CLASSIFYING τ-TILTING MODULES OVER NAKAYAMA ALGEBRAS

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Abstract. In this article, we study τ-tilting modules over Nakayama algebras. We establish bijection between τ-tilting modules and triangulations of a regular polygon with a puncture.

Throughout this article, Λ is a basic connected finite dimensional algebra over an algebraically closed field K and by a module we mean a finite dimensional right module. For a Λ-module M with a minimal projective presentation $P^{-1} \xrightarrow{p} P^0 \to M \to 0$, we define a Λ-module τM by an exact sequence

$$0 \to \tau M \to \nu P^{-1} \xrightarrow{\nu p} \nu P^0,$$

where $\nu := \mathrm{Hom}_K(\mathrm{Hom}_\Lambda(-, \Lambda), K)$. In this article, the following quivers are useful:

$$\vec{A}_n : n \xrightarrow{\alpha_{n-1}} n{-}1 \xrightarrow{\alpha_{n-2}} \cdots \xrightarrow{\alpha_2} 2 \xrightarrow{\alpha_1} 1$$

$$\vec{\Delta}_n : \begin{tikzcd} & 2 \arrow[dl, "\alpha_1"'] & 3 \arrow[l, "\alpha_2"'] & \\ 1 \arrow[dr, "\alpha_n"'] & & & \vdots \arrow[ul, "\alpha_3"'] \\ & n \arrow[r, "\alpha_{n-1}"'] & n{-}1 \arrow[ur, "\alpha_{n-2}"'] & \end{tikzcd}$$

In representation theory of finite dimensional algebra, tilting modules play an important role because they induce derived equivalences. Recently, the authors in [2] introduced the notion of τ-tilting modules, which is a generalization of (classical) tilting modules. They showed there are close relationships between τ-tilting modules and some important notions: *torsion classes, silting complexes, and cluster-tilting objects.* For background and results of τ-tilting modules, we refer to [2].

Our aim of this article is to give a generalization of the following well-known result. A Λ-module M is called *tilting* if $\mathrm{pd}M \leq 1$, $\mathrm{Ext}^1_\Lambda(M, M) = 0$ and $|M| = |\Lambda|$, where $\mathrm{pd}M$ is the projective dimension and $|M|$ the number of nonisomorphic indecomposable direct summands of M.

Theorem 1. *Let $\Lambda = K\vec{A}_n$ be a path algebra. Then there is a one-to-one correspondence between*

(1) *the set* $\mathrm{tilt}\Lambda$ *of isomorphism classes of basic tilting Λ-modules,*
(2) *the set of triangulations of an $(n+2)$-regular polygon.*

The following theorem is our main result of this article. A Λ-module M is called *τ-rigid* if $\mathrm{Hom}_\Lambda(M, \tau M) = 0$. A τ-rigid Λ-module M is called *τ-tilting* if $|M| = |\Lambda|$.

The detailed version of this paper will be submitted for publication elsewhere.

Theorem 2. [1] *Let Λ be a Nakayama algebra with $n := |\Lambda|$. Assume that the Loewy length of every indecomposable projective Λ-module is at least n. Then there is a one-to-one correspondence between*

(1) *the set τ-tiltΛ of isomorphism classes of basic τ-tilting Λ-modules,*
(2) *the set $\mathcal{T}(n)$ of triangulations of an n-regular polygon with a puncture.*

First, recall the definition and basic properties of Nakayama algebras. An algebra Λ is said to be *Nakayama* if every indecomposable projective Λ-module and every indecomposable injective Λ-module are *uniserial* (*i.e*, it has a unique composition series). We give a characterization of Nakayama algebras by using quivers.

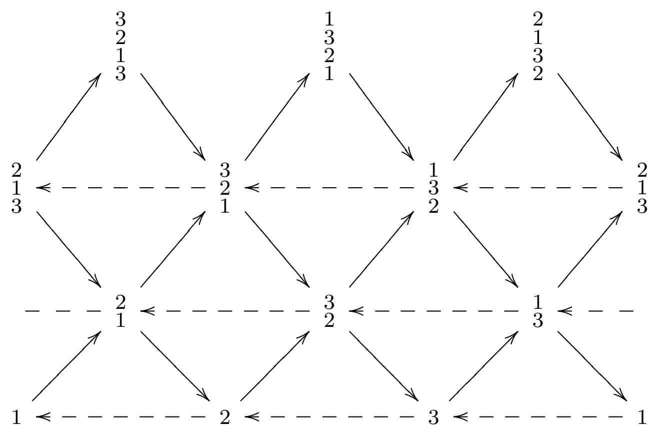
Proposition 3. [3, V.3.2] *A basic connected algebra Λ is Nakayama if and only if its quiver Q_Λ is one of $\vec{A}_n$ or $\vec{\Delta}_n$.*

In the following, we assume that Λ is a basic connected Nakayama algebra and $n := |\Lambda|$. We give a concrete description of indecomposable Λ-modules. We denote by P_i (respectively, S_i) the indecomposable projective (respectively, the simple) Λ-module corresponding to the vertex i in Q_Λ and $\ell(M)$ the Loewy length of a Λ-module M.

Proposition 4. [3, V.3.5, V.4.1 and V.4.2] *Let M be an indecomposable nonprojective Λ-module. Then there exists an indecomposable projective Λ-module P_i and an integer $1 \leq t < \ell(P_i)$ such that $M \simeq P_i/\mathrm{rad}^t P_i$. In this case, we have $\tau M \simeq P_{i-1}/\mathrm{rad}^t P_{i-1}$ and $t = \ell(M) = \ell(\tau M)$.*

Every indecomposable Λ-module M is uniquely determined, up to isomorphism, by its simple top S_i and the Loewy length $t = \ell(M)$. In this case, M has a unique composition series with the associated composition factors $S_i = S_{i_1}, S_{i_2}, \cdots, S_{i_t}$, where $i_1, \cdots, i_t \in \{1, 2, \cdots, n\}$ with $i_{j+1} = i_j - 1 (\mathrm{mod}\ n)$ for any j.

Let Λ_n^r be a self-injective Nakayama algebra with $n = |\Lambda|$ and the Loewy length r. The Auslander-Reiten quiver of Λ_3^4 is given by the following:



Then the following modules are all τ-tilting Λ_3^4-modules:

$$\begin{smallmatrix}1\\3\\2\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\\3\\2\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\\1\\3\end{smallmatrix}, \quad \begin{smallmatrix}1\\3\\2\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\\3\\2\end{smallmatrix} \oplus 1, \quad \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\\3\\2\end{smallmatrix} \oplus 1, \quad \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\\3\\2\end{smallmatrix} \oplus 2, \quad \begin{smallmatrix}3\\2\\1\\3\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\\3\\2\end{smallmatrix} \oplus 2,$$

$$\begin{smallmatrix}3\\2\\1\\3\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus 2, \quad \begin{smallmatrix}3\\2\\1\\3\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus 3, \quad \begin{smallmatrix}3\\2\\1\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\\2\\1\end{smallmatrix} \oplus 3, \quad \begin{smallmatrix}1\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\\2\\1\end{smallmatrix} \oplus 3, \quad \begin{smallmatrix}1\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\\2\\1\end{smallmatrix} \oplus 1.$$

By the example above, note that every τ-tilting module does not have an indecomposable Λ-module M with the Loewy length $\ell(M) = 3$ as a direct summand. This is always the case as the following result shows. By Proposition 4, we can easily understand the existence of homomorphisms between indecomposable Λ-modules.

Lemma 5. *Assume that $M = P_i/\mathrm{rad}^j P_i$ and $N = P_k/\mathrm{rad}^l P_k$. Then the following are equivalent.*

(1) $\mathrm{Hom}_\Lambda(M, N) \neq 0$.
(2) $i \in \{k, k-1, \cdots, k-l+1 \pmod n\}$ *and* $k-l+1 \in \{i, i-1, \cdots, i-j+1 \pmod n\}$.

Thus we have the following result.

Proposition 6. *Let M be an indecomposable nonprojective Λ-module. Then M is τ-rigid if and only if $\ell(M) < n$.*

Proof. By Proposition 4, we can assume that $M = P_i/\mathrm{rad}^j P_i$ and $\tau M = P_{i-1}/\mathrm{rad}^j P_{i-1}$. Then we have

$$\begin{aligned}\mathrm{Hom}_\Lambda(M, \tau M) \neq 0 &\overset{5}{\Leftrightarrow} \begin{cases} i \in \{i-1, i-2, \cdots, i+j \pmod n\} \\ i+j \in \{i, i-1, \cdots, i-j+1 \pmod n\} \end{cases} \\ &\Leftrightarrow \ell(M) \geq n.\end{aligned}$$

$\square$

To study a connection between τ-tilting modules and triangulations of an n-regular polygon with a puncture, recall the definition and basic properties of triangulations. Let $\mathcal{G}_n$ be an n-regular polygon with a puncture. We label the points of $\mathcal{G}_n$ counterclockwise around the boundary by $1, 2, \cdots, n \pmod n$. Let $i, j \in \mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$. An *inner* arc $\langle i, j\rangle$ in $\mathcal{G}_n$ is a path from the point i to the point j homotopic to the boundary path $i, i+1, \cdots, i+l = j \pmod n$ such that $1 < l \leq n$. Then we call i an *initial* point, j a *terminal* point and $\ell(\langle i, j\rangle) := l$ the *length* of the inner arc. A *projective* arc $\langle \bullet, j\rangle$ in $\mathcal{G}_n$ is a path from the puncture to the point j. Then we call j a *terminal* point. An *admissible* arc is an inner arc or a projective arc.

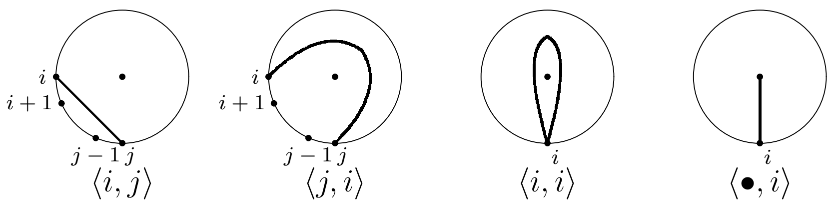


FIGURE 1. Admissible arcs in a polygon with a puncture

Two admissible arcs in $\mathcal{G}_n$ are called *compatible* if they do not intersect in $\mathcal{G}_n$ (except their initial and terminal points). A *triangulation* of $\mathcal{G}_n$ is a maximal set of distinct pairwise compatible admissible arcs. Note that the set of all projective arcs gives a triangulation of $\mathcal{G}_n$. Triangulations have the following property.

Proposition 7. *Each triangulation of $\mathcal{G}_n$ consists of exactly n admissible arcs and contains at least one projective arc.*

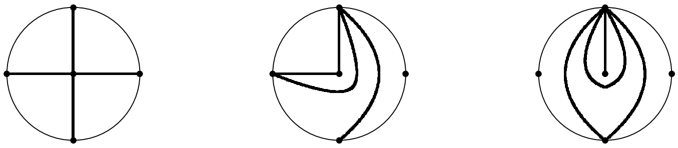


FIGURE 2. Triangulations of $\mathcal{G}_4$

We give a bijection between indecomposable τ-rigid Λ-modules and admissible arcs in $\mathcal{G}_n$. By Proposition 6, every indecomposable nonprojective τ-rigid Λ-module M is uniquely determined by its simple top S_j and its simple socle S_k. Thus we denote by $(k-2, j)$ the τ-rigid module M above. Moreover, we put $(\bullet, j) := P_j$ for any j. We denote by τ-rigidΛ the set of isomorphism classes of indecomposable τ-rigid Λ-modules.

Proposition 8. *The following hold.*

(1) *There is a bijection*

$$\tau\text{-rigid}\Lambda \longrightarrow \{\langle i,j\rangle \mid i,j \in \mathbb{Z}_n,\ \ell(\langle i,j\rangle) \leq \ell(P_j)\} \coprod \{\langle \bullet, i\rangle \mid i \in \mathbb{Z}_n\}$$

given by $(i,j) \mapsto \langle i,j\rangle$ for $i \in \{1,2,\cdots,n\} \sqcup \{\bullet\}$ and $j \in \{1,2,\cdots,n\}$.

(2) *For any $i,k \in \{1,2,\cdots,n\} \coprod \{\bullet\}$ and $j,l \in \{1,2,\cdots,n\}$, $(i,j) \oplus (k,l)$ is τ-rigid if and only if $\langle i,j\rangle$ and $\langle k,l\rangle$ are compatible.*

Proof. (1) By Proposition 6, every indecomposable τ-rigid Λ-module M is either a projective Λ-module or a Λ-module satisfying $\ell(M) < n$. Thus we can easily check the map is a well-defined bijection.

(2) It follows from Lemma 5. □

Instead of proving Theorem 2, we give a proof of the following theorem which is a generalization of Theorem 2. Let $\ell_i := \ell(P_i)$ for any $i \in \{1,2,\cdots,n\}$. We denote by $\mathcal{T}(n;\ell_1,\ell_2,\cdots,\ell_n)$ the subset of $\mathcal{T}(n)$ consisting of triangulations such that the length of every inner arc with the terminal point j is at most ℓ_j for any $j \in \{1,2,\cdots,n\}$.

Theorem 9. *Let Λ be a Nakayama algebra. Then the map in Proposition 8 induces a bijection*

$$\tau\text{-tilt}\Lambda \longrightarrow \mathcal{T}(n;\ell_1,\ell_2,\cdots,\ell_n).$$

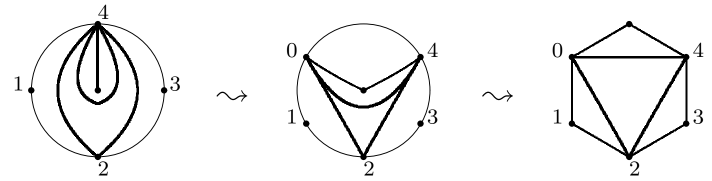
Proof. By Proposition 7 and 8, we can easily check that the map is a bijection. □

As an application of Theorem 9, we have the following corollary.

Corollary 10. *Let $\Lambda = K\vec{A}_n$ be a path algebra. Then the map in Proposition 8 induces a bijection*

$$\text{tilt}\Lambda \to \mathcal{U}(n) := \{X \in \mathcal{T}(n) \mid \langle \bullet, n\rangle \in X\}.$$

Note that $\mathcal{U}(n)$ can identify the set of triangulations of an $(n+2)$-regular polygon. As a result, we can recover Theorem 1.



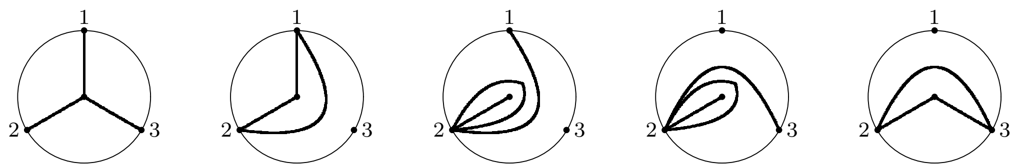
Proof. By Theorem 9, there is a bijection between τ-tiltΛ and $\mathcal{T}(n; 1, 2, \cdots, n)$. Since Λ is hereditary, we have tilt$\Lambda = \tau$-tiltΛ. We have only to show that

$$\mathcal{U}(n) = \mathcal{T}(n; 1, 2, \cdots, n)$$

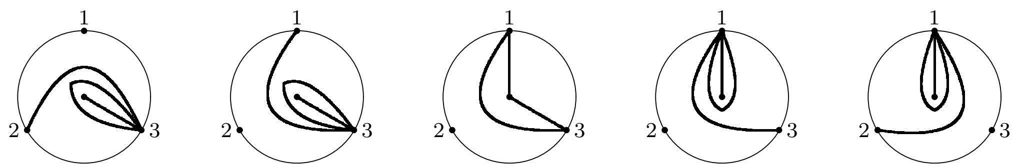
Indeed, assume that $X \in \mathcal{U}(n)$. Since X contains the projective arc $\langle \bullet, n \rangle$, we have $\ell(\langle i, j \rangle) \leq j$ for each inner arc $\langle i, j \rangle \in X$. Thus, we have $X \in \mathcal{T}(n; 1, 2, \cdots, n)$. Conversely, assume that $X \in \mathcal{T}(n; 1, 2, \cdots, n)$. Clearly, the projective arc $\langle \bullet, n \rangle$ is compatible with all admissible arc in X. Thus, we have $\langle \bullet, n \rangle \in X$, and hence $X \in \mathcal{U}(n)$. $\square$

Finally, we give an example of Theorem 9.

Example 11. Let $\Lambda := \Lambda_3^r$ be a self-injective Nakayama algebra with $r \geq 3$. Then we have the followng description of τ-tilting Λ-modules.



$P_1 \oplus P_2 \oplus P_3$ $P_1 \oplus P_2 \oplus 1$ $\begin{smallmatrix}2\\1\end{smallmatrix} \oplus P_2 \oplus 1$ $\begin{smallmatrix}2\\1\end{smallmatrix} \oplus P_2 \oplus 2$ $P_3 \oplus P_2 \oplus 2$



$P_3 \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus 2$ $P_3 \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus 3$ $P_3 \oplus P_1 \oplus 3$ $\begin{smallmatrix}1\\3\end{smallmatrix} \oplus P_1 \oplus 3$ $\begin{smallmatrix}1\\3\end{smallmatrix} \oplus P_1 \oplus 1$

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