodules. The material can be found in [40], but the reader is warned on the difference of indization of complexes with respect to that book. The *total tensor product of $T^{\bullet}$ and $M^{\bullet}$*, denoted $T^{\bullet} \otimes_A^{\bullet} M^{\bullet}$, is the complex of $B - C-$bimodules given as follows (see [40, Section 2.7]):

(1) $(T^{\bullet} \otimes_A^{\bullet} M^{\bullet})^n = \oplus_{i+j=n} T^i \otimes_A M^j$, for each $n \in \mathbb{Z}$.
(2) The differential $d^n : (T^{\bullet} \otimes_A^{\bullet} M^{\bullet})^n \longrightarrow (T^{\bullet} \otimes_A^{\bullet} M^{\bullet})^{n+1}$ takes $t \otimes m \rightsquigarrow d_T(t) \otimes m + (-1)^i t \otimes d_M(m)$, whenever $t \in T^i$ and $m \in M^j$

When $T^{\bullet}$ is fixed, the assignment $M^{\bullet} \rightsquigarrow T^{\bullet} \otimes_A^{\bullet} M^{\bullet}$ is the definition on objects of a k-linear functor $T^{\bullet} \otimes_A^{\bullet} ? : \mathcal{C}(A^{op} \otimes C) \longrightarrow \mathcal{C}(B^{op} \otimes C)$. It acts as $f \rightsquigarrow 1_T \otimes f$ on morphisms, where $(1_T \otimes f)(t \otimes m) = t \otimes f(m)$, whenever $t \in T^{\bullet}$ and $m \in M^{\bullet}$ are homogeneous elements. Note that, with a symmetric argument, when $M^{\bullet} \in \mathcal{C}(A^{op} \otimes C)$ is fixed, we also get another k-linear functor $? \otimes_A^{\bullet} M^{\bullet} : \mathcal{C}(B^{op} \otimes A) \longrightarrow \mathcal{C}(B^{op} \otimes C)$ which takes $T^{\bullet} \rightsquigarrow T^{\bullet} \otimes_A^{\bullet} M^{\bullet}$ and $g \rightsquigarrow g \otimes 1_M$. In this way, we get a bifunctor $? \otimes_A^{\bullet} ? : \mathcal{C}(B^{op} \otimes A) \times \mathcal{C}(A^{op} \otimes C) \longrightarrow \mathcal{C}(B^{op} \otimes C)$.

Suppose now that $N^{\bullet}$ is a complex of $C - A-$bimodules. The *total Hom complex* $\mathrm{Hom}_A^{\bullet}(T^{\bullet}, N^{\bullet})$ is the complex of $C - B-$bimodules given as follows (see [40, Section 2.7] or, in a more general context, [19, Section 1.2]):

(1) $\mathrm{Hom}_A^{\bullet}(T^{\bullet}, N^{\bullet})^n = \prod_{j-i=n} \mathrm{Hom}_A(T^i, N^j)$, for each $n \in \mathbb{Z}$.
(2) The differential $\mathrm{Hom}_A^{\bullet}(T^{\bullet}, N^{\bullet})^n \longrightarrow \mathrm{Hom}_A^{\bullet}(T^{\bullet}, N^{\bullet})^{n+1}$ takes $f \rightsquigarrow d_N \circ f - (-1)^n f \circ d_M$

The assigment $N^{\bullet} \rightsquigarrow \mathrm{Hom}_A^{\bullet}(T^{\bullet}, N^{\bullet})$ is the definition on objects of a k-linear functor $\mathrm{Hom}_A^{\bullet}(T^{\bullet}, ?) : \mathcal{C}(C^{op} \otimes A) \longrightarrow \mathcal{C}(C^{op} \otimes B)$. The functor acts on morphism as $\alpha \rightsquigarrow \alpha_*$,

where $\alpha_*(f) = \alpha \circ f$, for each homogeneous element $f \in \operatorname{Hom}_A^\bullet(T^\bullet, N^\bullet)$. Symmetrically, if $N^\bullet \in \mathcal{C}(C^{op} \otimes A)$ is fixed, then the assignment $T^\bullet \rightsquigarrow \operatorname{Hom}_A^\bullet(T^\bullet, N^\bullet)$ is the definition on objects of a functor $\operatorname{Hom}_A^\bullet(?, N^\bullet) : \mathcal{C}(B^{op} \otimes A)^{op} \longrightarrow \mathcal{C}(C^{op} \otimes B)$. It acts on morphisms as $\beta \rightsquigarrow \operatorname{Hom}_A^\bullet(\beta, N^\bullet) =: \beta^*$, where $\beta^*(f) = f \circ \beta$, for each homogeneous element $f \in \operatorname{Hom}_A^\bullet(T^\bullet, N^\bullet)$. In this way, we get a bifunctor $\operatorname{Hom}_A^\bullet(?, ?) : \mathcal{C}(B^{op} \otimes A) \times \mathcal{C}(C^{op} \otimes A) \longrightarrow \mathcal{C}(C^{op} \otimes B)$.

Obviously, we also get bifunctors:

$$?\otimes_B^\bullet? : \mathcal{C}(C^{op} \otimes B) \times \mathcal{C}(B^{op} \otimes A) \longrightarrow \mathcal{C}(C^{op} \otimes A), \qquad (X^\bullet, T^\bullet) \rightsquigarrow X^\bullet \otimes_B^\bullet T^\bullet;$$
$$\operatorname{Hom}_{B^{op}}^\bullet(?, ?) : \mathcal{C}(B^{op} \otimes C) \times \mathcal{C}(B^{op} \otimes A) \longrightarrow \mathcal{C}(C^{op} \otimes A),$$
$$(Y^\bullet, T^\bullet) \rightsquigarrow \operatorname{Hom}_{B^{op}}^\bullet(Y^\bullet, T^\bullet).$$

Proposition 3.1. *Let A, B and C be k-algebras. All the bifunctors $?\otimes_A^\bullet?$, $?\otimes_B^\bullet?$, $\operatorname{Hom}_A^\bullet(?, ?)$ and $\operatorname{Hom}_{B^{op}}^\bullet(?, ?)$ defined above preserve conflations (i.e. semi-split exact sequences) and contractible complexes on each variable. Moreover, if $T^\bullet$ is a complex of $B-A-$bimodules, then the following assertions hold:*

(1) *The pairs $(? \otimes_B^\bullet T^\bullet, \operatorname{Hom}_A^\bullet(T^\bullet, ?))$ and $(T^\bullet \otimes_A^\bullet ?, \operatorname{Hom}_{B^{op}}^\bullet(T^\bullet, ?))$ are adjoint pairs.*

(2) *The pair $(\operatorname{Hom}_{B^{op}}^\bullet(?, T^\bullet) : \mathcal{C}(B^{op} \otimes C) \longrightarrow \mathcal{C}(C^{op} \otimes A)^{op}, \operatorname{Hom}_A^\bullet(?, T^\bullet) : \mathcal{C}(C^{op} \otimes A)^{op} \longrightarrow \mathcal{C}(B^{op} \otimes C))$ is an adjoint pair.*

Proof. It is clear that the k-linear functors $T^\bullet \otimes_A^\bullet ? : \mathcal{C}(A^{op} \otimes C) \longrightarrow \mathcal{C}(B^{op} \otimes C)$, $\operatorname{Hom}_A^\bullet(T^\bullet, ?) : \mathcal{C}(C^{op} \otimes A) \longrightarrow \mathcal{C}(C^{op} \otimes B)$ and $\operatorname{Hom}_A^\bullet(?, T^\bullet) : \mathcal{C}(C^{op} \otimes A) \longrightarrow \mathcal{C}(B^{op} \otimes C)^{op}$, after forgetting about the differentials, induce corresponding functors between the associated graded categories. For instance, we have an induced k-linear functor $? \otimes_B^\bullet T^\bullet : (\operatorname{Mod}(C^{op} \otimes B))^{\mathbb{Z}} \longrightarrow (\operatorname{Mod}(C^{op} \otimes A))^{\mathbb{Z}}$. But then this latter functor preserves split exact sequences. This is exactly saying that $? \otimes_B^\bullet T^\bullet : \mathcal{C}(C^{op} \otimes B) \longrightarrow \mathcal{C}(C^{op} \otimes A)$ takes conflations to conflations. The corresponding argument works for all the other functors.

So all the bifunctors in the list preserve conflations on each component. That they also preserve contractible complexes on each variable, will follow from lemma 2.27 once we prove the adjunctions of assertions (1) and (2).

(1) This goes as in the usual adjunction between the tensor product and the Hom functor in module categories. Concretely, we define

$$\eta = \eta_{M,Y} : \mathcal{C}(C^{op} \otimes A)(M^\bullet \otimes_A^\bullet T^\bullet, Y^\bullet) \longrightarrow \mathcal{C}(C^{op} \otimes B)(M^\bullet, \operatorname{Hom}_A^\bullet(T^\bullet, Y^\bullet))$$

by the rule: $[\eta(f)(m)](t) = f(m \otimes t)$, whenever $m \in M^i$ and $t \in T^j$, for some $i, j \in \mathbb{Z}$. We leave to the reader the routine task of checking that $\eta(f)$ is a chain map $M^\bullet \longrightarrow \operatorname{Hom}_A^\bullet(T^\bullet, Y^\bullet)$. The naturallty of η is then proved as in modules.

(2) We need to define an isomorphism

$$\xi = \xi_{M,Y} : \mathcal{C}(B^{op} \otimes C)(Y^\bullet, \operatorname{Hom}_A^\bullet(M^\bullet, T^\bullet)) \longrightarrow \mathcal{C}(C^{op} \otimes A)^{op}(\operatorname{Hom}_{B^{op}}^\bullet(Y^\bullet, T^\bullet), M^\bullet) =$$
$$\mathcal{C}(C^{op} \otimes A)(M^\bullet, \operatorname{Hom}_{B^{op}}^\bullet(Y^\bullet, T^\bullet)),$$

natural on $M^\bullet \in \mathcal{C}(C^{op} \otimes A)$ and $Y^\bullet \in \mathcal{C}(B^{op} \otimes C)$. Our choice of ξ is identified by the equality $[\xi(f)(m)](y) = (-1)^{|m||y|} f(y)(m)$, for all homogeneous elements $f \in \mathcal{C}(B^{op} \otimes C)(Y^\bullet, \operatorname{Hom}_A^\bullet(M^\bullet, T^\bullet))$, $m \in M^\bullet$ and $y \in Y^\bullet$. $\qquad\square$

With the notation of last proposition, we adopt the following notation:

a) $? \otimes_B^{\mathbf{L}} T : \mathcal{D}(C^{op} \otimes B) \longrightarrow \mathcal{D}(C^{op} \otimes A)$ will denote the left derived functor of $? \otimes_B^{\bullet} T^{\bullet} : \mathcal{C}(C^{op} \otimes B) \longrightarrow \mathcal{C}(C^{op} \otimes A)$

b) $\mathbf{R}\mathrm{Hom}_A(T^{\bullet}, ?) : \mathcal{D}(C^{op} \otimes A) \longrightarrow \mathcal{D}(C^{op} \otimes B)$ will denote the right derived functor of $\mathrm{Hom}_A^{\bullet}(T^{\bullet}, ?) : \mathcal{C}(C^{op} \otimes A) \longrightarrow \mathcal{C}(C^{op} \otimes B)$

c) $\mathbf{R}\mathrm{Hom}_A(?, T^{\bullet}) : \mathcal{D}(C^{op} \otimes A)^{op} \longrightarrow \mathcal{D}(B^{op} \otimes C)$ will denote the right derived functor of $\mathrm{Hom}_A^{\bullet}(?, T^{\bullet}) : \mathcal{C}(C^{op} \otimes A)^{op} \longrightarrow \mathcal{D}(B^{op} \otimes C)$ or, equivalently, the left derived functor of $\mathrm{Hom}_A^{\bullet}(?, T^{\bullet}) : \mathcal{C}(C^{op} \otimes A) \longrightarrow \mathcal{D}(B^{op} \otimes C)^{op}$.

d) $\mathbf{T}_A(?, ?) : \mathcal{D}(B^{op} \otimes A) \times \mathcal{D}(A^{op} \otimes C) \longrightarrow \mathcal{D}(B^{op} \otimes C)$ will denote the composition

$$\mathcal{D}(B^{op} \otimes A) \times \mathcal{D}(A^{op} \otimes C) \overset{\mathbf{p}_{B^{op}\otimes A} \times \mathbf{p}_{B^{op}\otimes A}}{\longrightarrow} \mathcal{H}(B^{op} \otimes A) \times \mathcal{H}(A^{op} \otimes C) \overset{?\otimes_A^{\bullet}?}{\longrightarrow}$$
$$\mathcal{H}(B^{op} \otimes C) \overset{q}{\longrightarrow} \mathcal{D}(B^{op} \otimes C),$$

where the central arrow is induced by the bifunctor $\mathcal{C}(B^{op} \otimes A) \times \mathcal{C}(A^{op} \otimes C) \overset{?\otimes_A^{\bullet}?}{\longrightarrow} \mathcal{C}(B^{op} \otimes C)$ and is well-defined due to proposition 3.1. The bifunctor $\mathbf{T}_A(?, ?)$ is triangulated on each variable.

e) $\mathbf{H}_A(?, ?) : \mathcal{D}(B^{op} \otimes A)^{op} \times \mathcal{D}(C^{op} \otimes A) \longrightarrow \mathcal{D}(C^{op} \otimes B)$ is the composition

$$\mathcal{D}(B^{op} \otimes A)^{op} \times \mathcal{D}(C^{op} \otimes A) \overset{\mathbf{p}_{B^{op}\otimes A} \times \mathbf{i}_{C^{op}\otimes A}}{\longrightarrow} \mathcal{H}(B^{op} \otimes A)^{op} \times \mathcal{H}(C^{op} \otimes A) \overset{\mathrm{Hom}_A^{\bullet}(?,?)}{\longrightarrow}$$
$$\mathcal{H}(C^{op} \otimes B) \overset{q}{\longrightarrow} \mathcal{D}(C^{op} \otimes B),$$

which is a bifunctor which is triangulated on each variable.

e') $\mathbf{H}_{B^{op}}(?, ?) : \mathcal{D}(B^{op} \otimes C)^{op} \times \mathcal{D}(B^{op} \otimes A) \longrightarrow \mathcal{D}(C^{op} \otimes A)$ is a bifunctor, triangulated on both variables, which is defined in a similar way as that in e).

Of course, there are left-right symmetric versions $T^{\bullet} \otimes_A^{\mathbf{L}} ? : \mathcal{D}(A^{op} \otimes C) \longrightarrow \mathcal{D}(B^{op} \otimes C)$, $\mathbf{R}\mathrm{Hom}_{B^{op}}(T^{\bullet}, ?) : \mathcal{D}(B^{op} \otimes C) \longrightarrow \mathcal{D}(A^{op} \otimes C)$ and $\mathbf{R}\mathrm{Hom}_{B^{op}}(?, T^{\bullet}) : \mathcal{D}(B^{op} \otimes C)^{op} \longrightarrow \mathcal{D}(C^{op} \otimes A)$ of the functors in a), b) and c). Their precise definition is left to the reader.

As a direct consequence of propositions 2.28 and 3.1, we get:

Corollary 3.2. *Let A, B and C be k-algebras and let $T^{\bullet}$ be a complex of $B-A-$bimodules. With the notation above, the following pairs of triangulated functors are adjoint pairs:*

(1) $(? \otimes_B^{\mathbf{L}} T^{\bullet}, \mathbf{R}Hom_A(T^{\bullet}, ?))$;

(2) $(T^{\bullet} \otimes_A^{\mathbf{L}} ?, \mathbf{R}Hom_{B^{op}}(T^{\bullet}, ?))$;

(3) $(\mathbf{R}Hom_{B^{op}}(?, T^{\bullet}) : \mathcal{D}(B^{op} \otimes C) \longrightarrow \mathcal{D}(C^{op} \otimes A)^{op}, \mathbf{R}Hom_A(?, T^{\bullet}) : \mathcal{D}(C^{op} \otimes A)^{op} \longrightarrow \mathcal{D}(B^{op} \otimes C))$.

Definition 14. *We will adopt the following terminology, referred to the adjunctions of last corollary:*

(1) $\lambda : 1_{\mathcal{D}(C^{op}\otimes B)} \longrightarrow RHom_A(T^{\bullet}, ?) \circ (? \otimes_B^{\mathbf{L}} T)$ *and* $\delta : (? \otimes_B^{\mathbf{L}} T) \circ RHom_A(T^{\bullet}, ?) \longrightarrow 1_{\mathcal{D}(C^{op}\otimes A)}$ *will be the unit and the counit of the first adjunction.*

(2) $\rho : 1_{\mathcal{D}(A^{op}\otimes C)} \longrightarrow RHom_{B^{op}}(T^{\bullet}, ?) \circ (T^{\bullet} \otimes_A^{\mathbf{L}} ?)$ *and* $\phi : (T^{\bullet} \otimes_A^{\mathbf{L}} ?) \circ RHom_{B^{op}}(T^{\bullet}, ?) \longrightarrow 1_{\mathcal{D}(B^{op}\otimes C)}$ *are the unit and counit of the second adjunction.*

(3) $\sigma : 1_{\mathcal{D}(B^{op}\otimes C)} \longrightarrow \mathbf{R}Hom_A(?, T^{\bullet}) \circ \mathbf{R}Hom_{B^{op}}(?, T^{\bullet})$ *and* $\tau : 1_{\mathcal{D}(C^{op}\otimes A)} \longrightarrow \mathbf{R}Hom_{B^{op}}(?, T^{\bullet}) \circ \mathbf{R}Hom_A(?, T^{\bullet})$ *are the unit and the counit of the third adjunction (note that the last one is an arrow in the opposite direction when the functors are considered as endofunctors of $\mathcal{D}(C^{op} \otimes A)^{op}$).*

The following lemma is very useful:

Lemma 3.3. *In the situation of last definition, let us take $C = k$ and consider the following assertions:*

(1) $\lambda_B : B \longrightarrow \mathbf{R}Hom_A(T^\bullet, ? \otimes_B^{\mathbf{L}} T^\bullet)(B) \cong \mathbf{R}Hom_A(T^\bullet, ?)(T^\bullet)$ *is an isomorphism in* $\mathcal{D}(B)$;

(2) $\delta_{T^\bullet} : [\mathbf{R}Hom_A(T^\bullet, ?) \otimes_B^{\mathbf{L}} T^\bullet](T^\bullet) \longrightarrow T^\bullet$ *is an isomorphism in* $\mathcal{D}(A)$;

(3) $\sigma_B : B \longrightarrow [\mathbf{R}Hom_A(\mathbf{R}Hom_{B^{op}}(?, T^\bullet), T^\bullet)](B) \cong \mathbf{R}Hom_A(?, T^\bullet)(T^\bullet)$ *is an isomorphism in* $\mathcal{D}(B^{op})$;

(4) $\tau_{T^\bullet} : T^\bullet \longrightarrow [\mathbf{R}Hom_{B^{op}}(\mathbf{R}Hom_A(?, T^\bullet), T^\bullet)](T^\bullet)$ *is an isomorphism in* $\mathcal{D}(A)$.

Then the implications (2) $\Longleftarrow$ (1) $\Longleftrightarrow$ (3) $\Longrightarrow$ (4) *hold true.*

Proof. (1) $\Longrightarrow$ (2) Putting $F = ? \otimes_B^{\mathbf{L}} T^\bullet$ and $G = \mathbf{R}Hom_A(T^\bullet, ?)$, the truth of the implication is a consequence of the adjunction equation $1_{F(B)} = \delta_{F(B)} \circ F(\lambda_B)$ and the fact that $F(B) \cong T^\bullet$.

(3) $\Longrightarrow$ (4) also follows from the equations of the adjunction $(\mathbf{R}\mathrm{Hom}_{B^{op}}(?, T^\bullet), \mathbf{R}\mathrm{Hom}_A(?, T^\bullet))$ (see corollary 3.2).

(1) $\Longleftrightarrow$ (3) Let $\mathbf{p}_A, \mathbf{i}_A : \mathcal{D}(A) \longrightarrow \mathcal{H}(A)$ be the homotopically projective resolution functor and the homotopically injective resolution functor, respectively. By definition of the derived functors, we have $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?)(T^\bullet) = \mathrm{Hom}_A^\bullet(T^\bullet, \mathbf{i}_A T^\bullet)$ and $\mathbf{R}\mathrm{Hom}_A(?, T^\bullet)(T^\bullet) = \mathrm{Hom}_A^\bullet(\mathbf{p}_A T^\bullet, T^\bullet)$. Let then $\pi : \mathbf{p}_A T^\bullet \longrightarrow T^\bullet$ and $\iota : T^\bullet \longrightarrow \mathbf{i}_A T^\bullet$ be the canonical quasi-isomorphisms. We then have quasi-isomorphisms in $\mathcal{C}(k)$:

$$\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?)(T^\bullet) = \mathrm{Hom}_A^\bullet(T^\bullet, \mathbf{i}_A T^\bullet) \xrightarrow{\pi^*} \mathrm{Hom}_A^\bullet(\mathbf{p}_A T^\bullet, \mathbf{i}_A T^\bullet) \xleftarrow{\iota_*} \mathrm{Hom}_A^\bullet(\mathbf{p}_A T^\bullet, T^\bullet) \cong$$
$$\mathbf{R}\mathrm{Hom}_A(?, T^\bullet)(T^\bullet)$$

Note that λ_B and σ_B are the compositions:

$$\lambda_B : B \longrightarrow \mathrm{Hom}_A^\bullet(T^\bullet, T^\bullet) \xrightarrow{\tilde{\iota}} \mathrm{Hom}_A^\bullet(T^\bullet, \mathbf{i}_A T^\bullet) = \mathbf{R}\mathrm{Hom}_A(T^\bullet, ?)(T^\bullet),$$
$$\text{and}$$
$$\sigma_B : B \longrightarrow \mathrm{Hom}_A^\bullet(T^\bullet, T^\bullet) \xrightarrow{\tilde{\pi}} \mathrm{Hom}_A^\bullet(\mathbf{p}_A T^\bullet, T^\bullet) = \mathbf{R}\mathrm{Hom}_A(?, T^\bullet)(T^\bullet),$$

where the first arrow, in both cases, takes $b \rightsquigarrow \lambda_b : t \rightsquigarrow bt$, and the other arrows are $\tilde{\iota} = \mathrm{Hom}_A^\bullet(T^\bullet, \iota)$ and $\tilde{\pi} = \mathrm{Hom}_A^\bullet(\pi, T^\bullet)$.

A direct easy calculation shows that the equality $\pi^* \circ \lambda_B = \iota_* \circ \sigma_B$ holds in $\mathcal{C}(k)$. As a consequence, λ_B is a quasi-isomorphism if, and only if, so is σ_B. $\qquad\square$

We explicitly state the left-right symmetric of the previous lemma since it will be important for us:

Lemma 3.4. *In the situation of last definition, let us take $C = k$ and consider the following assertions:*

(1) $\rho_A : A \longrightarrow \mathbf{R}Hom_{B^{op}}(T^\bullet, T^\bullet \otimes_A^{\mathbf{L}} ?)(A) \cong \mathbf{R}Hom_{B^{op}}(T^\bullet, ?)(T^\bullet)$ *is an isomorphism in* $\mathcal{D}(A^{op})$;

(2) $\phi_T : [T^\bullet \otimes_A^{\mathbf{L}} \otimes \mathbf{R}Hom_A(T^\bullet, ?)](T^\bullet) \longrightarrow T^\bullet$ *is an isomorphism in* $\mathcal{D}(B^{op})$;

(3) $\tau_A : A \longrightarrow [\mathbf{R}Hom_{B^{op}}(\mathbf{R}Hom_A(?, T^\bullet), T^\bullet)](A) \cong \mathbf{R}Hom_{B^{op}}(?, T^\bullet)(T^\bullet)$ *is an isomorphism in* $\mathcal{D}(A)$;

(4) $\sigma_T : T^\bullet \longrightarrow [\mathbf{R}Hom_A(\mathbf{R}Hom_{B^{op}}(?, T^\bullet), T^\bullet)](T^\bullet)$ *is an isomorphism in* $\mathcal{D}(B^{op})$.

Then the implications (2) $\Longleftarrow$ (1) $\Longleftrightarrow$ (3) $\Longrightarrow$ (4) *hold true.*

3.2. **Homotopically flat complexes. Restrictions.** In order to deal with the derived tensor product, it is convenient to introduce a class of chain complexes which is wider than that of the homotopically projective ones.

Definition 15. *Let B be any algebra. A complex $F^\bullet \in \mathcal{C}(B)$ is called* homotopically flat *when the functor $F^\bullet \otimes_B^\bullet ? : \mathcal{C}(B) \longrightarrow \mathcal{C}(k)$ preserves acyclic complexes.*

The key points are assertion (3) and (4) of the following result.

Lemma 3.5. *Let A and B be k-algebras. The following assertions hold:*

(1) *If $Z^\bullet \in \mathcal{C}(B)$ is acyclic and homotopically flat, then the essential image of $Z^\bullet \otimes_B^\bullet ? : \mathcal{H}(B^{op}) \longrightarrow \mathcal{H}(k)$ consists of acyclic complexes;*

(2) *Each homotopically projective object of $\mathcal{H}(B)$ is homotopically flat;*

(3) *If $F^\bullet \overset{s}{\longrightarrow} M^\bullet$ is a quasi-isomorphism in $\mathcal{C}(B)$, where $F^\bullet$ is homotopically flat, then, for each $N^\bullet \in \mathcal{C}(B^{op} \otimes A)$, we have an isomorphicm $(? \otimes_B^{\mathbf{L}} N^\bullet)(M^\bullet) \cong F^\bullet \otimes_B^\bullet N^\bullet$ in $\mathcal{D}(A)$;*

(4) *If $F^\bullet$ is homotopically flat in $\mathcal{H}(B)$ and $N^\bullet$ is a complex of $B - A$-bimodules, then $(F^\bullet \otimes_B^{\mathbf{L}} ?)(N^\bullet) = F^\bullet \otimes_B^\bullet N^\bullet$ in $\mathcal{D}(A)$.*

Proof. All throughout the proof we will use the fact that $\mathcal{D}(B)$ (resp. $\mathcal{D}(B^{op})$) is compactly generated by $\{B\}$ (see proposition 3.13 below).

(1) The given functor $T := Z^\bullet \otimes_B^\bullet ?$ is triangulated and take acyclic complexes to acyclic complexes. Then it preserves quasi-isomorphisms. By construction of the derived category, we then get a unique triangulated functor $\bar{T} : \mathcal{D}(B^{op}) \longrightarrow \mathcal{D}(k)$ such that $\bar{T} \circ q_B = q_k \circ T$, where $q_B : \mathcal{H}(B^{op}) \longrightarrow \mathcal{D}(B^{op})$ and $q_k : \mathcal{H}(k) \longrightarrow \mathcal{D}(k)$ are the canonical functor.

We will prove that $\bar{T}$ is the zero functor and this will prove the assertion. We consider the full subcategory $\mathcal{X}$ of $\mathcal{D}(B^{op})$ consisting of the complexes $M^\bullet$ such that $\bar{T}(M^\bullet) = 0$. It is a triangulated subcategory, closed under taking arbitrary coproducts, which contains B. We then have $\mathcal{X} = \mathcal{D}(B)$.

(2) By [19, Theorem P], each homotopically projective complex is isomorphic in $\mathcal{H}(A)$ to a complex $P^\bullet$ which admits a countable filtration

$$P_0^\bullet \subset P_1^\bullet \subset ... \subset P_n^\bullet \subset ...,$$

in $\mathcal{C}(B)$, where the inclusions are inflations and where $P_0^\bullet$ and all the factors $P_n^\bullet / P_{n-1}^\bullet$ are direct summands of direct sums of stalk complexes of the form $B[k]$, with $k \in \mathbb{Z}$. We just need to check that this $P^\bullet$ is homotopically flat. Due to the fact that the bifunctor $? \otimes_B^\bullet ? : \mathcal{C}(B) \times \mathcal{C}(B^{op}) \longrightarrow \mathcal{C}(k)$ preserves direct limits on both variables, with an evident induction argument, the proof is easily reduced to the case when $P^\bullet = B[r]$, for some $r \in \mathbb{Z}$, in which case it is trivial.

(3) Let $P^\bullet := \mathbf{p}M^\bullet \overset{\pi}{\longrightarrow} M^\bullet$ be the homotopically projective resolution. Then $s^{-1} \circ \pi \in \mathcal{D}(B)(P^\bullet, F^\bullet) \cong \mathcal{H}(B)(P^\bullet, F^\bullet)$. Then we have a chain map $f : P^\bullet \longrightarrow F^\bullet$ such that $s \circ f = \pi$ in $\mathcal{H}(B)$. In particular, f is a quasi-isomoprhism between homotopically flat objects. Its cone is then an acyclic and homotopically flat complex. By assertion (1), we conclude that $f \otimes_B^\bullet 1_{N^\bullet}$ has an acyclic cone and, hence, it is a quasi-isomorphism, for each $N^\bullet \in \mathcal{C}(B^{op} \otimes A)$. We then have an isomorphism $(? \otimes_B^{\mathbf{L}} N^\bullet)(M^\bullet) = P^\bullet \otimes_B^\bullet N^\bullet \overset{f \otimes^\bullet 1_{N^\bullet}}{\longrightarrow} F^\bullet \otimes_B^\bullet N^\bullet$ in $\mathcal{D}(A)$.

(4) Let us consider the triangle in $\mathcal{H}(B^{op} \otimes A)$

$$\mathbf{p}_{B^{op} \otimes A} N^{\bullet} \xrightarrow{\pi} N^{\bullet} \longrightarrow Z^{\bullet} \xrightarrow{+}$$

afforded by the homotopically projective resolution of $N^{\bullet}$. Then $Z^{\bullet}$ is acyclic and the homotopically flat condition of $F^{\bullet}$ tells us that $(F^{\bullet} \otimes_B^{\mathbf{L}} ?)(N^{\bullet}) = F^{\bullet} \otimes_B^{\bullet} \mathbf{p}_{B^{op} \otimes A} N^{\bullet} \xrightarrow{1 \otimes \pi} F^{\bullet} \otimes_B^{\bullet} N^{\bullet}$ is a quasi-isomorphism and, therefore, an isomorphism in $\mathcal{D}(A)$. $\qquad \square$

In order to understand triangulated functors between derived categories of bimodules as 'one-sided' triangulated functors, it is important to know how homotopically projective or injective resolutions behave with respect to the restriction functors. The following result and its left-right symmetric are important in this sense.

Lemma 3.6. *Let A and B be algebras. The following assertions hold:*

(1) *If A is k-flat, then the forgetful functor $\mathcal{H}(A^{op} \otimes B) \longrightarrow \mathcal{H}(B)$ preserves homotopically injective complexes and takes homotopically projective complexes to homotopically flat ones.*

(2) *If A is k-projective, then the forgetful functor $\mathcal{H}(A^{op} \otimes B) \longrightarrow \mathcal{H}(B)$ preserves homotopically projective complexes.*

Proof. Assertion (2) and the part of assertion (1) concerning the preservation of homotopically injective complexes are proved in [32] in a much more general contex. Suppose now that A is k-flat and let $P^{\bullet} \in \mathcal{H}(A^{op} \otimes B)$ be homotopically projective. Using [19, Theorem P], we can assume that $P^{\bullet}$ admits a countable filtration

$$P_0^{\bullet} \subset P_1^{\bullet} \subset ... \subset P_n^{\bullet} \subset ...,$$

where the inclusions are inflations and where $P_0^{\bullet}$ and all quotients $P_n^{\bullet}/P_{n-1}^{\bullet}$ are direct summands of coproducts of stalk complexes of the form $A \otimes B[r]$. That A is k-flat implies that it is the direct limit in $\text{Mod} - k$ of a direct system of finitely generated free k-modules (see [26, Théoréme 1.2]). It follows that, in $\mathcal{C}(B)$, $A \otimes B[r]$ is a direct limit stalk complexes of the form $B^{(k)}[r]$, all of which are homotopically flat in $\mathcal{C}(B)$. Bearing in mind that the bifunctor $? \otimes_B^{\bullet} ?$ preserves direct limits on both variables, one gets that each stalk complex $A \otimes B[r]$ is homotopically flat in $\mathcal{C}(B)$, and then one easily proves by induction that each $P_n^{\bullet}$ in the filtration is homotopically flat in $\mathcal{C}(B)$. But, by definition, the direct limit in $\mathcal{C}(B)$ of homotopically flat complexes is homotopically flat. $\qquad \square$

3.3. **Classical derived functors as components of a bifunctor.** It would be a natural temptation to believe that if $T^{\bullet} \in \mathcal{C}(B^{op} \otimes A)$ and $M^{\bullet} \in \mathcal{C}(A^{op} \otimes C)$, then we have isomorphisms $\mathbf{T}_A(T^{\bullet}, M^{\bullet}) \cong (T^{\bullet} \otimes_A^{\mathbf{L}} ?)(M^{\bullet})$ and $\mathbf{T}_A(T^{\bullet}, M^{\bullet}) \cong (? \otimes_A^{\mathbf{L}} M^{\bullet})(T^{\bullet})$ in $\mathcal{D}(B^{op} \otimes C)$. Similarly, one could be tempted to believe that if $T^{\bullet}$ is as above and $N^{\bullet} \in \mathcal{C}(C^{op} \otimes A)$, then one has isomorphisms $\mathbf{H}_A(T^{\bullet}, N^{\bullet}) \cong \mathbf{R}\text{Hom}_A(T^{\bullet}, ?)(N^{\bullet})$ and $\mathbf{H}_A(T^{\bullet}, N^{\bullet}) \cong \mathbf{R}\text{Hom}_A(?, N^{\bullet})(T^{\bullet})$ in $\mathcal{D}(C^{op} \otimes B)$. However, we need extra hypotheses to guarantee that.

Proposition 3.7. *Let A, B and C be k-algebras and let ${}_B T_A^{\bullet}$, ${}_A M_C^{\bullet}$ and ${}_C N_A^{\bullet}$ complexes of bimodules over the indicated algebras. There exist canonical natural transformations of triangulated functors:*

(1) $\mathbf{T}_A(T^{\bullet}, ?) \longrightarrow T^{\bullet} \otimes_A^{\mathbf{L}} ? : \mathcal{D}(A^{op} \otimes C) \longrightarrow \mathcal{D}(B^{op} \otimes C)$;

(2) $\mathbf{T}_A(?, M) \longrightarrow ? \otimes_A^{\mathbf{L}} M^\bullet : \mathcal{D}(B^{op} \otimes A) \longrightarrow \mathcal{D}(B^{op} \otimes C)$;

(3) $\mathbf{R}Hom_A(T^\bullet, ?) \longrightarrow \mathbf{H}_A(T^\bullet, ?) : \mathcal{D}(C^{op} \otimes A) \longrightarrow \mathcal{D}(C^{op} \otimes B)$;

(4) $\mathbf{R}Hom_A(?, N^\bullet) \longrightarrow \mathbf{H}_A(?, N^\bullet) : \mathcal{D}(B^{op} \otimes A)^{op} \longrightarrow \mathcal{D}(C^{op} \otimes B)$

Moreover, the following assertions hold

a) *If C is k-flat, then the natural transformations 1 and 3 are isomorphism;*

b) *if B is k-flat, then the natural transformation 2 is an isomorphism;*

c) *if B is k-projective, then the natural transformation 4 is an isomorphism.*

Proof. (1) By definition, we have $\mathbf{T}_A(T^\bullet, M^\bullet) = (\mathbf{p}_{B^{op}\otimes A}T^\bullet)\otimes_A^\bullet(\mathbf{p}_{A^{op}\otimes C}M^\bullet)$ and $(T^\bullet\otimes_A^{\mathbf{L}}?)(M^\bullet) = T^\bullet \otimes_A^\bullet (\mathbf{p}_{A^{op}\otimes C}M^\bullet)$. If $\pi_T : (\mathbf{p}_{B^{op}\otimes A}T^\bullet) \longrightarrow T^\bullet$ is the homotopically projective resolution, we clearly have a chain map

$$\mathbf{T}_A(T^\bullet, M^\bullet) = (\mathbf{p}_{B^{op}\otimes A}T^\bullet) \otimes_A^\bullet (\mathbf{p}_{A^{op}\otimes C}M^\bullet) \xrightarrow{\pi_T\otimes^\bullet 1} T^\bullet \otimes_A^\bullet (\mathbf{p}_{A^{op}\otimes C}M^\bullet) = (T^\bullet\otimes_A^{\mathbf{L}}?)(M^\bullet)$$

That this map defines a natural transformation $\mathbf{T}_A(T^\bullet, ?) \longrightarrow T^\bullet\otimes_A^{\mathbf{L}}?$ is routine.

(2) follows as (1), by applying a left-right symmetric argument.

(3), (4) By definition again, we have $\mathbf{H}_A(T^\bullet, N^\bullet) = \mathrm{Hom}_A^\bullet(\mathbf{p}_{B^{op}\otimes A}T^\bullet, \mathbf{i}_{C^{op}\otimes A}N^\bullet)$, $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?)(N^\bullet) = \mathrm{Hom}_A^\bullet(T^\bullet, \mathbf{i}_{C^{op}\otimes A}N^\bullet)$ and $\mathbf{R}\mathrm{Hom}_A(?, N^\bullet)(T^\bullet) = \mathrm{Hom}_A^\bullet(\mathbf{p}_{B^{op}\otimes A}T^\bullet, N^\bullet)$. If π_T is as above and $\iota_N : N^\bullet \longrightarrow \mathbf{i}_{C^{op}\otimes A}N^\bullet$ is the homotopically injective resolution, we then have obvious chain maps

$$\mathrm{Hom}_A^\bullet(\mathbf{p}_{B^{op}\otimes A}T^\bullet, N^\bullet) \xrightarrow{(\iota_N)_*} \mathrm{Hom}_A^\bullet(\mathbf{p}_{B^{op}\otimes A}T^\bullet, \mathbf{i}_{C^{op}\otimes A}N^\bullet) \xleftarrow{\pi_T^*} \mathrm{Hom}_A^\bullet(T^\bullet, \mathbf{i}_{C^{op}\otimes A}N^\bullet).$$

It is again routine to see that they induce natural transformations of triangulated functors

$$\mathbf{R}\mathrm{Hom}_A(?, N^\bullet) \longrightarrow \mathbf{H}_A(?, N^\bullet) \text{ and } \mathbf{H}_A(T^\bullet, ?) \longleftarrow \mathbf{R}\mathrm{Hom}_A(T^\bullet, ?).$$

On the other hand, when C is k-flat, by lemma 3.6, we know that the forgetful functor $\mathcal{H}(A^{op} \otimes C) \longrightarrow \mathcal{H}(A^{op})$ (resp. $\mathcal{H}(C^{op} \otimes A) \longrightarrow \mathcal{H}(A)$) takes homotopically projective objects to homotopically flat objects (resp. preserves homotopically injective objects). In particular, the morphisms $\pi_T\otimes^\bullet 1$ and π_T^* considered above, are both quasi-isomorphisms, which gives assertion a). Assertion b) follows from the part of a) concerning the derived tensor product by a symmetric argument.

Finally, if B is k-projective, then, by lemma 3.6, the forgetful functor $\mathcal{H}(B^{op} \otimes A) \longrightarrow \mathcal{H}(A)$ preserves homotopically projective objects. Then the map $(\iota_N)_*$ above is a quasi-isomorphism.

$\square$

Lemma 3.8. *Let A, B and C be k-algebras. The following assertions hold:*

(1) *The assignment $(T^\bullet, X^\bullet) \rightsquigarrow Hom_{B^{op}}^\bullet(T^\bullet, B) \otimes_B^\bullet X^\bullet$ is the definition on objects of a bifunctor*

$$Hom_{B^{op}}^\bullet(?, B)\otimes_B^\bullet? : \mathcal{C}(B^{op} \otimes A)^{op} \times \mathcal{C}(B^{op} \otimes C) \longrightarrow \mathcal{C}(A^{op} \otimes C)$$

which preserves conflations and contractible complexes on each variable.

(2) *There is a natural transformation of bifunctors*

$$\psi : Hom_{B^{op}}^\bullet(?, B)\otimes_B^\bullet? \longrightarrow Hom_{B^{op}}^\bullet(?, ?).$$

Proof. Assertion (1) is routine and left to the reader. As for assertion (2), let us fix $T^\bullet \in \mathcal{C}(B^{op} \otimes A)$ and $X^\bullet \in \mathcal{C}(B^{op} \otimes C)$. We need to define a map

$$\psi = \psi_{T,X} : \text{Hom}^{\bullet}_{B^{op}}(T^{\bullet}, B) \otimes^{\bullet}_B X^{\bullet} \longrightarrow \text{Hom}^{\bullet}_{B^{op}}(T^{\bullet}, X^{\bullet}).$$

Let $f \in \text{Hom}^{\bullet}_{B^{op}}(T^{\bullet}, B)$ and $x \in X^{\bullet}$ be homogeneous elements, whose degrees are denoted by $|f|$ and $|x|$. We define $\psi(f \otimes x)(t) = (-1)^{|f| \cdot |x|} f(t)x$, for all homogeneous elements $f \in \text{Hom}^{\bullet}_{B^{op}}(T^{\bullet}, B)$, $x \in X^{\bullet}$ and $t \in T^{\bullet}$, and leave to the reader the routine task of checking that ψ is a chain map of complexes of $A - C-$bimodules which is natural on both variables. $\qquad \square$

Assertion 1 in the previous lemma allows us to define the following bifunctor, which is triangulated on both variables:

$$\mathbf{TH}(?,?) : \mathcal{D}(B^{op} \otimes A)^{op} \times \mathcal{D}(B^{op} \otimes C) \xrightarrow{\mathbf{P}_{B^{op} \otimes A} \times \mathbf{P}_{B^{op} \otimes C}}$$

$$\mathcal{H}(B^{op} \otimes A)^{op} \times \mathcal{H}(B^{op} \otimes C) \xrightarrow{\text{Hom}^{\bullet}_{B^{op}}(?,B) \otimes^{\bullet}_B ?} \mathcal{H}(A^{op} \otimes C) \xrightarrow{q} \mathcal{D}(A^{op} \otimes C).$$

We have the following property of this bifunctor.

Proposition 3.9. *Let A, B and C be k-algebras and let $T^{\bullet} \in \mathcal{C}(B^{op} \otimes A)$ and $Y^{\bullet} \in \mathcal{C}(B^{op} \otimes C)$ be complexes of bimodules. The natural transformation of lemma 3.8 induces:*

(1) *A natural transformation of bifunctors $\theta : \mathbf{TH}(?,?) \longrightarrow \mathbf{H}_{B^{op}}(?,?)$ and natural transformation $\theta_{Y^{\bullet}} : (? \otimes^{\mathbf{L}}_B Y^{\bullet}) \circ \mathbf{R}Hom_{B^{op}}(?, B) \longrightarrow \mathbf{R}Hom_{B^{op}}(?, Y^{\bullet})$ of triangulated functors $\mathcal{D}(B^{op} \otimes A) \longrightarrow \mathcal{D}(A^{op} \otimes C)$. When A is k-projective, there is a natural transformation $\upsilon_{Y^{\bullet}} : (? \otimes^{\mathbf{L}}_B Y^{\bullet}) \circ \mathbf{R}Hom_{B^{op}}(?, B) \longrightarrow \mathbf{TH}(?, Y^{\bullet})$ such that $\theta_{Y^{\bullet}} \cong \theta(?, Y^{\bullet}) \circ \upsilon_{Y^{\bullet}}$. If, in addition, C is k-flat, then $\upsilon_{Y^{\bullet}}$ is a natural isomorphism.*

(2) *When C is k-flat, θ induces a natural transformation $\theta^{T^{\bullet}} : \mathbf{R}Hom_{B^{op}}(T^{\bullet}, B) \otimes^{\mathbf{L}}_B ? \cong \mathbf{TH}(T^{\bullet}, ?) \longrightarrow \mathbf{H}_{B^{op}}(T^{\bullet}, ?) \cong \mathbf{R}Hom_{B^{op}}(T^{\bullet}, ?)$ of triangulated functors $\mathcal{D}(B^{op} \otimes C) \longrightarrow \mathcal{D}(A^{op} \otimes C)$.*

Proof. All throughout the proof, for simplicity, whenever $M^{\bullet}$ is a complex of bimodules, we will denote by $\pi = \pi_M : \mathbf{p}M^{\bullet} \longrightarrow M^{\bullet}$ and $\iota = \iota_M : M^{\bullet} \longrightarrow \mathbf{i}M^{\bullet}$, respectively, the homotopically projective resolution and the homotopically injective resolution, without explicitly mentioning over which algebras it is a complex of bimodules. In each case, it should be clear from the context which algebras they are.

(1) We define $\theta_{(T,Y)}$ as the composition

$$\mathbf{TH}(T^{\bullet}, Y^{\bullet}) = \text{Hom}^{\bullet}_{B^{op}}(\mathbf{p}T^{\bullet}, B) \otimes^{\bullet}_B \mathbf{p}Y^{\bullet} \xrightarrow{\psi} \text{Hom}^{\bullet}_{B^{op}}(\mathbf{p}T^{\bullet}, \mathbf{p}Y^{\bullet}) \xrightarrow{(\pi_Y)_*}$$

$$\text{Hom}^{\bullet}_{B^{op}}(\mathbf{p}T^{\bullet}, Y^{\bullet}) \xrightarrow{(\iota_Y)_*} \text{Hom}^{\bullet}_{B^{op}}(\mathbf{p}T^{\bullet}, \mathbf{i}Y^{\bullet}) = \mathbf{H}_{B^{op}}(T^{\bullet}, Y^{\bullet}).$$

Checking its naturality on both variables is left to the reader.

As definition of $\theta_{Y^{\bullet}}$, we take the composition:

$$[(? \otimes^{\mathbf{L}}_B Y) \circ \mathbf{R}Hom_{B^{op}}(?, B)](T^{\bullet}) = \mathbf{p}\text{Hom}^{\bullet}_{B^{op}}(\mathbf{p}T^{\bullet}, B) \otimes^{\bullet}_B Y^{\bullet} \xrightarrow{\pi \otimes^{\bullet} 1}$$

$$\text{Hom}^{\bullet}_{B^{op}}(\mathbf{p}T^{\bullet}, B) \otimes^{\bullet}_B Y^{\bullet} \xrightarrow{\psi} \text{Hom}^{\bullet}_{B^{op}}(\mathbf{p}T^{\bullet}, Y^{\bullet}) = \mathbf{R}Hom_{B^{op}}(?, Y^{\bullet})(T^{\bullet}).$$

When A is k-projective, we have a natural isomorphism $\mathbf{R}Hom_{B^{op}}(?, Y^{\bullet}) \cong \mathbf{H}_{B^{op}}(?, Y^{\bullet})$ (see proposition 3.7 c)). On the other hand, we have morphisms in $\mathcal{H}(A^{op} \otimes B)$

$$\mathbf{TH}(T^{\bullet}, Y^{\bullet}) = \text{Hom}^{\bullet}_{B^{op}}(\mathbf{p}T^{\bullet}, B) \otimes^{\bullet}_B \mathbf{p}Y^{\bullet} \xleftarrow{\pi_H \otimes^{\bullet} 1} \mathbf{p}\text{Hom}^{\bullet}_{B^{op}}(\mathbf{p}T^{\bullet}, B) \otimes^{\bullet}_B \mathbf{p}Y^{\bullet} \xrightarrow{1 \otimes^{\bullet} \pi_Y}$$

$$\mathbf{p}\text{Hom}^{\bullet}_{B^{op}}(\mathbf{p}T^{\bullet}, B) \otimes^{\bullet}_B Y^{\bullet} = [(? \otimes^{\mathbf{L}}_B Y) \circ \mathbf{R}Hom_{B^{op}}(?, B)](T^{\bullet}),$$

where π_H is taken for $H^\bullet := \mathrm{Hom}^\bullet_{B^{op}}(\mathbf{p}T^\bullet, B)$. The right morphism is a quasi-isomorphism since the k-projectivity of A guarantees that $\mathbf{p}\mathrm{Hom}^\bullet_{B^{op}}(\mathbf{p}T^\bullet, B)$ is homotopically projective in $\mathcal{H}(B)$. The evaluation of the natural transformation v_Y at $T^\bullet$ is $v_{Y,T} = (\pi_H \otimes^\bullet 1) \circ (1 \otimes^\bullet \pi_Y)^{-1} : [(? \otimes \otimes^{\mathbf{L}}_B Y) \circ \mathbf{R}\mathrm{Hom}_{B^{op}}(?, B)](T^\bullet) \longrightarrow \mathbf{TH}(T^\bullet, Y^\bullet)$. This is an isomorphism if, and only if, $\pi_H \otimes^\bullet 1$ is an isomorphism in $\mathcal{D}(A^{op} \otimes C)$. For this to happen it is enough that C be k-flat, for then $\mathbf{p}Y^\bullet$ is homotopically flat in $\mathcal{H}(B^{op})$.

The fact that, when A is k-projective, we have an equality $\theta_Y = \theta(?, Y) \circ v_Y$ follows from the commutativity in $\mathcal{H}(A^{op} \otimes B)$ of the following diagram:

$$
\begin{array}{ccc}
& & \left[(? \otimes^{\mathbf{L}}_B Y^\bullet) \circ \mathbf{R}\mathrm{Hom}_{B^{op}}(?, B)\right](T^\bullet) \\
& & \| \\
\mathbf{p}\mathrm{Hom}^\bullet_{B^{op}}(\mathbf{p}T^\bullet, B) \otimes_B \mathbf{p}Y^\bullet \xrightarrow{\;1 \otimes \pi_Y\;} & & \mathbf{p}\mathrm{Hom}^\bullet_{B^{op}}(\mathbf{p}T^\bullet, B \otimes_B Y^\bullet) \\
\downarrow{\scriptstyle \pi_H \otimes 1} & & \downarrow{\scriptstyle \pi_H \otimes 1} \\
\mathbf{TH}(T^\bullet, Y^\bullet) = \mathrm{Hom}_{B^{op}}(\mathbf{p}T^\bullet, B) \otimes \mathbf{p}Y^\bullet \xrightarrow{\;1 \otimes \pi_Y\;} & & \mathrm{Hom}_{B^{op}}(\mathbf{p}T^\bullet, B) \otimes Y^\bullet \\
\downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} \\
\mathrm{Hom}_{B^{op}}(\mathbf{p}T^\bullet, \mathbf{p}Y^\bullet) \xrightarrow{\;(\pi_Y)_*\;} & & \mathrm{Hom}_{B^{op}}(\mathbf{p}T^\bullet, Y^\bullet) = \mathbf{R}\mathrm{Hom}_{B^{op}}(?, Y^\bullet)(T^\bullet) \\
& & \downarrow{\scriptstyle (\iota_Y)_*} \\
& & \mathrm{Hom}_{B^{op}}(\mathbf{p}T^\bullet, \iota Y^\bullet) = \mathbf{H}_{B^{op}}(T^\bullet, Y^\bullet)
\end{array}
$$

(2) Note that, by definition of the involved functors, we have an equality $\mathbf{R}\mathrm{Hom}_{B^{op}}(T, B) \otimes^{\mathbf{L}}_B ? = \mathbf{TH}(T^\bullet, ?)$. Then assertion (2) is a consequence of assertion (1) and proposition 3.7 a). $\qquad\square$

Definition 16. *Let A and B be k-algebras. We shall denote by $(?)^*$ both functors $\mathbf{R}Hom_B(?, B) : \mathcal{D}(A^{op} \otimes B)^{op} \longrightarrow \mathcal{D}(B^{op} \otimes A)$ and $\mathbf{R}Hom_{B^{op}}(?, B) : \mathcal{D}(B^{op} \otimes A)^{op} \longrightarrow \mathcal{D}(A^{op} \otimes B)$.*

Note that, with the appropriate interpretation, the functors $(?)^*$ are adjoint to each other due to corollary 3.2. In particular, we have the corresponding unit $\sigma : 1 \longrightarrow (?)^{**}$ for the two possible compositions. The following result is now crucial for us:

Proposition 3.10. *Let A, B and C be k-algebras, let $T^\bullet$ be a complex of $B - A -$ bimodules such that $_BT^\bullet$ is compact in $\mathcal{D}(B^{op})$ and suppose that A is k-projective. The following assertions hold:*

(1) *For each complex $Y^\bullet$ of $B - C -$ bimodules, the map $\theta_Y(T^\bullet) : (? \otimes^{\mathbf{L}}_B Y)(T^{\bullet*}) = [(? \otimes^{\mathbf{L}}_B Y) \circ \mathbf{R}Hom_{B^{op}}(?, B)](T^\bullet) \longrightarrow \mathbf{R}Hom_{B^{op}}(?, Y^\bullet)(T^\bullet)$ is an isomorphism in $\mathcal{D}(A^{op} \otimes C)$;*

(2) *$T^{\bullet*}_B$ is compact in $\mathcal{D}(B)$ and he map $\sigma_T : T^\bullet \longrightarrow T^{\bullet**}$ is an isomophism in $\mathcal{D}(B^{op} \otimes A)$;*

(3) *When in addition C is k-flat, the following assertions hold:*

  (a) *There is a natural isomorphism of triangulated functors $T^{\bullet*} \otimes^{\mathbf{L}}_B ? \cong \mathbf{R}Hom_{B^{op}}(T^\bullet, ?) : \mathcal{D}(B^{op} \otimes C) \longrightarrow \mathcal{D}(A^{op} \otimes C)$;*

(b) *There is a natural isomorphism of triangulated functors* $? \otimes_B^{\mathbf{L}} T^\bullet \cong \mathbf{R}\mathit{Hom}_B(T^{\bullet*}, ?)$: $\mathcal{D}(C^{op} \otimes B) \longrightarrow \mathcal{D}(A^{op} \otimes C)$.

Proof. All throughout the proof we use the fact that $\mathrm{per}(B) = \mathrm{thick}_{\mathcal{D}(B)}(B)$ and similarly for B^{op} (see proposition 3.14 below).

(1) For any k-projective algebra A, let us put $F_A := (? \otimes_B^{\mathbf{L}} Y^\bullet) \circ \mathbf{R}\mathrm{Hom}_{B^{op}}(?, B)$ and $G_A := \mathbf{R}\mathrm{Hom}_{B^{op}}(?, Y^\bullet)$, which are triangulated functors $\mathcal{D}(B^{op} \otimes A)^{op} \longrightarrow \mathcal{D}(A^{op} \otimes C)$. Since the forgetful functors $\mathcal{H}(B^{op} \otimes A) \longrightarrow \mathcal{H}(B^{op}) = \mathcal{H}(B^{op} \otimes k)$ and $\mathcal{H}(A^{op} \otimes C) \longrightarrow \mathcal{H}(C) = \mathcal{H}(C \otimes k)$ preserve homotopically projective objects, we have the following commutative diagrams (one for F and another one for G), where the vertical arrows are the forgetful (or restriction of scalars) functors:

$$
\begin{array}{ccc}
\mathcal{D}(B^{op} \otimes A)^{op} & \xrightarrow[\ \ G_A\ \ ]{\ \ F_A\ \ } & \mathcal{D}(A^{op} \otimes C) \\
\downarrow{\scriptstyle U_{B^{op}}} & & \downarrow{\scriptstyle U_C} \\
\mathcal{D}(B^{op})^{op} & \xrightarrow[\ \ G_k\ \ ]{\ \ F_k\ \ } & \mathcal{D}(C)
\end{array}
$$

We shall denote $\theta_{Y^\bullet}(T^\bullet)$ by $\theta_{T^\bullet, Y^\bullet}^A : F_A(T^\bullet) \longrightarrow G_A(T^\bullet)$ to emphasize that there is one for each choice of a k-projective algebra A. Note that, by proposition 3.9, if C is not k-flat we cannot guarantee that $\theta_{?,?}^A$ is a natural transformation of bifunctors. Recall that $\theta_{T^\bullet, Y^\bullet}^A$ is the composition

$$
F_A(T^\bullet) = [(? \otimes_B^{\mathbf{L}} Y) \circ \mathbf{R}\mathrm{Hom}_{B^{op}}(?, B)](T^\bullet) = \mathbf{p}\mathrm{Hom}_{B^{op}}^\bullet(\mathbf{p}T^\bullet, B) \otimes_B^\bullet Y^\bullet \xrightarrow{\pi \otimes^\bullet 1}
$$
$$
\mathrm{Hom}_{B^{op}}^\bullet(\mathbf{p}T^\bullet, B) \otimes_B^\bullet Y^\bullet \xrightarrow{\psi} \mathrm{Hom}_{B^{op}}^\bullet(\mathbf{p}T^\bullet, Y^\bullet) = \mathbf{R}\mathrm{Hom}_{B^{op}}(?, Y^\bullet)(T^\bullet) = G_A(T^\bullet).
$$

When applying the forgetful functor $U_C : \mathcal{D}(A^{op} \otimes C) \longrightarrow \mathcal{D}(C)$, we obtain $U_C(\theta_{T,Y}^A) := (U_C \circ F_A)(T^\bullet) \longrightarrow (U_C \circ G_A)(T^\bullet)$. Due to the commutativity of the above diagram, this last morphism can be identified with the morphism $\theta_{T,Y}^k$, i.e., the version of θ_Y obtained when taking $A = k$. But observe that $\theta_{B,Y}^k$ is an isomorphism. This implies that $\theta_{X,Y}^k$ is an isomorphism, for each $X^\bullet \in \mathrm{thick}_{\mathcal{D}(B^{op})}(B) = \mathrm{per}(B^{op})$. It follows that $\theta_{T,Y}^k = U_C(\theta_{T,Y}^A)$ is an isomorphism since $_B T^\bullet \in \mathrm{per}(B^{op})$. But then $\theta_{T,Y}^A$ is an isomorphism because U_C reflects isomorphisms.

(2) Due to the k-projectivity of A, we have the following commutative diagram, where the lower horizontal arrow is the version of $(?)^*$ when $A = k$.

$$
\begin{array}{ccc}
\mathcal{D}(B^{op} \otimes A)^{op} & \xrightarrow{\ \ (?)^*\ \ } & \mathcal{D}(A^{op} \otimes B) \\
\downarrow{\scriptstyle U_{B^{op}}} & & \downarrow{\scriptstyle U_B} \\
\mathcal{D}(B^{op})^{op} & \xrightarrow{\ \ (?)^*\ \ } & \mathcal{D}(B)
\end{array}
$$

The task is hence reduced to check that the lower horizontal arrow preserves compact objects. That is a direct consequence of the fact that $(_B B)^* = \mathbf{R}\mathrm{Hom}_{B^{op}}(?, B)(B) \cong B_B$ and that the full subcategory of $\mathcal{D}(B^{op})$ consisting of the $X^\bullet$ such that $X^{\bullet*} \in \mathrm{per}(B)$ is a thick subcategory of $\mathcal{D}(B^{op})$.

In order to prove that $\sigma = \sigma_{T^\bullet}$ is an isomorphism there is no loss of generality in assuming that $A = k$. Note that the full subcategory $\mathcal{X}$ of $\mathcal{D}(B^{op})$ consisting of the $X^\bullet$ for which $\sigma_{X^\bullet} : X^\bullet \longrightarrow X^{\bullet**}$ is an isomorphism is a thick subcategory. We just need to prove that $_B B \in \mathcal{X}$ since $_B T^\bullet \in \mathrm{per}(B^{op})$. We do it by applying lemma 3.3 with $_B T^\bullet_A = {}_B B_B$. Indeed, $\lambda_B : B \longrightarrow \mathbf{R}\mathrm{Hom}_B(B, ? \otimes^{\mathbf{L}}_B B)(B)$ is an isomorphism in $\mathcal{D}(B)$.

(3) Assume now that C is k-flat. Due to assertion (2), assertions (3)(a) and (3)(b) are left-right symmetric. We then prove (3)(a). Proposition 3.9 allows us to identify $\theta_{Y^\bullet} = \theta_{?,Y^\bullet}$ and $\theta^{T^\cdot} = \theta_{T^\bullet,?}$, where $\theta : \mathbf{TH}(?,?) \longrightarrow \mathbf{H}_{B^{op}}(?,?)$ is the natural transformation of bifunctors given in that proposition. We then have the following chain of double implications:

$$\theta^{T^\bullet} \text{ natural isomorphism} \Longleftrightarrow \theta_{T^\bullet,Y^\bullet} \text{ isomorphism, for each } Y^\bullet \in \mathcal{D}(B^{op} \otimes C), \Longleftrightarrow$$
$$\theta^{Y^\bullet}(T^\bullet) \text{ isomorphism, for each } Y^\bullet \in \mathcal{D}(B^{op} \otimes C).$$

Now use assertion (1). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

As a direct consequence of the previous proposition, we have:

Corollary 3.11. *Let A and B be k-algebras, where A is k-projective, and denote by $\mathcal{L}_{B,A}$ (resp $\mathcal{R}_{A,B}$) the full subcategory of $\mathcal{D}(B^{op} \otimes A)$ (resp. $\mathcal{D}(A^{op} \otimes B)$) consisting of the objects $X^\cdot$ such that $_B X^\bullet$ (resp. $X^\bullet_B$) is compact in $\mathcal{D}(B^{op})$ (resp. $\mathcal{D}(B)$). Then $\mathcal{L}_{B,A}$ and $\mathcal{R}_{A,B}$ are thick subcategories of $\mathcal{D}(B^{op} \otimes A)$ and $\mathcal{D}(A^{op} \otimes B)$, respectively. Moreover, the assignments $X^\bullet \rightsquigarrow X^{\bullet*}$ define quasi-inverse triangulated dualities (=contravariant equivalences) $\mathcal{L}_{B,A} \overset{\cong}{\longleftrightarrow} \mathcal{R}_{A,B}$.*

3.4. **A few remarks on dg algebras and their derived categories.** In a few places in this paper, we will need some background on dg algebras and their derived categories. We just give an outline of the basic facts that we need. Interpreting dg algebras as dg categories with just one object, the material is a particular case of the development in [19] or [20].

A *dg algebra* is a ($\mathbb{Z}$-)graded algebra $A = \oplus_{p \in \mathbb{Z}} A^p$ together with a *differential*. This is a graded k-linear map $d : A \longrightarrow A$ of degree 1 such that $d(ab) = d(a)b + (-1)^{|a|}ad(b)$, for all homogeneous elements $a, b \in A$, where $|a|$ denotes the degree of a, and such that $d \circ d = 0$. In such case a (right) *dg A-module* is a graded A-module $M^\bullet = \oplus_{p \in \mathbb{Z}} M^p$ together with a k-linear map $d_M : M^\bullet \longrightarrow M^\bullet$ of degree $+1$, called the differential of $M^\bullet$, such that $d_M(xa) = d_M(x)a + (-1)^{|x|}xd(a)$, for all homogeneous elements $x \in M^\bullet$ and $a \in A$, and such that $d_M \circ d_M = 0$. It is useful to look at each dg A-module as a complex $...M^p \overset{d_M^p}{\longrightarrow} M^{p+1} \longrightarrow ..$ of k-modules with some extra properties. Note that an ordinary algebra is just a dg algebra with grading concentrated in degree 0, i.e. $A^p = 0$ for $p \neq 0$. In that case a dg A-module is just a complex of A-modules.

Let A be a dg algebra in the rest of this paragraph. We denote by $\mathrm{Gr} - A$ the category of graded A-modules (and morphisms of degree 0). Note that, when A is an ordinary algebra, we have $\mathrm{Gr} - A = (\mathrm{Mod} - A)^{\mathbb{Z}}$, with the notation of subsection 2.2. We next define a category $\mathcal{C}(A)$ whose objects are the dg A-modules as follows. A morphism $f : M^\bullet \longrightarrow N^\bullet$ in $\mathcal{C}(A)$ is a morphism in $\mathrm{Gr} - A$ which is a chain map of complex of k-modules, i.e., such that $f \circ d_M = d_M \circ f$. This category is abelian and comes with a canonical shifting $?[1] : \mathcal{C}(A) \longrightarrow \mathcal{C}(A)$ which comes from the canonical shifting of

$\mathrm{Gr} - A$, by defining $d_{M[1]}^n = -d_M^{n+1}$, for each $n \in \mathbb{Z}$. Note that we have an obvious faithful forgetful functor $\mathcal{C}(A) \longrightarrow \mathrm{Gr} - A$, which is also dense since we can interpret each graded A-module as an object of $\mathcal{C}(A)$ with zero differential. Viewing the objects of $C(A)$ as complexes of k-modules, we clearly have, for each $p \in \mathbb{Z}$, the *p-th homology functor* $H^p : \mathcal{C}(A) \longrightarrow \mathrm{Mod} - k$. A morphism $f : M^\bullet \longrightarrow N^\bullet$ in $\mathcal{C}(A)$ is called a *quasi-isomorphism* when $H^p(f)$ is an isomorphism, for all $p \in \mathbb{Z}$. A dg A-module $M^\bullet$ is called *acyclic* when $H^p(M) = 0$, for all $p \in \mathbb{Z}$.

Given any dg A-module $X^\bullet$, we denote by $P_X^\bullet$ the dg A-module which, as a graded A-module, is equal to $X^\bullet \oplus X^\bullet[1]$, and where the differential, viewed as a morphism $X^\bullet \oplus X^\bullet[1] = P_X^\bullet \longrightarrow P_X^\bullet[1] = X^\bullet[1] \oplus X^\bullet[2]$ in $\mathrm{Gr} - k$, is the 'matrix' $\begin{pmatrix} d_X & 1_{X[1]} \\ 0 & d_{X[1]} \end{pmatrix}$. Note that we have a canonical exact sequence $0 \to X^\bullet \longrightarrow P_X^\bullet \longrightarrow X^\bullet[1] \to 0$ in $\mathcal{C}(A)$, which splits in $\mathrm{Gr} - A$ but not in $\mathcal{C}(A)$. A morphism $f : M^\bullet \longrightarrow N^\bullet$ in $\mathcal{C}(A)$ is called *null-homotopic* when there exists a morphism $\sigma : M^\bullet \longrightarrow N^\bullet[-1]$ in $\mathrm{Gr} - A$ such that $\sigma \circ d_M + d_N \circ \sigma = f$.

The following is the fundamental fact (see [19] and [20]):

Proposition 3.12. *Let A be a dg algebra. The following assertions hold:*

(1) $\mathcal{C}(A)$ *has a structure of exact category where the conflations are those exact sequences which become split when applying the forgetful functor* $\mathcal{C}(A) \longrightarrow Gr - A$;

(2) *The projective objects with respect to this exact structure coincide with the injective ones, and they are the direct sums of dg A-modules of the form $P_X^\bullet$. In particular $\mathcal{C}(A)$ is a Frobenius exact category;*

(3) *A morphism $f : M^\bullet \longrightarrow N^\bullet$ in $\mathcal{C}(A)$ factors through a projective object if, an only it, it is* null-homotopic. *The stable category $\underline{\mathcal{C}(A)}$ with respect to the given exact structure is denoted by $\mathcal{H}(A)$ and called the* homotopy category of A. *It is a triangulated category, where ?[1] is the suspension functor and where the triangles are the images of conflations by the projection functor $p : \mathcal{C}(A) \longrightarrow \mathcal{H}(A)$;*

(4) *If Q denotes the class of quasi-isomorphisms in $\mathcal{C}(A)$ and $\Sigma := p(Q)$, then Σ is a multiplicative system in $\mathcal{H}(A)$ compatible with the triangulation and $\mathcal{C}(A)[Q^{-1}] \cong \mathcal{H}(A)[\Sigma^{-1}]$. In particular, this latter category is triangulated. It is denoted by $\mathcal{D}(A)$ and called the* derived category of A;

(5) *If $\mathcal{Z}$ denotes the full subcategory of $\mathcal{H}(A)$ consisting of the acyclic dg A-modules, then $\mathcal{Z}$ is a triangulated subcategory closed under taking coproducts and products in $\mathcal{H}(A)$. The category $\mathcal{D}(A)$ is equivalent, as a triangulated category, to the quotient category $\mathcal{H}(A)/\mathcal{Z}$.*

As in the case of an ordinary algebra, we do not have set-theoretical problems with the just defined derived category. The reason is the following:

Proposition 3.13. *Let A be a dg algebra and let $\mathcal{Z}$ be the full subcategory of $\mathcal{H}(A)$ consisting of acyclic dg A-modules. Then the pairs $(\mathcal{Z}, \mathcal{Z}^\perp)$ and $({}^\perp\mathcal{Z}, \mathcal{Z})$ are semi-orthogonal decompositions of $\mathcal{H}(A)$. In particular, the canonical functor $q : \mathcal{H}(A) \longrightarrow \mathcal{D}(A)$ has both a left adjoint $p_A : \mathcal{D}(A) \longrightarrow \mathcal{H}(A)$ and a right adjoint $i_A : \mathcal{D}(A) \longrightarrow \mathcal{H}(A)$.*

Moreover, the category $\mathcal{D}(A)$ is compactly generated by $\{A\}$.

As in the case of an ordinary algebra, we call $\mathbf{p}_A$ and $\mathbf{i}_A$ the *homotopically projective resolution functor* and the *homotopically injective resolution functor*, respectively. Also, the objects of $^\perp \mathcal{Z}$ are called *homotopically projective* and those of $\mathcal{Z}^\perp$ are called *homotopically injective*.

When A is a dg algebra, its *opposite dg algebra* A^{op} is equal to A as a graded k-module, but the multiplication is given by $b^o \cdot a^o = (-1)^{|a||b|}(ab)^o$. Here we denote by x^o any homogeneous element $x \in A$ when viewed as an element of A^{op}. Then, given dg algebras A and B, a dg $B - A-$module T is just a dg right $B^{op} \otimes A-$module, and these dg $B - A-$modules form the category $\mathcal{C}(B^{op} \otimes A)$.

With some care on the use of signs, the reader will have no difficulty in extending to dg algebras and their derived categories the results in the earlier paragraphs of this section, simply by defining correctly the total tensor and total Hom functors bifunctors. Indeed, if A, B and C are dg algebras and $_BT_A^\bullet$, $_AM_C^\bullet$ and $_CN_A^\bullet$ are dg bimodules, then $T^\bullet \otimes_A^\bullet M^\bullet$ is just the classical tensor product of a graded right A-module and a graded left A-module and the differential $d = d_{T \otimes_A^\bullet M} : (T \otimes_A^\bullet M) \longrightarrow (T \otimes_A^\bullet M)$ takes $t \otimes m \rightsquigarrow d_T(t) \otimes m + (-1)^{|t|}t \otimes d_M(m)$, for all homogeneous elements $t \in T^\bullet$ and $m \in M^\bullet$.

We define $\mathrm{Hom}_A^\bullet(T^\bullet, N^\bullet) = \oplus_{p \in \mathbb{Z}}\mathrm{Gr}(A)(M^\bullet, N^\bullet[p])$. This has an obvious structure of graded $C-B-$bimodule, where the homogeneous component of degree p is $\mathrm{Hom}_A^\bullet(T^\bullet, N^\bullet)^p = \mathrm{Gr}(A)(M^\bullet, N^\bullet[p])$, for each $p \in \mathbb{Z}$. We then define the differential $d : \mathrm{Hom}_A^\bullet(T^\bullet, N^\bullet) \longrightarrow \mathrm{Hom}_A^\bullet(T^\bullet, N^\bullet)$ by taking $f \rightsquigarrow d_N \circ f - (-1)^{|f|}f \circ d_M$, for each homogeneous element $f \in \mathrm{Hom}_A^\bullet(T^\bullet, N^\bullet)$.

The following is a very well-known fact (see [19, Theorem 5.3] and [34]):

Proposition 3.14. *Let A be a dg algebra. The compact objects of $\mathcal{D}(A)$ are precisely the objects of $\mathrm{thick}_{\mathcal{D}(A)}(A) =: \mathrm{per}(A)$. When A is an ordinary algebra, those are precisely the complexes which are quasi-isomorphic to bounded complexes of finitely generated projective A-modules.*

3.5. **Some special objects and a generalization of Rickard theorem.**

Definition 17. *Let $\mathcal{D}$ be a triangulated category with coproducts. An object T of $\mathcal{D}$ is called:*

   (1) exceptional *when $\mathcal{D}(T, T[p]) = 0$, for all $p \neq 0$;*
   (2) classical tilting *when T is exceptional and $\{T\}$ is a set of compact generators of $\mathcal{D}$;*
   (3) self-compact *when T is a compact object of $\mathrm{Tria}_{\mathcal{D}}(T)$.*

The following result is a generalization of Rickard theorem ([34]), which, as can be seen in the proof, is an adaptation of an argument of Keller that the authors used in [31].

Theorem 3.15. *Let A be a k-flat dg algebra and $T^\bullet$ be a self-compact and exceptional object of $\mathcal{D}(A)$. If $B = \mathrm{End}_{\mathcal{D}(A)}(T^\bullet)$ is the endomorphism algebra then, up to replacement of $T^\bullet$ by an isomorphic object in $\mathcal{D}(A)$, we can view $T^\bullet$ as a dg B-A-bimodules such that the restriction of $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ induces an equivalence of triangulated categories $\mathrm{Tria}_{\mathcal{D}(A)}(T^\bullet) \overset{\cong}{\longrightarrow} \mathcal{D}(B)$.*

Proof. We can assume that $T^\bullet$ is homotopically injective in $\mathcal{H}(A)$. Then $\hat{B} = \mathrm{End}_A^\bullet(T^\bullet) := \mathrm{Hom}_A^\bullet(T^\bullet, T^\bullet)$ is a dg algebra and $T^\bullet$ becomes a dg $\hat{B} - A-$bimodule in a natural way.

Moreover, by [31, Corollary 2.5], we get that $? \otimes_{\hat{B}}^{\mathbf{L}} T^{\bullet} : \mathcal{D}(\hat{B}) \longrightarrow \mathcal{D}(A)$ induces an equivalence of triangulated categories $\mathcal{D}(\hat{B}) \overset{\cong}{\longrightarrow} \mathcal{T} := \mathrm{Tria}_{\mathcal{D}(A)}(T^{\cdot})$.

On the other hand, at the level of homology, we have:

$$H^n(\hat{B}) = \mathcal{H}(\hat{B})(\hat{B}, \hat{B}[n]) \cong \mathcal{D}(\hat{B})(\hat{B}, \hat{B}[n]) \cong \mathcal{T}(T^{\bullet}, T^{\bullet}[n]) \cong \mathcal{D}(A)(T^{\bullet}, T^{\bullet}[n]),$$

for all $n \in \mathbb{Z}$. It follows that $H^n(\hat{B}) = 0$, for $n \neq 0$, while $H^0(\hat{B}) \cong B$. We take the canonical truncation of $\hat{B}$ at 0, i.e., the dg subalgebra $\tau^{\leq 0} \hat{B}$ of $\hat{B}$ given, as a complex of k-modules, by

$$... \hat{B}^{-n} \longrightarrow ... \longrightarrow \hat{B}^{-1} \longrightarrow \mathrm{Ker}(d^0) \longrightarrow 0...,$$

where $d^0 : \hat{B}^0 \longrightarrow \hat{B}^1$ is the 0-th differential of $\hat{B}$. Then $B = H^0(\tau^{\leq 0}\hat{B}) = H^0(\hat{B})$ and we have quasi-isomorphism of dg algebras $B \overset{p}{\longleftarrow} \tau^{\leq 0}\hat{B} \overset{j}{\hookrightarrow} \hat{B}$, where p and j are the projection and inclusion, respectively. Replacing $\hat{B}$ by $\tau^{\leq 0}\hat{B}$ and the dg bimodule $_{\hat{B}}T_A^{\bullet}$ by $_{\tau^{\leq 0}\hat{B}}T_A^{\bullet}$, we can assume, without loss of generality, that $\hat{B}^p = 0$, for all $p > 0$. We assume this in the sequel.

It is convenient now to take the homotopically projective resolution $\pi : \mathbf{p}_{\hat{B}^{op} \otimes A} T^{\bullet} \longrightarrow T^{\bullet}$ in $\mathcal{H}(\hat{B}^{op} \otimes A)$. We then replace $T^{\bullet}$ by $\mathbf{p}T^{\bullet} := \mathbf{p}_{\hat{B}^{op} \otimes A} T^{\bullet}$. In this way we lose the homotopically injective condition of $T_A^{\bullet}$ in $\mathcal{H}(A)$, but we win that $_{\hat{B}}T^{\bullet}$ is homotopically flat in $\mathcal{H}(\hat{B}^{op})$ (this follows by the extension of lemma 3.6 to dg algebras). Note, however, that $\pi : \mathbf{p}T^{\bullet} \longrightarrow T^{\bullet}$ induces a natural isomorphism $? \otimes_{\hat{B}}^{\mathbf{L}} \mathbf{p}T^{\bullet} \cong ? \otimes_{\hat{B}}^{\mathbf{L}} T^{\bullet}$ of triangulated functors $\mathcal{D}(\hat{B}) \longrightarrow \mathcal{D}(A)$.

We can view $p : \hat{B} \longrightarrow B$ as a quasi-isomorphism in $\mathcal{H}(\hat{B})$ and then $T^{\bullet} \cong \hat{B} \otimes_{\hat{B}}^{\bullet} T^{\bullet} \overset{p \otimes 1_T}{\longrightarrow} B \otimes_{\hat{B}}^{\bullet} T^{\bullet}$ is a quasi-isomorphism since $_{\hat{B}}T^{\bullet}$ is homotopically flat. By this same reason, we have that $B \otimes_{\hat{B}}^{\bullet} T^{\bullet} \cong B \otimes_{\hat{B}}^{\mathbf{L}} T^{\bullet}$ (see lemma 3.5). Then we get a composition of triangulated equivalences

$$\mathcal{D}(B) \overset{p_*}{\longrightarrow} \mathcal{D}(\hat{B}) \overset{? \otimes_{\hat{B}}^{\mathbf{L}} T^{\bullet}}{\longrightarrow} \mathcal{T},$$

where the first arrow is the restriction of scalars along the projection $p : \hat{B} \longrightarrow B$. This composition of equivalences is clearly identified with the functor $? \otimes_B^{\mathbf{L}} (B \otimes_{\hat{B}}^{\mathbf{L}} T^{\bullet}) = ? \otimes_B^{\mathbf{L}} (B \otimes_{\hat{B}}^{\bullet} T^{\bullet})$.

Replacing $\hat{T}^{\bullet}$ by $B \otimes_{\hat{B}}^{\bullet} T^{\bullet}$, we can then assume that $T^{\bullet}$ is a dg B-A-bimodule such that $? \otimes_B^{\mathbf{L}} T^{\bullet} : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ induces an equivalence $\mathcal{D}(B) \overset{\cong}{\longrightarrow} \mathrm{Tria}_{\mathcal{D}(A)}(T^{\bullet})$. A quasi-inverse of this equivalence is then the restriction of $\mathbf{R}\mathrm{Hom}_A(T^{\bullet}, ?)$ to $\mathrm{Tria}_{\mathcal{D}(A)}(T^{\bullet})$. $\qquad\square$

4. Main results

All throughout this section A and B are arbitrary k-algebras and $T^{\bullet}$ is a complex of $B - A-$bimodules. We want to give necessary and sufficient conditions of the functors $\mathbf{R}\mathrm{Hom}_A(T^{\bullet}, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ or $? \otimes_B^{\mathbf{L}} T^{\bullet} : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ to be fully faithful. We start with two very helpful auxiliary results.

The following is the final result of [32]. Note that condition 2 of the proposition was given only for $\mathcal{D}(B^{op})$, but this condition holds if, and only if, it holds for $\mathcal{D}(k)$ since the forgetful functor $\mathcal{D}(B^{op}) \longrightarrow \mathcal{D}(k)$ reflects isomorphisms.

Proposition 4.1. *Suppose that $H^p(T^\bullet) = 0$, for $p >> 0$. The following assertions are equivalent:*

1. *$T^\bullet$ is isomorphic in $\mathcal{D}(B^{op})$ to an upper bounded complex of finitely generated projective left B-modules;*
2. *The canonical morphism $B^\alpha \otimes_B^{\mathbf{L}} T^\bullet \longrightarrow (B \otimes_B^{\mathbf{L}} T^\bullet)^\alpha = T^{\bullet\alpha}$ is an isomorphism (in $\mathcal{D}(A)$, $\mathcal{D}(B^{op}$ or $\mathcal{D}(k))$, for each cardinal α.*

We can go further in the direction of the previous proposition. Recall that if $X^\bullet = (X^\bullet, d) \in \mathcal{C}(\mathcal{A})$ is any complex, where $\mathcal{A}$ is an abelian category, then its *left (resp. right) stupid truncation at m* is the complex $\sigma^{\leq m} X^\bullet = \sigma^{<m+1} X^\bullet$: $...X^k \xrightarrow{d^k} ...X^{m-1} \xrightarrow{d^{m-1}} X^m \longrightarrow 0 \longrightarrow 0...$ (resp. $\sigma^{\geq m} X^\bullet = \sigma^{>m-1} X^\bullet$: $...0 \longrightarrow 0 \longrightarrow X^m \xrightarrow{d^m} X^{m+1}...X^n \xrightarrow{d^n} ...$). Note that we have a conflation $0 \to \sigma^{>m} X^\bullet \hookrightarrow X^\bullet \longrightarrow \sigma^{\leq m} X^\bullet \to 0$ in $\mathcal{C}(\mathcal{A})$, and, hence, an induced triangle

$$\sigma^{\geq m} X^\bullet \longrightarrow X^\bullet \longrightarrow \sigma^{<m} X^\bullet \xrightarrow{+}$$

in $\mathcal{H}(\mathcal{A})$ and in $\mathcal{D}(\mathcal{A})$.

Proposition 4.2. *The following assertions are equivalent:*

1. *$_BT^\bullet$ is compact in $\mathcal{D}(B^{op})$;*
2. *The functor $? \otimes_B^{\mathbf{L}} T^\bullet : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ preserves products;*
3. *The functor $? \otimes_B^{\mathbf{L}} T^\bullet : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ has a left adjoint.*

Proof. $(2) \Longleftrightarrow (3)$ is a direct consequence of corollary 2.15.

$(1) \Longrightarrow (2)$ It is enough to prove that the composition $\mathcal{D}(B) \xrightarrow{?\otimes_B^{\mathbf{L}}T^\bullet} \mathcal{D}(A) \xrightarrow{\text{forgetful}} \mathcal{D}(k)$ preserves products since the forgetful functor preserves products and reflects isomorphisms.

Abusing of notation, we still denote by $? \otimes_B^{\mathbf{L}} T^\bullet$ the mentioned composition and note that, when doing so, we have isomorphisms $(?\otimes_B^{\mathbf{L}}N^\bullet)(X^\bullet) \cong \mathbf{T}_B(X^\bullet, N^\bullet) \cong (X^\bullet\otimes_B^{\mathbf{L}}?)(N^\bullet)$ in $\mathcal{D}(k)$, for all $X^\bullet \in \mathcal{D}(B)$ and $N^\bullet \in \mathcal{D}(B^{op})$ (see proposition 3.7). If $(X_i^\bullet)_{i\in I}$ is any family of objects of $\mathcal{D}(B)$, we consider the full subcategory $\mathcal{T}$ of $\mathcal{D}(B^{op})$ consisting of the $N^\bullet$ such that the canonical morphism $(\prod_{i\in I} X_i^\bullet) \otimes_B^{\mathbf{L}} N^\bullet \longrightarrow \prod_{i\in I} X_i^\bullet \otimes_B^{\mathbf{L}} N^\bullet$ is an isomorphism. It is a thick subcategory of $\mathcal{D}(B^{op})$ which contains $_BB$. Then it contains $\text{per}(B^{op})$ and, in particular, it contains $_BT^\bullet$.

$(2) \Longrightarrow (1)$ Without loss of generality, we can assume that $A = k$. The canonical morphism $\coprod_{p\in\mathbb{Z}} B[p] \longrightarrow \prod_{p\in\mathbb{Z}} B[p]$ is an isomorphism in $\mathcal{D}(B^{op})$. By applying $? \otimes_B^{\mathbf{L}} T^\bullet$ and using the hypothesis, we then have an isomorphism in $\mathcal{D}(k)$

$$\coprod_{p\in\mathbb{Z}} T^\bullet[p] \cong (? \otimes_B^{\mathbf{L}} T^\bullet)(\coprod_{p\in\mathbb{Z}} B[p]) \xrightarrow{\cong} (? \otimes_B^{\mathbf{L}} T^\bullet)(\prod_{p\in\mathbb{Z}} B[p]) \cong \prod_{p\in\mathbb{Z}}(B[p] \otimes_B^{\mathbf{L}} T^\bullet) \cong \prod_{p\in\mathbb{Z}} T^\bullet[p],$$

which can be identified with the canonical morphism from the coproduct to the product. It follows that the canonical map

$$\coprod_{p \in \mathbb{Z}} H^p(T^\bullet) \cong H^0(\coprod_{p \in \mathbb{Z}} T^\bullet[p]) \longrightarrow H^0(\prod_{p \in \mathbb{Z}} T^\bullet[p]) \cong \prod_{p \in \mathbb{Z}} H^p(T^\bullet)$$

is an isomorphism. This implies that $H^p(T^\bullet) = 0$, for all but finitely many $p \in \mathbb{Z}$.

By proposition 4.1, replacing $T^\bullet$ by its homotopically projective resolution in $\mathcal{H}(B^{op})$, we can assume without loss of generality that $T^\bullet$ is an upper bounded complex of finitely generated projective left B-modules. Let us put $m := \min\{p \in \mathbb{Z} : H^p(T) \neq 0\}$. We then consider the triangle in $\mathcal{D}(B^{op})$ induced by the stupid truncation at m

$$\sigma^{\geq m}T^\bullet \longrightarrow T^\bullet \longrightarrow \sigma^{<m}T^\bullet \overset{+}{\longrightarrow}.$$

Then $\sigma^{\geq m}T^\bullet$ is compact in $\mathcal{D}(B^{op})$ while $\sigma^{<m}T^\bullet$ has homology concentrated in degree $m-1$. Then we have an isomorphism $\sigma^{<m}T^\bullet \cong M[1-m]$ in $\mathcal{D}(B^{op})$, where $M = H^{m-1}(\sigma^{<m}T^\bullet)$. By the implication 1) $\Longrightarrow$ 2) we know that $? \otimes_B^{\mathbf{L}} \sigma^{\geq m}T^\bullet : \mathcal{D}(B) \longrightarrow \mathcal{D}(k)$ preserves products and, by hypothesis, also $? \otimes_B^{\mathbf{L}} T^\bullet : \mathcal{D}(B) \longrightarrow \mathcal{D}(k)$ does. It follows that $? \otimes_B^{\mathbf{L}} \sigma^{<m}T^\bullet \cong ? \otimes_B^{\mathbf{L}} M[1-m] : \mathcal{D}(B) \longrightarrow \mathcal{D}(k)$ preserves products.

Note that M admits admits a projective resolution with finitely generated terms, namely, the canonical quasi-isomorphism $P^\bullet := \sigma^{<m}T^\bullet[m-1] \longrightarrow M = M[0]$. Therefore our task reduces to check that if M is a left B module which admits a projective resolution with finitely generated terms and such that $? \otimes_B^{\mathbf{L}} M : \mathcal{D}(B) \longrightarrow \mathcal{D}(k)$ preserves products, then M has finite projective dimension. For that it is enough to prove that there is an integer $n \geq 0$ such that $\operatorname{Tor}_{n+1}^B(?, M) \equiv 0$. Indeed, if this is proved then $\Omega^n(M) := \operatorname{Im}(d^{-n} : P^{-n} \longrightarrow P^{-n+1})$ will be a flat module, and hence projective (see [26, Corollaire 1.3]), thus ending the proof.

Let us assume by way of contradiction that $\operatorname{Tor}_n^B(?, M) \not\equiv 0$, for all $n > 0$. For each such n, choose a right B-module X_n such that $\operatorname{Tor}_n^B(X_n, M) \neq 0$. Then the canonical morphism $\coprod_{n>0} X_n[-n] \longrightarrow \prod_{n>0} X_n[-n]$ is an isomorphism in $\mathcal{D}(B)$. Our hypothesis then guarantees that the canonical morphism

$$\coprod_{n>0}(X_n[-n] \otimes_B^{\mathbf{L}} M) \cong (\coprod_{n>0} X_n[-n]) \otimes_B^{\mathbf{L}} M \cong (\prod_{n>0} X_n[-n]) \otimes_B^{\mathbf{L}} M \longrightarrow$$
$$\prod_{n>0}(X_n[-n] \otimes_B^{\mathbf{L}} M)$$

is an isomorphism. When applying the 0-homology functor H^0, we obtain an isomorphism

$$\coprod_{n>0} \operatorname{Tor}_n^B(X_n, M) \cong \coprod_{n>0} H^0(X_n[-n] \otimes_B^{\mathbf{L}} M) \longrightarrow \prod_{n>0} H^0(X_n[-n] \otimes_B^{\mathbf{L}} M) \cong$$
$$\prod_{n>0} \operatorname{Tor}_n^B(X_n, M)$$

which is identified with the canonical morphism from the coproduct to the product in $\operatorname{Mod} - K$. It follows that $\operatorname{Tor}_n^B(X_n, M) = 0$, for almost all $n > 0$, which is a contradiction. $\qquad\square$

Remark 4.3. *The argument in the last two paragraphs of the proof of proposition 4.2 was communicated to us by Rickard, to whom we deeply thank for it. When passing to the context of dg algebras or even dg categories, the implication (1) $\Longrightarrow$ (2) in that proposition still holds (see [32]), essentially with the same proof. However, we do not know if (2) $\Longrightarrow$ (1) holds for dg algebras A and B.*

4.1. **Statements and proofs.**

Proposition 4.4. *Let* $\delta : (? \otimes^{\mathbf{L}}_B T^\bullet) \circ \mathbf{R}\,Hom_A(T^\bullet, ?) \longrightarrow 1_{\mathcal{D}(A)}$ *be the counit of the adjoint pair* $(? \otimes^{\mathbf{L}}_B T^\bullet, \mathbf{R}\,Hom_A(T^\bullet, ?))$. *The following assertions are equivalent:*

(1) $\mathbf{R}\,Hom_A(T^\bullet, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ *is fully faithful;*
(2) *The map* $\delta_A : [(? \otimes^{\mathbf{L}}_B T) \circ \mathbf{R}\,Hom_A(T, ?)](A) \longrightarrow A$ *is an isomorphism in* $\mathcal{D}(A)$ *and the functor* $(? \otimes^{\mathbf{L}}_B T) \circ \mathbf{R}\,Hom_A(T, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(A)$ *preserves coproducts.*

In that case, the functor $? \otimes^{\mathbf{L}}_B T^\bullet$ *induces a triangulated equivalence* $\mathcal{D}(B)/Ker(? \otimes^{\mathbf{L}}_B T^\bullet) \overset{\cong}{\longrightarrow} \mathcal{D}(A).$

Proof. Assertion (1) is equivalent to saying that $\delta : (? \otimes^{\mathbf{L}}_B T^\bullet) \circ \mathbf{R}\,Hom_A(T^\bullet, ?) \longrightarrow 1_{\mathcal{D}(A)}$ is a natural isomorphism (see the dual of [17, Proposition II.7.5]). As a consequence, the implication (1) $\Longrightarrow$ (2) is automatic. Conversely, if assertion (2) holds, then the full subcategory $\mathcal{T}$ of $\mathcal{D}(A)$ consisting of the $M^\bullet \in \mathcal{D}(A)$ such that $\delta_{M^\bullet}$ is an isomorphism is a triangulated subcategory closed under taking coproducts and containing A. It follows that $\mathcal{T} = \mathcal{D}(A)$, so that assertion (1) holds.

For the final statement, note that $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?)$ gives an equivalence of triangulated categories $\mathcal{D}(A) \overset{\cong}{\longrightarrow} \mathrm{Im}(\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?))$. On the other hand, by proposition 2.12, we know that $(\mathrm{Ker}(? \otimes^{\mathbf{L}}_B T^\bullet), \mathrm{Im}(\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?)))$ is a semi-orthogonal decomposition of $\mathcal{D}(B)$. Then, by proposition 2.13, we have a triangulated equivalence $\mathcal{D}(B)/\mathrm{Ker}(? \otimes^{\mathbf{L}}_B T^\bullet) \cong \mathrm{Im}(\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?))$. We then get a triangulated equivalence $\mathcal{D}(B)/\mathrm{Ker}(? \otimes^{\mathbf{L}}_B T^\bullet) \overset{\cong}{\longrightarrow} \mathcal{D}(A)$, which is easily seen to be induced by $? \otimes^{\mathbf{L}}_B T^\bullet$. $\qquad \square$

We now pass to study the recollement situations where one of the fully faithful functors is $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?)$.

Corollary 4.5. *Let* $T^\bullet$ *a complex of* $B - A-bimodules$. *The following assertions hold:*

(1) *There is a triangulated category* $\mathcal{D}'$ *and a recollement* $\mathcal{D}(A) \equiv \mathcal{D}(B) \equiv \mathcal{D}'$, *with* $i_* = \mathbf{R}\,Hom_A(T^\bullet, ?)$;
(2) *There is a triangulated category* $\mathcal{D}'$ *and a recollement* $\mathcal{D}(A) \equiv \mathcal{D}(B) \equiv \mathcal{D}'$, *with* $i^* = ? \otimes^{\mathbf{L}}_B T^\bullet$;
(3) $T^\bullet_A$ *is compact in* $\mathcal{D}(A)$ *and* $\delta_A : [(? \otimes^{\mathbf{L}}_B T^\bullet) \circ \mathbf{R}\,Hom_A(T^\bullet, ?)](A) \longrightarrow A$ *is an isomorphism in* $\mathcal{D}(A)$, *where* δ *is the counit of the adjoint pair* $(? \otimes^{\mathbf{L}}_B T^\bullet, \mathbf{R}\,Hom_A(T^\bullet, ?))$.

Proof. (1) $\Longleftrightarrow$ (2) is clear.

(1) $\Longrightarrow$ (3) If the recollement exists, then $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ is fully faithful, so that δ_A is an isomorphism. Moreover, $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?)$ preserves coproducts since it it is a left adjoint functor. This preservation of coproducts is equivalent to having $T^\bullet_A \in \mathrm{per}(A)$.

(3) $\Longrightarrow$ (1) We clearly have that the functor $(? \otimes^{\mathbf{L}}_B T^\bullet) \circ \mathbf{R}\mathrm{Hom}_A(T^\bullet, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(A)$ preserves coproducts. Then assertion (2) of proposition 4.4 holds, so that $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?)$ is fully faithful. On the other hand, by proposition 2.14, we get that $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ has a right adjoint, so that assertion (1) holds. $\qquad \square$

Theorem 4.6. *Let* $_B T^\bullet_A$ *be a complex of* $B - A-bimodules$. *Consider the following assertions:*

(1) *There is a recollement $\mathcal{D}' \equiv \mathcal{D}(B) \equiv \mathcal{D}(A)$, with $j_* = \mathbf{R}Hom_A(T^\bullet, ?)$, for some triangulated category (which is equivalent to $\mathcal{D}(C)$, where C is a dg algebra);*

(2) *There is a recollement $\mathcal{D}' \equiv \mathcal{D}(B) \equiv \mathcal{D}(A)$, with $j^* = j^! =? \otimes_B^{\mathbf{L}} T^\bullet$, for some triangulated category (which is equivalent to $\mathcal{D}(C)$, where C is a dg algebra);*

(3) *The following three conditions hold:*
   (a) *The counit map $\delta_A : [(? \otimes_B^{\mathbf{L}} T^\bullet) \circ \mathbf{R}Hom_A(T^\bullet, ?)](A) \longrightarrow A$ is an isomorphism;*
   (b) *The functor $(? \otimes_B^{\mathbf{L}} T^\bullet) \circ \mathbf{R}Hom_A(T^\bullet, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(A)$ preserves coproducts;*
   (c) *The functor $? \otimes_B^{\mathbf{L}} T^\bullet : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ preserves products.*

(4) *$_BT^\bullet$ is compact and exceptional in $\mathcal{D}(B^{op})$ and the canonical algebra morphism $A \longrightarrow End_{\mathcal{D}(B^{op})}(T^\bullet)^{op}$ is an isomorphism.*

The implications $(1) \Longleftrightarrow (2) \Longleftrightarrow (3) \Longleftarrow (4)$ hold true and, when A is k-projective, all assertions are equivalent. Moreover, if B is k-flat, then the dg algebra C can be chosen together with a homological epimorphism $f : B \longrightarrow C$ such that i_* is the restriction of scalars $f_* : \mathcal{D}(C) \longrightarrow \mathcal{D}(B)$.

Proof. $(1) \Longleftrightarrow (2)$ is clear.

$(1) \Longleftrightarrow (3)$ By proposition 2.19, we know that the recollement in (3) exists if, and only if, $? \otimes_B^{\mathbf{L}} T^\bullet : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ has a left adjoint and $\mathbf{R}Hom_A(T^\bullet, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ is fully faithful. Apply now corollary 2.15 and proposition 4.4.

$(4) \Longrightarrow (3)$ Condition (3)(c) follows from proposition 4.2. By proposition 4.4, proving conditions (3)(a) and (3)(b) is equivalent to proving that $\mathbf{R}Hom_A(T^\bullet, ?)$ is fully faithful. This is in turn equivalent to proving that the counit $\delta : (? \otimes_B^{\mathbf{L}} T^\bullet) \circ \mathbf{R}Hom_A(T^\bullet, ?) \longrightarrow 1_{\mathcal{D}(A)}$ is a natural isomorphism.

In order to prove this, we apply proposition 3.10 to $T^\bullet$, when viewed as a complex of left B-modules (equivalently, of $B - k-$bimodules). Then $T^{\bullet*}$ is obtained from $T^\bullet$ by applying $\mathbf{R}Hom_{B^{op}}(?, B) : \mathcal{D}(B^{op})^{op} \longrightarrow \mathcal{D}(B)$ and, similarly, we obtain $T^{\bullet**}$ from $T^{\bullet*}$. By the mentioned proposition, we know that $T^\bullet \cong T^{\bullet**}$ in $\mathcal{D}(B^{op})$. Moreover, applying assertion (1) of that proposition, with A and C replaced by k and A, respectively, and putting $Y^\bullet = T^\bullet \in \mathcal{C}(B^{op} \otimes A)$, we obtain isomorphisms in $\mathcal{D}(A)$:

$$T^{\bullet*} \otimes_B^{\mathbf{L}} T^\bullet := [(? \otimes_B^{\mathbf{L}} T^\bullet) \circ \mathbf{R}Hom_{B^{op}}(?, B)](T^\bullet) \cong \mathbf{R}Hom_{B^{op}}(?, T^\bullet)(T^\bullet) \cong A_A,$$

where the last isomorphism follows from the exceptionality of $_BT^\bullet$ in $\mathcal{D}(B^{op})$ and the fact that the canonical algebra morphism $A \longrightarrow End_{\mathcal{D}(B^{op})}(T^\bullet)^{op}$ is an isomorphism.

Using the previous paragraph, proposition 3.10 and adjunction, for each object $M^\bullet \in \mathcal{D}(A)$ we get a chain of isomorphisms in $\mathcal{D}(k)$:

$$[(? \otimes_B^{\mathbf{L}} T^\bullet) \circ \mathbf{R}Hom_A(T^\bullet, ?)](M^\bullet) = (? \otimes_B^{\mathbf{L}} T^\bullet)(\mathbf{R}Hom_A(T^\bullet, M^\bullet)) \xrightarrow{\cong}$$
$$\mathbf{R}Hom_B(T^{\bullet*}, \mathbf{R}Hom_A(T^\bullet, M^\bullet)) \xrightarrow{\cong} \mathbf{R}Hom_A(T^{\bullet*} \otimes_B^{\mathbf{L}} T^\bullet, M^\bullet) \xrightarrow{\cong} \mathbf{R}Hom_A(A, M^\bullet) \xrightarrow{\cong}$$
$$M^\bullet.$$

It is routine now to see that the composition of these isomorphisms is obtained from the counit map $\delta_{M^\bullet} : [(? \otimes_B^{\mathbf{L}} T^\bullet) \circ \mathbf{R}Hom_A(T^\bullet, ?)](M^\bullet) \longrightarrow M^\bullet$ by applying the forgetful functor $\mathcal{D}(A) \longrightarrow \mathcal{D}(k)$. Since this last functor reflects isomorphisms we get that δ is a natural isomorphism, so that $\mathbf{R}Hom_A(T^\bullet, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ is a fully faithful functor.

$(1) - (3) \Longrightarrow (4)$ (Assuming that A is k-projective). By proposition 4.2, we know that $_BT^\bullet \in per(B^{op})$. Then, by proposition 3.10, we have a natural isomorphism $? \otimes_B^{\mathbf{L}} T^\bullet \cong$

$\mathbf{R}\mathrm{Hom}_B(T^{\bullet*}, ?)$ of triangulated functors $D(B) \longrightarrow \mathcal{D}(A)$, where now we are considering $T^{\bullet*}$ as obtained from $T^{\bullet}$ by applying $(?)^* := \mathbf{R}\mathrm{Hom}_{B^{op}}(?, B) : \mathcal{D}(B^{op} \otimes A)^{op} \longrightarrow \mathcal{D}(A^{op} \otimes B)$. By lemma 2.16, the fully faithful condition of $\mathbf{R}\mathrm{Hom}_A(T^{\bullet}, ?)$ implies the same condition for $? \otimes^{\mathbf{L}}_A T^{\bullet*}$. Then corollary 4.14 below says that $T^{\bullet*}_B$ is compact and exceptional in $\mathcal{D}(B)$ and the canonical algebra morphism $A \longrightarrow \mathrm{End}_{\mathcal{D}(B)}(T^{\bullet*})$ is an isomorphism.

Note now that, due to the k-projectivity of A, when applying to $_BT^{\bullet}$ the functor $\mathbf{R}\mathrm{Hom}_{B^{op}}(?, B) : \mathcal{D}(B^{op})^{op} \longrightarrow \mathcal{D}(B)$, we obtain an object isomorphic to $T^{\bullet*}_B$ in $\mathcal{D}(B)$, and conversely. Applying now corollary 3.11 with $A = k$, we have that $_BT^{\bullet} \cong {}_BT^{\bullet**}$ is exceptional in $\mathcal{D}(B^{op})$ and that the algebra map $A \longrightarrow \mathrm{End}_{\mathcal{D}(B^{op})}(T^{\bullet})^{op} \cong \mathrm{End}_{\mathcal{D}(B)}(T^{\bullet*})$ is an isomorphism.

For the final statement note that, when the recollement of assertions (1) or (2) exists, $\mathcal{D}'$ has a compact generator, namely $i^!(A)$. Then, by [19, Theorem 4.3], we know that $\mathcal{D}' \cong \mathcal{D}(C)$, for some dg algebra C. When B is k-flat, the fact that this dg algebra can be chosen together with a homological epimorphism $f : B \longrightarrow C$ satisfying the requirements is a direct consequence of [31, Theorem 4]. □

Remark 4.7. *Due to the results in [32], except for the implication* $(1) - (3) \implies (4)$, *theorem 4.6 is also true in the context of dg categories, with the proof adapted. In that case its implication* $(4) \implies (3)$ *partially generalizes [6, Theorem 4.3] in the sense that we explicitly prove that the recollement exists with* $j^* = j^! = ? \otimes^{\mathbf{L}}_B T^{\bullet}$. *Note, however, that if* $T^{\bullet*}$ *is the dg* $A - B-$*module obtained from* $T^{\bullet}$ *by application of the functor* $\mathbf{R}Hom_{B^{op}}(?, B) : \mathcal{D}(B^{op} \otimes A) \longrightarrow \mathcal{D}(A^{op} \otimes B)$, *we cannot guarantee that the left adjoint of* $? \otimes^{\mathbf{L}}_B T^{\bullet} : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ *is (naturally isomorphic to)* $? \otimes^{\mathbf{L}}_A T^{\bullet*}$. *Due to the version of proposition 3.10 for dg algebras, we can guarantee that when* A *is assumed to be* k-*projective. Note that, for the entire theorem 4.6 to be true in the context of dg algebras(or even dg categories), one only needs to prove that the implication* $(2) \implies (1)$ *of proposition 4.2 holds in this more general context. The rest of the proof of theorem 4.6 can be extended without problems.*

Example 4.8. *Let A be a hereditary Artin algebra and let S be a non-projective simple module. Then $T = A \oplus S$ is a right A-module, so that T becomes a $B - A-$bimodule, where $B = End(T_A) \cong \begin{pmatrix} A & 0 \\ S & D \end{pmatrix}$, where $D = End(S_A)$. There are a recollement $\mathcal{D}(A) \equiv \mathcal{D}(B) \equiv \mathcal{D}'$, with $i_* = \mathbf{R}Hom_A(T, ?)$, and a recollement $\mathcal{D}'' \equiv \mathcal{D}(B) \equiv \mathcal{D}(A)$, with $j_* = \mathbf{R}Hom_A(T, ?)$, for some triangulated categories $\mathcal{D}'$ and $\mathcal{D}''$. However T_A is not exceptional in $\mathcal{D}(A)$.*

Proof. It is well-known that $\mathrm{Ext}^1_A(S, S) = 0$, which implies that $\mathrm{Ext}^1_A(T, T) \cong \mathrm{Ext}^1_A(S, A) \neq 0$ and, hence, that T_A is not exceptional in $\mathcal{D}(A)$.

We denote by e_i $(i = 1, 2)$ the canonical idempotents of B. We readily see that $_BT \cong Be_1$, that $\mathrm{Hom}_A(T, A) \cong e_1B$ and that $\mathrm{Ext}^1_A(T, A)$ is isomorphic to $\begin{pmatrix} 0 & \mathrm{Ext}^1_A(S, A) \end{pmatrix}$, when viewed as a right B-module in the usual way (see, e.g., [4, Proposition III.2.2]). In particular, we have $\mathrm{Ext}^1_A(T, A)e_1 = 0$. We then get a triangle in $\mathcal{D}(B)$:

$$e_1B[0] \longrightarrow \mathbf{R}\mathrm{Hom}_A(T, A) \longrightarrow \mathrm{Ext}^1_A(T, A)[-1] \overset{+}{\longrightarrow}.$$

When applying $? \otimes_B^{\mathbf{L}} T =? \otimes_B^{\mathbf{L}} Be_1 =? \otimes_B Be_1$, we get a triangle in $\mathcal{D}(A)$:

$$A = e_1 Be_1[0] \longrightarrow \mathbf{R}\mathrm{Hom}_A(T, A) \otimes_B^{\mathbf{L}} T \longrightarrow \mathrm{Ext}_A^1(T, A)e_1[-1] = 0 \overset{+}{\longrightarrow}.$$

We then get an isomorphism $A \overset{\cong}{\longrightarrow} \mathbf{R}\mathrm{Hom}_A(T, A) \otimes_B^{\mathbf{L}} T$ in $\mathcal{D}(A)$, which is easily seen to be inverse to δ_A. Then assertion (3) of corollary 4.5 holds.

On the other hand, also condition (4) of theorem 4.6 holds. $\qquad\square$

Although the exceptionality property is not needed, it helps to extract more information about $T^\bullet$, when $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?)$ is fully faithful. The following is an example:

Proposition 4.9. *Let $T^\bullet$ be a complex of $B - A-$ bimodules such that $\mathbf{R}Hom_A(T^\bullet, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ is fully faithful. The following assertions hold:*
  (1) *If $\mathcal{D}(A)(T^\bullet, T^\bullet[n]) = 0$, for all but finitely many $n \in \mathbb{Z}$, then $H^p(T^\bullet) = 0$, for all but finitely many $p \in \mathbb{Z}$;*
  (2) *If $T_A^\bullet$ is exceptional in $\mathcal{D}(A)$ and the algebra morphism $B \longrightarrow End_{\mathcal{D}(A)}(T^\bullet)$ is an isomorphism, then $_B T^\bullet$ is isomorphic in $\mathcal{D}(B^{op})$ to an upper bounded complex of finitely generated projective left B-modules (with bounded homology).*

Proof. (1) Let us put $X^\bullet = \mathbf{R}\mathrm{Hom}_A(T^\bullet, T^\bullet)$. The hypothesis says that $H^n(X^\bullet) = 0$, for all but finitely many $n \in \mathbb{Z}$. In particular the canonical morphism $f : \coprod_{n \in \mathbb{Z}} X^\bullet[n] \longrightarrow \prod_{n \in \mathbb{Z}} X^\bullet[n]$ is an isomorphism in $\mathcal{D}(B)$ since $H^p(f)$ is an isomorphism, for all $p \in \mathbb{Z}$. On the other hand, due to the adjunction equations and the fact the the counit $\delta : (? \otimes_B^{\mathbf{L}} T^\bullet) \circ \mathbf{R}\mathrm{Hom}_A(T^\bullet, ?) \longrightarrow 1_{\mathcal{D}(A)}$ is a natural isomorphism, the unit $\lambda : 1_{\mathcal{D}(B)} \longrightarrow \mathbf{R}\mathrm{Hom}_A(T^\bullet, ?) \circ (? \otimes_B^{\mathbf{L}} T^\bullet)$ satisfies that $\lambda_{M^\bullet} \otimes_B^{\mathbf{L}} 1_{T^\bullet} := (? \otimes_B^{\mathbf{L}} T^\bullet)(\lambda_{M^\bullet})$ is an isomorphism, for each $M^\bullet \in \mathcal{D}(B)$. Applying this to the map $\lambda_B : B \longrightarrow [\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?) \circ (? \otimes_B^{\mathbf{L}} T^\bullet)](B) = X^\bullet$, we conclude that $\lambda_B \otimes_B^{\mathbf{L}} 1_{T^\bullet} : T^\bullet \cong B \otimes_B^{\mathbf{L}} T^\bullet \longrightarrow X^\bullet \otimes_B^{\mathbf{L}} T^\bullet$ is an isomorphism in $\mathcal{D}(A)$.

We now have the following chain of isomorphisms in $\mathcal{D}(A)$:

$$\coprod_{n \in \mathbb{Z}} T^\bullet[n] \overset{\cong}{\longrightarrow} \coprod_{n \in \mathbb{Z}} (X^\bullet \otimes_B^{\mathbf{L}} T^\bullet[n]) \cong (\coprod_{n \in \mathbb{Z}} X^\bullet[n]) \otimes_B^{\mathbf{L}} T^\bullet \overset{f \otimes 1}{\longrightarrow} (\prod_{n \in \mathbb{Z}} X^\bullet[n]) \otimes_B^{\mathbf{L}} T^\bullet \cong$$
$$(\prod_{n \in \mathbb{Z}} \mathbf{R}\mathrm{Hom}_A(T^\bullet, T^\bullet[n])) \otimes_B^{\mathbf{L}} T^\bullet \overset{\cong}{\longrightarrow} \mathbf{R}\mathrm{Hom}_A(T^\bullet, \prod_{n \in \mathbb{Z}} T^\bullet[n]) \otimes_B^{\mathbf{L}} T^\bullet \overset{\delta}{\longrightarrow} \prod_{n \in \mathbb{Z}} T^\bullet[n].$$

It is routine to check that the composition of these isomormorphisms is the canonical morphism from the coproduct to the product. Arguing now as in the proof of proposition 4.2, we conclude that $H^p(T^\bullet) = 0$, for all but finitely many $p \in \mathbb{Z}$.

(2) The counit gives an isomorphism

$$B^\alpha \otimes_B^{\mathbf{L}} T^\bullet \cong \mathbf{R}\mathrm{Hom}_A(T^\bullet, T^\bullet)^\alpha \otimes_B^{\mathbf{L}} T^\bullet \overset{\cong}{\longrightarrow} \mathbf{R}\mathrm{Hom}_A(T^\bullet, T^{\bullet \alpha}) \otimes_B^{\mathbf{L}} T^\bullet \overset{\delta}{\longrightarrow} T^{\bullet \alpha},$$

for each cardinal α. Now apply proposition 4.1 to end the proof. $\qquad\square$

Unlike the case of $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?)$, one-sided exceptionality is a consequence of the fully faithfulness of $? \otimes_B^{\mathbf{L}} T^\bullet$.

Proposition 4.10. *Let $T^\bullet$ be a complex of $B - A-$bimodules. Consider the following assertions:*
  (1) *$? \otimes_B^{\mathbf{L}} T^\bullet : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ is fully faithful;*

(2) $T_A^\bullet$ is exceptional in $\mathcal{D}(A)$, the canonical algebra morphism $B \longrightarrow End_{\mathcal{D}(A)}(T^\bullet)$ is an isomorphism, and the functor $\mathbf{R}Hom_A(T^\bullet, ?) \circ (? \otimes_B^{\mathbf{L}} T^\bullet) : \mathcal{D}(B) \longrightarrow \mathcal{D}(B)$ preserves coproducts.

(3) $T_A^\bullet$ is exceptional and self-compact in $\mathcal{D}(A)$, and the canonical algebra morphism $B \longrightarrow End_{\mathcal{D}(A)}(T^\bullet)$ is an isomorphism

(4) $T_A^\bullet$ satisfies the following conditions:
   (a) The canonical morphism of algebras $B \longrightarrow End_{\mathcal{D}(A)}(T^\bullet)$ is an isomorphism;
   (b) For each cardinal α, the canonical morphism $\mathcal{D}(A)(T^\bullet, T^\bullet)^{(\alpha)} \longrightarrow \mathcal{D}(A)(T^\bullet, T^{\bullet(\alpha)})$ is an isomorphism and $\mathcal{D}(A)(T^\bullet, T^{\bullet(\alpha)}[p]) = 0$, for all $p \in \mathbf{Z} \setminus \{0\}$;
   (c) $Susp_{\mathcal{D}(A)}(T^\bullet)^\perp \cap Tria_{\mathcal{D}(A)}(T^\bullet)$ is closed under taking coproducts.

The implications $(1) \Longleftrightarrow (2) \Longleftrightarrow (3) \Longrightarrow (4)$ hold true. When $T_A^\bullet$ is quasi-isomorphic to a bounded complex of projective A-modules, all assertions are equivalent.

Proof. Let $\lambda : 1_{\mathcal{D}(B)} \longrightarrow \mathbf{R}Hom_A(T^\bullet, ?) \circ (? \otimes_B^{\mathbf{L}} T^\bullet)$ be the unit of the adjoint pair $(? \otimes_B^{\mathbf{L}} T^\bullet, \mathbf{R}Hom_A(T^\bullet, ?))$. Assertion (1) is equivalent to saying that λ is a natural isomorphism.

$(1) \Longrightarrow (2)$ In particular, $\lambda_B : B \longrightarrow [\mathbf{R}Hom_A(T^\bullet, ?) \circ (? \otimes_B^{\mathbf{L}} T^\bullet)](B) =: \mathbf{R}Hom_A(T^\bullet, B \otimes_B^{\mathbf{L}} T^\bullet) \cong \mathbf{R}Hom_A(T^\bullet, T^\bullet)$ is an isomorphism. This implies that $T^\bullet$ is exceptional and the 0-homology map $B = H^0(B) \longrightarrow H^0(\mathbf{R}Hom_A(T^\bullet, T^\bullet)) = \mathcal{D}(A)(T^\bullet, T^\bullet)$ is an isomorphism. But this latter map is easily identified with the canonical algebra morphism $B \longrightarrow End_{\mathcal{D}(A)}(T^\bullet)$. That $\mathbf{R}Hom_A(T^\bullet, ?) \circ (? \otimes_B^{\mathbf{L}} T^\bullet) : \mathcal{D}(B) \longrightarrow \mathcal{D}(B)$ preserves coproducts is automatic since this functor is naturally isomorphic to the identity.

$(2) \Longrightarrow (1)$ The full subcategory $\mathcal{D}$ of $\mathcal{D}(B)$ consisting of the objects $X^\bullet$ such that $\lambda_{X^\bullet}$ is an isomorphism is a triangulated subcategory, closed under coproducts, which contains B_B. Then we have $\mathcal{D} = \mathcal{D}(B)$.

$(1), (2) \Longrightarrow (3)$ The functor $? \otimes_B^{\mathbf{L}} T^\bullet : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ induces a triangulated equivalence $\mathcal{D}(B) \overset{\cong}{\longrightarrow} Tria_{\mathcal{D}(A)}(T^\bullet)$. The exceptionality (in $\mathcal{D}(A)$ or in $Tria_{\mathcal{D}(A)}(T^\bullet)$) and the compactness of $T^\bullet$ in $Tria_{\mathcal{D}(A)}(T^\bullet)$ follow from the exceptionality and compactness of B in $\mathcal{D}(B)$. Moreover, we get an algebra isomorphism:

$$B \cong End_{\mathcal{D}(B)}(B) \overset{\cong}{\longrightarrow} End_{\mathcal{D}(A)}(B \otimes_B^{\mathbf{L}} T^\bullet) \cong End_{\mathcal{D}(A)}(T^\bullet).$$

$(3) \Longrightarrow (2)$ The functor $\mathbf{R}Hom_A(T^\bullet, ?) \circ (? \otimes_B^{\mathbf{L}} T^\bullet) : \mathcal{D}(B) \longrightarrow \mathcal{D}(B)$ coincides with the following composition:

$$\mathcal{D}(B) \overset{? \otimes_B^{\mathbf{L}} T^\bullet}{\longrightarrow} Tria_{\mathcal{D}(A)}(T^\bullet) \overset{\mathbf{R}Hom_A(T^\bullet, ?)}{\longrightarrow} \mathcal{D}(B).$$

The self-compactness of $T^\bullet$ implies that the second functor in this composition preserves coproducts. It then follows that the composition itself preserves coproducts since so does $? \otimes_B^{\mathbf{L}} T^\bullet$.

$(3) \Longrightarrow (4)$ Condition (4)(a) is in the hyphtesis, and from the self-compactness and the exceptionality of $T^\bullet$ conditions (4)(b) and (4)(c) follow immediately.

$(4) \Longrightarrow (3)$ Without loss of generality, we assume that that $T^\bullet$ is a bounded complex of projective A-modules in $\mathcal{C}^{\leq 0}(A)$. We just need to prove the self-compactness of $T^\bullet$ in $\mathcal{D}(A)$. We put $\mathcal{U} := Susp_{\mathcal{D}(A)}(T^\bullet)$ and $\mathcal{T} := Tria_{\mathcal{D}(A)}(T^\bullet)$. Note that $(\mathcal{U}, \mathcal{U}^\perp[1])$ is a t-structure in $\mathcal{D}(A)$ and that $\mathcal{U}^\perp$ consists of the $Y^\bullet \in \mathcal{D}(A)$ such that $\mathcal{D}(A)(T^\bullet[k], Y^\bullet) = 0$, for all $k \geq 0$ (see proposition 2.26). Let $(X_i^\bullet)_{i \in I}$ be any family of objects in $\mathcal{T}$. We want

to check that the canonical morphism $\coprod_{i\in I}\mathcal{D}(A)(T^\bullet, X_i^\bullet) \longrightarrow \mathcal{D}(A)(T^\bullet, \coprod_{i\in I} X_i^\bullet)$ is an isomorphism, which is equivalent to proving that it is an epimorphism. We consider the triangle associated to the t-structure $(\mathcal{U}, \mathcal{U}^\perp[1])$

$$\coprod_{i\in I}\tau_\mathcal{U}(X_i) \longrightarrow \coprod_{i\in I} X_i \longrightarrow \coprod_{i\in I}\tau^{\mathcal{U}^\perp}(X_i) \overset{+}{\longrightarrow}.$$

This triangle is in $\mathcal{T}$ because its central and left terms are in $\mathcal{T}$. Note also that we have $\mathcal{U} \subseteq \mathcal{D}^{\leq 0}(A)$. In particular, $\tau_\mathcal{U}(X_i)$ is in $\mathcal{D}^{\leq 0}(A)$, for each $i \in I$.

From [30, Theorem 3 and its proof], we know that the inclusion $\mathcal{T} \cap \mathcal{D}^-(A) \hookrightarrow \mathcal{D}^-(A)$ has a right adjoint and that $T^\bullet$ is a compact object of $\mathcal{T} \cap \mathcal{D}^-(A)$. That is, the functor $\mathcal{D}(A)(T^\bullet, ?)$ preserves coproducts of objects in $\mathcal{T} \cap \mathcal{D}^-(A)$ whenever the coproduct is in $\mathcal{D}^-(A)$. In our case, this implies that the canonical morphism $\coprod_{i\in I}\mathcal{D}(A)(T^\bullet, \tau_\mathcal{U}(X_i^\bullet)) \longrightarrow \mathcal{D}(A)(T^\bullet, \coprod_{i\in I}\tau_\mathcal{U}(X_i))$ is an isomorphism. On the other hand, condition 4.c says that $\mathcal{D}(A)(T^\bullet, \coprod_{i\in I}\tau^{\mathcal{U}^\perp}(X_i^\bullet)) = 0$. It follows that the induced map $\mathcal{D}(A)(T, \coprod_{i\in I}\tau_\mathcal{U}(X_i)) \longrightarrow \mathcal{D}(A)(T, \coprod_{i\in I} X_i)$ is an epimorphism. We then get the following commutative square:

$$\begin{array}{ccc}
\coprod_{i\in I}\mathcal{D}(A)(T^\bullet, \tau_\mathcal{U} X_i^\bullet) & \overset{\sim}{\longrightarrow} & \coprod_{i\in I}\mathcal{D}(A)(T^\bullet, X_i^\bullet) \\
\downarrow\wr & & \downarrow \\
\mathcal{D}(A)(T^\bullet, \coprod_{i\in I}\tau_\mathcal{U} X_i^\bullet) & \longrightarrow & \mathcal{D}(A)(T^\bullet, \coprod_{i\in I} X_i^\bullet)
\end{array}$$

Its upper horizontal and its left vertical arrows are isomorphism, while the lower horizontal one is an epimorphism. It follows that the right vertical arrow is an epimorphism. $\qquad\square$

Recall that if A and B are dg algebras and $f : B \longrightarrow A$ is morphism of dg algebras, then f is called a *homological epimorphism* when the morphism $(? \otimes_B^\mathbf{L} A)(A) \longrightarrow A$ in $\mathcal{D}(A)$, defined by the multiplication map $A \otimes_B^\bullet A \longrightarrow A$, is an isomorphism. This is also equivalent to saying that the left-right symmetric version $(A \otimes_B^\mathbf{L} ?)(A) \longrightarrow A$ is an isomorphism $\mathcal{D}(A^{op})$ (see [33]). When A and B are ordinary algebras, f is a homological epimorphism if, and only if, $\text{Tor}_i^B(A, A) = 0$, for $i \neq 0$, and the multiplication map $A \otimes_B A \longrightarrow A$ is an isomorphism.

Remark 4.11. *Under the hypothesis that $T_A^\bullet$ is quasi-isomorphic to a bounded complex of projective modules, we do not know if condition (4)(c) is needed for the implication (4) $\implies$ (3) to hold (see the questions in the next subsection). On the other hand, if in that same assertion one replaces $Susp_{\mathcal{D}(A)}(T^\bullet)^\perp \cap Tria_{\mathcal{D}(A)}(T^\bullet)$ by just $Susp_{\mathcal{D}(A)}(T^\bullet)^\perp$, then the implication (3) $\implies$ (4) need not be true, as the following example shows.*

Example 4.12. *The functor $? \otimes_\mathbb{Q}^\mathbf{L} \mathbb{Q} : \mathcal{D}(\mathbb{Q}) \longrightarrow \mathcal{D}(\mathbb{Z})$ is fully faithful, but $Susp_{\mathcal{D}(\mathbb{Z})}(Q)^\perp$ is not closed under taking coproducts in $\mathcal{D}(\mathbb{Z})$.*

Proof. The fully faithful condition of $? \otimes_\mathbb{Q}^\mathbf{L} \mathbb{Q}$ follows from theorem 4.13 below since the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is a homological epimorphism. On the other hand, if I is the set of prime natural numbers and we consider the family of stalk complexes $(\mathbb{Z}_p[1])_{p\in I}$, we clearly have that $\mathcal{D}(\mathbb{Z})(Q[k], \mathbb{Z}_p[1]) = 0$, for all $p \in I$ and integers $k \geq 0$, so that $\mathbb{Z}_p[1] \in Susp_{\mathcal{D}(\mathbb{Z})}(\mathbb{Q})^\perp$.

We claim that $\mathcal{D}(\mathbb{Z})(\mathbb{Q}, \coprod_{p\in I}\mathbb{Z}_p[1]) = \text{Ext}_\mathbb{Z}^1(\mathbb{Q}, \coprod_{p\in I}\mathbb{Z}_p) \neq 0$. Indeed, consider the map $\epsilon : \mathbb{Q}/\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z}$ given by multiplication by the 'infinite product' of all prime natural

numbers. Concretely, if $\frac{a}{p_1^{m_1}\cdot\ldots\cdot p_t^{m_t}}$ is a fraction of integers, where $m_j > 0$ and $p_j \in I$, for all $j = 1, ..., t$, then $\epsilon(\frac{a}{b} + \mathbb{Z}) = \frac{a}{p_1^{m_1-1}\cdot\ldots\cdot p_t^{m_t-1}} + \mathbb{Z}$. It is clear that $\mathrm{Ker}(\epsilon)$ is the subgroup of $\mathbb{Q}/\mathbb{Z}$ generated by the elements $\frac{1}{p} + \mathbb{Z}$, with $p \in I$, which is clearly isomorphic to $\coprod_{p \in I} \mathbb{Z}_p$. In particular $\mathrm{Ext}_{\mathbb{Z}}^1(\mathbb{Q}, \coprod_{p \in I} \mathbb{Z}_p)$ is isomorphic to the cokernel of the induced map $\epsilon_* : \mathrm{Hom}_{\mathbb{Z}}(Q, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(Q, \mathbb{Q}/\mathbb{Z})$, which takes $\alpha \rightsquigarrow \epsilon \circ \alpha$. The reader is invited to check that the canonical projection $\pi : \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z}$ is not in the image of ϵ_*, thus proving our claim. $\qquad\square$

We know pass to study the recollement situations in which one of the fully faithful functors is $? \otimes_B^{\mathbf{L}} T^\bullet$.

Theorem 4.13. *Let $T^\bullet$ be a complex of $B - A-$bimodules. Consider the following assertions:*

(1) *There is a recollement $\mathcal{D}(B) \equiv \mathcal{D}(A) \equiv \mathcal{D}'$, for some triangulated category $\mathcal{D}'$, where $i_* = ? \otimes_B^{\mathbf{L}} T^\bullet$;*

(2) *There is a recollement $\mathcal{D}(B) \equiv \mathcal{D}(A) \equiv \mathcal{D}'$, for some triangulated category $\mathcal{D}'$, where $i^! = \mathbf{R}Hom_A(T^\bullet, ?)$;*

(3) *$T_A^\bullet$ is exceptional, self-compact, the canonical algebra morphism $B \longrightarrow End_{\mathcal{D}(A)}(T^\bullet)$ is an isomorphism and the functor $? \otimes_B^{\mathbf{L}} T^\bullet : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ preserves products;*

(4) *$T_A^\bullet$ is exceptional, self-compact, the canonical algebra morphism $B \longrightarrow End_{\mathcal{D}(A)}(T^\bullet)$ is an isomorphism and $Tria_{\mathcal{D}(A)}(T^\bullet)$ is closed under taking products in $\mathcal{D}(A)$.*

(5) *$T_A^\bullet$ is exceptional, self-compact, the canonical algebra morphism $B \longrightarrow End_{\mathcal{D}(A)}(T^\bullet)$ is an isomorphism and $_B T^\bullet \in per(B^{op})$.*

(6) *There is a dg algebra $\hat{A}$, a homological epimorphism of dg algebras $f : A \longrightarrow \hat{A}$ and a classical tilting object $\hat{T}^\bullet \in \mathcal{D}(\hat{A})$ such that the following conditions hold:*

    (a) *$T_A^\bullet \cong f_*(\hat{T}^\bullet)$, where $f_* : \mathcal{D}(\hat{A}) \longrightarrow \mathcal{D}(A)$ is the restriction of scalars functor;*

    (b) *the canonical algebra morphism $B \longrightarrow End_{\mathcal{D}(A)}(T) \cong End_{\mathcal{D}(\hat{A})}(\hat{T})$ is an isomorphism.*

Then the implications $(1) \Longleftrightarrow (2) \Longleftrightarrow (3) \Longleftrightarrow (4) \Longleftrightarrow (5) \Longleftarrow (6)$ hold true and, when A is k-flat, all assertions are equivalent.

Proof. $(1) \Longleftrightarrow (2)$ is clear.

Note that the recollement of assertions (1) or (2) exists if, and only if, the functor $? \otimes_B^{\mathbf{L}} T^\bullet$ is fully faithful and has a left adjoint.

$(1) \Longleftrightarrow (3)$ is then a direct consequence of theorem 4.10 and corollary 2.15.

$(1), (3) \Longrightarrow (4)$ The functor $? \otimes_B^{\mathbf{L}} T^\bullet$ induces an equivalence of triangulated categories $\mathcal{D}(B) \xrightarrow{\cong} \mathcal{T} := \mathrm{Tria}_{\mathcal{D}(A)}(T^\bullet)$. The fact that $? \otimes_B^{\mathbf{L}} T^\bullet$ has a left adjoint functor implies that also the inclusion functor $j : \mathcal{T} \hookrightarrow \mathcal{D}(A)$ has a left adjoint. But then j preserves products, so that $\mathcal{T}$ is closed under taking products in $\mathcal{D}(A)$.

$(4) \Longrightarrow (3)$ By proposition 4.10 and its proof, we know that $? \otimes_B^{\mathbf{L}} T^\bullet$ is a fully faithful functor which establishes an equivalence of triangulated categories $\mathcal{D}(B) \xrightarrow{\cong} \mathcal{T} := \mathrm{Tria}_{\mathcal{D}(A)}(T^\bullet)$. In particular, this equivalence of categories preserves products. This, together with the fact that $\mathcal{T}$ is closed under taking products in $\mathcal{D}(A)$, implies that $? \otimes_B^{\mathbf{L}} T^\bullet$ preserves products.

(3) $\iff$ (5) is a direct consequence of proposition 4.2.

(6) $\implies$ (4) By the properties of homological epimorphisms (see [31, Section 4]), the functor $f_* : \mathcal{D}(\hat{A}) \longrightarrow \mathcal{D}(A)$ is fully faithful (and triangulated). It then follows that

$$\mathrm{Hom}_{\mathcal{D}(A)}(T^\bullet, T^\bullet[p]) = \mathrm{Hom}_{\mathcal{D}(A)}(f_*(\hat{T}^\bullet), f_*(\hat{T}^\bullet)[p]) \cong \mathrm{Hom}_{\mathcal{D}(\hat{A})}(\hat{T}^\bullet, \hat{T}^\bullet[p]) = 0,$$

for all integers $p \neq 0$. By condition (6)(b), it also follows that the canonical algebra morphism $B \longrightarrow \mathrm{End}_{\mathcal{D}(A)}(T^\bullet)$ is an isomorphism.

On the other hand, f_* defines an equivalence $\mathcal{D}(\hat{A}) \xrightarrow{\cong} \mathrm{Im}(f_*)$. Moreover $\hat{T}^\bullet$ is a compact generator of $\mathcal{D}(\hat{A})$, which implies that $T^\bullet_A$ is a compact generator of $\mathrm{Im}(f_*)$. Since the last one is a full triangulated subcategory of $\mathcal{D}(A)$ closed under taking coproducts we get that $\mathrm{Im}(f_*) = \mathrm{Tria}_{\mathcal{D}(A)}(T^\bullet)$. But $\mathrm{Im}(f_*)$ is also closed under taking products in $\mathcal{D}(A)$ (see [31, Section 4]). Then $T^\bullet_A$ is self-compact and $\mathrm{Tria}_{\mathcal{D}(A)}(T^\bullet)$ is closed under taking products in $\mathcal{D}(A)$.

(1), (4) $\implies$ (6) (assuming that A is k-flat) We have seen above that the inclusion functor $\mathcal{T} := \mathrm{Tria}_{\mathcal{D}(A)}(T^\bullet) \hookrightarrow \mathcal{D}(A)$ has a left adjoint. By proposition 2.11, its dual and by proposition 2.26, we get that $(^\perp\mathcal{T}, \mathcal{T}, \mathcal{T}^\perp)$ is a TTF triple in $\mathcal{D}(A)$. By [31, Theorem 4], there exists a dg algebra $\hat{A}$ and a homological epimorphism of dg algebras $f : A \longrightarrow \hat{A}$ such that $\mathcal{T} = \mathrm{Im}(f_*)$. Bearing in mind that f_* is fully faithful, it induces an equivalence of categories $f_* : \mathcal{D}(\hat{A}) \xrightarrow{\cong} \mathcal{T}$. If now $\hat{T}^\bullet \in \mathcal{D}(\hat{A})$ is an object such that $f_*(\hat{T}^\bullet) \cong T^\bullet_A$, then $\hat{T}^\bullet$ is a compact generator of $\mathcal{D}(\hat{A})$ such that $\mathcal{D}(\hat{A})(\hat{T}^\bullet, \hat{T}^\bullet[p]) \cong \mathcal{D}(A)(T^\bullet, T^\bullet[p])$, for each $p \in \mathbb{Z}$. Now all requirements for $\hat{T}^\bullet$ are easily verified. $\qquad\square$

The following result deeply extends [11, Lemma 4.2].

Corollary 4.14. *The following assertions are equivalent for a complex $T^\bullet$ of $B-A-$bimodules:*
  (1) *There is a recollement $\mathcal{D}' \equiv \mathcal{D}(A) \equiv \mathcal{D}(B)$ such that $j_! =? \otimes^{\mathbf{L}}_B T^\bullet$, for some triangulated category $\mathcal{D}'$ (which is equivalent to $\mathcal{D}(C)$, where C is a dg algebra);*
  (2) *There is a recollement $\mathcal{D}' \equiv \mathcal{D}(A) \equiv \mathcal{D}(B)$ such that $j^! = j^* = \mathbf{R}Hom_A(T^\bullet, ?)$, for some triangulated category $\mathcal{D}'$ (which is equivalent to $\mathcal{D}(C)$, where C is a dg algebra)*
  (3) *$T^\bullet_A$ is compact and exceptional in $\mathcal{D}(A)$ and the canonical algebra morphism $B \longrightarrow \mathrm{End}_{\mathcal{D}(A)}(T^\bullet)$ is an isomorphism.*

In such case, if A is k-flat, then C can be chosen together with a homological epimorphism $f : A \longrightarrow C$ such that i_* is the restriction of scalars $f_* : \mathcal{D}(C) \longrightarrow \mathcal{D}(A)$.

Proof. (1) $\iff$ (2) is clear.

(1) $\iff$ (3) The mentioned recollement exists if, and only if, $? \otimes^{\mathbf{L}}_B T$ is fully faithful and $\mathbf{R}Hom_A(T^\bullet, ?)$ has a right adjoint. Using proposition 4.10 and corollary 2.15, the existence of such recollement is equivalent to assertion (3). This is because that $\mathbf{R}Hom_A(T^\bullet, ?)$ preserves coproducts if, and only if, $T^\bullet_A$ is compact in $\mathcal{D}(A)$.

The statements about the dg algebra C follow as at the end of the proof of theorem 4.6. $\qquad\square$

We then get the following consequence (compare with [14, Theorem 2]).

Corollary 4.15. *Let $T^\bullet$ be a complex of $B - A-$bimodules, where A and B are ordinary algebras. If there is a recollement $\mathcal{D}' \equiv \mathcal{D}(A) \equiv \mathcal{D}(B)$, with $j_! =? \otimes_B^{\mathbf{L}} T^\bullet$, for some triangulated category $\mathcal{D}'$, then there is a recollement $\mathcal{D}'' \equiv \mathcal{D}(A^{op}) \equiv \mathcal{D}(B^{op})$, with $j^! = j^* = T^\bullet \otimes_A^{\mathbf{L}}?$, for a triangulated category $\mathcal{D}''$. When A is k-projective, the converse is also true.*

Proof. It is a direct consequence of corollary 4.14 and the left-right symmetric version of theorem 4.6. $\qquad\square$

The following is a rather general result on the existence of recollements.

Proposition 4.16. *Let Λ be any dg algebra and $U^\bullet$, $V^\bullet$ be exceptional objects of $\mathcal{D}(\Lambda)$. Put $A = End_{\mathcal{D}(\Lambda)}(U^\bullet)$ and $B = End_{\mathcal{D}(\Lambda)}(V^\bullet)$. If $V^\bullet \in thick_{\mathcal{D}(\Lambda)}(U^\bullet)$, then there is a recollement $\mathcal{D}' \equiv \mathcal{D}(A) \equiv \mathcal{D}(B)$, for some triangulated category $\mathcal{D}'$ (which is equivalent to $\mathcal{D}(C)$, for some dg algebra C).*

Proof. We choose $U^\bullet$ and $V^\bullet$ to be homotopically projective. We then put $\hat{A} = End_\Lambda^\bullet(U^\bullet)$ and $\hat{B} = End_\Lambda^\bullet(V^\bullet)$, which are dg algebras whose associated homology algebras are concentrated in degree 0 and satisfy that $H^0(\hat{A}) \cong A$ and $H^0(\hat{B}) \cong B$. Then we know that $\mathcal{D}(\hat{A}) \cong \mathcal{D}(A)$ and $\mathcal{D}(\hat{B}) \cong \mathcal{D}(B)$.

We next put $T^\bullet = Hom_\Lambda^\bullet(U^\bullet, V^\bullet)$ which is a dg $\hat{B} - \hat{A}-$bimodule. It is easy to see that $Hom_\Lambda^\bullet(U^\bullet, ?) = \mathbf{R}Hom_\Lambda(U^\bullet, ?) : \mathcal{D}(\Lambda) \longrightarrow \mathcal{D}(\hat{A})$ induces an equivalence of of triangulated categories $thick_{\mathcal{D}(\Lambda)}(U^\bullet) \overset{\cong}{\longrightarrow} thick_{\mathcal{D}(\hat{A})}(\hat{A}) = per(\hat{A})$. In particular, we get that $T_{\hat{A}}^\bullet$ is compact in $\mathcal{D}(\hat{A})$. We claim now that there is a recollement $\mathcal{D}' \equiv \mathcal{D}(\hat{A}) \equiv \mathcal{D}(\hat{B})$, where $j_! =? \otimes_{\hat{B}}^{\mathbf{L}} T^\bullet$, an this will end the proof.

The desired recollement exists if, and only if, $? \otimes_{\hat{B}}^{\mathbf{L}} T^\bullet$ is fully faithful and $\mathbf{R}Hom_{\hat{A}}(T^\bullet, ?) : \mathcal{D}(\hat{A}) \longrightarrow \mathcal{D}(\hat{B})$ has a right adjoint. This second condition holds due to corollary 2.15. As for the first condition, we just need to check that the unit $\lambda : 1_{\mathcal{D}(\hat{B})} \longrightarrow \mathbf{R}Hom_{\hat{A}}(T^\bullet, ?) \circ (? \otimes_{\hat{B}}^{\mathbf{L}} T^\bullet)$ is a natural isomorphism. For that, we take the full subcategory $\mathcal{X}$ of $\mathcal{D}(\hat{B})$ consisting of the objects $X^\bullet$ such that $\lambda_{X^\bullet}$ is an isomorphism. It is clearly a triangulated subcategory closed under taking coproducts. But if we apply to $\lambda_{\hat{B}} : \hat{B} \longrightarrow [\mathbf{R}Hom_{\hat{A}}(T^\bullet, ?) \circ (? \otimes_{\hat{B}}^{\mathbf{L}} T^\bullet)](\hat{B}) \cong \mathbf{R}Hom_{\hat{A}}(T^\bullet, T^\bullet)$ the homology functor, we obtain a map

$$H^p(\lambda_{\hat{B}}) : 0 = H^p(\hat{B}) \longrightarrow H^p(\mathbf{R}Hom_{\hat{A}}(T^\bullet, T^\bullet)) \cong \mathcal{D}(\hat{A})(T^\bullet, T^\bullet[p]) \cong \mathcal{D}(\Lambda)(V^\bullet, V^\bullet[p]) = 0,$$

so that $H^p(\lambda_B)$ is an isomorphism, for each $p \in \mathbb{Z} \setminus \{0\}$. As for $p = 0$, we have

$$H^0(\lambda_{\hat{B}}) : B \cong H^0(\hat{B}) \longrightarrow H^0(\mathbf{R}Hom_{\hat{A}}(T^\bullet, T^\bullet)) \cong \mathcal{D}(\hat{A})(T^\bullet, T^\bullet) \cong \mathcal{D}(\Lambda)(V^\bullet, V^\bullet) \cong B,$$

which also an isomorphism. This proves $\lambda_{\hat{B}}$ is an isomorphism in $\mathcal{D}(\hat{B})$, so that $\hat{B} \in \mathcal{X}$. It follows that $\mathcal{X} = \mathcal{D}(\hat{B})$. $\qquad\square$

The following is a particular case of last proposition. Compare with [11, Theorem 1.3]

Example 4.17. *Let Λ be an algebra, let V be an injective cogenerator of $Mod - \Lambda$ and let $B = End_\Lambda(V)$ be its endomorphism algebra. Suppose that U is a Λ-module satisfying the following two conditions:*

    (1) *$Ext_\Lambda^p(U, U) = 0$, for all $p > 0$;*
    (2) *There is an exact sequence $0 \to U^{-n} \longrightarrow ... \longrightarrow U^{-1} \longrightarrow U^0 \longrightarrow V \to 0$, where $U^k \in add_{Mod-\Lambda}(U)$, for each $k = -n, ..., -1, 0$.*

Then there exists a recollement $\mathcal{D}' \equiv \mathcal{D}(A) \equiv \mathcal{D}(B)$, for some triangulated category $\mathcal{D}'$, where $A = End_\Lambda(U)$. Moreover, a slight modification of the proof of last proposition shows that the recollement can be chosen in such a way that $j_! = ? \otimes_B^{\mathbf{L}} T$, where $T = Hom_\Lambda(U, V)$, which is a $B - A-$bimodule.

Note that when Λ is an algebra with Morita duality (e.g. an Artin algebra), the algebras Λ and A are Morita equivalent, and so $\mathcal{D}(A) \cong \mathcal{D}(\Lambda)$.

We will end the section by studying an interesting case of fully faithfulness of $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?)$, where two recollement situations come at once.

Theorem 4.18. *Let $T^\bullet$ be a complex of $B - A-$bimodules such that $T_A^\bullet$ is exceptional in $\mathcal{D}(A)$ and the algebra morphism $B \longrightarrow End_{\mathcal{D}(A)}(T^\bullet)$ is an isomorphism. The following assertion are equivalent:*

    (1) *$_BT^\bullet$ is compact and exceptional in $\mathcal{D}(B^{op})$ and the canonical algebra morphism $A \longrightarrow End_{\mathcal{D}(B^{op})}(T^\bullet)^{op}$ is an isomorphism;*
    (1') *There is a recollement $\mathcal{D}' \equiv \mathcal{D}(B^{op}) \equiv \mathcal{D}(A^{op})$, with $j_! = T^\bullet \otimes_A^{\mathbf{L}} ?$, for some triangulated category $\mathcal{D}'$ (which is equivalent to $\mathcal{D}(C)$, for some dg algebra C);*
    (2) *$\mathbf{R}Hom_A(T^\bullet, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ is fully faithful and preserves compact objects;*
    (3) *A_A is in the thick subcategory of $\mathcal{D}(A)$ generated by $T_A^\bullet$;*
    (4) *$? \otimes_B^{\mathbf{L}} T^\bullet : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ has a fully faithful left adjoint;*
    (5) *There is a recollement $\mathcal{D}' \equiv \mathcal{D}(B) \equiv \mathcal{D}(A)$, with $j_* = \mathbf{R}Hom_A(T^\bullet, ?)$, for triangulated category $\mathcal{D}'$ (which is equivalent to $\mathcal{D}(C)$, for some dg algebra C).*

When B is k-flat, the dg algebra in conditions (1') and (5) can be chosen together with a homological epimorphism of dg algebras $f : B \longrightarrow C$ such that i_* is the restriction of scalars $f_* : \mathcal{D}(C) \longrightarrow \mathcal{D}(B)$.

Proof. Note that the exceptionality of $T_A^\bullet$ plus the fact that the algebra morphism $B \longrightarrow End_{\mathcal{D}(A)}(T^\bullet)$ is an isomorphism is equivalent to saying that the unit map $\lambda_B : B \longrightarrow [\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?) \circ (? \otimes_B^{\mathbf{L}} T^\bullet)](B) = \mathbf{R}\mathrm{Hom}_A(T^\bullet, T^\bullet)$ is an isomorphism in $\mathcal{D}(B)$.

(1) $\Longleftrightarrow$ (1') follows from the left-right symmetric version of corollary 4.14.

(1) $\Longrightarrow$ (3) The hypothesis implies that the unit map $\rho_A : A \longrightarrow [\mathbf{R}\mathrm{Hom}_{B^{op}}(T^\bullet, ?) \circ (T^\bullet \otimes_A^{\mathbf{L}} ?)](A)$ is an isomorphism in $\mathcal{D}(A^{op})$. Then, using lemma 3.4 and its terminology, we know that τ_A is an isomorphism in $\mathcal{D}(A)$, which implies that $A_A \cong \mathbf{R}\mathrm{Hom}_{B^{op}}(?, T^\bullet)(T^\bullet)$. The fact that $_BT^\bullet \in \mathrm{per}(B^{op}) = \mathrm{thick}_{\mathcal{D}(B^{op})}(B)$ implies then that $A_A \in \mathrm{thick}_{\mathcal{D}(A)}([\mathbf{R}\mathrm{Hom}_{B^{op}}(?, T^\bullet)](B)) \subseteq \mathrm{thick}_{\mathcal{D}(A)}(T^\bullet)$.

(3) $\Longrightarrow$ (1) Since λ_B is an isomorphism, using lemma 3.3 and its terminology, we get that $\tau_{T^\bullet} : T^\bullet \longrightarrow [\mathbf{R}\mathrm{Hom}_{B^{op}}(\mathbf{R}\mathrm{Hom}_A(?, T^\bullet), T^\bullet)](T^\bullet)$ is an isomorphism in $\mathcal{D}(A)$. Since $A_A \in \mathrm{thick}_{\mathcal{D}(A)}(T^\bullet)$ we get that τ_A is an isomorphism in $\mathcal{D}(A)$ which, by lemma 3.4, implies that ρ_A is an isomorphism. That is, $_BT^\bullet$ is exceptional in $\mathcal{D}(B^{op})$ and the algebra morphism $A \longrightarrow End_{\mathcal{D}(B^{op})}(T^\bullet)^{op}$ is an isomorphism.

On the other hand, the fact that $A_A \in \text{thick}_{\mathcal{D}(A)}(T^\bullet)$ implies that $_B T^\bullet \cong \mathbf{R}\text{Hom}_A(?, T^\bullet)(A)$ is in $\text{thick}_{\mathcal{D}(B^{op})}(\mathbf{R}\text{Hom}_A(?, T^\bullet)(T^\bullet)) = \text{thick}_{\mathcal{D}(B^{op})}(B) = \text{per}(B^{op})$.

$(1') \Longrightarrow (5)$ is the left-right symmetric version of corollary 4.15.

$(4) \Longleftrightarrow (5)$ Apply proposition 2.19 to the functor $G =: ? \otimes_B^{\mathbf{L}} T^\bullet : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$.

$(4) \Longrightarrow (3)$ Let $L : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ be a fully faithful left adjoint of $? \otimes_B^{\mathbf{L}} T^\bullet$. One easily sees that L preserves compact objects. Moreover, the unit $1_{\mathcal{D}(A)} \longrightarrow (? \otimes_B^{\mathbf{L}} T^\bullet) \circ L$ of the adjunction $(L, ? \otimes_B^{\mathbf{L}} T^\bullet)$ is a natural isomorphism. It follows that $A \cong [(? \otimes_B^{\mathbf{L}} T^\bullet) \circ L](A) = (? \otimes_B^{\mathbf{L}} T^\bullet)(L(A)) \in (? \otimes_B^{\mathbf{L}} T^\bullet)(\text{per}(B)) = (? \otimes_B^{\mathbf{L}} T^\bullet)(\text{thick}_{\mathcal{D}(B)}(B)) \subseteq \text{thick}_{\mathcal{D}(A)}((? \otimes_B^{\mathbf{L}} T^\bullet)(B)) = \text{thick}_{\mathcal{D}(A)}(T^\bullet)$.

$(5), (3) \Longrightarrow (2)$ From assertion (5) we get that $\mathbf{R}\text{Hom}_A(T^\bullet, ?)$ is fully faithful. On the other hand, the fact that $A_A \in \text{thick}_{\mathcal{D}(A)}(T^\bullet)$ and that λ_B is an isomorphism imply that $\mathbf{R}\text{Hom}_A(T^\bullet ?)(A) \in \text{thick}_{\mathcal{D}(B)}(\mathbf{R}\text{Hom}_A(T^\bullet, ?)(T^\bullet)) = \text{thick}_{\mathcal{D}(B)}(B) = \text{per}(B)$. It follows from this that $\mathbf{R}\text{Hom}_A(T^\bullet, ?)$ takes objects of $\text{per}(A) = \text{thick}_{\mathcal{D}(A)}(A)$ to objects of $\text{per}(B)$, thus proving assertion (2).

$(2) \Longrightarrow (3)$ Bearing in mind that the counit $\delta : (? \otimes_B^{\mathbf{L}} T^\bullet) \circ \mathbf{R}\text{Hom}_A(T^\bullet, ?) \longrightarrow 1_{\mathcal{D}(A)}$ is a natural isomorphism and that $\mathbf{R}\text{Hom}_A(T^\bullet, ?)(A)$ is a compact object of $\mathcal{D}(B)$, we get:

$$A \cong [? \otimes_B^{\mathbf{L}} T^\bullet) \circ \mathbf{R}\text{Hom}_A(T^\bullet, ?)](A) = (? \otimes_B^{\mathbf{L}} T^\bullet))(\mathbf{R}\text{Hom}_A(T^\bullet, A)) \in$$
$$(? \otimes_B^{\mathbf{L}} T^\bullet)(\text{per}(B)) = (? \otimes_B^{\mathbf{L}} T^\bullet)(\text{thick}_{\mathcal{D}(B)}(B)) \subseteq \text{thick}_{\mathcal{D}(A)}(B \otimes_B^{\mathbf{L}} T^\bullet) = \text{thick}_{\mathcal{D}(A)}(T^\bullet).$$

$\square$

Remark 4.19. *The precursor of theorem 4.18 is* [5, Theorem 2.2]*, where the authors prove that if T_A is a good tilting module (see definition 18) and $B = End(T_A)$, then condition (2) in our theorem holds. It is a consequence of theorem 4.18 (see corollary 5.5 below) that the converse is also true when one assume that T_A has finite projective dimension and $Ext_A^p(T, T^{(\alpha)}) = 0$, for all integers $p > 0$ and all cardinals α. Another consequence (see corollary 5.5 and example 5.6(1)) is that there are right A-modules, other than the good tilting ones, for which the equivalent conditions of the theorem holds. In the case of good 1-tilting modules, it was proved in* [10, Theorem 1.1] *that the dg algebra C can be chosen to be an ordinary algebra.*

The corresponding of the implication $(1) \Longrightarrow (5)$ in our theorem was proved in [41, Theorem 1] *for dg algebras over field. This result and its converse is then covered by the extension of theorem 4.18 to the context of dg categories, which is proved in* [32]*.*

4.2. **Some natural questions.** As usual, $T^\bullet$ is a complex of $B - A-$bimodules. After the previous subsection, some natural questions arise, starting with the questions 1.2 of the introduction. Our next list of examples gives negative answers to all questions 1.2.

For question 1.2(1)(a), the following is a counterexample:

Example 4.20. *Let T_A be a good tilting module (see definition 18) which is not finitely generated (e.g. $\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}$ as $\mathbb{Z}$-module) and let $B = End(T_A)$ be its endomorphism algebra. The functor $\mathbf{R}Hom_A(T, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ is fully faithful, but there is no recollement $\mathcal{D}(A) \equiv \mathcal{D}(B) \equiv \mathcal{D}'$, with $i_* = \mathbf{R}Hom_A(T, ?)$, for any triangulated category $\mathcal{D}'$.*

Proof. That $\mathbf{R}\text{Hom}_A(T, ?)$ is fully faithful follows from corollary 5.5 below. On the other hand, if the desired recollement $\mathcal{D}(A) \equiv \mathcal{D}(B) \equiv \mathcal{D}'$ existed, then, by corollary 4.5, we would have that T_A is compact in $\mathcal{D}(A)$, and this is not the case. $\square$

For question 1.2(1.b), the following is a counterexample.

Example 4.21. *If $f : B \longrightarrow A$ is a homological epimorphism of algebras, and we take $T = {}_BA_A$, then $\mathbf{R}Hom_A(A, ?) = f_* : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ is fully faithful, but there need not exist a recollement $\mathcal{D}' \equiv \mathcal{D}(B) \equiv \mathcal{D}(A)$, with $j_* = \mathbf{R}Hom_A(A, ?)$, for any triangulated category $\mathcal{D}'$.*

Proof. That $\mathbf{R}Hom_A(A, ?) = f_*$ is fully faithful follows from the properties of homological epimorphisms. If the mentioned recollement exists, then the functor $? \otimes_B^{\mathbf{L}} A : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ preserves products and, by proposition 4.2, ${}_BA$ is compact in $\mathcal{D}(B^{op})$. There are obvious homological epimorphisms which do not satisfy this last property. $\qquad\square$

For question 1.2(2)(a), the following is a counterexample, inspired by theorem 4.13:

Example 4.22. *Let A be an algebra and let P be a finitely generated projective right A-module such that P is not finitely generated as a left module over $B := End_A(P)$. Then $? \otimes_B^{\mathbf{L}} P : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ is fully faithful, but there is no recollement $\mathcal{D}(B) \equiv \mathcal{D}(A) \equiv \mathcal{D}'$, with $i_* = ? \otimes_B^{\mathbf{L}} P = ? \otimes_B P$, for any triangulated category $\mathcal{D}'$.*

Concretely, if k is a field, V is an infinite dimensional k-vector space and $A = End_k(V)^{op}$, then $? \otimes_k^{\mathbf{L}} V = ? \otimes_k V : \mathcal{D}(k) \longrightarrow \mathcal{D}(A)$ is fully faithful, but does not define the mentioned recollement.

Proof. The final statement follows directly from the first part since V is a simple projective right A-module such that $End_A(V) \cong k$. On the other hand, we get from proposition 4.10 that $? \otimes_B^{\mathbf{L}} P : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ is fully faithful and, since ${}_BP$ is not compact in $\mathcal{D}(B^{op})$, theorem 4.13 implies that the recollement does not exist.
$\qquad\square$

As a counter example to question 1.2(2)(b), we have:

Example 4.23. *The functor $? \otimes_{\mathbb{Q}}^{\mathbf{L}} \mathbb{Q} = ? \otimes_{\mathbb{Q}} Q : \mathcal{D}(\mathbb{Q}) \longrightarrow \mathcal{D}(\mathbb{Z})$ is fully faithful (see example 4.12), but there is no recollement $\mathcal{D}' \equiv \mathcal{D}(\mathbb{Z}) \equiv \mathcal{D}(\mathbb{Q})$, with $j_! = ? \otimes_{\mathbb{Q}} Q$, for any triangulated category $\mathcal{D}'$.*

Proof. If the recollement existed, then, by corollary 4.14, $\mathbb{Q}$ would be compact in $\mathcal{D}(\mathbb{Z})$, which is absurd. $\qquad\square$

But, apart from questions 1.2, there are some other natural questions whose answer we do not know even in the case of a bimodule.

Questions 4.24.   (1) *(Motivated by proposition 4.10) Suppose that $T_A^{\bullet}$ is isomorphic in $\mathcal{D}(A)$ to a bounded complex of projective right A-modules, that the canonical morphism $Hom_{\mathcal{D}(A)}(T^{\bullet}, T^{\bullet})^{(\alpha)} \longrightarrow Hom_{\mathcal{D}(A)}(T^{\bullet}, T^{\bullet(\alpha)})$ is an isomoprhism and that $Hom_{\mathcal{D}(A)}(T^{\bullet}, T^{\bullet(\alpha)}[p]) = 0$, for all cardinals α and all integers $p \neq 0$. Is $T_A^{\bullet}$ self-compact in $\mathcal{D}(A)$?.*
    (2) *(Motivated by proposition 4.10) Suppose that $? \otimes_B T^{\bullet} : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ is fully faithful. Is $H^p(T^{\bullet}) = 0$ for $p >> 0$?. Is $T_A^{\bullet}$ quasi-isomorphic to a bounded complex of projective right A-modules?*

Remark 4.25. *The converse of question 4.24(2) has a negative answer, even for a bounded complex of projective A-modules. For instance, if P is a projective generator of $Mod - A$ which is not finitely generated, then $Tria_{\mathcal{D}(A)}(P) = \mathcal{D}(A)$ and P is not compact in this category. It follows from proposition 4.10 that $? \otimes_B^{\mathbf{L}} P : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ is not fully faithful, where $B = End(P_A)$.*

Our next question concerns the relationship between proposition 4.4 and theorem 4.18. That $\mathbf{R}\mathrm{Hom}_A(T^\bullet, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ be fully faithful does not imply that it preserves compact objects (see example 4.21). The correct question to answer, for which we do not have an aswer, is the following:

Question 4.26. *Suppose that $T_A^\bullet$ is exceptional in $\mathcal{D}(A)$, the canonical algebra morphism $B \longrightarrow End_{\mathcal{D}(A)}(T^\bullet)$ is an isomorphism and that $\mathbf{R}Hom_A(T^\bullet, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ is fully faithful. Due to theorems 4.6 and 4.18, each of the following questions has an affirmative answer if, and only if, so do the other ones, but we do not know the answer:*

   (1) Does $\mathbf{R}Hom_A(T^\bullet, ?)$ preserve compact objects?;
   (2) Is A_A in $thick_{\mathcal{D}(A)}(T^\bullet)$?;
   (3) Is $_B T^\bullet$ compact in $\mathcal{D}(B^{op})$?

Note that, by proposition 4.9, $_B T^\bullet$ is isomorphic in $\mathcal{D}(B^{op})$ to an upper bounded complex of finitely generated projective left B-modules with bounded homology.

In next section, we will show that, in case $T^\bullet$ is a $B - A-$bimodule, the question has connections with Wakamatsu tilting problem.

5. The case of a bimodule

5.1. Re-statement of the main results. For the convenience of the reader, we make explicit what some results of the previous section say in the particular case when $T^\bullet = T$ is just a $B - A-$bimodule. The statements show a close connection with the theory of (not necessarily finitely generated) tilting modules.
   Recall:

Definition 18. *Consider the following conditions for an A-module T:*

   a) *T has finite projective dimension;*
   a') *T admits a finite projective resolution with finitely generated terms;*
   b) *There is an exact sequence $0 \to A \longrightarrow T^0 \longrightarrow T^1 \longrightarrow .... \longrightarrow T^m \to 0$ in $Mod - A$, with $T^i \in Add(T)$ for $i = 0, 1, ..., m$;*
   b') *There is an exact sequence $0 \to A \longrightarrow T^0 \longrightarrow T^1 \longrightarrow .... \longrightarrow T^m \to 0$ in $Mod - A$, with $T^i \in add(T)$ for $i = 0, 1, ..., m$;*
   c) *$Ext_A^p(T, T^{(\alpha)}) = 0$, for all integers $p > 0$ and all cardinals α.*

T is called a n-tilting module when conditions a), b) and c) hold and $pd_A(T) = n$. Such a tilting module is classical n-tilting *when it satisfies a'), b') and c) and it is called a* good n-tilting *module when it satisfies conditions a), b') and c). We will simply say that T is tilting (resp. classical tilting, resp. good tilting) when it is n-tilting (resp. classical n-tilting, resp. good n-tilting), for some $n \in \mathbf{N}$.*

Remark 5.1. *Note that T is a classical n-tilting module if, and only if, it satisfies conditions a'), b') and $Ext_A^p(T, T) = 0$, for all integers $p > 0$.*

When $\text{Ext}_A^p(T,T) = 0$, for all $p > 0$, it is proved in [32] that the condition that $A_A \in \text{thick}_{\mathcal{D}(A)}(T)$ is equivalent condition b') in Definition 18.

In the rest of the subsection, unless otherwise stated, T will be a $B - A-$bimodule and all statements are given for it. The following result is then a direct consequence of proposition 4.10:

Corollary 5.2. *Consider the following assertions:*
1. $\otimes_B^{\mathbf{L}} T : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ *is fully faithful;*
2. $\text{Ext}_A^p(T,T) = 0$, *for all $p > 0$, the canonical algebra morphism $B \longrightarrow End_A(T)$ is an isomorphism and T_A is a compact object of $Tria_{\mathcal{D}(A)}(T)$.*
3. *The following conditions hold:*
   (a) *the canonical map $B^{(\alpha)} \longrightarrow Hom_A(T,T)^{(\alpha)} \longrightarrow Hom_A(T, T^{(\alpha)})$ is an isomorphism, for all cardinals α*
   (b) $\text{Ext}_A^p(T, T^{(\alpha)}) = 0$, *for all cardinals α and integers $p > 0$;*
   (c) *for each family $(X_i^\bullet)_{i \in I}$ in $Tria_{\mathcal{D}(A)}(T)$ such that $\mathcal{D}(A)(T[k], X_i^\bullet) = 0$, for all $k \geq 0$ and all $i \in I$, one has that $\mathcal{D}(A)(T, \coprod_{i \in I} X_i^\bullet) = 0$.*

The implications $(1) \Longleftrightarrow (2) \Longrightarrow (3)$ hold true. When T_A has finite projective dimension, all assertions are equivalent.

The next result is then a consequence of theorem 4.13:

Corollary 5.3. *The following assertions are equivalent:*
1. *There is a recollement $\mathcal{D}(B) \equiv \mathcal{D}(A) \equiv \mathcal{D}'$ with $i_* =? \otimes_B^{\mathbf{L}} T$, for some triangulated category $\mathcal{D}'$;*
2. $\text{Ext}_A^p(T,T) = 0$, *for all $p > 0$, the canonical algebra morphism $B \longrightarrow End_A(T)$ is an isomorphism, T is a compact object of $Tria_{\mathcal{D}(A)}(T)$ and this subcategory is closed under taking products in $\mathcal{D}(A)$.*
3. $\text{Ext}_A^p(T,T) = 0$, *for all $p > 0$, the canonical algebra morphism $B \longrightarrow End_A(T)$ is an isomorphism, T is a compact object of $Tria_{\mathcal{D}(A)}(T)$ and T admits a finite projective resolution with finitely generated terms as a left B-module.*
   When A is k-flat, these conditions are also equivalent to:
4. *There is a dg algebra $\hat{A}$, a homological epimorphism of dg algebras $f : A \longrightarrow \hat{A}$ and a classical tilting object $\hat{T}^\bullet \in \mathcal{D}(\hat{A})$ such that*
   (a) $f_*(\hat{T}^\bullet) \cong T_A$, *where $f_* : \mathcal{D}(\hat{A}) \longrightarrow \mathcal{D}(A)$ is the restriction of scalars functor;*
   (b) *The canonical algebra morphism $B \longrightarrow End_A(T) \cong End_{\mathcal{D}(\hat{A})}(\hat{T}^\bullet)$ is an isomorphism.*

The next result is a direct consequence of corollaries 4.14 and 4.15:

Corollary 5.4. *Consider the following assertions for the $B - A-$bimodule T:*
1. T_A *admits a finite projective resolution with finitely generated terms, $\text{Ext}_A^p(T,T) = 0$, for all $p > 0$, and the algebra morphism $B \longrightarrow End(T_A)$ is an isomorphism;*
2. *There is recollement $\mathcal{D}' \equiv D(A) \equiv \mathcal{D}(B)$, with $j_! =? \otimes_B^{\mathbf{L}} T$, for some triangulated category $\mathcal{D}'$ (which is equivalent to $\mathcal{D}(C)$, for some dg algebra C);*
3. *There is a recollement $\mathcal{D}' \equiv D(A^{op}) \equiv \mathcal{D}(B^{op})$, with $j^! = j^* = T \otimes_A^{\mathbf{L}} ?$, for some triangulated category $\mathcal{D}'$ (which is equivalent to $\mathcal{D}(C^{op})$, for some dg algebra C).*

Then the implications (1) ⟺ (2) ⟹ (3) *hold true. When A is k-projective, all assertions are equivalent.*

The next result is a direct consequence of theorem 4.18 and the definition of good tilting module.

Corollary 5.5. *Let T be a right A-module such that $Ext_A^p(T,T) = 0$, for all $p > 0$, and let $B = End_A(T)$. The following assertions are equivalent:*

(1) *$Ext_{B^{op}}^p(T,T) = 0$, for all $p > 0$, the canonical algebra morphism $A \longrightarrow End_{B^{op}}(T)^{op}$ is an isomorphism and T admits a finite projective resolution with finitely generated terms as a left B-module;*

(2) *$\mathbf{R}Hom_A(T,?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ is fully faithful and preserves compact objects;*

(3) *There exists an exact sequence $0 \to A \longrightarrow T^0 \longrightarrow T^1... \longrightarrow T^n \to 0$ in $Mod - A$, with $T^k \in add(T)$ for each $k = 0, 1, ..., n$;*

(4) *$? \otimes_B^{\mathbf{L}} : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ has a fully faithful left adjoint.*

When in addition T_A has finite projective dimension and $Ext_A^p(T, T^{(\alpha)}) = 0$, for all cardinals α and all integers $p > 0$, the above conditions are also equivalent to:

(5) *T is a good tilting right A-module.*

The last results show that the fully faithful condition of the classical derived functors associated to an exceptional module is closely related to tilting theory. However, this relationship tends to be tricky, as the following examples show. They are explained in detail in the final part of [32].

Examples 5.6.   (1) *If A is a non-Noetherian hereditary algebra and I is an injective cogenerator of $Mod - A$ containing an isomorphic copy of each cyclic module, then $T = E(A) \oplus \frac{E(A)}{A} \oplus I$ satisfies the conditions (1)-(5) of corollary 5.5, but $Ext_A^1(T, T^{(\aleph_0)}) \neq 0$. Hence T is not a tilting A-module.*

(2) *Let A be a right Noetherian right hereditary algebra such that $Hom_A(E(A/A, E(A))) = 0$ and $E(A)/A$ contains an indecomposable summand with infinite multiplicity. If I is the direct sum of one isomorphic copy of each indecomposable summand of $E(A)/A$, then $T = E(A) \oplus I$ is a 1-tilting module such that $\mathbf{R}Hom_A(T,?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ is not fully faithful. The Weyl algebra $A_1(k) = k < x, y > /(xy - yx - 1)$ over the field k is an example where the situation occurs.*

(3) *If A is a hereditary Artin algebra, T is a finitely generated projective right A-module which is not a generator and $B = End_A(T)$, then T admits a finite projective resolution with finitely generated terms as a left B-module, but $\mathbf{R}Hom_A(T,?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ is not fully faithful. Indeed $\mathbf{R}Hom_A(T,?)$ preserves compact objects, but condition (3) of last corollary does not hold.*

5.2. **Connection with Wakamatsu tilting problem.** In this subsection we show a connection of question 4.26 with a classical problem in Representation Theory.

Definition 19. *Let T_A be a module and $B = End(T_A)$. Consider the following conditions:*

(1) *T_A admits a projective resolution with finitely generated terms;*

(2) *$Ext_A^p(T,T) = 0$, for all $p > 0$;*

(3) *There exists an exact sequence $0 \to A \to T^0 \to T^1 \to ... \to T^n \to ...$ such that*

(a) $T^i \in add(T_A)$, for all $i \geq 0$;

(b) The functor $Hom_A(?, T)$ leaves the sequence exact.

(4) There exists an exact sequence $0 \to A \to T^0 \to ... \to T^n \to 0$, with $T^i \in add(T_A)$, for all $i \geq 0$.

We shall say that T_A is

a) Wakamatsu tilting *when (1), (2) and (3) hold;*

b) semi-tilting *when (1), (2) and (4) hold;*

c) generalized Wakamatsu tilting *when (2) and (3) hold;*

d) generalized semi-tilting *when (2) and (4) hold.*

Remark 5.7. *Each classical tilting module is (generalized) semi-tilting and each (generalized) semi-tilting module is (generalized) Wakamatsu tilting.*

Proposition 5.8. *Let T be a Wakamatsu tilting right A-module. The following assertions are equivalent:*

(1) *T is classical tilting;*

(2) *T is semi-tilting of finite projective dimension;*

(3) *T has finite projective dimension, both as a right A-module and as left module over $B = End(T_A)$.*

Proof. $(1) \implies (2)$ is clear.

$(2) \implies (3)$ By hypothesis, we have that $pd(T_A) < \infty$. On the other hand, a finite projective resolution for $_B T$ is obtained by applying the functor $Hom_A(?, T)$ to the exact sequence $0 \to A \to T^0 \to ... \to T^n \to 0$, with $T^i \in add(T_A)$ given in the definition of semi-tilting module.

$(3) \implies (1)$ This is known (see [27, Section 4]). $\square$

Question 5.9. 1. *Is statement (1) of last proposition true, for all Wakamatsu tilting modules?.*

2. *We can ask an intermediate question, namely: is each Wakamatsu tilting module a semi-tilting one?.*

Remark 5.10. *The answer to question 1 is negative in general (see [39, Example 3.1]). However it is still an open question, known as* Wakamatsu tilting problem, *whether each Wakamatsu tilting module of finite projective dimension is classical tilting. Note that, by proposition 5.8, an affirmative answer to question 2 above implies an affirmative answer to Wakamatsu problem and, conversely.*

It turns out that question 2 is related to question 4.26, as the following result shows:

Proposition 5.11. *Let us assume that $Ext_A^p(T, T) = 0$, for all $p > 0$, and that $RHom_A(T, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ is fully faithful, where $B = End(T_A)$. Consider the following assertions:*

(1) *$RHom_A(T, ?)$ preserves compact objects;*

(2) *T_A is a generalized semi-tilting module;*

(3) *T_A is a generalized Wakamatsu tilting module;*

(4) *The structural algebra homomorphism $A \longrightarrow End_{B^{op}}(T)^{op}$ is an isomorphism and $Ext_{B^{op}}^p(T, T) = 0$, for all $p > 0$.*

Then the implications $(1) \Longleftrightarrow (2) \Longrightarrow (3) \Longleftrightarrow (4)$ hold true.

Proof. (1) $\Longleftrightarrow$ (2) is a direct consequence of corollary 5.5 and the definition of generalized semi-tllting module.

(2) $\Longrightarrow$ (3) is clear.

(3) $\Longrightarrow$ (4) Let us fix an exact sequence $0 \to A \longrightarrow T^0 \xrightarrow{d^0} T^1 \xrightarrow{d^1} ... \xrightarrow{d^{n-1}} T^n \xrightarrow{d^n} ...$ (*), with $T^i \in \mathrm{add}(T)$, for all $i \geq 0$. As shown in the proof of proposition 5.8, when we apply to it the functor $\mathrm{Hom}_A(?, T) : \mathrm{Mod} - A \longrightarrow B - \mathrm{Mod}$, we obtain a projective resolution of ${}_BT \cong \mathrm{Hom}_A(A, T)$. Bearing in mind that the canonical natural transformation $\sigma : 1_{\mathrm{Mod}-A} \longrightarrow \mathrm{Hom}_{B^{op}}(\mathrm{Hom}_A(?, T), T)$ is an isomorphism, when evaluated at a module $T' \in \mathrm{add}(T)$, when we apply $\mathrm{Hom}_{B^{op}}(?, T)$ to that projective resolution of ${}_BT$, we obtain a sequence

$$0 \to \mathrm{Hom}_{B^{op}}(T, T) \longrightarrow T^0 \xrightarrow{d^0} T^1 \xrightarrow{d^1} ... \xrightarrow{d^{n-1}} T^n \xrightarrow{d^n} ....$$

This sequence is exact due to the left exactness of $\mathrm{Hom}_{B^{op}}(?, T)$ and to the exactness of the sequence (*) and, hence, both sequences are isomorphic. Then assertion (4) holds.

(4) $\Longrightarrow$ (3) By proposition 4.9, we know that ${}_BT$ admits a projective resolution with finitely generated terms, say

$$...P^{-n} \longrightarrow ... \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow {}_BT \to 0. \qquad (**)$$

The hypotheses imply that, when we apply to it the functor $\mathrm{Hom}_{B^{op}}(?, T)$, we obtain an exact sequence in $\mathrm{Mod} - A$

$$0 \to A \longrightarrow \mathrm{Hom}_{B^{op}}(P^0, T) \longrightarrow \mathrm{Hom}_{B^{op}}(P^1, T) \longrightarrow ... \longrightarrow \mathrm{Hom}_{B^{op}}(P^{-n}, T) \longrightarrow ....$$

Note that $\mathrm{Hom}_{B^{op}}(P^{-i}, T)$ is a direct summand of $\mathrm{Hom}_{B^{op}}(B^{(r)}, T) \cong T_A^{(r)}$, for some $r \in \mathbb{N}$, so that $\mathrm{Hom}_{B^{op}}(P^{-i}, T) =: T^i$ is in $\mathrm{add}(T_A)$, for each $i \geq 0$. Note also that the canonical natural transformation $\sigma : 1_{B-\mathrm{Mod}} \longrightarrow \mathrm{Hom}_A(\mathrm{Hom}_{B^{op}}(?, T), T)$ is an isomorphism when evaluated at any finitely generated projective left B-module, because $\mathrm{Hom}_A(T, T) \cong {}_BB$. It follows from this and the fact that $\mathrm{Ext}_A^p(T, T) = 0$, for all $p > 0$, that when we apply $\mathrm{Hom}_A(?, T)$ to the last exact sequence we obtain, up to isomorphism, the initial projective resolution (**). Then the exact sequence

$$0 \to A_A \longrightarrow T^0 \longrightarrow T^1 \longrightarrow ... \longrightarrow T^n \longrightarrow ...$$

is kept exact when applying $\mathrm{Hom}_A(?, T)$. Therefore T_A is a generalized Wakamatsu tilting module. $\square$

As an immediate consequence, we get:

Corollary 5.12. *Each of the following statements is true if, and only if, so is the other:*

(1) *If T_A is a generalized Wakamatsu tilting module such that $\mathbf{R}Hom_A(T, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ is fully faithful, where $B = End_A(T)$, then T_A is generalized semi-tilting.*

(2) *Let ${}_BT_A$ be a bimodule such that $Ext_A^p(T, T) = 0 = Ext_{B^{op}}^p(T, T)$, for all $p > 0$ and the algebra morphisms $B \longrightarrow End_A(T)$ and $A \longrightarrow End_{B^{op}}(T)^{op}$ are isomorphisms. If the functor $\mathbf{R}Hom_A(T, ?) : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ is fully faithful, then it preserves compact objects.*

References

[1] L. Alonso, A. Jeremías, M. Saorín, *Compactly generated t-structures on the derived category of a Noetherian ring*, J. Algebra bf 324(4) (2010), 313–346.

[2] L. Alonso, A. Jeremías, M.J. Souto, *Localization in categories of complexes and unbounded resolutions*, Canad. J. Math. **52**(2) (2000), 225–247.

[3] L. Alonso, A. Jeremías, M.J. Souto, *Constructions of t-structures and equivalences of derived categories*, Trans. Amer. Math. Soc. **355**(6) (2003), 2523–2543.

[4] M. Auslander, I. Reiten, S.O. Smal∅, Representation theory of Artin algebras, Cambridge Stud. Adv. Maths **36**, Cambridge Univ. Press (1995).

[5] S. Bazzoni, F. Mantese, A. Tonolo, *Derived equivalences induced by infinitely generated n-tilting modules*, Proc. Amer. Math. Soc. **139**(12) (2011), 4225–4234.

[6] S. Bazzoni, A. Pavarin, *Recollements from partial tilting complexes*, J. Algebra **388** (2013), 338–363.

[7] A.A. Belinson, J. Bernstein, P. Deligne, Faisceaux pervers, Astérisque **100** (1982).

[8] T. Bühler, *Exact categories*, Expo. Math. **28**(1) (2010), 1–69.

[9] S.U. Chase, *Direct products of modules*, Trans. Amer. Math. Soc. **97** (1960), 457–473.

[10] H. Chen, C. Xi, *Good tilting modules and recollements of derived module categories*, Proc. London Math. Soc. **104**(5) (2012), 959–996.

[11] H. Chen, C. Xi, *Ringel modules and homological subcategories*, (2012). Preprint available at arxiv.org/abs/1206.0522

[12] R. Colpi, C. Menini *On the structure of ⋆-modules*, J. Algebra **158** (1993), 400–419.

[13] P. Gabriel, M. Zisman, Calculus of fractions and Homotopy Theory, Springer-Verlag New York (1967).

[14] Y. Hang, *Recollements and Hochschild theory*, J. Algebra **397** (2014), 535–547.

[15] D. Happel, *On the derived category of a finite dimensional algebra*, Comment. Math. Helvet. **62**(1) (1987), 339–389.

[16] D. Happel, Triangulated categories in the representation theory of finite dimensional algebras, London Math. Soc. Lect Note Ser. **119**. Cambridge Univ. Press (1988).

[17] P.J. Hilton, U. Stambach, A course in Homological Algebra, Springer-Verlag (1971).

[18] B. Keller, *Chain complexes and stable categories*, Manuscr. Math. **67** (1990), 379–417.

[19] B. Keller, *Deriving DG categories*, Ann. Sci. E. Norm Sup **27**(1) (1994), 63–102.

[20] B. Keller, *On differential graded categories*, Int. Congress of Mathematics (Madrid 2006), Vol II, 151–190. Eur. Math. Soc. Zurich (2006).

[21] B. Keller, Derived categories and their uses, Chapter of *Handbook of Algebra*, volume 1. Elsevier (1996). Editor M. Hazewinkel.

[22] B. Keller, P. Nicolás, *Weight structures and simple dg modules for positive dg algebras*, Int. Math. Res. Notices (2012) doi: 10.1093/imrn/ms009.IMRNsite

[23] B. Keller, D. Vossieck, *Aisles in derived categories*, Bull. Soc. Math. Belg. Ser. A **40**(2) (1988), 239–253.

[24] H. Krause, *A Brown representability theorem via coherent functors*, Topology **41** (2002), 853–861.

[25] H. Krause, *Localization theory for triangulated categories*, pp 161-235. In 'Triangulated categories', by T. Holm, P. Jorgensen and R. Rouquier (edts). London Math. Soc. Lect. Not. Ser. **375**. Cambridge Univ. Press (2010).

[26] D. Lazard, *Autour de la platitude*, Bull. Soc. Math. France **97** (1969), 81–128.

[27] F. Mantese, I. Reiten, *Wakamatsu tilting modules*, J. Algebra **278** (2004), 532–552.

[28] J. Miyachi, *Localization of triangulated categories and derived categories*, J. Algebra **141**(2) (1991), 463–483.

[29] A. Neeman, Triangulated categories, Annals of Math. Stud. **148**. Princeton Univ. Press (2001).

[30] P. Nicolás, M. Saorín, *Lifting and restricting recollement data*, Appl. Categ. Struct. **19**(3) (2011).

[31] P. Nicolás, M. Saorín, *Parametrizing recollement data for triangulated categories*, J. Algebra **322** (2009), 1220–1250.

[32] P. Nicolás, M. Saorín, *Generalized tilting theory*, Preprint available at arXiv:1208.2803

[33] D. Pauksztello, *Homological epimorphisms of differential graded algebras*, Comm. Algebra **37**(7) (2009), 2337–2350.

[34] J. Rickard, *Morita theory for derived categories*, J. London Math. Soc. **39** (1989), 436–456.

[35] J. Rickard, *Derived equivalences as derived functors*, J. London Math. Soc. **43**(2) (1991), 37–48.

[36] M.J. Souto Salorio, *On the cogeneration of t-structures*, Arch. Math. **83** (2004), 113–122.

[37] N. Spaltenstein, *Resolutions of unbounded complexes*, Comp. Math. **65**(2) (1988), 121–154.

[38] J.L. Verdier, Des catégories dérivées des catégories abéliennes, Asterisque **239** (1996).

[39] T. Wakamatsu, *Stable equivalence for self-injective algebras and a generalization of tilting modules*, J. Algebra **134** (1990), 298–325.

[40] C.A. Weibel, An introduction to Homological Algebra, Cambridge Stud. Adv. Maths **38**. Cambridge Univ. Press (1994).

[41] D. Yang, *Recollements from generalized tilting*, Proc. Amer. Math. Soc. **140**(1) (2012), 83–91.

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