HEARTS OF TWIN COTORSION PAIRS ON EXACT CATEGORIES

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ABSTRACT. In the papers of Nakaoka, he introduced the notion of hearts of (twin) cotorsion pairs on triangulated categories and showed that they have structures of (semi-) abelian categories. We study in this article a twin cotorsion pair $(\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})$ on an exact category $\mathcal{B}$ with enough projectives and injectives and introduce the notion of the heart. The heart of a twin cotorsion pair on $\mathcal{B}$ is semi-abelian. Moreover, the heart of a single cotorsion pair is abelian. These results are analog of Nakaoka's results in triangulated categories.

1. INTRODUCTION

The cotorsion pairs were first introduced by Salce, and it has been deeply studied in the representation theory during these years, especially in tilting theory and Cohen-Macaulay modules. Recently, the cotorsion pair are also studied in triangulated categories [3], in particular, Nakaoka introduced the notion of hearts of cotorsion pairs and showed that the hearts are abelian categories [6]. This is a generalization of the hearts of t-structure in triangulated categories [1] and the quotient of triangulated categories by cluster tilting subcategories [4]. Moreover, he generalized these results to a more general setting called twin cotorsion pair [7].

The aim of this paper is to give similar results for cotorsion pairs on Quillen's exact categories, which plays an important role in representation theory. An important class of exact categories is given by Frobenius categories, which gives most of important triangulated categories appearing in representation theory. We consider a *cotorsion pair* in an exact category, which is a pair $(\mathcal{U},\mathcal{V})$ of subcategories of an exact category $\mathcal{B}$ satisfying $\mathrm{Ext}^1_{\mathcal{B}}(\mathcal{U},\mathcal{V})=0$ (*i.e.* $\mathrm{Ext}^1_{\mathcal{B}}(U,V)=0$, $\forall U\in\mathcal{U}$ and $\forall V\in\mathcal{V}$) and any $B\in\mathcal{B}$ admits two short exact sequences $V_B\rightarrowtail U_B\twoheadrightarrow B$ and $B\rightarrowtail V^B\twoheadrightarrow U^B$ where $V_B,V^B\in\mathcal{V}$ and $U_B,U^B\in\mathcal{U}$. Let

$$\mathcal{B}^+:=\{B\in\mathcal{B}\mid \mathcal{U}_B\in\mathcal{V}\},\quad \mathcal{B}^-:=\{B\in\mathcal{B}\mid \mathcal{V}^B\in\mathcal{U}\}.$$

We define the *heart* of $(\mathcal{U},\mathcal{V})$ as the quotient category

$$\underline{\mathcal{H}}:=(\mathcal{B}^+\cap\mathcal{B}^-)/(\mathcal{U}\cap\mathcal{V}).$$

Now we state our first main result, which is an analogue of [6, Theorem 6.4].

Theorem 1. *Let $(\mathcal{U},\mathcal{V})$ be a cotorsion pair on an exact category $\mathcal{B}$ with enough projectives and injectives. Then $\underline{\mathcal{H}}$ is abelian.*

The detailed version [5] of this paper has been submitted for publication.

Moreover, following Nakaoka, we consider pairs of cotorsion pairs $(\mathcal{S},\mathcal{T})$ and $(\mathcal{U},\mathcal{V})$ in $\mathcal{B}$ such that $\mathcal{S}\subseteq\mathcal{U}$, we also call such a pair a *twin cotorsion pair.* The notion of hearts is generalized to such pairs, and our second main result is the following, which is an analogue of [7, Theorem 5.4].

Theorem 2. *Let $(\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})$ be a twin cotorsion pair on $\mathcal{B}$. Then the heart of this cotorsion pair is semi-abelian.*

2. Notations

For briefly review of the important properties of exact categories, we refer to [5, §2]. For more details, we refer to [2].

Throughout this paper, let $\mathcal{B}$ be a Krull-Schmidt exact category with enough projectives and injectives. Let $\mathcal{P}$ (resp. $\mathcal{I}$) be the full subcategory of projectives (resp. injectives) of $\mathcal{B}$.

Definition 3. Let $\mathcal{U}$ and $\mathcal{V}$ be full additive subcategories of $\mathcal{B}$ which are closed under direct summands. We call $(\mathcal{U},\mathcal{V})$ a *cotorsion pair* if it satisfies the following conditions:

(a) $\mathrm{Ext}^1_{\mathcal{B}}(\mathcal{U},\mathcal{V})=0$.
(b) For any object $B\in\mathcal{B}$, there exits two short exact sequences

$$V_B\rightarrowtail U_B\twoheadrightarrow B,\quad B\rightarrowtail V^B\twoheadrightarrow U^B$$

satisfying $U_B,U^B\in\mathcal{U}$ and $V_B,V^B\in\mathcal{V}$.

By definition of a cotorsion pair, we can immediately conclude:

Proposition 4. *Let $(\mathcal{U},\mathcal{V})$ be a cotorsion pair of $\mathcal{B}$, then*

(a) *B belongs to $\mathcal{U}$ if and only if $\mathrm{Ext}^1_{\mathcal{B}}(B,\mathcal{V})=0$.*
(b) *B belongs to $\mathcal{V}$ if and only if $\mathrm{Ext}^1_{\mathcal{B}}(\mathcal{U},B)=0$.*
(c) *$\mathcal{U}$ and $\mathcal{V}$ are closed under extension.*
(d) *$\mathcal{P}\subseteq\mathcal{U}$ and $\mathcal{I}\subseteq\mathcal{V}$.*

Definition 5. A pair of cotorsion pairs $(\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})$ on $\mathcal{B}$ is called a *twin cotorsion pair* if it satisfies:

$$\mathcal{S}\subseteq\mathcal{U}.$$

By the definition of the cotorsion pair and Proposition 4 this condition is equivalent to $\mathrm{Ext}^1_{\mathcal{B}}(\mathcal{S},\mathcal{V})=0$, and also to $\mathcal{V}\subseteq\mathcal{T}$.

Definition 6. For any twin cotorsion pair $(\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})$, put

$$\mathcal{W}:=\mathcal{T}\cap\mathcal{U}.$$

(a) $\mathcal{B}^+$ is defined to be the full subcategory of $\mathcal{B}$, consisting of objects B which admits a short exact sequence

$$V_B\rightarrowtail U_B\twoheadrightarrow B$$

where $U_B\in\mathcal{W}$ and $V_B\in\mathcal{V}$.

(b) $\mathcal{B}^-$ is defined to be the full subcategory of $\mathcal{B}$, consisting of objects B which admits a short exact sequence

$$B \rightarrowtail T^B \twoheadrightarrow S^B$$

where $T^B \in \mathcal{W}$ and $S^B \in \mathcal{S}$.

Definition 7. Let $(\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})$ be a twin cotorsion pair of $\mathcal{B}$, we denote the quotient of $\mathcal{B}$ by $\mathcal{W}$ as $\underline{\mathcal{B}} := \mathcal{B}/\mathcal{W}$. For any subcategory $\mathcal{C}$ of $\mathcal{B}$, we denote by $\underline{\mathcal{C}}$ the subcategory of $\underline{\mathcal{B}}$ consisting of the same objects as $\mathcal{C}$. Put

$$\mathcal{H} := \mathcal{B}^+ \cap \mathcal{B}^-.$$

Since $\mathcal{H} \supseteq \mathcal{W}$, we have an additive full quotient subcategory

$$\underline{\mathcal{H}} := \mathcal{H}/\mathcal{W}$$

which we call the *heart* of twin cotorsion pair $(\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})$.
The heart of a cotorsion pair $(\mathcal{U},\mathcal{V})$ is defined to be the heart of twin cotorsion pair $(\mathcal{U},\mathcal{V})$, $(\mathcal{U},\mathcal{V})$.

3. Main results

A additive category is called preabelian if every morphism has both kernel and cokernel.

Proposition 8. *For any twin cotorsion pair $(\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})$, its heart $\underline{\mathcal{H}}$ is preabelian.*

A preabelian category is abelian if every monomorphism is a kernel and every epimorphism is a cokernel. For a single cotorsion pair, we have the following result:

Theorem 9. *For any cotorsion pair $(\mathcal{U},\mathcal{V})$ on $\mathcal{B}$, its heart $\underline{\mathcal{H}}$ is an abelian category.*

For the hearts of twin cotorsion pairs, we can not get the same result.

Definition 10. A preabelian category $\mathcal{A}$ is called *left semi-abelian* if in any pull-back diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\beta"'] & B \arrow[d, "\gamma"] \\ C \arrow[r, "\delta"'] & D \end{tikzcd}$$

in $\mathcal{A}$, α is an epimorphism whenever δ is a cokernel. *Right semi-abelian* is defined dually. $\mathcal{A}$ is called *semi-abelian* if it is both left and right semi-abelian.

We can prove the following theorem.

Theorem 11. *For any twin cotorsion pair $(\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})$, its heart $\underline{\mathcal{H}}$ is semi-abelian.*

4. examples

In this section we give several examples of twin cotorsion pair, and we also give some view of the relation between the heart of a cotorsion pair and the hearts of its two components.

Recall that $\mathcal{M}$ is *cluster tilting* if it satisfies the following conditions:

(a) $\mathcal{M}$ is contravariantly finite and covariantly finite in $\mathcal{B}$.

(b) $\mathcal{M}^{\perp_1} = \mathcal{M}$.
(c) ${}^{\perp_1}\mathcal{M} = \mathcal{M}$.

Proposition 12. *A subcategory $\mathcal{M}$ in $\mathcal{B}$ is cluster tilting if and only if $(\mathcal{M}, \mathcal{M})$ is a cotorsion pair on $\mathcal{B}$.*

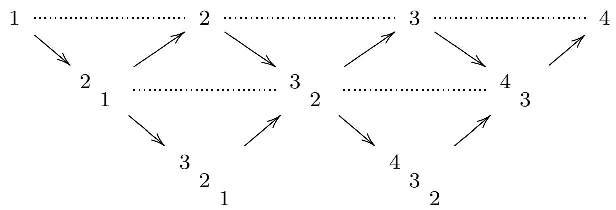
Proposition 13. *If $\mathcal{M}$ is a cluster tilting subcategory, then the heart of $(\mathcal{M}, \mathcal{M})$ is $\mathcal{B}/\mathcal{M}$.*

In the following examples, we denote by "$\circ$" in a quiver the objects belong to a subcategory and by "$\cdot$" the objects do not.

Example 14. Let Λ be the path algebra of the following quiver

$$1 \longleftarrow 2 \longleftarrow 3 \longleftarrow 4$$

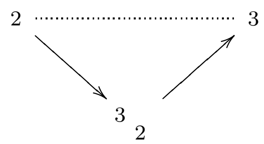
then we obtain the AR-quiver of $\operatorname{mod}\Lambda$.



Let $\mathcal{M} = \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^1_{\mathcal{B}}(X, \Lambda) = 0\}$, then $(\mathcal{M}, \mathcal{M}^{\perp_1})$ is a cotorsion pair on $\operatorname{mod}\Lambda$. But

$$\mathcal{M} = \begin{array}{ccccccc} \circ & & \cdot & & \cdot & & \circ \\ & \circ & & \cdot & & \circ & \\ & & \circ & & \circ & & \end{array}$$

which consisting of all the direct sums of indecomposable projectives and indecomposable injectives. We observe that in fact $\mathcal{M} = \mathcal{M}^{\perp_1}$ and hence it is a cluster tilting subcategory. And the quiver of the quotient category $(\operatorname{mod}\Lambda)/\mathcal{M}$ is



which is equivalent to the AR-quiver of A_2.

Example 15. Take the notion of the former example, Let

$$\mathcal{M}' = \begin{array}{ccccccc} \circ & & \cdot & & \cdot & & \cdot \\ & \circ & & \cdot & & \circ & \\ & & \circ & & \circ & & \end{array}$$

then $(\mathcal{M}', \mathcal{M}'^{\perp_1})$ is a cotorsion pair and

$$\mathcal{M}'^{\perp_1} = \begin{array}{ccccccc} \circ & & \cdot & & \circ & & \circ \\ & \circ & & \cdot & & \circ & \\ & & \circ & & \circ & & \end{array}$$

hence it contains Λ. Obviously it is closed under extension and contravariantly finite, then $(\mathcal{M}'^{\perp_1}, (\mathcal{M}'^{\perp_1})^{\perp_1})$ is also a cotorsion pair on $\operatorname{mod}\Lambda$ and

$$(\mathcal{M}'^{\perp_1})^{\perp_1} = \begin{array}{ccccccc} \circ & & \cdot & & \cdot & & \circ \\ & \cdot & & \cdot & & \circ & \\ & & \circ & & \circ & & \end{array}$$

Thus we get a twin cotorsion pair

$$(\mathcal{M}', \mathcal{M}'^{\perp_1}), (\mathcal{M}'^{\perp_1}, (\mathcal{M}'^{\perp_1})^{\perp_1})$$

The heart of this twin cotorsion pair is $(\operatorname{mod}\Lambda)/\mathcal{M}'^{\perp_1}$, and the AR-quiver of it is ${\scriptstyle 2} \to {\scriptstyle \begin{smallmatrix}3\\2\end{smallmatrix}}$. Thus it is not abelian.

Example 16. Let Λ be the k-algebra given by the quiver

$$\begin{tikzcd} & 1 \arrow[dl, "\beta"'] & \\ 3 \arrow[rr, "\gamma"'] & & 2 \arrow[ul, "\alpha"'] \end{tikzcd}$$

and bound by $\alpha\beta = 0$ and $\beta\gamma\alpha = 0$. Then the AR-quiver of $\operatorname{mod}\Lambda$ is given by

$$\begin{tikzcd}[column sep=small, row sep=small] 1 \arrow[dr] \arrow[rr, dotted, no head] & & 2 \arrow[dr] \arrow[rr, dotted, no head] & & 3 \arrow[dr] \arrow[rr, dotted, no head] & & 1 \\ & \begin{smallmatrix}2\\1\end{smallmatrix} \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & \begin{smallmatrix}3\\2\end{smallmatrix} \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & \begin{smallmatrix}1\\3\end{smallmatrix} \arrow[ur] & \\ & & \begin{smallmatrix}3\\2\\1\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}1\\3\\2\end{smallmatrix} \arrow[ur] & & \end{tikzcd}$$

Here, the first and the last columns are identified. Let

$$\mathcal{S} = \begin{array}{ccccccc} \cdot & & \circ & & \cdot & & \cdot \\ & \circ & & \cdot & & \circ & \\ & & \circ & & \circ & & \end{array} \quad \mathcal{T} = \begin{array}{ccccccc} \cdot & & \cdot & & \circ & & \cdot \\ & \circ & & \cdot & & \circ & \\ & & \circ & & \circ & & \end{array}$$

and

$$\mathcal{U} = \begin{array}{ccccccc} \circ & & \circ & & \cdot & & \circ \\ & \circ & & \cdot & & \circ & \\ & & \circ & & \circ & & \end{array} \quad \mathcal{V} = \begin{array}{ccccccc} \cdot & & \cdot & & \cdot & & \cdot \\ & \circ & & \cdot & & \circ & \\ & & \circ & & \circ & & \end{array}$$

The heart of cotorsion pair $(\mathcal{S}, \mathcal{T})$ is $\operatorname{add}({\scriptstyle 1})$ and the heart of cotorsion pair $(\mathcal{U}, \mathcal{V})$ is $\operatorname{add}({\scriptstyle 3})$. But when we consider the twin cotorsion pair $(\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})$, we get $\mathcal{W} = \mathcal{V}$ and

$$(\operatorname{mod}\Lambda)^-/\mathcal{W} = \operatorname{add}({\scriptstyle 1} \oplus {\scriptstyle 2}) \text{ and } (\operatorname{mod}\Lambda)^+/\mathcal{W} = \operatorname{add}({\scriptstyle 3})$$

hence its heart is zero.

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