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Kazutoshi Koike
Okinawa National College of Technology

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Organizing Committee of The Symposium on Ring Theory and Representation Theory

The Symposium on Ring Theory and Representation Theory has been held annually in Japan and the Proceedings have been published by the organizing committee. The first Symposium was organized in 1968 by H. Tominaga, H. Tachikawa, M. Harada and S. Endo. After their retirement, a new committee was organized in 1997 for managing the Symposium and committee members are listed in the web page

http://fuji.cec.yamanashi.ac.jp/ring/h-of-ringsymp.html.

The present members of the committee are H. Asashiba (Shizuoka Univ.), S. Ikehata (Okayama Univ.), S. Kawata (Osaka City Univ.) and I. Kikumasa (Yamaguchi Univ.).

The Proceedings of each Symposium is edited by program organizer. Anyone who wants these Proceedings should ask to the program organizer of each Symposium or one of the committee members.

The Symposium in 2013 will be held at Tokyo University of Science for Oct. 12 (Sat.) —14 (Mon.) and the program will be arranged by I. Kikumasa (Yamaguchi Univ.).

Concerning several information on ring theory and representation theory of group and algebras containing schedules of meetings and symposiums as well as ring mailing list service for registered members, you should refer to the following ring homepage, which is arranged by M. Sato (Yamanashi Univ.):

http://fuji.cec.yamanashi.ac.jp/ring/ (in Japanese)
(Mirror site: www.cec.yamanashi.ac.jp/ring/)

http://fuji.cec.yamanashi.ac.jp/ring/japan/ (in English)

Shûichi Ikehata
Okayama Japan
February, 2013

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Preface

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Kazutoshi Koike
Nago, Japan
February, 2013

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On the existence of a bypass in the Auslaner-Reiten quiver

τ-TILTING MODULES FOR SELF-INJECTIVE NAKAYAMA ALGEBRAS

TAKAHIDE ADACHI

ABSTRACT. In this paper, we study τ-tilting modules over Nakayama algebras. First, for self-injective Nakayama algebras, we give a classification of τ-tilting modules. Secondly, for Nakayama algebras, we give a combinatorial method to provide Hasse quivers of support τ-tilting modules.

1. INTRODUCTION

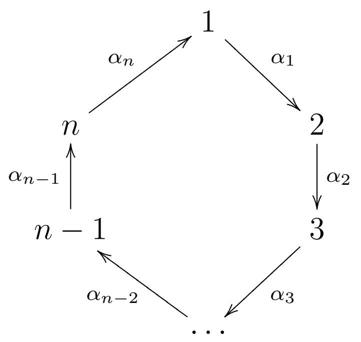
In tilting theory of algebras, tilting modules are important objects. As a way to construct tilting modules, there is the notion of tilting mutations introduced by Riedtmann-Schofield [8]. Roughly speaking, tilting mutations are operations which construct new tilting modules by replacing indecomposable direct summands of given tilting modules. However, it is known that tilting mutations have the following disadvantage. Namely, any basic almost complete tilting module can be completed to a basic tilting module in at most two different ways [8, 9]. This means that tilting mutations are not always defined. To overcome the disadvantage of tilting modules, the notion of τ-tilting modules was introduced in [2]. The authors showed any basic almost support τ-tilting module can be completed to a basic support τ-tilting module in exactly two different ways. Moreover, for a given algebra Λ, it is shown that there are bijections between support τ-tilting Λ-modules, two-term silting complexes for Λ (see [1, 7]), and cluster-tilting objects in a 2-CY triangulated category $\mathcal{C}$ if Λ is an associated 2-CY tilted algebra to $\mathcal{C}$ (see [4, 6]). Thus it is important to give a classification of support τ-tilting Λ-modules.

In this paper, we study τ-tilting modules over Nakayama algebras. First, we classify τ-tilting modules over self-injective Nakayama algebras. We shall give a bijection between τ-tilting modules and proper support τ-tilting modules. In this case, proper support τ-tilting modules are reduced to tilting modules over path algebras of Dynkin quivers of type A. A classification of tilting modules of the path algebras is well-known (*e.g.* triangulations of polygons). Thus we can easily obtain proper support τ-tilting modules.

Secondly, we give a combinatorial method to provide Hasse quivers of support τ-tilting modules over Nakayama algebras. Then Rejection Lemma of Drozd-Kirichenko plays important role. The rejection lemma gives a connection of indecomposable modules between an algebra and its factor algebra by some ideal. Any Nakayama algebra is given by a sequence of Drozd-Kirichenko rejection from some semisimple algebra. We study a connection of support τ-tilting modules between two algebras of Drozd-Kirichenko rejection. Using the connection, we construct Hasse quivers of Nakayama algebras from some semisimple algebra.

The detailed version of this paper will be submitted for publication elsewhere.

Notation. Throughout this paper, K is an algebraically closed field, and Λ is a basic finite dimensional K-algebra. We denote by modΛ the category of finitely generated right Λ-modules, and by indΛ the set of isomorphism classes of indecomposable Λ-modules. For two sets X and Y, we denote by $X \sqcup Y$ the disjoint union of X and Y. We denote by C_n the cyclic quiver and by $\vec{A}_n$ the Dynkin quiver of type A with linear orientation:



$$n \xleftarrow{\alpha_{n-1}} n-1 \xleftarrow{\alpha_{n-2}} \cdots \xleftarrow{\alpha_2} 2 \xleftarrow{\alpha_1} 1$$

2. Preliminaries

Let Λ be a basic finite dimensional K-algebra with a complete set $\{e_1, e_2, \cdots, e_n\}$ of primitive orthogonal idempotents, and $E_\Lambda := \{\sum_{j\in J} e_j \mid \emptyset \neq J \subset \{1, 2, \cdots, n\}\}$. For a module $M \in$ modΛ, we denote by $|M|$ the number of nonisomorphic indecomposable direct summands of M. We write by τ_Λ the Auslander-Reiten translation of Λ, and by $\langle e \rangle$ a two-sided ideal of Λ generated by $e \in \Lambda$.

In this section, we recall definitions and basic properties of τ-tilting modules.

Definition 1. Let Λ be a finite dimensional K-algebra, and $M \in$ modΛ a module.

(1) We call M *τ-rigid* Λ-module if $\mathrm{Hom}_\Lambda(M, \tau_\Lambda M) = 0$.
(2) We call M *τ-tilting* Λ-module if it is τ-rigid and $|M| = |\Lambda|$.
(3) We call M *support τ-tilting* Λ-module if there exists an idempotent $e \in \Lambda$ such that M is a τ-tilting $(\Lambda/\langle e \rangle)$-module. In this case, if $e \neq 0$, we call M *proper support τ-tilting* Λ-module.

In the rest of the paper, we denote by tiltΛ (respectively, τ-tiltΛ, sτ-tiltΛ, psτ-tiltΛ) the set of isomorphism classes of basic tilting (respectively, τ-tilting, support τ-tilting, proper support τ-tilting) Λ-modules.

Lemma 2. [2, Proposition 2.3] *For any proper support τ-tilting Λ-module M, there uniquely exists an idempotent $e \in E_\Lambda$ such that M is a τ-tilting $(\Lambda/\langle e \rangle)$-module. We write by e_M the above idempotent e.*

The following is straightforward.

Proposition 3. *The following hold.*

(1) τ-tilt$\Lambda =$ tiltΛ *if Λ is a hereditary algebra.*
(2) sτ-tilt$\Lambda = \tau$-tilt$\Lambda \sqcup$ psτ-tiltΛ.
(3) psτ-tilt$\Lambda = \displaystyle\bigsqcup_{e\in E_\Lambda} \tau\text{-tilt}(\Lambda/\langle e \rangle)$.

By the proposition above, we have important observations.

Remark 4. We can decompose sτ-tiltΛ as the disjoint union of τ-tiltΛ and psτ-tiltΛ. Moreover, proper support τ-tilting Λ-modules are reduced to τ-tilting modules over smaller algebras. To determine sτ-tiltΛ, it is thus important to construct τ-tilting Λ-modules.

The following lemma will be useful.

Lemma 5. [2, Lemma 2.1] *Let I be a two-sided ideal of Λ, and $M, N \in \mathrm{mod}(\Lambda/I)$. Then the following hold.*

(1) *If $\mathrm{Hom}_\Lambda(N, \tau_\Lambda M) = 0$, then $\mathrm{Hom}_{\Lambda/I}(N, \tau_{\Lambda/I} M) = 0$.*
(2) *Assume that $I = \langle e \rangle$ for an idempotent $e \in \Lambda$. Then $\mathrm{Hom}_\Lambda(N, \tau_\Lambda M) = 0$ if and only if $\mathrm{Hom}_{\Lambda/I}(N, \tau_{\Lambda/I} M) = 0$.*

We call $M \in \mathrm{mod}\Lambda$ *almost support* τ-tilting Λ-module if there exists an idempotent $e \in \Lambda$ such that M is a τ-rigid $(\Lambda/\langle e \rangle)$-module and $|M| = |\Lambda| - |e\Lambda| - 1$.

Proposition 6. [2, Theorem 2.18] *Any basic almost support τ-tilting Λ-module can be completed to a basic support τ-tilting module in exactly two different ways.*

For any $M, N \in$ sτ-tiltΛ, we write $M \geq N$ if $\mathrm{Fac}(M) \supseteq \mathrm{Fac}(N)$.

Proposition 7. [2, Theorem 2.7] *Let Λ be a finite dimensional K-algebra. Then $\geq$ gives a partial order on* sτ-tiltΛ.

By the proposition above, we have an associated Hasse quiver. We recall Hasse quivers.

Definition 8. We define the *Hasse quiver* of sτ-tiltΛ as follows:

- The vertices set is sτ-tiltΛ.
- We draw an arrow from M to N if $M > N$ and there exists no $L \in$ sτ-tiltΛ such that $M > L > N$.

We denote by Γ(sτ-tiltΛ) the Hasse quiver of sτ-tiltΛ.

3. Main result I

In this section, we study τ-tilting modules over self-injective Nakayama algebras. As an application of this section, we can easily obtain support τ-tilting modules over self-injective Nakayama algebras.

Throughout this section, the following notation is used. Let $\Lambda := \Lambda_n^r$ be a connected self-injective Nakayama algebra with $|\Lambda| = n$ and the Loewy length $\ell(\Lambda) = r$. Then we have $\Lambda \simeq KC_n/R^r$, where C_n is the cyclic quiver and R is the arrow ideal of KC_n (see [3, V.3.8 Proposition]).

We define an automorphism $\phi : \Lambda \to \Lambda$ by $\phi(e_i) = e_{i+1}$ and $\phi(\alpha_i) = \alpha_{i+1}$ for any $i \in \{1, 2, \cdots, n\}$. Then ϕ induces a functor as follows.

Lemma 9. *The automorphism $\phi : \Lambda \to \Lambda$ induces an equivalence of categories $\Phi : \mathrm{mod}\Lambda \to \mathrm{mod}\Lambda$ such that $\Phi(e_i\Lambda) \simeq e_{i+1}\Lambda$ for any $i \in \{1, 2, \cdots, n\}$. Moreover, for any nonprojective module $M \in \mathrm{mod}\Lambda$, we have $\Phi(M) \simeq \tau M$.*

Let Ψ be a quasi-inverse of Φ. Then we have $\Psi(e_i\Lambda) \simeq e_{i-1}\Lambda$ and $\Psi(M) \simeq \tau^- M$ for any $i \in \{1, 2, \cdots, n\}$ and nonprojective module $M \in \mathrm{mod}\Lambda$. By Remark 4, it is important to construct τ-tilting modules for given an algebra. Our main result of this section is to construct τ-tilting Λ-modules from proper support τ-tilting Λ-module. Proper support τ-tilting Λ-modules are reduced to tilting modules over path algebras of Dynkin quivers of type A with linear orientation. A classification of tilting $(K\vec{A}_l)$-modules is already well-known for any integer $l > 0$. Indeed, there is a bijection

$$\mathrm{tilt}(K\vec{A}_l) \longleftrightarrow \{ \text{ triangulations of } (l+2)\text{-gon } \}.$$

Thus we can easily obtain proper support τ-tilting modules over a self-injective Nakayama algebra.

In the rest of the paper, we denote by $\mathrm{mod}_{\mathrm{np}}\Lambda$ the full subcategory of $\mathrm{mod}\Lambda$ consisting Λ-modules which does not have nonzero projective direct summands. we let $\mathrm{ps}\tau\text{-tilt}_{\mathrm{np}}\Lambda := \mathrm{ps}\tau\text{-tilt}\Lambda \cap \mathrm{mod}_{\mathrm{np}}\Lambda$, and $\tau\text{-tilt}_{\mathrm{np}}\Lambda := \tau\text{-tilt}\Lambda \cap \mathrm{mod}_{\mathrm{np}}\Lambda$. We decompose $M \in \mathrm{mod}\Lambda$ as $M = M_{\mathrm{np}} \oplus M_{\mathrm{pr}}$, where $M_{\mathrm{np}} \in \mathrm{mod}_{\mathrm{np}}\Lambda$ and M_{pr} is a maximal projective direct summand of M.

we state our main theorem of this section.

Theorem 10. *Let* $\Lambda := \Lambda_n^r$.

(1) *There is a bijection*

$$\tau\text{-tilt}\Lambda \longleftrightarrow \mathrm{ps}\tau\text{-tilt}_{\mathrm{np}}\Lambda$$

given by $\tau\text{-tilt}\Lambda \ni M \mapsto M_{\mathrm{np}} \in \mathrm{ps}\tau\text{-tilt}_{\mathrm{np}}\Lambda$ *and* $\mathrm{ps}\tau\text{-tilt}_{\mathrm{np}}\Lambda \ni M \mapsto M \oplus \Phi(e_M\Lambda) \in \tau\text{-tilt}\Lambda$.

(2) *Moreover, if* $r \geq n$, *we have* $\mathrm{ps}\tau\text{-tilt}_{\mathrm{np}}\Lambda = \mathrm{ps}\tau\text{-tilt}\Lambda$. *Namely, (1) gives a bijection*

$$\tau\text{-tilt}\Lambda \longleftrightarrow \mathrm{ps}\tau\text{-tilt}\Lambda.$$

As an immediate consequence of Theorem 10, we have the following corollary.

Corollary 11. *The following hold.*

(1) *If* $r \geq n$, *we have*

$$\begin{aligned} \mathrm{s}\tau\text{-tilt}\Lambda &= \{M,\ M \oplus \Phi(e_M\Lambda) \mid M \in \mathrm{ps}\tau\text{-tilt}\Lambda\} \\ &= \bigsqcup_{e \in E_\Lambda} \{M,\ M \oplus \Phi(e\Lambda) \mid M \in \mathrm{tilt}(\Lambda/\langle e \rangle)\}. \end{aligned}$$

(2) *If* $r < n$, *we have*

$$\mathrm{s}\tau\text{-tilt}\Lambda = (\mathrm{ps}\tau\text{-tilt}\Lambda \setminus \mathrm{ps}\tau\text{-tilt}_{\mathrm{np}}\Lambda) \sqcup \{M,\ M \oplus \Phi(e_M\Lambda) \mid M \in \mathrm{ps}\tau\text{-tilt}_{\mathrm{np}}\Lambda\}.$$

In the rest of this section, we give the proof of Theorem 10.

Proposition 12. *If* M *is in* $\mathrm{ps}\tau\text{-tilt}_{\mathrm{np}}\Lambda$, *then* $M \oplus \Phi(e_M\Lambda)$ *is a* τ*-tilting* Λ*-module.*

Proof. Let $M \in \mathrm{mod}_{\mathrm{np}}\Lambda$ be a τ-tilting $(\Lambda/\langle e \rangle)$-module, where $e := e_M \in E_\Lambda$. Thus M is a τ-rigid Λ-module by Lemma 5. Moreover we have

$$\mathrm{Hom}_\Lambda(\Phi(e\Lambda), \tau_\Lambda M) \simeq \mathrm{Hom}_\Lambda(\Psi\Phi(e\Lambda), \Psi(\tau_\Lambda M)) \simeq \mathrm{Hom}_\Lambda(e\Lambda, M) = 0$$

and

$$|M \oplus \Phi(e\Lambda)| = |M| + |\Phi(e\Lambda)| = |M| + |e\Lambda| = |\Lambda|$$

by Lemma 9 and $M \in \mathrm{mod}_{\mathrm{np}}\Lambda$. Thus $M \oplus \Phi(e_M\Lambda)$ is a τ-tilting Λ-module. $\square$

Conversely, we shall construct a proper support τ-tilting Λ-module for a given τ-tilting Λ-module.

Proposition 13. *Assume that $M \in \mathrm{mod}\Lambda$ is not in $\mathrm{mod}_{\mathrm{np}}\Lambda$. If M is a τ-tilting Λ-module, then M_{np} is a proper support τ-tilting Λ-module.*

Proof. Let M is a τ-tilting Λ-module and not in $\mathrm{mod}_{\mathrm{np}}\Lambda$. We decompose M as $M = M_{\mathrm{np}} \oplus M_{\mathrm{pr}}$ and assume $M_{\mathrm{pr}} = e\Lambda$, where $e \in E_\Lambda$ is an idempotent. Then M_{np} is trivially a τ-rigid Λ-module. Since M is a τ-tilting Λ-module, we have

$$\mathrm{Hom}_\Lambda(\phi^{-1}(e)\Lambda, M_{\mathrm{np}}) \simeq \mathrm{Hom}_\Lambda(\Phi(\phi^{-1}(e)\Lambda), \Phi(M_{\mathrm{np}})) \simeq \mathrm{Hom}_\Lambda(e\Lambda, \tau_\Lambda M) = 0$$

by Lemma 9, and

$$|M_{\mathrm{np}}| = |M| - |e\Lambda| = |\Lambda| - |\phi^{-1}(e)\Lambda|.$$

Thus M_{np} is a τ-tilting $(\Lambda/\langle\phi^{-1}(e)\rangle)$-module or proper support τ-tilting Λ-module by Lemma 5. $\square$

By Proposition 12 and 13, there is a bijection

$$\tau\text{-tilt}\Lambda \setminus \tau\text{-tilt}_{\mathrm{np}}\Lambda \longleftrightarrow \mathrm{ps}\tau\text{-tilt}_{\mathrm{np}}\Lambda.$$

To complete the proof of Theorem 10, we have only to show that any τ-tilting Λ-module always has a nonzero projective Λ-module as a direct summand.

We need the following lemma.

Lemma 14. *Let $X, Y \in \mathrm{mod}\Lambda$ be indecomposable with the Loewy length $\ell(X) \geq \ell(Y)$, and P_X a projective cover of X. Then $\mathrm{Hom}_\Lambda(X, Y) = 0$ if and only if $\mathrm{Hom}_\Lambda(P_X, Y) = 0$.*

Proposition 15. *Each τ-tilting Λ-module has a nonzero projective Λ-module as a direct summand.*

Proof. Let $M = X \oplus N$ be a τ-tilting Λ-module such that X is indecomposable and the Loewy length $\ell(X) \geq \ell(N)$. In particular, we have $\ell(N) = \ell(\tau N)$ because Λ is Nakayama. By the definition, N is an almost support τ-tilting Λ-module. Assume that M has no projective Λ-module as a direct summand. Since M is τ-rigid, $\mathrm{Hom}_\Lambda(X, \tau N)$ vanishes. By Lemma 14, we have $\mathrm{Hom}_\Lambda(P_X, \tau N) = 0$, where P_X is a projective cover of X. Since Λ is a Nakayama algebra, P_X is indecomposable. Therefore we have

$$|P_X \oplus N| = |P_X| + |N| = |P_X| + |M| - |X| = |M| = |\Lambda|.$$

Namely, $P_X \oplus N$ is a τ-tilting Λ-module. Moreover, N is a support τ-tilting Λ-module by Proposition 13. This means that almost support τ-tilting Λ-module N has pairwise nonisomorphic 3 support τ-tilting Λ-modules $N, X \oplus N$ and $P_X \oplus N$. By Proposition 6, this is contradiction. $\square$

Now we are ready to prove Theorem 10.

Proof of Theorem 10. (1) It follows from Proposition 12, 13 and 15.

(2) One can show that any proper support τ-tilting Λ-module has no projective Λ-module as a direct summand. $\square$

As an application of Theorem 10, we can easily calculate τ-tilting modules over self-injective Nakayama algebras.

Finally, we give a example.

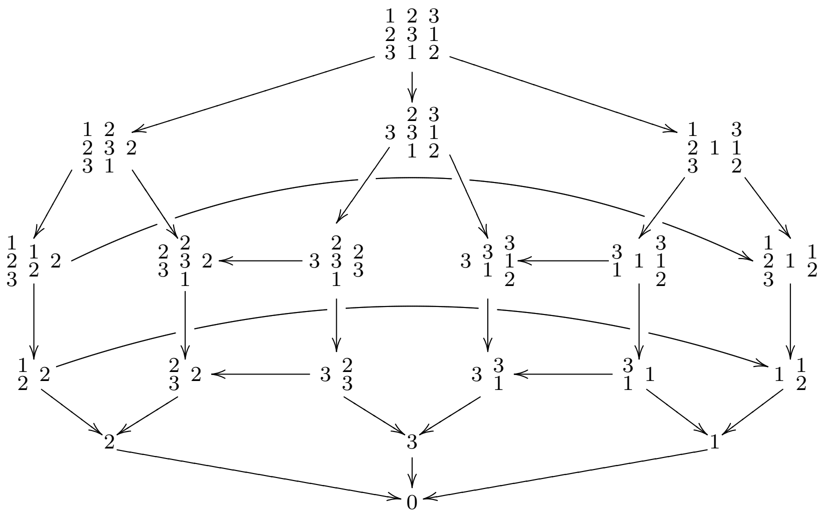
Example 16. Let $\Lambda := \Lambda_3^3$. To obtain τ-tilting Λ-modules, we need to know factor algebras $\Lambda/\langle e\rangle$ for any idempotent $e \in E_\Lambda$. Indeed, we have $\Lambda/\langle e_i\rangle \simeq K\vec{A}_2$, $\Lambda/\langle e_i + e_j\rangle \simeq K\vec{A}_1$, and $\Lambda/\langle e_1+e_2+e_3\rangle = \{0\}$ for $i, j \in \{1, 2, 3\}$. Thus proper support τ-tilting modules are given as follows:

$$\begin{aligned}
\tau\text{-tilt}(\Lambda/\langle e_3\rangle) &= \operatorname{tilt}(K\vec{A}_2) = \{\begin{smallmatrix}1\\2\end{smallmatrix} \oplus 2, \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1\}\\
\tau\text{-tilt}(\Lambda/\langle e_2 + e_3\rangle) &= \operatorname{tilt}(K\vec{A}_1) = \{1\}\\
\tau\text{-tilt}(\Lambda/\langle e_1 + e_2 + e_3\rangle) &= \{0\}
\end{aligned}$$

and cyclic permutation. By Theorem 10, we have

$$\begin{aligned}
\mathrm{s}\tau\text{-tilt}\Lambda = \{\, &\{0\}, 1, 2, 3, \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 2, \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1, \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 3, \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 2, \begin{smallmatrix}3\\1\end{smallmatrix} \oplus 1, \begin{smallmatrix}3\\1\end{smallmatrix} \oplus 3\,\}\\
\sqcup \{\, &\begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\\2\end{smallmatrix}, 1 \oplus \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\\2\end{smallmatrix}, 2 \oplus \begin{smallmatrix}2\\3\\1\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\\3\end{smallmatrix}, 3 \oplus \begin{smallmatrix}3\\1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\\1\end{smallmatrix},\\
&\begin{smallmatrix}1\\2\end{smallmatrix} \oplus 2 \oplus \begin{smallmatrix}1\\2\\3\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1 \oplus \begin{smallmatrix}1\\2\\3\end{smallmatrix}, \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}2\\3\\1\end{smallmatrix}, \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 2 \oplus \begin{smallmatrix}2\\3\\1\end{smallmatrix}, \begin{smallmatrix}3\\1\end{smallmatrix} \oplus 1 \oplus \begin{smallmatrix}3\\1\\2\end{smallmatrix}, \begin{smallmatrix}3\\1\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}3\\1\\2\end{smallmatrix}\,\}
\end{aligned}$$

and the Hasse quiver $\Gamma(\mathrm{s}\tau\text{-tilt}\Lambda)$ as follows:



4. Main result II

In this section, we give a combinatorial method to provide Hasse quivers of support τ-tilting modules over Nakayama algebras. Then Rejection Lemma of Drozd-Kirichenko plays important role.

Let Λ be a finite dimensional K-algebra (not necessarily Nakayama). The following lemma is called Rejection Lemma of Drozd-Kirichenko[5].

Lemma 17 (Rejection Lemma of Drozd-Kirichenko). *Let Λ be a finite dimensional K-algebra, and Q a projective-injective indecomposable summand of Λ. Then the following hold.*

(1) *$I := \operatorname{soc}(Q)$ is a two-sided ideal of Λ.*

(2) *There exists a one-to-one correspondence between* $\mathrm{ind}(\Lambda/I)$ *and* $\mathrm{ind}(\Lambda)\setminus\{Q\}$.

From now, we always assume that Q is a projective-injective indecomposable summand of Λ, and $I := \mathrm{soc}(Q)$. In the rest of the paper, we denote by $\mathrm{s}\tau\text{-tilt}_{Q/I}(\Lambda/I)$ (respectively, $\mathrm{s}\tau\text{-tilt}^{I}(\Lambda/I)$) the subset of $\mathrm{s}\tau\text{-tilt}(\Lambda/I)$ consisting Λ-modules which have Q/I as a direct summand (respectively, does not have I as a composition factor). We let $\mathrm{s}\tau\text{-tilt}^{I}_{Q/I}(\Lambda/I) := \mathrm{s}\tau\text{-tilt}_{Q/I}(\Lambda/I)\cap\mathrm{s}\tau\text{-tilt}^{I}(\Lambda/I)$ and $\mathrm{s}\tau\text{-tilt}^{I}_{Q/I}\Lambda := \{M \in \mathrm{s}\tau\text{-tilt}\Lambda \mid \mathrm{bas}(M\otimes_\Lambda \Lambda/I) \in \mathrm{s}\tau\text{-tilt}^{I}_{Q/I}(\Lambda/I)\}$, where $\mathrm{bas}(X)$ means a basic part of $X \in \mathrm{mod}\Lambda$.

The following theorem is very important.

Theorem 18. *Let* Λ *be a finite dimensional* K*-algebra,* Q *be a projective-injective indecomposable summand of* Λ*, and* $I := \mathrm{soc}(Q)$.

(1) *The map* $M \mapsto \mathrm{bas}(M\otimes_\Lambda \Lambda/I)$ *gives a surjection*

$$\mathrm{s}\tau\text{-tilt}\Lambda \twoheadrightarrow \mathrm{s}\tau\text{-tilt}(\Lambda/I)$$

which preserves the partial orders. Moreover, the restriction gives a bijection

$$\mathrm{s}\tau\text{-tilt}\Lambda \setminus \mathrm{s}\tau\text{-tilt}^{I}_{Q/I}\Lambda \longleftrightarrow \mathrm{s}\tau\text{-tilt}(\Lambda/I)\setminus \mathrm{s}\tau\text{-tilt}^{I}_{Q/I}(\Lambda/I)$$

where the inverse is given by

$$\mathrm{s}\tau\text{-tilt}_{Q/I}(\Lambda/I)\setminus \mathrm{s}\tau\text{-tilt}^{I}_{Q/I}(\Lambda/I) \ni Q/I\oplus U \mapsto Q\oplus U \in \mathrm{s}\tau\text{-tilt}\Lambda\setminus \mathrm{s}\tau\text{-tilt}^{I}_{Q/I}\Lambda$$
$$\mathrm{s}\tau\text{-tilt}(\Lambda/I)\setminus \mathrm{s}\tau\text{-tilt}_{Q/I}(\Lambda/I) \ni N \mapsto N \in \mathrm{s}\tau\text{-tilt}\Lambda\setminus \mathrm{s}\tau\text{-tilt}^{I}_{Q/I}\Lambda.$$

(2) *We have*

$$\mathrm{s}\tau\text{-tilt}\Lambda = (\mathrm{s}\tau\text{-tilt}\Lambda\setminus \mathrm{s}\tau\text{-tilt}^{I}_{Q/I}\Lambda)\sqcup\{N, Q\oplus N \mid N \in \mathrm{s}\tau\text{-tilt}^{I}_{Q/I}(\Lambda/I)\}.$$

By Theorem 18, we can recover $\mathrm{s}\tau\text{-tilt}\Lambda$ from $\mathrm{s}\tau\text{-tilt}(\Lambda/I)$. Moreover, since the map preserves the partial orders, Hasse quivers of $\mathrm{s}\tau\text{-tilt}\Lambda$ and $\mathrm{s}\tau\text{-tilt}(\Lambda/I)$ are almost same. Thus, as a result of Theorem 18, we have two corollaries for a construction of the Hasse quiver $\Gamma(\mathrm{s}\tau\text{-tilt}\Lambda)$.

If any $M \in \mathrm{s}\tau\text{-tilt}_{Q/I}(\Lambda/I)$ has I as a composition factor, we have $\mathrm{s}\tau\text{-tilt}^{I}_{Q/I}\Lambda = \emptyset$. Thus we have a bijection between $\mathrm{s}\tau\text{-tilt}\Lambda$ and $\mathrm{s}\tau\text{-tilt}(\Lambda/I)$.

Corollary 19. *If* Q/I *has* I *as a composition factor, then the map of Theorem 18 is a bijection. In particular, there exists a quiver isomorphism*

$$\Gamma(\mathrm{s}\tau\text{-tilt}\Lambda) \longrightarrow \Gamma(\mathrm{s}\tau\text{-tilt}(\Lambda/I)).$$

Assume that $X \geq N$ in $\mathrm{s}\tau\text{-tilt}(\Lambda/I)$ and $N \in \mathrm{s}\tau\text{-tilt}_{Q/I}(\Lambda/I)$. Then we remark that X is also in $\mathrm{s}\tau\text{-tilt}_{Q/I}(\Lambda/I)$.

Corollary 20. $\Gamma(\mathrm{s}\tau\text{-tilt}\Lambda)$ *is obtained from* $\Gamma(\mathrm{s}\tau\text{-tilt}(\Lambda/I))$ *by the following two steps: First we replace any arrow* $X \to N$ *in* $\Gamma(\mathrm{s}\tau\text{-tilt}(\Lambda/I))$ *satisfying* $N \in \mathrm{s}\tau\text{-tilt}^{I}_{Q/I}(\Lambda/I)$ *as follows:*

- *If X is in* $\mathrm{s}\tau\text{-tilt}_{Q/I}(\Lambda/I)$ *but not in* $\mathrm{s}\tau\text{-tilt}^{I}_{Q/I}(\Lambda/I)$,

$$\textit{replacing}\quad \begin{tikzcd} X \arrow[dd] \\ \\ N \end{tikzcd} \quad \textit{by a subquiver} \quad \begin{tikzcd} Q\oplus (X/(Q/I)) \arrow[d] \\ Q\oplus N \arrow[d] \\ N \end{tikzcd}\ .$$

- *If X is in* $\mathrm{s}\tau\text{-tilt}^{I}_{Q/I}(\Lambda/I)$,

$$\textit{replacing}\quad \begin{tikzcd} X \arrow[dd] \\ \\ N \end{tikzcd} \quad \textit{by a subquiver} \quad \begin{tikzcd} & Q\oplus X \arrow[dl] \arrow[dr] & \\ X \arrow[dr] & & Q\oplus N. \arrow[dl] \\ & N & \end{tikzcd}$$

Finally we replace other vetices by the bijection of Theorem 18(1).

From now, we assume that Λ is Nakayama with $n = |\Lambda|$. Let Q be a projective-injective indecomposable summand of Λ, and $I := \mathrm{soc}(Q)$. If the Loewy length of Q is bigger than n or $\ell(Q/I) \geq n$, then Q/I is sincere. Namely, Q/I has I as a composition factor. Then we have a quiver isomorphism $\Gamma(\mathrm{s}\tau\text{-tilt}\Lambda) \to \Gamma(\mathrm{s}\tau\text{-tilt}(\Lambda/I))$ by Corollary 19.

On the other hand, if the Loewy length of Q is not bigger than n or $\ell(Q/I) < n$, then Q/I does not have I as a composition factor. In this case, by Corollary 20, we can construct the Hasse quiver of Λ from Λ/I.

Since Nakayama algebras have a projective-injective indecomposable module and its factor algebras is also Nakayama (see [3, V.3.3 Lemma and V.3.4 Lemma]), we can iteratively apply the rejection lemma to Nakayama algebras.

Let $\Lambda_0 := \Lambda$ be a Nakayama algebra with $n = |\Lambda|$. By iteratively applying the rejection lemma, we have a sequence of Nakayama algebras

$$\cdots \twoheadrightarrow \Lambda_{-2} \twoheadrightarrow \Lambda_{-1} \twoheadrightarrow \Lambda_0 \twoheadrightarrow \Lambda_1 \twoheadrightarrow \cdots \twoheadrightarrow \Lambda_m = K^n$$

such that $\Lambda_i := \Lambda_{i-1}/I_{i-1}$ and Λ_m is a semisimple algebra K^n, where Q_i is a projective-injective indecomposable Λ_i-module, $I_i := \mathrm{soc}(Q_i)$ and $m > 0$ is an integer. Thus we always can construct the Hasse quiver of any Nakayama algebra from some semisimple algebra by the observation above.

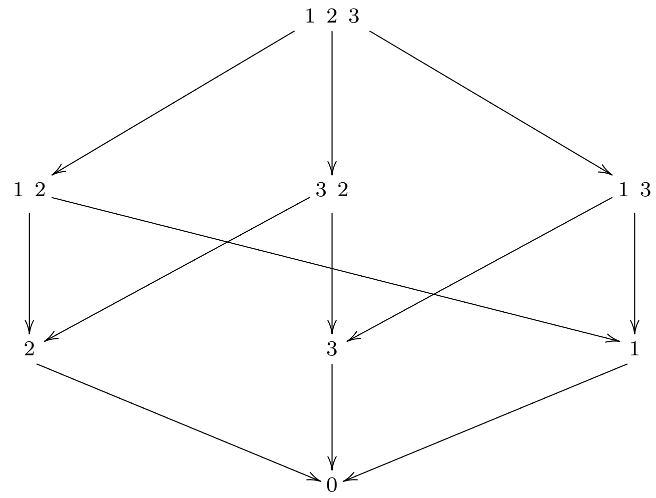
Theorem 21. *Let Λ be a Nakayama algebra with $n = |\Lambda|$. Then $\Gamma(\mathrm{s}\tau\text{-tilt}\Lambda)$ is obtained from $\Gamma(\mathrm{s}\tau\text{-tilt}(K^n))$.*

Example 22. Let $\Lambda_0 := \Lambda_3^3$ be a self-injecitve Nakayama algebra. Then we have a sequence of Nakayama algebras

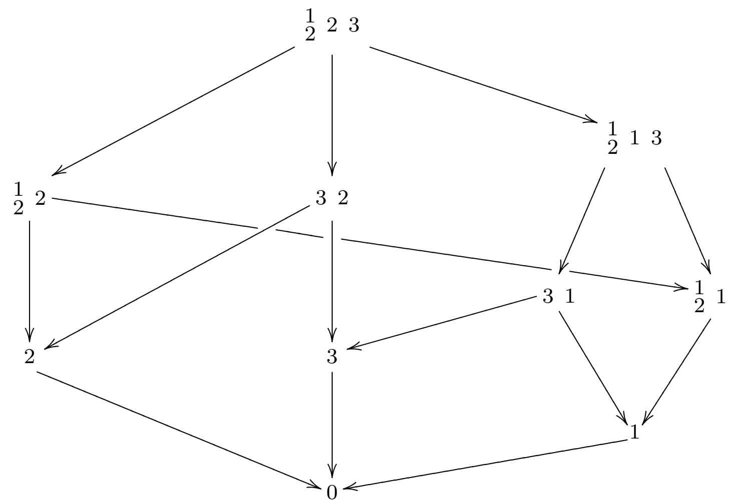
$$\Lambda_0 \twoheadrightarrow \Lambda_1 \twoheadrightarrow \Lambda_2 \twoheadrightarrow \Lambda_3 \twoheadrightarrow \Lambda_4 \twoheadrightarrow \Lambda_5 \twoheadrightarrow \Lambda_6$$

by the rejection lemma. Thus we have Hasse quivers from K^3 to Λ_3^3 by Theorem 21.

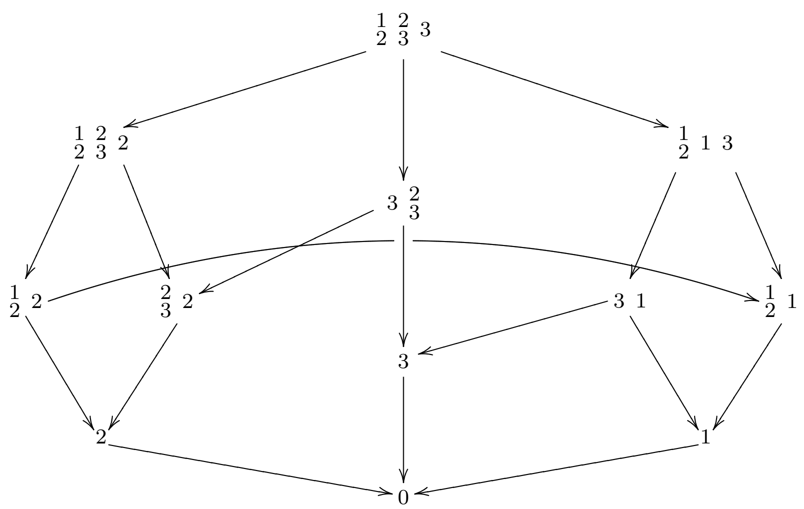
(1) $\Lambda_6 = K^3$



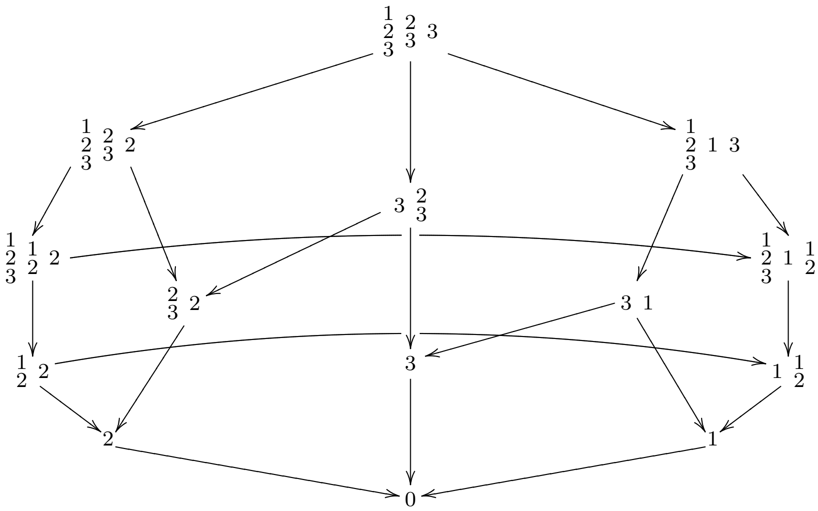
(2) $\Lambda_5 = K(\ 3 \qquad 2 \longleftarrow 1\ )$



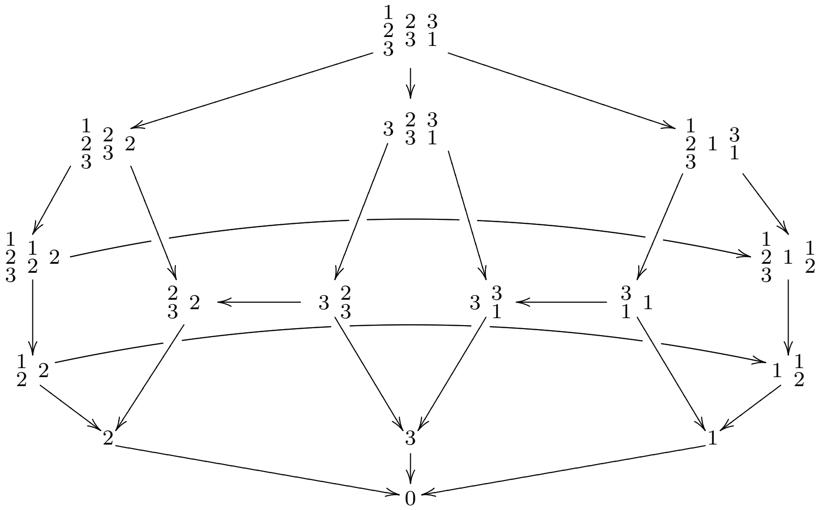
(3) $\Lambda_4 = K(\ 3 \xleftarrow{\alpha_2} 2 \xleftarrow{\alpha_1} 1\ )/\langle \alpha_1\alpha_2 \rangle$



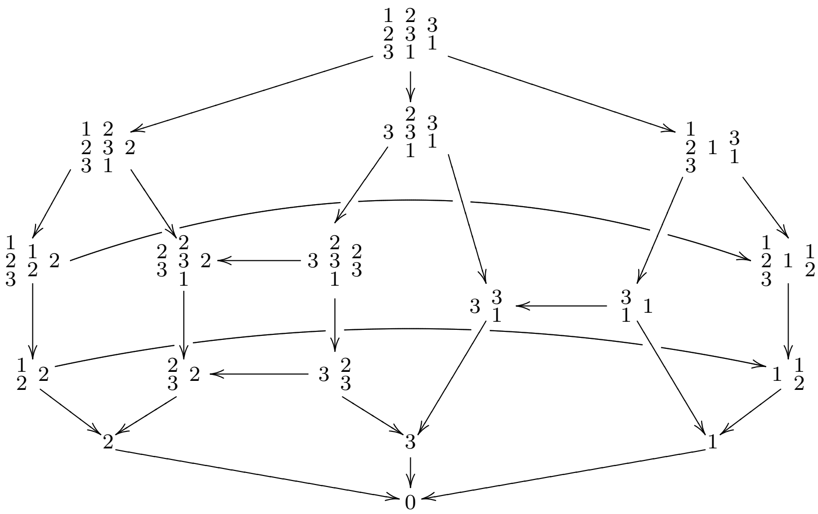
(4) $\Lambda_3 = K(\ 3 \xleftarrow{\alpha_2} 2 \xleftarrow{\alpha_1} 1\ )$



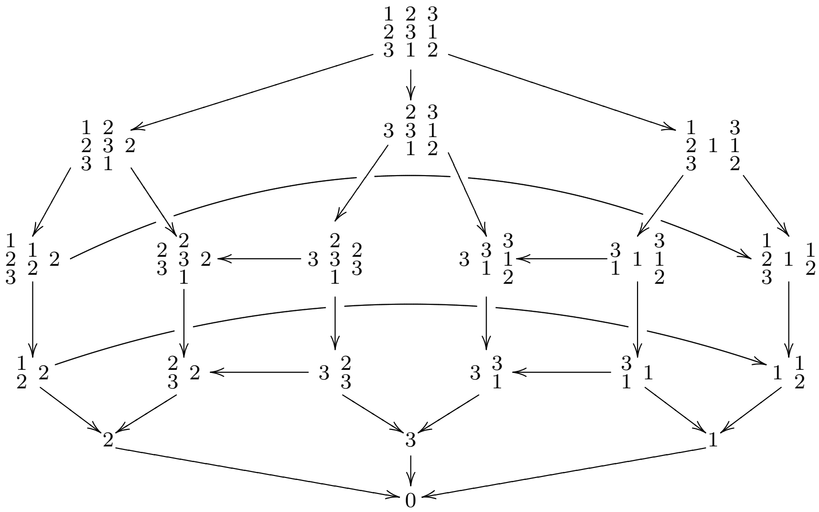
(5) $\Lambda_2 = KC_n/\langle \alpha_2\alpha_3, \alpha_3\alpha_1 \rangle$



(6) $\Lambda_1 = KC_n/\langle \alpha_1\alpha_2\alpha_3, \alpha_3\alpha_1 \rangle$



(7) $\Lambda_0 = \Lambda_3^3$



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DIMENSIONS OF TRIANGULATED CATEGORIES WITH RESPECT TO SUBCATEGORIES

TOKUJI ARAYA

Abstract. We introduce the concept of the dimension of a triangulated category with respect to a fixed full subcategory. For the bounded derived category of an abelian category, upper bounds of the dimension with respect to a contravariantly finite subcategory are given. Our methods not only recover some known results on the dimensions of derived categories in the sense of Rouquier, but also apply to various commutative and non-commutative noetherian rings.

1. Introduction

This is a joint work with T. Aihara, O. Iyama, R. Takahashi and M. Yoshiwaki [1]. The notion of the dimension of a triangulated category has been introduced by Rouquier [14] based on work of Bondal and Van den Bergh [9] on Brown representability. It measures how many extensions are needed to build the triangulated category out of a single object, up to finite direct sum, direct summand and shift. First of all, we recall its definition.

Definition 1. Let $\mathcal{T}$ be a triangulated category and $\mathcal{X}, \mathcal{Y}$ be subcategories of $\mathcal{T}$.

(1) We denote by $\mathcal{X} * \mathcal{Y}$ the subcategory of $\mathcal{T}$ consisting of objects M that admit triangles $X \to M \to Y \to X[1]$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. Then $(\mathcal{X} * \mathcal{Y}) * \mathcal{Z} = \mathcal{X} * (\mathcal{Y} * \mathcal{Z})$ holds by octahedral axiom.

(2) Set $\langle \mathcal{X} \rangle := \operatorname{add}\{X[i] \mid X \in \mathcal{X}, i \in \mathbb{Z}\}$. For a positive integer n, let
$$\langle \mathcal{X} \rangle_n^{\mathcal{T}} = \langle \mathcal{X} \rangle_n := \operatorname{add}(\underbrace{\langle \mathcal{X} \rangle * \langle \mathcal{X} \rangle * \cdots * \langle \mathcal{X} \rangle}_{n}).$$
Clearly $\langle \mathcal{X} \rangle_n$ is closed under shifts. For an object M of $\mathcal{T}$, we set
$$\langle M \rangle_n := \langle \operatorname{add} M \rangle_n.$$

(3) The *(triangle) dimension* of $\mathcal{T}$ is defined as
$$\operatorname{tri.dim} \mathcal{T} := \inf\{n \geq 0 \mid \mathcal{T} = \langle M \rangle_{n+1},\ \exists M \in \mathcal{T}\}.$$

We give an example.

Example 2. Let R be an artinian local ring with a maximal ideal $\mathfrak{m}$ and a residue class field $k = R/\mathfrak{m}$. Since R is artin, there exists a positive integer ℓ such that $\mathfrak{m}^\ell = 0$. In this case, We have $\operatorname{tri.dim} \mathsf{D}^{\mathrm{b}}(\operatorname{mod} R) \leq \ell - 1$. Indeed, let X be a bounded complex on R. Then the short exact sequence $0 \to \mathfrak{m}^i X \to \mathfrak{m}^{i-1} X \to \mathfrak{m}^{i-1}X/\mathfrak{m}^i X \to 0$ of complexes induces the exact triangle $\mathfrak{m}^i X \to \mathfrak{m}^{i-1} X \to \mathfrak{m}^{i-1}X/\mathfrak{m}^i X \to \mathfrak{m}^i X[1]$ for each i. Since $\mathfrak{m}^{i-1}X/\mathfrak{m}^i X$ is annihilated by $\mathfrak{m}$, it is isomorphic to $\bigoplus_i k^{\oplus}[i]$, and we have $\mathfrak{m}^{i-1}X/\mathfrak{m}^i X \in \langle k \rangle_1$. On

The detailed version of this paper will be submitted for publication elsewhere.

the other hand, we can see that $\mathfrak{m}^{\ell-i}X$ belongs to $\langle k\rangle_i$ by induction on i. Thus we get $X = \mathfrak{m}^0X$ belongs to $\langle k\rangle_\ell$.

We give the definition of the *(triangle) dimension* of triangulated category with respect to a subcategory.

Definition 3. Let $\mathcal{T}$ be a triangulated category and $\mathcal{X}$ be a fullsubcategory of $\mathcal{T}$. Then we define

$$\mathcal{X}\text{-}\operatorname{tri.dim}\mathcal{T} := \inf\{n \geq 0 \mid \mathcal{T} = \langle\mathcal{X}\rangle_{n+1}\}.$$

2. Main results

First of this section, we give some basic definitions and preliminary results.

Let $\mathcal{X}$ be an additive category. An $\mathcal{X}$-module is an additive contravariant functor from $\mathcal{X}$ to the category of abelian groups. A morphism between $\mathcal{X}$-modules is a natural transformation. For any object $X \in \mathcal{X}$, the functor $\operatorname{Hom}_{\mathcal{X}}(-, X)$ is an $\mathcal{X}$-module. We say that an $\mathcal{X}$-module F is *finitely presented* if there is an exact sequence $\operatorname{Hom}_{\mathcal{X}}(-, X_1) \to \operatorname{Hom}_{\mathcal{X}}(-, X_0) \to F \to 0$ with $X_0, X_1 \in \mathcal{X}$ [3, 16]. The category of finitely presented $\mathcal{X}$-modules is denoted by $\operatorname{mod}\mathcal{X}$. The assignment $X \mapsto \operatorname{Hom}_{\mathcal{X}}(-, X)$ makes a fully faithful functor $\mathcal{X} \to \operatorname{mod}\mathcal{X}$, which is called the *Yoneda embedding* of $\mathcal{X}$.

We recall here a well-known criterion for $\operatorname{mod}\mathcal{X}$ to be abelian. Let $\mathcal{X}$ be an additive category and $f : X \to Y$ be a morphism in $\mathcal{X}$. A morphism $g : Z \to X$ in $\mathcal{X}$ is called a *pseudo-kernel* if $\operatorname{Hom}_{\mathcal{X}}(-, Z) \to \operatorname{Hom}_{\mathcal{X}}(-, X) \to \operatorname{Hom}_{\mathcal{X}}(-, Y)$ is exact on $\mathcal{X}$. We say that $\mathcal{X}$ *has pseudo-kernels* if all morphisms in $\mathcal{X}$ have pseudo-kernels.

Proposition 4. [4] *Let $\mathcal{X}$ be an additive category. Then* $\operatorname{mod}\mathcal{X}$ *is an abelian category if and only if $\mathcal{X}$ has pseudo-kernels.*

We give a class of additive categories having pseudo-kernels. We say that a subcategory $\mathcal{X}$ of an additive category $\mathcal{A}$ is *contravariantly finite* if for any object $M \in \mathcal{A}$ there exist $X \in \mathcal{X}$ and a morphism $f : X \to M$ such that $\operatorname{Hom}_{\mathcal{A}}(X', f)$ is surjective for all $X' \in \mathcal{X}$ [8].

Example 5. Let $\mathcal{A}$ be an additive category and $\mathcal{X}$ be a contravariantly finite subcategory of $\mathcal{A}$. If $\mathcal{A}$ has pseudo-kernels, then $\mathcal{X}$ also has pseudo-kernels. Hence if $\mathcal{A}$ is an abelian category, then so is $\operatorname{mod}\mathcal{X}$.

Let $\mathcal{A}$ be an abelian category and $\mathcal{X}$ be a subcategory of $\mathcal{A}$. We say that $\mathcal{X}$ *generates* $\mathcal{A}$ if for any object M of $\mathcal{A}$ there is an epimorphism $X \to M$ with $X \in \mathcal{X}$.

Now we can state the main result.

Theorem 6. *Let $\mathcal{A}$ be an abelian category and $\mathcal{X}$ a contravariantly finite subcategory which generates $\mathcal{A}$. Then there is an inequality $\mathcal{X}\text{-}\operatorname{tri.dim}\mathsf{D}^{\mathrm{b}}(\mathcal{A}) \leq \operatorname{gl.dim}(\operatorname{mod}\mathcal{X})$.*

In representation theory, the notion of tilting modules/complexes plays an important role to control derived categories [13]. Its dual notion of cotilting modules was studied by Auslander and Reiten as a non-commutative generalization of canonical modules over commutative rings [5, 6, 7]. Now, we apply the results above to rings admitting cotilting modules. Let us begin with recalling the definition of a cotilting module.

Definition 7. Let A be a noetherian ring and T be a finitely generated A-module. Denote by $\mathcal{X}_T$ the subcategory of $\operatorname{mod} A$ consisting of modules X with $\operatorname{Ext}^i_A(X,T) = 0$ for all $i > 0$. We call T *cotilting* if it satisfies the following three conditions.

(1) The injective dimension of the A-module T is finite.
(2) $\operatorname{Ext}^i_A(T,T) = 0$ for all $i > 0$ (i.e., $T \in \mathcal{X}_T$).
(3) For any $X \in \mathcal{X}_T$, there exists an exact sequence $0 \to X \to T' \to X' \to 0$ in $\operatorname{mod} A$ with $T' \in \operatorname{add} T$ and $X' \in \mathcal{X}_T$.

Example 8. (1) Let R be a commutative Cohen-Macaulay local ring with a canonical module ω_R. We denote by $\mathrm{CM}(R)$ the category of maximal Cohen-Macaulay R-modules. Then ω_R is a cotilting module over R and $\mathcal{X}_{\omega_R} = \mathrm{CM}(R)$ holds. Let Λ be an R-order. For any tilting Λ^{op}-module T in the sense of Miyashita [12] with $T \in \mathrm{CM}(R)$, the Λ-module $\operatorname{Hom}_R(T,\omega_R)$ is cotilting. For the cotilting Λ-module $\omega_\Lambda := \operatorname{Hom}_R(\Lambda,\omega_R)$ it holds that $\mathcal{X}_{\omega_\Lambda} = \mathrm{CM}(\Lambda)$.

(2) Let Λ be an Iwanaga-Gorenstein ring. Then Λ is a cotilting module over Λ, and hence $\mathcal{X}_\Lambda = \mathrm{CM}(\Lambda)$.

Let R and Λ be as above. We set $A := R$ or Λ. Let T be a cotilting A-module. It comes from Auslander-Buchweitz approximation theory [5], we can see that the subcategory $\mathcal{X}_T$ of $\operatorname{mod} A$ is a contravariantly finite subcategory which generates $\operatorname{mod} A$.

Immediately we have the following inequality, which is a special case of [10].

Proposition 9. *Let T be a cotilting module of A. Then one has*

$$\operatorname{gl.dim}(\operatorname{mod} \mathcal{X}_T) \leq \max\{2, \operatorname{inj.dim} T\}.$$

Let R be a commutative Cohen-Macaulay local ring with a canonical module ω_R. Since the injective dimension of ω_R is equal to the Krull dimension of R, we obtain the following corollary.

Corollary 10. *Let R be a commutative Cohen-Macaulay local ring with a canonical module. Then one has*

$$\mathrm{CM}(R)\text{-}\operatorname{tri.dim} \mathsf{D}^{\mathrm{b}}(\operatorname{mod} R) \leq \max\{1, \dim R\}.$$

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DERIVED EQUIVALENCE CLASSIFICATION OF GENERALIZED MULTIFOLD EXTENSIONS OF PIECEWISE HEREDITARY ALGEBRAS OF TREE TYPE

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Abstract. We give a derived equivalence classification of algebras of the form $\hat{A}/\langle\phi\rangle$ for some piecewise hereditary algebra A of tree type and some automorphism ϕ of $\hat{A}$ such that $\phi(A^{[0]}) = A^{[n]}$ for some positive integer n.

Introduction

Throughout this note we fix an algebraically closed field $\Bbbk$, and assume that all algebras are basic and finite-dimensional $\Bbbk$-algebras and that all categories are $\Bbbk$-categories.

Let A be an algebra and n a positive integer. Then an algebra of the form $T_\psi^n(A) := \hat{A}/\langle\hat{\psi}\nu_A^n\rangle$ for some automorphism ψ of A is called a *twisted n-fold extension* of A. Further an algebra of the form $\hat{A}/\langle\phi\rangle$ for some automorphism ϕ of $\hat{A}$ with jump n is called a *generalized n-fold extension* of A, where ϕ is called an automorphism with *jump n* if $\phi(A^{[0]}) = A^{[n]}$. Since obviously $\hat{\psi}\nu_A^n$ is an automorphism with jump n, we see that twisted n-fold extensions are generalized n-fold extensions. An algebra is called a *generalized* (*resp. twisted*) *multifold extension* if it is a generalized (resp. twisted) n-fold extension for some positive integer n. In [3], we gave the derived equivalence classification of twisted multifold extensions of piecewise hereditary algebras of tree type by giving a complete invariant. In this note we extend this result to generalized multifold extensions of piecewise hereditary algebras of tree type.

1. Preliminaries

For a category R we denote by R_0 and R_1 the class of objects and morphisms of R, respectively. A category R is said to be *locally bounded* if it satisfies the following:

- Distinct objects of R are not isomorphic;
- $R(x, x)$ is a local algebra for all $x \in R_0$;
- $R(x, y)$ is finite-dimensional for all $x, y \in R_0$; and
- The set $\{y \in R_0 \mid R(x, y) \neq 0 \text{ or } R(y, x) \neq 0\}$ is finite for all $x \in R_0$.

A category is called *finite* if it has only a finite number of objects.

A pair (A, E) of an algebra A and a complete set $E := \{e_1, \dots, e_n\}$ of orthogonal primitive idempotents of A can be identified with a locally bounded and finite category R by the following correspondences. Such a pair (A, E) defines a category $R_{(A,E)} := R$ as follows: $R_0 := E$, $R(x, y) := yAx$ for all $x, y \in E$, and the composition of R is

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defined by the multiplication of A. Then the category R is locally bounded and finite. Conversely, a locally bounded and finite category R defines such a pair (A_R, E_R) as follows: $A_R := \bigoplus_{x,y\in R_0} R(x,y)$ with the usual matrix multiplication (regard each element of A as a matrix indexed by R_0), and $E_R := \{(\mathbb{1}_x\delta_{(i,j),(x,x)})_{i,j\in R_0} \mid x \in R_0\}$. We always regard an algebra A as a locally bounded and finite category by fixing a complete set A_0 of orthogonal primitive idempotents of A.

Definition 1.1. Let A be an algebra.

(1) The *repetition* $\hat{A}$ of A is a $\Bbbk$-category defined as follows ($\hat{A}$ turns out to be locally bounded):

- $\hat{A}_0 := A_0 \times \mathbb{Z} = \{x^{[i]} := (x,i) \mid x \in A_0, i \in \mathbb{Z}\}$.
- $\hat{A}(x^{[i]}, y^{[j]}) := \begin{cases} \{f^{[i]} \mid f \in A(x,y)\} & \text{if } j = i, \\ \{\phi^{[i]} \mid \phi \in DA(y,x)\} & \text{if } j = i+1, \\ 0 & \text{otherwise,} \end{cases}$ for all $x^{[i]}, y^{[j]} \in \hat{A}_0$.
- For each $x^{[i]}, y^{[j]}, z^{[k]} \in \hat{A}_0$ the composition $\hat{A}(y^{[j]}, z^{[k]}) \times \hat{A}(x^{[i]}, y^{[j]}) \to \hat{A}(x^{[i]}, z^{[k]})$ is given as follows.
  - (i) If $i = j, j = k$, then this is the composition of A $A(y,z)\times A(x,y) \to A(x,z)$.
  - (ii) If $i = j, j+1 = k$, then this is given by the right A-module structure of DA: $DA(z,y) \times A(x,y) \to DA(z,x)$.
  - (iii) If $i+1 = j, j = k$, then this is given by the left A-module structure of DA: $A(y,z) \times DA(y,x) \to DA(z,x)$.
  - (iv) Otherwise, the composition is zero .

(2) We define an automorphism ν_A of $\hat{A}$, called the *Nakayama automorphism* of $\hat{A}$, by $\nu_A(x^{[i]}) := x^{[i+1]}$, $\nu_A(f^{[i]}) := f^{[i+1]}$, $\nu_A(\phi^{[i]}) := \phi^{[i+1]}$ for all $i \in \mathbb{Z}, x \in A_0, f \in A_1, \phi \in \bigcup_{x,y\in A_0} DA(y,x)$.

(3) For each $n \in \mathbb{Z}$, we denote by $A^{[n]}$ the full subcategory of $\hat{A}$ formed by $x^{[n]}$ with $x \in A$, and by $\mathbb{1}^{[n]} : A \xrightarrow{\sim} A^{[n]} \hookrightarrow \hat{A}, x \mapsto x^{[n]}$, the embedding functor.

We cite the following from [3, Lemma 2.3].

Lemma 1.2. *Let $\psi\colon A \to B$ be an isomorphism of algebras. Denote by $\psi_x^y\colon A(y,x) \to B(\psi y, \psi x)$ the isomorphism defined by ψ for all $x, y \in A$. Define $\hat{\psi}\colon \hat{A} \to \hat{B}$ as follows.*

- *For each $x^{[i]} \in \hat{A}$, $\hat{\psi}(x^{[i]}) := (\psi x)^{[i]}$;*
- *For each $f^{[i]} \in \hat{A}(x^{[i]}, y^{[i]})$, $\hat{\psi}(f^{[i]}) := (\psi f)^{[i]}$; and*
- *For each $\phi^{[i]} \in \hat{A}(x^{[i]}, y^{[i+1]})$, $\hat{\psi}(\phi^{[i]}) := (D((\psi_x^y)^{-1})(\phi))^{[i]} = (\phi \circ (\psi_x^y)^{-1})^{[i]}$.*

Then

(1) *$\hat{\psi}$ is an isomorphism.*

(2) *Given an isomorphism $\rho\colon \hat{A} \to \hat{B}$, the following are equivalent.*

  (a) *$\rho = \hat{\psi}$;*

  (b) *ρ satisfies the following.*

    (i) *$\rho\nu_A = \nu_B\rho$;*

    (ii) *$\rho(A^{[0]}) = A^{[0]}$;*

(iii) *The diagram*

$$\begin{array}{ccc} A & \xrightarrow{\psi} & B \\ {\scriptstyle \mathbf{1}^{[0]}}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{1}^{[0]}} \\ A^{[0]} & \xrightarrow[\rho]{} & B^{[0]} \end{array}$$

is commutative; and

(iv) $\rho(\phi^{[0]}) = (\phi \circ (\psi_x^y)^{-1})^{[0]}$ *for all* $x, y \in A$ *and all* $\phi \in DA(y, x)$.

An algebra is called a *tree algebra* if its ordinary quiver is an oriented tree. Let R be a locally bounded category with the Jacobson radical J and with the ordinary quiver Q. Then by definition of Q there is a bijection $f\colon Q_0 \to R_0, x \mapsto f_x$ and injections $\bar{a}_{y,x}\colon Q_1(x,y) \to J(f_x, f_y)/J^2(f_x, f_y)$ such that $\bar{a}_{y,x}(Q_1(x,y))$ forms a basis of $J(f_x, f_y)/J^2(f_x, f_y)$, where $Q_1(x, y)$ is the set of arrows from x to y in Q for all $x, y \in Q_0$. For each $\alpha \in Q_1(x, y)$ choose $a_{y,x}(\alpha) \in J(f_x, f_y)$ such that $a(\alpha) + J^2(f_x, f_y) = \bar{a}_{y,x}(\alpha)$. Then the pair (f, a) of the bijection f and the family a of injections $a_{y,x}\colon Q_1(x, y) \to J(f_x, f_y)$ $(x, y \in Q_0)$ uniquely extends to a full functor $\Phi\colon \Bbbk Q \to R$, which is called a *display functor* for R.

A path μ from y to x in a quiver with relations (Q, I) is called *maximal* if $\mu \notin I$ but $\alpha\mu, \mu\beta \in I$ for all arrows $\alpha, \beta \in Q_1$. For a k-vector space V with a basis $\{v_1, \dots, v_n\}$ we denote by $\{v_1^*, \dots, v_n^*\}$ the basis of DV dual to the basis $\{v_1, \dots, v_n\}$. In particular if $\dim_k V = 1$, $v^* \in DV$ is defined for all $v \in V\backslash\{0\}$.

Lemma 1.3. *Let A be a tree algebra and $\Phi : \Bbbk Q \to A$ a display functor with $I := \operatorname{Ker}\Phi$. Then*

(1) Φ *uniquely induces the display functor* $\hat{\Phi} : \Bbbk\hat{Q} \to \hat{A}$ *for* $\hat{A}$, *where*

(i) $\hat{Q} = (\hat{Q}_0, \hat{Q}_1, \hat{s}, \hat{t})$ *is defined as follows:*

- $\hat{Q}_0 := Q_0 \times \mathbb{Z} = \{x^{[i]} := (x, i) \mid x \in Q_0, i \in \mathbb{Z}\}$,
- $Q_1 \times \mathbb{Z} := \{\alpha^{[i]} := (\alpha, i) \mid \alpha \in Q_1, i \in \mathbb{Z}\}$,
  $\hat{Q}_1 := (Q_1 \times \mathbb{Z}) \sqcup \{\mu^{*[i]} \mid \mu$ *is a maximal path in* $(Q, I), i \in \mathbb{Z}\}$,
- $\hat{s}(\alpha^{[i]}) := s(\alpha)^{[i]}$, $\hat{t}(\alpha^{[i]}) := t(\alpha)^{[i]}$ *for all* $\alpha^{[i]} \in Q_1 \times \mathbb{Z}$, *and if* μ *is a maximal path from* y *to* x *in* (Q, I) *then,* $\hat{s}(\mu^{*[i]}) := x^{[i]}$, $\hat{t}(\mu^{*[i]}) := y^{[i+1]}$.

(ii) $\hat{\Phi}$ *is defined by* $\hat{\Phi}(x^{[i]}) := (\Phi x)^{[i]}$, $\hat{\Phi}(\alpha^{[i]}) := (\Phi\alpha)^{[i]}$, *and* $\hat{\Phi}(\mu^{*[i]}) := (\Phi(\mu)^*)^{[i]}$ *for all* $i \in \mathbb{Z}$, $x \in Q_0$, $\alpha \in Q_1$ *and maximal paths* μ *in* (Q, I).

(2) *We define an automorphism* ν_Q *of* $\hat{Q}$ *by* $\nu_Q(x^{[i]}) := x^{[i+1]}$, $\nu_Q(\alpha^{[i]}) := \alpha^{[i+1]}$, $\nu_Q(\mu^{*[i]}) := \mu^{*[i+1]}$ *for all* $i \in \mathbb{Z}, x \in Q_0, \alpha \in Q_1$, *and maximal paths* μ *in* (Q, I).

(3) $\operatorname{Ker}\hat{\Phi}$ *is equal to the ideal* $\hat{I}$ *defined by the full commutativity relations on* $\hat{Q}$ *and the zero relations* $\mu = 0$ *for those paths* μ *of* $\hat{Q}$ *for which there is no path* $\hat{t}(\mu) \rightsquigarrow \nu_Q(\hat{s}(\mu))$. (*Therefore note that if a path* $\alpha_n \cdots \alpha_1$ *is in* I, *then* $\alpha_n^{[i]} \cdots \alpha_1^{[i]}$ *is in* $\hat{I}$ *for all* $i \in \mathbb{Z}$.)

Let R be a locally bounded category. A morphism $f\colon x \to y$ in R_1 is called a *maximal nonzero morphism* if $f \neq 0$ and $fg = 0, hf = 0$ for all $g \in \operatorname{rad} R(z, x), h \in \operatorname{rad} R(y, z), z \in R_0$.

Lemma 1.4. *Let A be an algebra and $x^{[i]}, y^{[j]} \in \hat{A}_0$. Then there exists a maximal nonzero morphism in $\hat{A}(x^{[i]}, y^{[j]})$ if and only if $y^{[j]} = \nu_A(x^{[i]})$.*

Proof. This follows from the fact that $\hat{A}(\text{-}, x^{[i+1]}) \cong D\hat{A}(x^{[i]}, \text{-})$ for all $i \in \mathbb{Z}, x \in A_0$. $\square$

Lemma 1.5. *Let A be an algebra. Then the actions of $\phi\nu_A$ and $\nu_A\phi$ coincide on the objects of $\hat{A}$ for all $\phi \in \operatorname{Aut}(\hat{A})$.*

Proof. Let $x^{[i]} \in \hat{A}_0$. Then there is a maximal nonzero morphism in $\hat{A}(x^{[i]}, \nu_A(x^{[i]}))$ by Lemma 1.4. Since ϕ is an automorphism of $\hat{A}$, there is a maximal nonzero morphism in $\hat{A}(\phi(x^{[i]}), \phi(\nu_A(x^{[i]})))$. Hence $\phi(\nu_A(x^{[i]})) = \nu_A(\phi(x^{[i]}))$ by the same lemma. $\square$

The following is immediate by the lemma above.

Proposition 1.6. *Let A be an algebra, n an integer, and ϕ an automorphism of $\hat{A}$. Then the following are equivalent:*

(1) *ϕ is an automorphism with jump n;*
(2) *$\phi(A^i) = A^{[i+n]}$ for some integer i; and*
(3) *$\phi(A^j) = A^{[j+n]}$ for all integers j.*

In the sequel, we always assume that n is a positive integer when we consider a morphism with jump n. Let Q be a quiver. We denote by $\bar{Q}$ the underlying graph of Q, and call Q *finite* if both Q_0 and Q_1 are finite sets. Each automorphism of Q is regarded as an automorphism of $\bar{Q}$ preserving the orientation of Q, thus $\operatorname{Aut}(Q)$ can be regarded as a subgroup of $\operatorname{Aut}(\bar{Q})$. Suppose now that Q is a finite oriented tree. Then it is also known that $\operatorname{Aut}(Q) \leq \operatorname{Aut}_0(\bar{Q}) := \{f \in \operatorname{Aut}(\bar{Q}) \mid \exists x \in Q_0, f(x) = x\}$. We say that Q is an *admissibly oriented* tree if $\operatorname{Aut}(Q) = \operatorname{Aut}_0(\bar{Q})$. We quote the following from [3, Lemma 4.1]:

Lemma 1.7. *For any finite tree T there exists an admissibly oriented tree Q with a unique source such that $\bar{Q} = T$.*

We recall the following (cf. [3, Section 4.1]):

Definition 1.8. Let R be a locally bounded category. The *formal additive hull* $\operatorname{add} R$ of R is a category defined as follows.

- $(\operatorname{add} R)_0 := \{\bigoplus_{i=1}^n x_i := (x_1, \dots, x_n) \mid n \in \mathbb{N},\ x_1, \dots, x_n \in R_0\}$;
- For each $x = \bigoplus_{i=1}^m x_i, y = \bigoplus_{j=1}^m y_i \in (\operatorname{add} R)_0$,

$$(\operatorname{add} R)(x, y) := \{(\mu_{j,i})_{j,i} \mid \mu_{j,i} \in R(x_i, y_j) \text{ for all } i = 1, \dots, m, j = 1, \dots, n\}; \text{and}$$

- The composition is given by the matrix multiplication.

It is well known that the *Yoneda functor* $Y_R \colon \operatorname{add} R \to \operatorname{prj} R, \bigoplus_{i=1}^n x_i \mapsto \bigoplus_{i=1}^n R(\text{-}, x_i)$ is an equivalence. Let $F \colon R \to S$ be a functor of locally bounded categories. Then F naturally induces functors $\operatorname{add} F \colon \operatorname{add} R \to \operatorname{add} S$ and $\tilde{F} := \mathcal{K}^{\mathrm{b}}(\operatorname{add} F) \colon \mathcal{K}^{\mathrm{b}}(\operatorname{add} R) \to \mathcal{K}^{\mathrm{b}}(\operatorname{add} S)$, which are isomorphisms if F is an isomorphism.

2. Reduction to hereditary tree algebras

Proposition 2.1. *Let A be a piecewise hereditary algebra of tree type $\bar{Q}$ for an admissibly oriented tree Q, and n a positive integer. Then we have the following:*

(1) *For any $\phi \in \mathrm{Aut}(\hat{A})$ with jump n, there exists some $\psi \in \mathrm{Aut}(\widehat{\Bbbk Q})$ with jump n such that $\hat{A}/\langle\phi\rangle$ is derived equivalent to $\widehat{\Bbbk Q}/\langle\psi\rangle$; and*

(2) *If we set $\phi' := \nu_{\hat{A}}^n \hat{\phi}_0 \in \mathrm{Aut}(\hat{A})$, where $\phi_0 := (\mathbb{1}^{[0]})^{-1}\nu^{-n}\phi|_{A^{[0]}}\mathbb{1}^{[0]}$, then there exists some $\psi' \in \mathrm{Aut}(\widehat{\Bbbk Q})$ with jump n such that $\hat{A}/\langle\phi'\rangle$ is derived equivalent to $\widehat{\Bbbk Q}/\langle\psi'\rangle$, and that the actions of ψ and ψ' coincide on the objects of $\widehat{\Bbbk Q}$.*

Proof. (1) We set $\phi_i := (\mathbb{1}^{[i]})^{-1}\nu^{-n}\phi|_{A^{[i]}}\mathbb{1}^{[i]} \in \mathrm{Aut}(A)$ for all $i \in \mathbb{Z}$. By [3, Lemma 5.4], there exists a tilting triple $(A, E, \Bbbk Q)$ with an isomorphism $\zeta\colon E \to \Bbbk Q$ such that E is $\langle\tilde{\eta}\rangle$-stable up to isomorphisms for all $\eta \in \mathrm{Aut}(A)$. In particular, E is $\langle\tilde{\phi}_i\rangle$-stable up to isomorphisms for all $i \in \mathbb{Z}$. Then $(\hat{A}, \hat{E}, \widehat{\Bbbk Q})$ is a tilting triple with an isomorphism $\hat{\zeta}$ by [1, Theorem 1.5] and the following holds.

Claim 1. $\hat{E}$ is $\langle\tilde{\phi}\rangle$-stable up to isomorphisms.

Indeed for each $T \in E_0$ and $i \in \mathbb{Z}$, we have

$$\begin{aligned} \tilde{\phi}\tilde{\mathbb{1}}^{[i]}(T) &= \tilde{\nu}^n\tilde{\nu}^{-n}\tilde{\phi}\tilde{\mathbb{1}}^{[i]}(T) \\ &= \tilde{\nu}^n\tilde{\mathbb{1}}^{[i]}\tilde{\phi}_i(T) \\ &= \tilde{\mathbb{1}}^{[i+n]}\tilde{\phi}_i(T). \end{aligned} \tag{2.1}$$

Since E is $\langle\tilde{\phi}_i\rangle$-stable up to isomorphisms, there is some $T' \in E$ such that $T' \cong \tilde{\phi}_i(T)$, and hence $\tilde{\mathbb{1}}^{[i+n]}\tilde{\phi}_i(T) \cong \tilde{\mathbb{1}}^{[i+n]}(T') \in \hat{E}$, as desired.

By [3, Remark 3.5], we have a $\langle\tilde{\phi}\rangle$-stable tilting subcategory $\hat{E}'$ and an isomorphism $\theta\colon \hat{E}' \xrightarrow{\sim} \hat{E}$. Therefore by [2, Proposition 5.4] $\hat{A}/\langle\phi\rangle$ and $\hat{E}'/\langle\tilde{\phi}\rangle$ are derived equivalent. If we set $\psi := (\hat{\zeta}\theta)\tilde{\phi}(\hat{\zeta}\theta)^{-1}$, then (2.1) shows that ψ is an automorphism with jump n, and that $\hat{E}'/\langle\tilde{\phi}\rangle \cong \widehat{\Bbbk Q}/\langle\psi\rangle$. Hence $\hat{A}/\langle\phi\rangle$ and $\widehat{\Bbbk Q}/\langle\psi\rangle$ are derived equivalent.

(2) Note that ϕ' is also an automorphism with jump n. By the same argument we see that $\hat{E}$ is also $\langle\tilde{\phi}'\rangle$-stable up to isomorphisms; there exists a $\langle\tilde{\phi}'\rangle$-stable tilting subcategory $\hat{E}''$ and an isomorphism $\theta'\colon \hat{E}'' \xrightarrow{\sim} \hat{E}$; and $\hat{A}/\langle\phi'\rangle$ and $\hat{E}''/\langle\tilde{\phi}'\rangle$ are derived equivalent. Set $\psi' := (\hat{\zeta}\theta')\tilde{\phi}'(\hat{\zeta}\theta')^{-1}$, then ψ' is an automorphism with jump n, $\hat{E}''/\langle\tilde{\phi}'\rangle \cong \widehat{\Bbbk Q}/\langle\psi'\rangle$, and $\hat{A}/\langle\phi'\rangle$ and $\widehat{\Bbbk Q}/\langle\psi'\rangle$ are derived equivalent. Now for $i = 0$ (2.1) shows that $\tilde{\phi}\tilde{\mathbb{1}}^{[0]}(T) = \tilde{\mathbb{1}}^{[n]}\tilde{\phi}_0(T)$ for all $T \in E_0$. Since $\phi'_0 = \phi_0$, the same calculation shows that $\tilde{\phi}'\tilde{\mathbb{1}}^{[0]}(T) = \tilde{\mathbb{1}}^{[n]}\tilde{\phi}_0(T)$ for all $T \in E_0$. Thus the actions of $\tilde{\phi}$ and $\tilde{\phi}'$ coincide on the objects of $E^{[0]}$, which shows that the actions of ψ and ψ' coincide on the objects of $\Bbbk Q^{[0]}$. Hence by Lemma 1.5 their actions coincide on the objects of $\widehat{\Bbbk Q}$. □

3. Hereditary tree algebras

Remark 3.1. Let Q be an oriented tree.

(1) We may identify $\widehat{\Bbbk Q} = \Bbbk\hat{Q}/\hat{I}$ as stated in Lemma 1.3, and we denote by $\overline{\mu}$ the morphism $\mu + \hat{I}$ in $\widehat{\Bbbk Q}$ for each morphism μ in $\Bbbk\hat{Q}$.

(2) Let $x, y \in \hat{Q}_0$. Since $\hat{I}$ contains full commutativity relations, we have $\dim_{\Bbbk} \widehat{\Bbbk Q}(x, y) \leq 1$, and in particular $\hat{Q}$ has no double arrows.

(3) Let $\alpha\colon x \to y$ be in $\hat{Q}_1$ and $\phi \in \mathrm{Aut}(\widehat{\Bbbk Q})$. Then there exists a unique arrow $\phi x \to \phi y$ in $\hat{Q}$, which we denote by $(\hat{\pi}\phi)(\alpha)$, and we have $\phi(\overline{\alpha}) = \phi_\alpha \overline{(\hat{\pi}\phi)(\alpha)} \in \widehat{\Bbbk Q}(\phi x, \phi y)$ for a unique $\phi_\alpha \in \Bbbk^\times := \Bbbk \setminus \{0\}$. This defines an automorphism $\hat{\pi}\phi$ of $\hat{Q}$, and thus a group homomorphism $\hat{\pi} : \mathrm{Aut}(\widehat{\Bbbk Q}) \to \mathrm{Aut}(\hat{Q})$.

(4) Similarly, let $\alpha\colon x \to y$ be in Q_1 and $\psi \in \mathrm{Aut}(\Bbbk Q)$. Then there exists a unique arrow $\psi x \to \psi y$ in Q, which we denote by $(\pi\psi)(\alpha)$. This defines an automorphism $\pi\psi$ of Q, and thus a group homomorphism $\pi : \mathrm{Aut}(\Bbbk Q) \to \mathrm{Aut}(Q)$.

We cite the following from [3, Proposition 7.4].

Proposition 3.2. *Let R be a locally bounded category, and g, h automorphisms of R acting freely on R. If there exists a map $\rho\colon R_0 \to \Bbbk^\times$ such that $\rho(y)g(f) = h(f)\rho(x)$ for all morphisms $f\colon x \to y$ in R, then $R/\langle g\rangle \cong R/\langle h\rangle$.* □

Definition 3.3. (1) For a quiver $Q = (Q_0, Q_1, s, t)$ we set $Q[Q_1^{-1}]$ to be the quiver

$$Q[Q_1^{-1}] := (Q_0, Q_1 \sqcup \{\alpha^{-1} \mid \alpha \in Q_1\}, s', t'),$$

where $s'|_{Q_1} := s$, $t'|_{Q_1} := t$, $s'(\alpha^{-1}) := t(\alpha)$ and $t'(\alpha^{-1}) := s(\alpha)$ for all $\alpha \in Q_1$. A *walk* in Q is a path in $Q[Q_1^{-1}]$.

(2) Suppose that Q is a finite oriented tree. Then for each $x, y \in Q_0$ there exists a unique shortest walk from x to y in Q, which we denote by $w(x, y)$. If $w(x, y) = \alpha_n^{\varepsilon_n} \cdots \alpha_1^{\varepsilon_1}$ for some $\alpha_1, \cdots, \alpha_n \in Q_1$ and $\varepsilon_1, \dots, \varepsilon_n \in \{1, -1\}$, then we define a subquiver $W(x, y)$ of Q by $W(x, y) := (W(x, y)_0, W(x, y)_1, s', t')$, where $W(x, y)_0 := \{s(\alpha_i), t(\alpha_i) \mid i = 1, \dots, n\}$, $W(x, y)_1 := \{\alpha_1, \dots, \alpha_n\}$, and s', t' are restrictions of s, t to $W(x, y)_1$, respectively. Since Q is an oriented tree, $w(x, y)$ is uniquely recovered by $W(x, y)$. Therefore we can identify $w(x, y)$ with $W(x, y)$, and define a *sink* and a *source* of $w(x, y)$ as those in $W(x, y)$.

Proposition 3.4. *Let Q be a finite oriented tree and ϕ, ψ automorphisms of $\widehat{\Bbbk Q}$ acting freely on $\widehat{\Bbbk Q}$. If the actions of ϕ and ψ coincide on the objects of $\widehat{\Bbbk Q}$, then there exists a map $\rho\colon (\hat{Q}_0 =)\ \widehat{\Bbbk Q}_0 \to \Bbbk^\times$ such that $\rho(y)\psi(f) = \phi(f)\rho(x)$ for all morphisms $f\colon x \to y$ in $\widehat{\Bbbk Q}$. Hence in particular, $\widehat{\Bbbk Q}/\langle\phi\rangle$ is isomorphic to $\widehat{\Bbbk Q}/\langle\psi\rangle$.*

Proof. Assume that the actions of $\phi, \psi \in \mathrm{Aut}(\widehat{\Bbbk Q})$ coincides on the objects of $\widehat{\Bbbk Q}$. Then ϕ and ψ induce the same quiver automorphism $q = \hat{\pi}\phi = \hat{\pi}\psi$ of $\hat{Q}$, and there exist $(\phi_\alpha)_{\alpha\in\hat{Q}_1}, (\psi_\alpha)_{\alpha\in\hat{Q}_1} \in (k^\times)^{\hat{Q}_1}$ such that for each $\alpha \in \hat{Q}_1$ we have

$$\phi(\overline{\alpha}) = \phi_\alpha \overline{q(\alpha)}, \quad \psi(\overline{\alpha}) = \psi_\alpha \overline{q(\alpha)}.$$

For each path $\lambda = \alpha_n \cdots \alpha_1$ in $\hat{Q}$ with $\alpha_1, \dots, \alpha_n \in \hat{Q}_1$ we set $\phi_\lambda := \phi_{\alpha_n} \cdots \phi_{\alpha_1}$. Then we have

$$\phi(\overline{\lambda}) = \phi_\lambda \overline{q(\lambda)},$$

where $q(\lambda) := q(\alpha_n)\cdots q(\alpha_1)$ because $\phi(\overline{\alpha_n})\cdots\phi(\overline{\alpha_1}) = \phi_{\alpha_n}\cdots\phi_{\alpha_1}\overline{q(\alpha_n)\cdots q(\alpha_1)}$.

To show the statement we may assume that $\psi_\alpha = 1$ for all $\alpha \in \hat{Q}_1$. Since for each $x, y \in \hat{Q}_0$ the morphism space $\Bbbk\widehat{Q}(x, y)$ is at most 1-dimensional and has a basis of the form $\overline{\mu}$ for some path μ, it is enough to show that there exists a map $\rho : \hat{Q}_0 \to \Bbbk^\times$ satisfying the following condition:

$$\rho(v^{[j]}) = \phi_\beta \rho(u^{[i]}) \quad \text{for all } \beta : u^{[i]} \to v^{[j]} \text{ in } \hat{Q}_1. \tag{3.1}$$

We define a map ρ as follows:
Fix a maximal path $\mu\colon y \rightsquigarrow x$ in Q. Then x is a sink and y is a source in Q. We can write μ as $\mu = \alpha_l \cdots \alpha_1$ for some $\alpha_1, \dots, \alpha_l \in Q_1$. First we set $\rho(x^{[0]}) := 1$. By induction on $0 \le i \in \mathbb{Z}$ we define $\rho(x^{[i]})$ and $\rho(x^{[-i]})$ by the following formulas:

$$\rho(x^{[i+1]}) := \phi_{\mu^{[i+1]}}\phi_{\mu^{*[i]}}\rho(x^{[i]}), \tag{3.2}$$

$$\rho(x^{[i-1]}) := \phi^{-1}_{\mu^{*[i-1]}}\phi^{-1}_{\mu^{[i]}}\rho(x^{[i]}). \tag{3.3}$$

Now for each $i \in \mathbb{Z}$ and $u \in Q_0$ if $w(u, x) = \beta_m^{\varepsilon_m} \cdots \beta_1^{\varepsilon_1}$ for some $\beta_1, \dots, \beta_m \in Q_1$ and $\varepsilon_1, \dots, \varepsilon_m \in \{1, -1\}$, then we set

$$\rho(u^{[i]}) := \phi^{-\varepsilon_1}_{\beta_1^{[i]}} \cdots \phi^{-\varepsilon_m}_{\beta_m^{[i]}} \rho(x^{[i]}). \tag{3.4}$$

We have to verify the condition (3.1).

Case 1. $\beta = \alpha^{[i]}\colon u^{[i]} \to v^{[i]}$ for some $i \in \mathbb{Z}$, and $\alpha\colon u \to v$ in Q_1. Since Q is an oriented tree, we have either $w(u, x) = w(v, x)\alpha$ or $w(v, x) = w(u, x)\alpha^{-1}$. In either case we have $\rho(v^{[i]}) = \phi_{\alpha^{[i]}}\rho(u^{[i]})$ by the formula (3.4).

Case 2. Otherwise, we have $\beta = \lambda^{*[i]} : u^{[i]} \to v^{[i+1]}$ for some maximal path $\lambda\colon v \rightsquigarrow u$ in Q and $i \in \mathbb{Z}$. In this case the condition (3.1) has the following form:

$$\rho(v^{[i+1]}) = \phi_{\lambda^{*[i]}}\rho(u^{[i]}). \tag{3.5}$$

Two paths are said to be *parallel* if they have the same source and the same target. We prepare the following for the proof.

Claim 2. If ζ and η are parallel paths in $\hat{Q}$, then we have $\phi_\zeta = \phi_\eta$.

Indeed, since $\zeta - \eta \in \hat{I}$, we have $\phi(\overline{\zeta}) = \phi(\overline{\eta})$, which shows

$$\phi_\zeta\overline{q(\zeta)} = \phi_\eta\overline{q(\eta)}.$$

Here we have $\overline{q(\zeta)} = \psi(\overline{\zeta}) = \psi(\overline{\eta}) = \overline{q(\eta)}$, and $\psi(\overline{\zeta}) \neq 0$ because $\overline{\zeta} \neq 0$. Hence $\phi_\zeta = \phi_\eta$, as required.

We now set $d(a, b)$ to be the number of sinks in $w(a, b)$ for all $a, b \in Q_0$. By induction on $d(y, v)$ we can show that the condition (3.5) holds. □

4. Main result

Theorem 4.1. *Let A be a piecewise hereditary algebra of tree type and ϕ an automorphism of $\hat{A}$ with jump n. Then $\hat{A}/\langle\phi\rangle$ and $T^n_{\phi_0}(A)$ are derived equivalent, where we set $\phi_0 := (\mathbb{1}^{[0]})^{-1}\nu^{-n}\phi|_{A^{[0]}}\mathbb{1}^{[0]}$.*

Proof. Let T be the tree type of A. Then by Lemma 1.7 there exists an admissibly oriented tree Q with $\bar{Q} = T$. We set $\phi' := \nu_A^n \hat{\phi}_0$ $(= \hat{\phi}_0 \nu_A^n)$. Then $T_{\phi_0}^n(A) = \hat{A}/\langle \phi' \rangle$. By Proposition 2.1(2) there exist some $\psi, \psi' \in \mathrm{Aut}(\widehat{\Bbbk Q})$ both with jump n such that $\hat{A}/\langle \phi \rangle$ (resp. $\hat{A}/\langle \phi' \rangle$) is derived equivalent to $\widehat{\Bbbk Q}/\langle \psi \rangle$ (resp. $\widehat{\Bbbk Q}/\langle \psi' \rangle$), and the actions of ψ and ψ' coincide on the objects of $\widehat{\Bbbk Q}$. Then by Proposition 3.4 we have $\widehat{\Bbbk Q}/\langle \psi \rangle \cong \widehat{\Bbbk Q}/\langle \psi' \rangle$. Hence $\hat{A}/\langle \phi \rangle$ and $T_{\phi_0}^n(A)$ are derived equivalent. □

Definition 4.2. Let Λ be a generalized n-fold extension of a piecewise hereditary algebra A of tree type T, say $\Lambda = \hat{A}/\langle \phi \rangle$ for some $\phi \in \mathrm{Aut}(A)$ with jump n. Further let Q be an admissibly oriented tree with $\bar{Q} = T$. Then by Proposition 2.1 there exists $\psi \in \mathrm{Aut}(\widehat{\Bbbk Q})$ with jump n such that $\hat{A}/\langle \phi \rangle$ is derived equivalent to $\widehat{\Bbbk Q}/\langle \psi \rangle$. We define the (*derived equivalence*) *type* $\mathrm{type}(\Lambda)$ of Λ to be the triple $(T, n, \overline{\pi}(\psi_0))$, where $\psi_0 := (\mathbb{1}^{[0]})^{-1} \nu_{\Bbbk Q}^{-n} \psi|_{(\Bbbk Q)^{[0]}} \mathbb{1}^{[0]}$ and $\overline{\pi}(\psi_0)$ is the conjugacy class of $\pi(\psi_0)$ in $\mathrm{Aut}(T)$. $\mathrm{type}(\Lambda)$ is uniquely determined by Λ.

By Theorem 4.1, we can extend the main theorem in [3] as follows.

Theorem 4.3. *Let Λ, Λ' be generalized multifold extensions of piecewise hereditary algebras of tree type. Then the following are equivalent:*

(i) Λ *and* Λ' *are derived equivalent.*
(ii) Λ *and* Λ' *are stably equivalent.*
(iii) $\mathrm{type}(\Lambda) = \mathrm{type}(\Lambda')$.

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DECOMPOSING TENSOR PRODUCTS FOR CYCLIC AND DIHEDRAL GROUPS

ERIK DARPÖ AND CHRISTOPHER C. GILL

Abstract. We give a new formula for the decomposition of a tensor product of indecomposable modules of cyclic two-groups. This formula is also shown to describe the decomposition of tensor products of an important class of modules of dihedral two-groups.

1. Introduction

In this note, we give a new, closed formula for the decomposition of a tensor product of indecomposable modules of cyclic 2-groups, and show how this formula also describes the decomposition of tensor products of a class of D_{2^l}-modules. The problem of decomposing such a tensor product of modules of cyclic p-groups in characteristic p has been treated by several authors (e.g. [4, 6, 5, 1]). However, to date, all solutions have been recursive, and rather involved. Concentrating on the case $p = 2$ is a simplification which makes it possible to give a closed decomposition formula.

Our interest in this problem originated in the study of tensor products of modules of dihedral 2-groups. Thus, we show that the decomposition formula for modules of cyclic 2-groups also describes the decompositions of tensor products of the D_{2^l}-modules induced from the maximal cyclic subgroup.

Throughout this text, k denotes a field of characteristic 2. The dihedral group of order $2q$ is written as $D_{2q} = \langle \sigma, \tau \mid \sigma^2 = \tau^2 = (\sigma\tau)^q = 1 \rangle$. Here q will always be a 2-power, $q \geqslant 2$. The unique cyclic subgroup of index 2 in D_{2q} is $H_q = \langle \sigma\tau \rangle \lhd D_{2q}$.

The indecomposable modules of kC_q are classified by their dimensions; that is, up to isomorphism, for each $i \in \{1, \dots, q\}$ there exists a unique indecomposable kC_q-module of dimension i. Fix a set of representatives $\{V_i\}_{i \leqslant q}$ such that $\dim V_i = i$. Every projective indecomposable module is isomorphic to V_q, and the tensor product of a projective with any other module is again projective. We recall that every non-projective kC_q-module is Ω-periodic of period at most 2. Indeed, for each $i < q$, the formula $\Omega(V_i) \simeq V_{q-i}$ holds.

There is a unique projection $C_{2q} \twoheadrightarrow C_q$. This surjection, via the usual inflation operation, induces a full embedding of module categories $\operatorname{mod} kC_q \hookrightarrow \operatorname{mod} kC_{2q}$, $V_i \mapsto V_i$, respecting the tensor product. Thus V_i is viewed as a module for all cyclic 2-groups of order greater than or equal to i.

2. Decomposition formula for tensor products of modules of cyclic 2-groups

The following result makes it possible to compute the decomposition of a tensor product of any two kC_q-modules recursively.

This paper is a summary of results that will be published elsewhere.

Proposition 1. *Let $i, j \leqslant q$. Then $V_i \otimes V_j \simeq \Omega(V_{q-i} \otimes V_j) \oplus \max\{i+j-q, 0\} V_q$.*

If $q/2 \leq i < q$ then $q - i \leq q/2$ hence, by applying Proposition 1, we can transfer the problem of finding the decomposition of $V_i \otimes V_j$ to the smaller module category $\operatorname{mod} kC_{q/2}$. This gives an inductive process which halts when one of the factors is projective, in which case the product can be immediately computed. Example 2 below illustrates the procedure. To avoid any ambiguity, we write Ω_q to indicate the Heller translate in $\operatorname{mod} kC_q$.

Example 2. Consider the module $V_{18} \otimes V_6$, a tensor product of indecomposable modules of kC_{32}. Applying Proposition 1, we see that

$$V_{18} \otimes V_6 \simeq \Omega_{32}(V_{14} \otimes V_6). \tag{2.1}$$

Viewing $V_{14} \otimes V_6$ as a module for C_{16} and again applying Proposition 1, we obtain

$$V_{14} \otimes V_6 \simeq \Omega_{16}(V_2 \otimes V_6) \oplus 4V_{16}. \tag{2.2}$$

Now $V_2 \otimes V_6 \in \operatorname{mod} kC_8$, and

$$V_2 \otimes V_6 \simeq \Omega_8(V_2 \otimes V_2). \tag{2.3}$$

In $\operatorname{mod} kC_2$, V_2 is projective, so $V_2 \otimes V_2 \simeq 2V_2$. Applying in turn Equations (2.3), (2.2) and (2.1), we obtain the decomposition

$$\begin{aligned} V_{18} \otimes V_6 &\simeq \Omega_{32}(\Omega_{16}(\Omega_8(2V_2)) \oplus 4V_{16}) \\ &\simeq 2V_{22} \oplus 4V_{16}. \end{aligned}$$

The idea behind our decomposition formula is to record the successive applications of Proposition 1 in numerical sequences, which are then used to compute the indecomposable summands of the tensor product. Let x be any positive integer. Set $v(x) = \min\{y \in \mathbb{N} \mid 2^y \geqslant x\}$ and $x' = 2^{v(x)} - x$. A sequence $(x_n)_{n \geqslant 0}$ is defined recursively by $x_0 = x$ and $x_{n+1} = x'_n$. Let $r \in \mathbb{N}$ be the first number such that x_r is a 2-power. Then $(x_n)_{n=0}^r$ is strictly decreasing, whereas $x_n = 0$ for all $n > r$.

Now, given $i, j \in \mathbb{N}$, set $[i,j]_0 = (i_0, j_0) = (i, j)$ and, if $[i,j]_n = (i_a, j_b)$,

$$[i,j]_{n+1} = \begin{cases} (i_{a+1}, j_b) & \text{if } i_a \geqslant j_b, \\ (i_a, j_{b+1}) & \text{if } i_a < j_b. \end{cases} \tag{2.4}$$

This defines a sequence $([i,j]_n)_{n=0}^w = \left(([i,j]_n^{(1)}, [i,j]_n^{(2)}) \right)_{n=0}^w$, where w is the smallest number such that $\max\left\{[i,j]_w^{(1)}, [i,j]_w^{(2)}\right\}$ is a 2-power. Now, set $m_n = 2^{v(x_n)}$, for $x_n = \max\left\{[i,j]_n^{(1)}, [i,j]_n^{(2)}\right\}$, $n \in \{0, \ldots, w\}$. Finally, for all $n \leqslant w$, let

$$\alpha_n = \max\left\{0\,,\ [i,j]_n^{(1)} + [i,j]_n^{(2)} - m_n\right\} \quad \text{and} \tag{2.5}$$

$$\beta_n = \sum_{u=0}^{n} (-1)^u m_u\,. \tag{2.6}$$

Theorem 3. *For all $i, j \in \mathbb{N}$,*

$$V_i \otimes V_j \simeq \bigoplus_{n=0}^{w} \alpha_n V_{\beta_n} .$$

It may be noted that while the numbers i_n are, for simplicity of presentation, recursively defined, they may all be read off from the binary expansion of the number i in a non-recursive manner.

Example 4. Consider the case $i = 20$ and $j = 51$. We have

$$i_0 = 20, \qquad i_1 = 32 - i_0 = 12, \qquad i_2 = 16 - i_1 = 4,$$

and

$$j_0 = 51, \qquad j_1 = 64 - j_0 = 13, \qquad j_2 = 16 - j_1 = 3, \qquad j_3 = 4 - j_2 = 1.$$

Now we can define all sequences needed for the application of Theorem 3. First, the sequence $[i, j]$ consists of pairs (i_a, j_b), formed by applying the equation (2.4) above:

$$[i,j]_0 = (20, 51), \quad [i,j]_1 = (20, 13), \quad [i,j]_2 = (12, 13), \quad [i,j]_3 = (12, 3), \quad [i,j]_4 = (4, 3);$$

m_n is the smallest 2-power greater than or equal to the two components of $[i, j]_n$:

$$m_0 = 64, \qquad m_1 = 32, \qquad m_2 = 16, \qquad m_3 = 16, \qquad m_4 = 4;$$

$\alpha_n = [i, j]_n^{(1)} + [i, j]_n^{(2)} - m_n$ if this number is positive, otherwise $\alpha_n = 0$:

$$\alpha_0 = 7, \qquad \alpha_1 = 1, \qquad \alpha_2 = 9, \qquad \alpha_3 = 0, \qquad \alpha_4 = 3;$$

β_n is the alternating sum of the numbers $m_1, \ldots, m_n$:

$$\beta_0 = 64, \qquad \beta_1 = 32, \qquad \beta_2 = 48, \qquad \beta_3 = 32, \qquad \beta_4 = 36.$$

With Theorem 3, we conclude that

$$V_{20} \otimes V_{51} \simeq 7V_{64} \oplus V_{32} \oplus 9V_{48} \oplus 3V_{36} .$$

3. Application: pseudoprojective modules of dihedral 2-groups

It turns out that Theorem 3 can be used to describe tensor products of a class of modules of dihedral 2-groups. These are the so-called *pseudoprojective* modules, given as

$M(A_lB_l, 1)$ for some $l \in \mathbb{N}$ (see [3] for definition of the relevant notation). The pseudoprojective modules are band modules, given by schemas in the following way:

$$\begin{tikzcd}
& k \arrow[dl, "X"'] \arrow[dr, "Y"] & \\
k \arrow[d, "Y"'] & & k \arrow[d, "X"] \\
\vdots \arrow[d] & & \vdots \arrow[d] \\
k \arrow[dr] & & k \arrow[dl] \\
& k &
\end{tikzcd}$$

We shall use M_d to denote the pseudoprojective module of dimension d, in other words, $M(A_lB_l^{-1}, 1) \simeq M_{2l}$.

The pseudoprojective modules are precisely the kD_{2q}-modules that are induced from the maximal cyclic subgroup $H_q \lhd D_{2q}$:

Proposition 5. *For each $i \in \{1, \ldots, q\}$, the induced module $V_i{\uparrow}_{H_q}^{D_{2q}}$ is isomorphic to M_{2i}.*

Applying Mackey's tensor product theorem (see e.g. [2, Corollary 3.3.5(i)]),

$$M_{2i} \otimes M_{2j} \simeq 2\,(V_i \otimes V_j){\uparrow}^{D_{2q}} \simeq 2\left(\bigoplus_{n=0}^{w} \alpha_n V_{\beta_n}\right)\Bigg\uparrow^{D_{2q}} \simeq \bigoplus_{n=0}^{w} 2\alpha_n M_{2\beta_n}\,.$$

Similarly, for $V_{2i}, V_{2j} \in \operatorname{mod} kC_{2q}$,

$$V_{2i} \otimes V_{2j} \simeq V_i{\uparrow}_{C_q}^{C_{2q}} \otimes V_j{\uparrow}_{C_q}^{C_{2q}} \simeq 2\left(\bigoplus_{n=0}^{w} \alpha_n V_{\beta_n}\right)\Bigg\uparrow_{C_q}^{C_{2q}} \simeq \bigoplus_{n=0}^{w} 2\alpha_n V_{2\beta_n}\,.$$

It follows that the decompositions of tensor products $V_{2i} \otimes V_{2j}$ and $M_{2i} \otimes M_{2j}$ are governed by the same formula. This proves the following result.

Corollary 6. *For any even numbers $i, j \in \mathbb{N}$, the decomposition formula*

$$M_i \otimes M_j \simeq \bigoplus_{n=0}^{w} \alpha_n V_{\beta_n}$$

holds, with the numbers α_n and β_n defined by Equations (2.5) and (2.6) respectively.

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QUOTIENTS OF EXACT CATEGORIES BY CLUSTER TILTING SUBCATEGORIES AS MODULE CATEGORIES

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Abstract. We prove that some subquotient categories of exact categories are abelian. This generalizes a result by Koenig-Zhu in the case of (algebraic) triangulated categories. As a particular case, if an exact category $\mathcal{B}$ with enough projectives and injectives has a cluster tilting subcategory $\mathcal{M}$, then $\mathcal{B}/[\mathcal{M}]$ is abelian. More precisely, it is equivalent to the category of finitely presented modules over $\underline{\mathcal{M}}$.

1. Introduction

Recently, cluster tilting theory (see for example [1, 3, 6]) permitted to construct abelian categories from some triangulated categories. In this survey we sketch out the method we introduced in [2] to generalize this observation to exact categories.

Recall that an exact category is Frobenius if it has enough projectives and injectives and they coincide. From Happel [4, Theorem 2.6], the stable category of a Frobenius category has a structure of a triangulated category. On the other hand, by Keller-Reiten [7, Proposition 2.1], in the 2-Calabi-Yau case and then Koenig-Zhu [8, Theorem 3.3] in the general case, one can pass from triangulated categories to abelian categories by factoring out any cluster tilting subcategory. Combining these two results, we deduce that the quotient of a Frobenius category by a cluster tilting subcategory is abelian. Thus, this observation gives rise to a natural question: is the quotient of an exact category by a cluster tilting subcategory abelian? As we will see, it turns out to be true.

This new result seems *a priori* less surprising than the one in triangulated categories because these ones are intuitively further to abelian categories. Nevertheless, most triangulated categories appearing in representation theory turn out to be in fact *algebraic* (*i.e.* stable categories of Frobenius categories). In this respect, the case of exact categories can be seen as a generalization of the result concerning triangulated categories, as well as a more natural version.

2. Notations

Let $\mathcal{B}$ be a Krull-Schmidt exact category with enough projectives and injectives and $\mathcal{M}$ be a full *rigid* subcategory of $\mathcal{B}$ (*i.e.* $\mathrm{Ext}^1_{\mathcal{B}}(X,X) = 0$ for any $X \in \mathcal{M}$).

Denote by $\mathcal{P}$ (resp. $\mathcal{I}$) the subcategory of projective (resp. injective) objects in $\mathcal{B}$. For any object $X, Y \in \mathcal{B}$ and a full subcategory $\mathcal{C}$ of $\mathcal{B}$, denote by $[\mathcal{C}](X,Y)$ the set of morphisms in $\mathrm{Hom}_{\mathcal{B}}(X,Y)$ which factor through objects of $\mathcal{C}$. If $\mathcal{P} \subseteq \mathcal{C}$ (resp. $\mathcal{I} \subseteq \mathcal{C}$),

The detailed version [2] of this paper has been submitted for publication.

the *(co-)stable category* $\underline{\mathcal{C}}$ (resp. $\overline{\mathcal{C}}$) of $\mathcal{C}$ is the quotient category $\mathcal{C}/[\mathcal{P}]$ (resp. $\mathcal{C}/[\mathcal{I}]$), *i.e.* the category which has the same objects than $\mathcal{C}$ and morphisms are defined as

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) := \mathrm{Hom}_{\mathcal{C}}(X,Y)/[\mathcal{P}](X,Y)$$
$$(\text{resp. } \overline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) := \mathrm{Hom}_{\mathcal{C}}(X,Y)/[\mathcal{I}](X,Y)).$$

Denote by Mod$\mathcal{C}$ the category of contravariant additive functors from $\mathcal{C}$ to mod$k$ for any category $\mathcal{C}$ where k is a field. Let mod$\mathcal{C}$ be the full subcategory of Mod$\mathcal{C}$ consisting of objects A admitting an exact sequence:

$$\mathrm{Hom}_{\mathcal{C}}(-,C_1) \xrightarrow{\beta} \mathrm{Hom}_{\mathcal{C}}(-,C_0) \xrightarrow{\alpha} A \to 0$$

where $C_0, C_1 \in \mathcal{C}$.

Denote by $\overline{\Omega}\mathcal{M}$ the class of objects $X \in \mathcal{B}$ such that there exists a short exact sequence

$$0 \to M \to I \to X \to 0$$

where $M \in \mathcal{M}$, and I is injective.

Denote by $\mathcal{M}_L$ (resp. $\mathcal{M}_R$) the subcategory of objects X which admit short exact sequences

$$0 \to X \xrightarrow{d^0} M^0 \xrightarrow{d^1} M^1 \to 0 \quad (\text{resp. } 0 \to M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} X \to 0)$$

with $M^0, M^1, M_0, M_1 \in \mathcal{M}$. In this case, d^0 (resp. d_0) is a left (resp. right) $\mathcal{M}$-approximation of X.

3. Two quotient category: $\mathcal{M}_L/[\mathcal{M}]$ and $\mathcal{M}_R/[\overline{\Omega}\mathcal{M}]$

3.1. **Quotient category of $\mathcal{M}_L/[\mathcal{M}]$ by a rigid subcategory $\mathcal{M}$.** In this subsection, we assume that $\mathcal{M}$ is a rigid subcategory of $\mathcal{B}$ which contains $\mathcal{P}$. Now we consider the functor

$$\begin{aligned} H:\ \mathcal{M}_L &\to \mathrm{Mod}\underline{\mathcal{M}} \\ X &\mapsto \mathrm{Ext}^1_{\mathcal{B}}(-,X)|_{\mathcal{M}} \end{aligned}$$

Let $\pi : \mathcal{M}_L \to \mathcal{M}_L/[\mathcal{M}]$ be the projection functor. By definition of a rigid subcategory, $HX = 0$ if $X \in \mathcal{M}$. Hence, by the universal property of π, there exists a functor $F : \mathcal{M}_L/[\mathcal{M}] \to \mathrm{Mod}\underline{\mathcal{M}}$ such that $F\pi = H$. From the following lemma we can see directly that $F(X) \in \mathrm{mod}\underline{\mathcal{M}}$:

Lemma 1. *For any short exact sequence*

$$0 \to X \xrightarrow{d^0} M^0 \xrightarrow{d^1} M^1 \to 0$$

where $M^0, M^1 \in \mathcal{M}$, there is an exact sequence in Mod$\underline{\mathcal{M}}$

$$\underline{\mathrm{Hom}}_{\mathcal{M}}(-,M^0) \to \underline{\mathrm{Hom}}_{\mathcal{M}}(-,M^1) \to FX \to 0.$$

The functor F induces the equivalence we want:

Theorem 2. *The functor $F : \mathcal{M}_L/[\mathcal{M}] \to \mathrm{mod}\underline{\mathcal{M}}$ is an equivalence of categories.*

Moreover, we have the following corollary:

Corollary 3. *If $\mathcal{M}$ is rigid and contravariantly finite, then $\mathcal{M}_L/[\mathcal{M}]$ is abelian.*

3.2. **Quotient category of $\mathcal{M}_R$ by $\overline{\Omega}\mathcal{M}$.** In this subsection we assume that $\mathcal{M}$ is a rigid subcategory of $\mathcal{B}$ which contains $\mathcal{I}$.

We denote

$$\begin{aligned} K : \mathcal{M}_R &\to \mathrm{Mod}\overline{\mathcal{M}} \\ X &\mapsto \overline{\mathrm{Hom}}_{\mathcal{B}}(-, X). \end{aligned}$$

Let $\pi' : \mathcal{M}_R \to \mathcal{M}_R/[\overline{\Omega}\mathcal{M}]$ be the projection functor. By the universal property of π', there is a functor $G : \mathcal{M}_R/[\overline{\Omega}\mathcal{M}] \to \mathrm{Mod}\overline{\mathcal{M}}$ such that $G\pi' = K$. From the lemma we can see that $GX \in \mathrm{mod}\overline{\mathcal{M}}$:

Lemma 4. *For every short exact sequence*

$$0 \to M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} X \to 0$$

where $M_1, M_0 \in \mathcal{M}$, there is an exact sequence

$$\overline{\mathrm{Hom}}_{\mathcal{M}}(-, M_1) \to \overline{\mathrm{Hom}}_{\mathcal{M}}(-, M_0) \to GX \to 0.$$

The functor G also gives an equivalence:

Theorem 5. *The functor $G : \mathcal{M}_R/[\overline{\Omega}\mathcal{M}] \to \mathrm{mod}\overline{\mathcal{M}}$ is an equivalence of categories.*

If we denote $\overline{\mathcal{M}}^{\perp} = \{X \in \mathcal{M}_R \mid \overline{\mathrm{Hom}}_{\mathcal{B}}(\mathcal{M}, X) = 0\}$, we get the following corollary:

Corollary 6. *We have $\overline{\Omega}\mathcal{M} = \overline{\mathcal{M}}^{\perp}$.*

4. Case of n-cluster tilting subcategories and AR translation

For a subcategory $\mathcal{C}$ of $\mathcal{B}$, we define

$$\begin{aligned} {}^{\perp_m}\mathcal{C} &= \{X \in \mathcal{B} \,|\, \forall i \in \{1, \dots, m\}, \mathrm{Ext}^i_{\mathcal{B}}(X, \mathcal{C}) = 0\} \\ \text{and} \quad \mathcal{C}^{\perp_m} &= \{X \in \mathcal{B} \,|\, \forall i \in \{1, \dots, m\}, \mathrm{Ext}^i_{\mathcal{B}}(\mathcal{C}, X) = 0\}. \end{aligned}$$

Recall that $\mathcal{M}$ is called *n-cluster tilting*, if it satisfies the following conditions:

(1) $\mathcal{M}$ is contravariantly finite and covariantly finite in $\mathcal{B}$,
(2) $\mathcal{M} = \mathcal{M}^{\perp_{n-1}}$,
(3) $\mathcal{M} = {}^{\perp_{n-1}}\mathcal{M}$.

The previous results concern categories $\mathcal{M}_L$ and $\mathcal{M}_R$ which have not good properties in general. From now on, we suppose that $\mathcal{M}$ is n-cluster tilting for some integer $n \geq 2$ (see [5, 6]). Thus, the properties of $\mathcal{M}_L$ and $\mathcal{M}_R$ becomes much clearer:

Proposition 7. *The following equalities hold:*

$${}^{\perp_{n-2}}\mathcal{M} = \mathcal{M}_L \quad \textit{and} \quad \mathcal{M}^{\perp_{n-2}} = \mathcal{M}_R.$$

By this proposition, we obtain that both $\mathcal{M}_L$ and $\mathcal{M}_R$ are exact subcategories of $\mathcal{B}$. In particular we get

Corollary 8. *If $\mathcal{M}$ is 2-cluster tilting then $\mathcal{B}/[\mathcal{M}] \simeq \mathrm{mod}\underline{\mathcal{M}}$ is abelian.*

Now, we assume that $\mathcal{B}$ has an AR translation $\tau : \underline{B} \to \overline{B}$ with reciprocal τ^-. Following [5], we define $(n-1)$-AR translations

$$\tau_{n-1} : \underline{{}^{\perp_{n-2}}\mathcal{P}} \to \overline{\mathcal{I}^{\perp_{n-2}}} \quad \text{and} \quad \tau_{n-1}^- : \overline{\mathcal{I}^{\perp_{n-2}}} \to \underline{{}^{\perp_{n-2}}\mathcal{P}}$$

by $\tau_{n-1} = \tau\Omega^{n-2}$ and $\tau_{n-1}^- = \tau\overline{\Omega}^{n-2}$ (where Ω is the syzygy functor). In fact, the only property we need for these functors is that, if $X \in {}^{\perp_{n-2}}\mathcal{P}$ and $Y \in \mathcal{I}^{\perp_{n-2}}$, the following functorial isomorphisms hold:

(1) $\mathrm{Ext}_{\mathcal{B}}^{n-1}(X,Y) \simeq \mathrm{D}\overline{\mathrm{Hom}}_{\mathcal{B}}(Y, \tau_{n-1}X) \simeq \mathrm{D}\underline{\mathrm{Hom}}_{\mathcal{B}}(\tau_{n-1}^- Y, X)$,

(2) $\forall i \in \{1, 2, ..., n-2\}$,
$\mathrm{Ext}_{\mathcal{B}}^{n-1-i}(X,Y) \simeq \mathrm{D}\mathrm{Ext}_{\mathcal{B}}^{i}(Y, \tau_{n-1}X) \simeq \mathrm{D}\mathrm{Ext}_{\mathcal{B}}^{i}(\tau_{n-1}^- Y, X)$

where $D = \mathrm{HomExt}_k(-,k)$. This is a weak version of [5, Theorem 1.5].

From this, we deduce easily that τ_{n-1} induces an equivalence from $\underline{{}^{\perp_{n-2}}\mathcal{M}}$ to $\overline{\mathcal{M}^{\perp_{n-2}}}$ the inverse of which is $\tau_{n-1}^{-1} = \tau_{n-1}^-$.

Remark that

$$\begin{aligned} X \in \mathcal{M} &\Leftrightarrow \mathrm{Ext}_{\mathcal{B}}^{i}(X, \mathcal{M}) = 0,\ \forall i \in \{1, 2, ..., n-1\} \\ &\Leftrightarrow \begin{cases} \overline{\mathrm{Hom}}_{\mathcal{B}}(\mathcal{M}, \tau_{n-1}X) = 0 \\ \mathrm{Ext}_{\mathcal{B}}^{i}(\mathcal{M}, \tau_{n-1}X) = 0 \quad \text{for all } i \in \{1, 2, ..., n-2\} \end{cases} \\ &\Leftrightarrow \tau_{n-1}X \in \mathcal{M}^{\perp_{n-2}} \cap \overline{\mathcal{M}}^{\perp}. \end{aligned}$$

Moreover, as $\overline{\mathcal{M}}^{\perp} = \overline{\Omega}\mathcal{M} \subseteq \mathcal{M}^{\perp_{n-2}}$, $X \in \mathcal{M} \Leftrightarrow \tau_{n-1}X \in \overline{\mathcal{M}}$.

Now $X \in \mathcal{P}$ implies that $\mathrm{Ext}_{\mathcal{B}}^{n-1}(X, \mathcal{B}) = 0$, then $\overline{\mathrm{Hom}}_{\mathcal{B}}(\mathcal{B}, \tau_{n-1}X) = 0$, which means $\tau_{n-1}X \in \mathcal{I}$. Dually $X \in \mathcal{I}$ implies that $\tau_{n-1}^{-1}X \in \mathcal{P}$. Hence $X \in \mathcal{P} \Leftrightarrow \tau_{n-1}X \in \mathcal{I}$. We get the following proposition:

Proposition 9. *The functor τ_{n-1} induces an equivalence from $\underline{\mathcal{M}}$ to $\overline{\overline{\Omega}\mathcal{M}}$ and an equivalence from ${}^{\perp_{n-2}}\mathcal{M}/[\mathcal{M}]$ to $\mathcal{M}^{\perp_{n-2}}/[\overline{\Omega}/\mathcal{M}]$.*

Denote by $\overline{\Omega}^{-1}$ the inverse of $\overline{\Omega} : \overline{\mathcal{M}} \to \overline{\overline{\Omega}\mathcal{M}}$. Then we have

Corollary 10. *The compositions $\tau_{n-1}^{-1} \circ \overline{\Omega}$ and $\overline{\Omega}^{-1} \circ \tau_{n-1}$ induce mutually inverse equivalences between $\overline{\mathcal{M}}$ and $\underline{\mathcal{M}}$.*

According to this corollary, we can define reciprocal equivalences:

(1) $\mu : \mathrm{Mod}\underline{\mathcal{M}} \to \mathrm{Mod}\overline{\mathcal{M}}$, $\mu(C) = C \circ \tau_{n-1}^{-1} \circ \overline{\Omega}$,

(2) $\mu^{-1} : \mathrm{Mod}\overline{\mathcal{M}} \to \mathrm{Mod}\underline{\mathcal{M}}$, $\mu^{-1}(C') = C' \circ \overline{\Omega}^{-1} \circ \tau_{n-1}$.

Thus we have:

Proposition 11. *The functors μ and μ^{-1} induce mutually inverse equivalences between* $\mathrm{mod}\underline{\mathcal{M}}$ *and* $\mathrm{mod}\overline{\mathcal{M}}$.

Finally we give:

Theorem 12. *If $\mathcal{B}$ has an $(n-1)$–AR translation τ_{n-1}, then we have a diagram which is commutative up to the equivalence*

$$\begin{tikzcd} {}^{\perp_{n-2}}\mathcal{M}/[\mathcal{M}] \arrow[r, "F"] \arrow[d, "\tau_{n-1}"'] & \mathrm{mod}\underline{\mathcal{M}} \arrow[d, "\mu"] \\ \mathcal{M}^{\perp_{n-2}}/[\overline{\Omega}\mathcal{M}] \arrow[r, "G"'] & \mathrm{mod}\overline{\mathcal{M}}. \end{tikzcd}$$

By duality, if we denote by $\mathrm{mod}'\underline{\mathcal{M}}$ (resp. $\mathrm{mod}'\overline{\mathcal{M}}$) the category of finitely copresented modules over $\underline{\mathcal{M}}$ (resp. $\overline{\mathcal{M}}$), we get the following commutative diagram:

$$\begin{tikzcd} {}^{\perp_{n-2}}\mathcal{M}/[\Omega\mathcal{M}] \arrow[r, "\sim"] \arrow[d, "\tau_{n-1}"'] & \mathrm{mod}'\underline{\mathcal{M}} \arrow[d, "\wr"] \\ \mathcal{M}^{\perp_{n-2}}/[\mathcal{M}] \arrow[r, "\sim"] & \mathrm{mod}'\overline{\mathcal{M}} \end{tikzcd}$$

where $\Omega\mathcal{M}$ the class of objects $X \in \mathcal{B}$ such that there exists a short exact sequence

$$0 \to X \to P \to M \to 0$$

with $M \in \mathcal{M}$ and P projective.

5. Example

In this section, we explain an example coming directly from representation theory (Auslander algebras).

Let Λ be the Auslander algebra of $k\vec{A}_3$. That is kQ/R where Q is the following quiver

$$\begin{tikzcd} & & 3 \arrow[dl] & & \\ & 5 \arrow[dl] \arrow[rr, dotted, no head] & & 2 \arrow[ul] \arrow[dl] & \\ 6 \arrow[rr, dotted, no head] & & 4 \arrow[ul] \arrow[rr, dotted, no head] & & 1 \arrow[ul] \end{tikzcd}$$

and the ideal of relations R is generated by the mesh relations symbolized by dashed lines. Then, using the method introduced in [6, §1], one can compute a cluster tilting subcategory $\mathcal{M}$ of $\mathrm{mod}\Lambda$, and the quiver of $\mathcal{M}$ is given in Figure 1.

We can also calculate $\overline{\Omega}\mathcal{M}$ easily since in this case

$$\overline{\Omega}\mathcal{M} = \overline{\mathcal{M}}^{\perp} = \{X \in \mathrm{mod}\Lambda \mid \overline{\mathrm{Hom}}_{\Lambda}(\mathcal{M}, X) = 0\}.$$

In this example, the quiver of $\mathrm{mod}\Lambda/[\mathcal{M}]$ is the following.

$$\begin{tikzcd} & \begin{smallmatrix} & 3 \\ 5 & \end{smallmatrix} \arrow[dr] & & & & \begin{smallmatrix} 2 & \\ & 3 \end{smallmatrix} \arrow[dr] & \\ 5 \arrow[ur] \arrow[rr, dotted, no head] & & \begin{smallmatrix} 4 & & 3 \\ & 5 & \end{smallmatrix} \arrow[dr] \arrow[rr, dotted, no head] & & \begin{smallmatrix} & 2 & \\ 4 & & 3 \end{smallmatrix} \arrow[ur] \arrow[rr, dotted, no head] & & 2 \\ & & & 3 \arrow[ur] & & & \end{tikzcd}$$

The quiver of $\underline{\mathcal{M}}$ is the following.

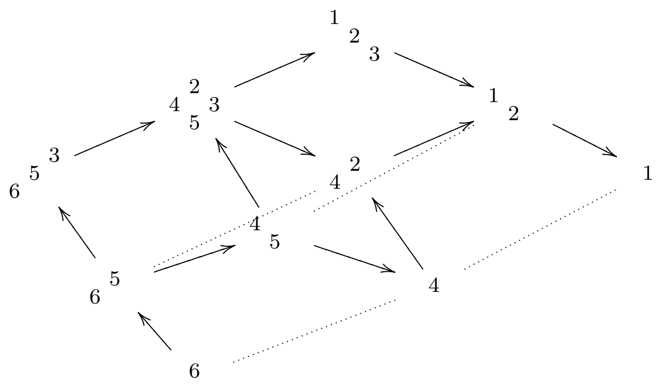
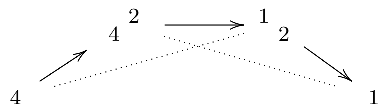


Figure 1. Quiver of $\mathcal{M}$



As expected, we obtain that $\text{mod}\Lambda/[\mathcal{M}] \simeq \text{mod}\underline{\mathcal{M}}$. One can also calculate and check the equivalence $\text{mod}\Lambda/[\overline{\mathcal{M}}^{\perp}] \simeq \text{mod}\overline{\mathcal{M}}$.

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WEAKLY SECTIONAL PATHS AND BYPASSES IN THE AUSLANDER-REITEN QUIVER

TAKAHIKO FURUYA

Abstract. We show that if a weakly sectional path in the Auslander-Reiten quiver of an artin algebra is a bypass, then it is precisely a sectional path.

1. 序 (準備)

本論文を通じて、K を可換アルティン環とし、A を K 上のアルティン多元環とする ([1])。$\operatorname{mod} A$ で有限生成右 A-加群の成す圏を表し、$\tau = D\mathrm{Tr}$ および $\tau^- = \mathrm{Tr}D$ で $\operatorname{mod} A$ におけるアウスランダー・ライテン移動を表す。また Γ_A で A のアウスランダー・ライテンクイバーを表す。

以降、任意の直既約加群 X $(\in \operatorname{mod} A)$ に対して、X を含む同型類を再び X で記す。$\Omega = X_n \to X_{n-1} \to \cdots \to X_0$ $(n \geq 1)$ を Γ_A における道とする。このとき整数 i $(1 \leq i \leq n-1)$ が Ω のフックであるとは、$\tau X_{i-1} = X_{i+1}$ となるときを言う。また Ω が sectional path であるとは、Ω がフックを持たないとき、つまり、任意の j $(1 \leq j \leq n-1)$ に対して $\tau X_{j-1} \neq X_{j+1}$ となるときを言う。さらに Ω が pre-sectional path であるとは、i $(1 \leq i \leq n-1)$ がフックならば $\tau X_{i-1} = X_{i+1}$ であるときを言う ([7])。明らかに sectional path は pre-sectional path である。

本論文の目的は、以下に述べる bypass の性質を調べる事である。

Definition 1 ([2, 3])**.** $X \to Y$ を Γ_A における矢とし、$n \geq 2$ を整数とする。このとき Γ_A の道 $X = X_0 \to X_1 \to \cdots \to X_n = Y$ が矢 $X \to Y$ の bypass であるとは、$X_1 \neq Y$ かつ $X_{n-1} \neq X$ であるときを言う。また、bypass が sectional path であるとき、その bypass を sectional bypass と呼ぶ。

Remark 2. bypass は [3] で最初に導入された道であるが、文献 [2] にある定義と [3] にある定義はわずかに異なる。本論文では [2] における定義を採用している。

Remark 3. 以下の事が示されている:

(1) Γ_A の oriented cycle を含まない成分における矢の bypass は sectional bypass である ([3])。
(2) A が有限表現型のとき、Γ_A は sectional bypass を持たない ([3])。
(3) Γ_A が sectional bypass を持つとき、Γ_A の sectional bypass を持つ左または右安定成分が存在する ([2])。

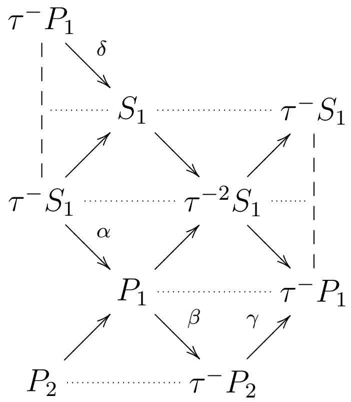
次に、sectional ではない bypass および sectional bypass の例をそれぞれ挙げておく:

Example 4. (1) K を代数閉体とし、Γ を次のクイバーとする:

$$\begin{tikzcd} 1 \arrow[loop left, "a"] \arrow[r, "b"] & 2 \end{tikzcd}$$

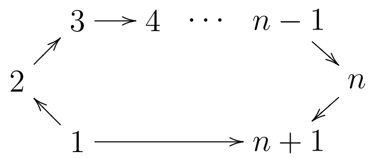
The detailed version of this paper will be submitted for publication elsewhere.

$I = \langle a^2 \rangle$ を道多元環 $K\varGamma$ のイデアルとする。$A := K\varGamma/I$ と置く。そうすると A は有限表現型であり、$\varGamma_A$ は oriented cycle を持つ次の translation クイバーである ([2, 3])。

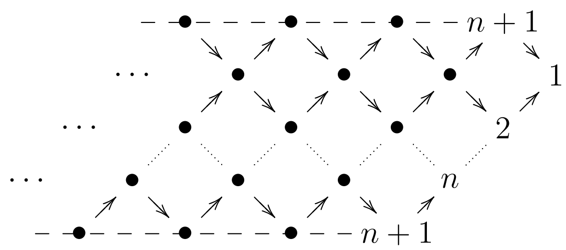


このとき、道 $\alpha\beta\gamma\delta$ は矢 $\tau^- S_1 \to S_1$ の sectional ではない bypass である。

(2) K を代数閉体とし、$n \geq 2$ を整数とする。$\varGamma$ を次の $\widetilde{\mathbb{A}}_n$ 型のクイバーとする:



$A := K\varGamma$ とする。そうすると、$\varGamma_A$ の前入射成分は次のような左安定成分となる。



この成分には、無限に sectional bypass が存在している。例えば道 $n+1 \to n \to \cdots \to 2 \to 1$ は矢 $n+1 \to 1$ の sectional bypass である。

上記の Remark 3 で述べたように、すでに sectional bypass に関するいくつかの事実が示されている。ここでは、bypass が sectional path の一般化である weakly sectional path ([4]) である場合を考察する。

2. Weakly sectional bypass

主結果を述べる前に、weakly sectional path の定義を述べておく。$\varGamma_A$ の矢 $X \to Y$ に対して、その付値を (d_{XY}, d'_{XY}) で表す。(つまり、d_{XY} は Y に対する右概分裂写像の定義域を直既分解したときに現れる X の個数、d'_{XY} は X に対する左概分裂写像の値域を直既分解したときに現れる Y の個数。)$\varGamma_A$ の道 $\varOmega = X_n \to X_{n-1} \to \cdots \to X_0$ $(n \geq 1)$ に対して、集合 $J_\varOmega$ を

$$J_\varOmega := \{1 \leq j \leq n-1 \mid j \text{ は } \varOmega \text{ のフックで、} d_{X_{j+1}X_j} = 1 \text{ を満たす}\}$$

で定める。明らかに、Ω が pre-sectional path になる必要十分条件は $J_\Omega = \emptyset$ である。

Definition 5 ([6])**.** n を正の整数とし、$\Omega = X_n \to X_{n-1} \to \cdots \to X_0$ を Γ_A の道とする。Ω が weakly sectional path とは、ある $\operatorname{mod} A$ における直既約加群の集合 $\{M_j\}_{j \in J_\Omega}$ が存在して、次の条件が成立するときを言う。

(1) $j-2 \notin J_\Omega$ である任意の $j \in J_\Omega$ に対して、$X_j \oplus M_j \oplus \tau X_{j-2}$ は X_{j-1} の右概分裂写像の定義域における直和因子。(ここで $1 \in J_\Omega$ のとき、τX_{-1} を $\operatorname{mod} A$ における直既約加群とする。)

(2) $j-2 \in J_\Omega$ である任意の $j \in J_\Omega$ に対して、$X_j \oplus M_j \oplus \tau X_{j-2} \oplus \tau M_{j-2}$ は X_{j-1} の右概分裂写像の定義域における直和因子。

(3) $j-2 \in J_\Omega$ である任意の $0 \leq j \leq n$ に対して、$X_j \oplus \tau X_{j-2} \oplus \tau M_{j-2}$ は X_{j-1} の右概分裂写像の定義域における直和因子。

Remark 6. (1) 明らかに pre-sectional path は weakly sectional path である。

(2) [4, 6] において、weakly sectional path の性質がいくつか述べられているが、特に任意の weakly sectional path は oriented cycle ではないことが示されている。

(3) [4, 6] では無限の長さの weakly sectional path が定義されている。また、上記の定義における集合 $\{M_j\}_{j \in J_\Omega}$ を Ω の support と呼んでいる。

本論文の主結果は次の通りである:

Theorem 7 ([5])**.** *weakly sectional path* が *bypass* のとき、それは *sectional path* である。(すなわち *weakly sectional bypass* は *sectional path* である。)

pre-sectional path は weakly sectional path なので、直ちに次を得る:

Corollary 8. *pre-sectional path* が *bypass* のとき、それは *sectional path* である。

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SKEW REES RINGS WHICH ARE MAXIMAL ORDERS

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ABSTRACT. Let R be a Noetherian prime Goldie ring, σ be an automorphism of R and X be an invertible ideal of R. In this paper, we define the $(\sigma;X)-$maximal order and show that a skew Rees ring $R[Xt;\sigma]$ is a maximal order if and only if R is a $(\sigma;X)-$maximal order, which is proved by using the complete description of $v-$ideals of $R[Xt;\sigma]$. We give some examples of $(\sigma;X)-$maximal orders which are not maximal orders (event not $\sigma-$maximal orders) and also of $\sigma-$maximal orders but not $(\sigma;X)-$maximal orders.

1. INTRODUCTION

Throughout this paper, R is a Noetherian prime ring with quotient ring Q (in another word, R is a Noetherian order in a simple Artinian ring Q), σ is an automorphism of R and X is an invertible ideal of R.

Put

$$S = R[Xt,\sigma] = R \oplus Xt \oplus X^2t^2 \oplus \dots . \oplus X^nt^n \oplus \dots$$

which is a subset of the skew polynomial ring $R[t,\sigma]$ in an indeterminate t. If S is a ring, then it is called a *skew Rees ring* associated to X. In this case, S and $R[t;\sigma]$ have the same quotient ring $Q(S) = Q(R[t;\sigma])$ which is a simple Artinian ring.

The aim of this paper is to obtain a necessary and sufficient conditions for S to be a maximal order and to describe the structure of $v-$ideals of S (Theorem 9 and Proposition 11). As applications, we give a necessary and sufficient conditions for S to be a generalized Asano ring and a unique factorization ring in the sense of [1], respectively (Corollary 12). These are done by using a complete description of $v-$ideals in $Q(S)$.

Furthermore we give some examples of rings which are $(\sigma;X)-$maximal orders but not maximal orders (even not $\sigma-$maximal orders). This means S is a maximal order but $R[t;\sigma]$ is not a maximal order. We also give examples of rings which are $\sigma-$maximal orders but not $(\sigma;X)-$maximal orders.

Generalized Rees rings were studied in [8] and [15] under *PI* conditions and in the book [16], they summarized them from torsion theoretical view points under *PI* conditions. Recently Akalan proved in [2] that if R is generalized Asano ring with *PI* conditions, then so is S, which motivates us to study skew Rees rings. Note we do not assume in this paper that R satisfies *PI* conditions.

In [2] Akalan defined generalized Dedekind prime ring R. It turns out that R is a generalized Dedekind ring if and only if it is a maximal order and any $v-$ideal is invertible. In this paper, we say that R is a *generalized Asano ring* if it is a generalized Dedekind ring in the sense of [2], because one-sided $v-$ideals are not necessarily projective.

We refer the readers to the books [12] or [13] for order theory.

The detailed version of this paper will be submitted for publication elsewhere.

2. $(\sigma;X)-$MAXIMAL ORDERS

First we introduce some notation. For any (fractional) right $R-$ideal I and left $R-$ideal J, let

$$(R:I)_l = \{q \in Q \mid qI \subseteq R\} \text{ and } (R:J)_r = \{q \in Q \mid Jq \subseteq R\}$$

which is a left (right) $R-$ideal, respectively and

$$I_v = (R:(R:I)_l)_r \quad \text{and} \quad {}_vJ = (R:(R:J)_r)_l,$$

which is a right (left) $R-$ideal containing $I(J)$. $I(J)$ is called a *right (left) $v-$ideal* if $I_v = I$ $({}_vJ = J)$. In case I is a two-sided $R-$ideal, it is said to be a *$v-$ideal* if $I_v = I = {}_vI$, and if $I \subseteq R$, we just say I is a *$v-$ideal* of R. An $R-$ideal A is said to be *$v-$invertible* if ${}_v((R:A)_lA) = R = (A(R:A)_r)_v$. We start with the following elementary lemma, which is frequently used in the paper.

Lemma 1. *Let A be an $R-$ideal and I be a right $R-$ideal.*

(1) *If A is $v-$invertible, then $O_r(A) = R = O_l(A)$ and $(R:A)_l = A^{-1} = (R:A)_r$, where $A^{-1} = \{q \in Q \mid AqA \subseteq A\}$.*
(2) *$(IA_v)_v = (IA)_v$. If A is $v-$invertible, then $(I_vA_v)_v = (IA)_v$.*

The following proposition is one of the crucial properties which shows a relation between ideals of R and of S.

Proposition 2. (1) *$S = R[Xt;\sigma]$ is a ring if and only if $\sigma(X) = X$. In this case, S is also Noetherian.*
(2) *Suppose $\sigma(X) = X$.*
  (i) *Let $\mathfrak{a}$ be an deal of R. Then*

$$\mathfrak{a}[Xt;\sigma] = \mathfrak{a} \oplus \mathfrak{a}Xt \oplus \mathfrak{a}X^2t^2 \oplus \ldots \oplus \mathfrak{a}X^nt^n \oplus \ldots$$

  is an ideal of S if and only if $X\sigma(\mathfrak{a}) = \mathfrak{a}X$.
  (ii) *Let $\mathfrak{a}$ be an $R-$ideal in Q with $X\sigma(\mathfrak{a}) = \mathfrak{a}X$. Then $\mathfrak{a}[Xt;\sigma]$ is an $S-$ideal in $Q(S)$.*

In the remainder of this paper, we assume that $S = R[Xt;\sigma]$ is a ring and put $T = Q[t;\sigma]$, the skew polynomial ring over Q. Note that T is a principal ideal ring ([3, Corollary 6.2.2] or [12, Corollary 2.3.7]) and we use this property to study $S-$ideal.

Lemma 3. *Let I be a right $S-$ideal and J be a left $S-$ideal. Then*

(1) *$(T:IT)_l = T(S:I)_l$ and $(T:TJ)_r = (S:J)_rT$.*
(2) *$(IT)_v = I_vT$ and ${}_v(TJ) = T_vJ$.*
(3) *If I' is a right ideal of T, then $I' = (I' \cap S)T$. If I' is an essential right ideal, then $(I' \cap S)_v = I' \cap S$.*

It is very important to investigate prime v-ideals P of S and there are two case whether $P \cap R$ is (0) or not. In case $P \cap R = (0)$, we have the following by using Lemma 3.

Lemma 4. *Let $T = Q[t;\sigma]$. There is a $(1-1)-$correspondence between*

$$\mathrm{Spec}_0(S) = \{P : \text{ prime ideal of } S \mid P \cap R = (0)\} \quad \textit{and} \quad \mathrm{Spec}(T)$$

via $P \longmapsto PT$, $P' \longmapsto P' \cap S$. In particular, P is a $v-$ideal.

To express the case $P\cap R\neq(0)$, we need some preliminaries. Let $\mathfrak{a}$ be a right $R-$ideal. Then $\mathfrak{a}[Xt;\sigma]=\mathfrak{a}\oplus\mathfrak{a}Xt\oplus\ldots\oplus\mathfrak{a}X^nt^n\oplus\ldots$ is a right $S-$ideal. Similarly for any left $R-$ideal $\mathfrak{b}$, $S\mathfrak{b}=\mathfrak{b}\oplus tX\mathfrak{b}\oplus\ldots\oplus t^nX^n\mathfrak{b}\oplus\ldots$ is a left $S-$ideal.

Lemma 5. *Let $\mathfrak{a}$ be a right $R-$ideal and $\mathfrak{b}$ be a left $R-$ideal. Then*

$$(S:\mathfrak{a}[Xt;\sigma])_l=S(R:\mathfrak{a})_l \text{ and } (S:S\mathfrak{b})_r=(R:\mathfrak{b})_rS$$

In particular, $(\mathfrak{a}[Xt;\sigma])_v=\mathfrak{a}_v[Xt;\sigma]$ *and* ${}_v(S\mathfrak{b})=S_v\mathfrak{b}$.

It is well known that σ is naturally extended to an automorphism of $Q(R[t;\sigma])$ by $\sigma(f(t))=tf(t)t^{-1}$ for any $f(t)\in R[t;\sigma]$. Note that σ induces an automorphism of S. Let $\mathfrak{a}$ be an ideal of R. We showed in Proposition 2 that $\mathfrak{a}[Xt;\sigma]$ is an ideal of S if and only if $X\sigma(\mathfrak{a})=\mathfrak{a}X$ which is crucial property for S to be a maximal order. In general, a subset I of $Q(S)$ is said to be $(\sigma;X)-$*invariant* if $X\sigma(I)=IX$.

R is said to be a $(\sigma;X)-$*maximal order* if $O_l(\mathfrak{a})=R=O_r(\mathfrak{a})$ for any $(\sigma;X)-$invariant ideal of R. If R is a $(\sigma;X)-$maximal order, then it is proved that $O_l(\mathfrak{a})=R=O_r(\mathfrak{a})$ for any $(\sigma;X)-$invariant $R-$ideal $\mathfrak{a}$. Hence $(R:\mathfrak{a})_l=\mathfrak{a}^{-1}=(R:\mathfrak{a})_r$ where $\mathfrak{a}^{-1}=\{q\in Q \mid \mathfrak{a}q\mathfrak{a}\subseteq\mathfrak{a}\}$ and $\mathfrak{a}_v=\mathfrak{a}^{-1-1}={}_v\mathfrak{a}$ follows.

Let $D_{\sigma,X}(R)$ be the set of all $(\sigma;X)-$invariant $v-$ideals. For any $\mathfrak{a},\mathfrak{b}\in D_{\sigma,X}(R)$, we define $\mathfrak{a}\circ\mathfrak{b}=(\mathfrak{a}\mathfrak{b})_v$. Then we have the following whose proof is similar to one in the maximal orders ([12, (2.1.2)]).

Proposition 6. *Let R be a $(\sigma;X)-$maximal order in Q. Then $D_{\sigma,X}(R)$ is an Abelian group generated by maximal $(\sigma;X)-$invariant $v-$ideals of R.*

The following lemmas show how to obtain prime ideals of S from ideals of R and how to connect ideals of S with ideals of R.

Lemma 7. *Suppose R is a $(\sigma;X)-$maximal order in Q. Let $\mathfrak{p}$ be a maximal $(\sigma;X)-$invariant $v-$ideal of R. Then $P=\mathfrak{p}[Xt;\sigma]$ is a prime ideal and it is a $v-$ideal.*

Lemma 8. *Suppose R is a $(\sigma;X)-$maximal order in Q. Let A be an ideal of S with $A=A_v$ and $\mathfrak{a}=A\cap R\neq(0)$. Then*

(1) *A and $\mathfrak{a}$ are $(\sigma;X)-$invariant.*
(2) *$A=\mathfrak{a}[Xt;\sigma]$ and is $v-$invertible.*

Theorem is proved by mainly using Lemmas 3 and 8.

Theorem 9. *Let R be a Noetherian prime ring with its quotient ring Q, σ be an automorphism of R and $S=R[Xt;\sigma]$ be a skew Rees ring associated to X, where X is an invertible ideal with $\sigma(X)=X$. Then R is a $(\sigma;X)-$maximal order if and only if $S=R[Xt;\sigma]$ is a maximal order in $Q(S)$.*

3. Applications, Examples and Conjectures

As applications of Theorem 9, we give a necessary and sufficient conditions for S to be a generalized Asano ring and a unique factorization ring (a UFR). Furthermore we give Noetherian prime rings which are $(\sigma;X)-$maximal orders (but not maximal orders) and $(\sigma;X)-$maximal orders (but not $\sigma-$maximal orders) where an order R is called a $\sigma-$*maximal order* if for any ideal $\mathfrak{a}$ with $\sigma(\mathfrak{a})=\mathfrak{a}$, $O_l(\mathfrak{a})=R=O_r(\mathfrak{a})$.

If R is a $(\sigma;X)-$maximal order, then S is a maximal order and so $D(S)$, the set of all $v-$ideals in $Q(S)$, is an Abelian group generated by prime $v-$ideals of S (see [12, Theorem 2.1.2]). Note that any maximal $v-$ideal of S is a prime $v-$ideal and the converse is also true. The set of principal $S-$ideals in $Q(S)$ is a subgroup $P(S)$ of $D(S)$. The factor group $D(S)/P(S)$ is called the *class group* of S and denoted by $C(S)$. Similarly $P_{\sigma,X}(R)$, the set of $(\sigma;X)-$invariant principal $R-$ideals in Q is a subgroup of $D_{\sigma,X}(R)$ and $C_{\sigma,X}(R)=D_{\sigma,X}(R)/P_{\sigma,X}(R)$ is called the $(\sigma;X)-$*class group* of R.

First we describe the structure of $v-$ideals in $Q(S)$ as follows (this is proved by using Lemma 8 and [12, (2.3.11)]):

Proposition 10. *Suppose R is a $(\sigma;X)-$maximal order and let A be a $v-$ideal in $Q(S)$. Then $A=t^n w\mathfrak{a}[Xt;\sigma]$ for some $\mathfrak{a}\in D_{\sigma,X}(R)$, $w\in Z(Q(T))$ the center of $Q(T)$ and n is an integer.*

The statement (1) of Proposition 11 follows from Lemmas 3 and 8. To prove the second statement, consider the mapping $\varphi: D_{\sigma,X}(R)\to D(S)$ given by $\varphi(\mathfrak{a})=\mathfrak{a}[Xt;\sigma]$ for any $\mathfrak{a}\in D_{\sigma,X}(R)$.

Proposition 11. *Suppose R is a $(\sigma;X)-$maximal order. Then*

(1) $D(S)\cong D_{\sigma,X}(R)\oplus D(T)$.

(2) $C(S)\cong C_{\sigma,X}(R)$.

An order R is called a *generalized Asano ring* (a *G-Asano ring*) if it is a maximal order and every $v-$ ideal of R is invertible. Similarly R is called a *generalized* $(\sigma;X)-$*Asano ring (a* $G-(\sigma;X)-$*Asano ring)* if it is a $(\sigma;X)-$maximal order and every $(\sigma;X)-$invariant $v-$ideals of R is invertible. If R is a $G-(\sigma;X)-$Asano ring, then S is a $G-$Asano ring by Proposition 10. The converse is also true which is proved by using Lemma 5.

In [1], they defined a non-commutative unique factorization ring (a UFR). It turns out that an order is a UFR if and only if it is a maximal order and every $v-$ideal is principal. We can define, in an obvious way, the concept of a $(\sigma;X)-$UFR and it follows from Proposition 11 that R is a $(\sigma;X)-$UFR if and only if $C_{\sigma,X}(R)=(0)$. Hence we have

Corollary 12. (1) *R is a $G-$ $(\sigma;X)-$Asano ring if and only if $S=R[Xt;\sigma]$ is a $G-$Asano ring.*

(2) *R is a $(\sigma;X)-$UFR if and only if S is a UFR.*

Now we give some examples of $(\sigma;X)-$maximal orders but not maximal orders (even not $\sigma-$maximal orders). We also give examples of $\sigma-$maximal orders but not $(\sigma;X)-$maximal orders. The first example is a trivial case.

Example 1. Any Noetherian maximal order R is a $(\sigma;X)-$maximal order and a $\sigma-$maximal order. Hence S and $R[t;\sigma]$ are maximal orders (Theorem 9 and [12, Theorem 2.3.19]).

Let R be an HNP ring satisfying the following conditions :

(a) There is a cycle $\mathfrak{m}_1,\mathfrak{m}_2,...,\mathfrak{m}_n$ $(n\geq 2)$ such that $\mathfrak{p}=\mathfrak{m}_1\cap\mathfrak{m}_2\cap...\cap\mathfrak{m}_n$ is principal, say $\mathfrak{p}=aR=Ra$ for some $a\in\mathfrak{p}$.

(b) Any maximal ideal different from $\mathfrak{m}_i(1\leq i\leq n)$ is invertible.

See [1] for examples of HNP rings satisfying conditions (a) and (b). Define an automorphism σ of R by $\sigma(r) = ara^{-1}$ for $r \in R$. Then it follows from [1] that

(1) $\sigma(\mathfrak{m}_1) = \mathfrak{m}_2, ..., \sigma(\mathfrak{m}_n) = \mathfrak{m}_1$ and
(2) $\sigma(\mathfrak{n}) = \mathfrak{n}$ for all maximal ideals $\mathfrak{n}$ with $\mathfrak{n} \neq \mathfrak{m}_i$ $(1 \leq i \leq n)$.

Example 2. Suppose R is an HNP ring with the conditions (a) and (b).

(1) Put $X = \mathfrak{n}_1^{e_1}...\mathfrak{n}_k^{e_k}$, where $\mathfrak{n}_j$ are maximal ideals different from $\mathfrak{m}_i$ $(1 \leq i \leq n)$. Then R is a $(\sigma;X)-$maximal order which is not a maximal order (in fact, it is a $G-(\sigma;X)-$Asano ring as well as a $\sigma-G-$Asano ring), but it is a $\sigma-G-$Asano ring. Hence S and $R[t;\sigma]$ are $G-$Asano rings.
(2) Put $X = \mathfrak{p}$. Then
  (i) If $n = 2$, then R is not a $(\sigma;X)-$maximal order and so S is not a maximal order.
  (ii) If $n \geq 3$, then R is a $(\sigma;X)-$maximal order and so S is a maximal order (in fact, it is a $G-$Asano ring).

As in Example 2, put $X = \mathfrak{p}$. Then since $\sigma(\mathfrak{m}_i) = X\mathfrak{m}_iX^{-1}$, we have $X\sigma^{-1}(\mathfrak{m}_i) = \mathfrak{m}_iX$ and so R is not a $(\sigma^{-1};X)-$ maximal order. Hence we have

Remark 1 Under the same notation and assumptions as in Example 2(2), $S_1 = R[Xt;\sigma^{-1}]$ is not a maximal order and $R[t;\sigma^{-1}]$ is a maximal order.

Next we give examples of rings which are $(\sigma;X)-$maximal orders but not $\sigma-$maximal orders.

Let k be a field with automorphism σ and let $K = \begin{pmatrix} k & k \\ k & k \end{pmatrix}$, the ring of 2×2 matrices over k. Then we can extend σ to an automorphism of K by $\sigma(q) = \begin{pmatrix} \sigma(a) & \sigma(b) \\ \sigma(c) & \sigma(d) \end{pmatrix}$, where $q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let $U = K[x;\sigma]$ and $I = eK + xU$, where $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then I is a $\sigma-$invariant maximal right ideal of U with $UI = U$. We consider $R = \{u \in U \mid uI \subseteq I\}$, the idealizer of I. By [13, Theorem 5.5.10], R is an HNP ring and I is an idempotent maximal ideal of R. We note that $R = K(1-e) + eK + xU$. R has another idempotent maximal ideal $J = K(1-e) + xU$, which is a $\sigma-$invariant maximal left ideal of U with $JU = U$. Put $X = I \cap J = eK(1-e) + xU$. Since $O_r(I) = U = O_l(J)$ and $O_r(J) = x^{-1}(eK(1-e)) + R = O_l(I)$, $\{I,J\}$ is a cycle and X is an invertible ideal of R by [5, Proposition 2.5].

Example 3. Under the same notation and assumptions,

(1) R is not a $\sigma-$maximal order and $R[t;\sigma]$ is not a maximal order.
(2) R is a $(\sigma;X)-$maximal order and S is a maximal order (in fact, S is a $G-$Asano ring). Furthermore
  (i) If σ is of infinite order, then XS and XtS are only prime $v-$ideals of S.
  (ii) If σ is of finite order, say n, then there are infinite number of prime $v-$ideals of S.

Remark 2 There exist some examples of maximal orders which are not $G-$Asano rings ([2, Example 3.4] and [11, Example]).

Remark 3 In Examples 2 and 3, the rings are all HNP rings. However, by using examples in [10] we can provide $(\sigma;X)-$maximal orders which are neither HNP rings nor maximal orders. We will show them in detail in the forth-coming paper.

Finally we introduce a conjecture concerning skew Rees rings.

Problem Let $S = R[Xt;\sigma,\delta]$ be a subset of an Ore extension $R[t;\sigma,\delta]$, where δ is a left σ-derivation of R. Then what is a necessary and sufficient condition for S to be a maximal order or a generalized Asano ring?

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ON TORIC RINGS ARISING FROM CYCLIC POLYTOPES

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Abstract. Let d and n be positive integers with $n \geq d+1$ and $\mathcal{P} \subset \mathbb{R}^d$ an integral cyclic polytope of dimension d with n vertices. For a field K, let $K[\mathcal{P}] = K[\mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}}]$ denote its associated semigroup K-algebra, where $\mathcal{A}_{\mathcal{P}} = \{(1, \alpha) \in \mathbb{R}^{d+1} : \alpha \in \mathcal{P}\} \cap \mathbb{Z}^{d+1}$. In this draft, we study when $K[\mathcal{P}]$ is normal or very ample. Moreover, we also consider the problem when $K[\mathcal{P}]$ is Cohen–Macaulay by discussing Serre's condition (R_1) and we give a complete characterization when $K[\mathcal{P}]$ is Gorenstein. In addition, we investigate the normality of the other semigroup K-algebra $K[Q]$ arising from an integral cyclic polytope, where Q is a semigroup generated only with its vertices.

1. Introduction

This draft is based on a joint work with Takayuki Hibi, Lukas Katthän and Ryota Okazaki.

The cyclic polytope is one of the most distinguished polytopes and played the essential role in the classical theory of convex polytopes. Let d and n be positive integers with $n \geq d+1$ and $\tau_1, \ldots, \tau_n$ real numbers with $\tau_1 < \cdots < \tau_n$. We write $C_d(\tau_1, \ldots, \tau_n) \subset \mathbb{R}^d$ for the convex hull of $\{(\tau_i, \tau_i^2, \ldots, \tau_i^d) \in \mathbb{R}^d : i = 1, \ldots, n\}$. The convex polytope $C_d(\tau_1, \ldots, \tau_n) \subset \mathbb{R}^d$ is called a *cyclic polytope* of dimension d with n vertices. In particular, we say that it is an *integral cyclic polytope* if $\tau_1, \ldots, \tau_n$ are all integers. A cyclic polytope is a simplicial polytope and its combinatorial type is independent of a choice of $\tau_1, \ldots, \tau_n$. Moreover, it is well known that a cyclic polytope is a convex polytope which attains the upper bound in the Upper Bound Theorem.

In this draft, we focus on *integral* cyclic polytopes and discuss some properties on toric rings arising from integral cyclic polytopes.

In general, for an integral convex polytope $\mathcal{P}$, let $\mathcal{P}^* \subset \mathbb{R}^{N+1}$ be the convex hull of $\{(1, \alpha) \in \mathbb{R}^{N+1} : \alpha \in \mathcal{P}\}$ and $\mathcal{A}_{\mathcal{P}} = \mathcal{P}^* \cap \mathbb{Z}^{N+1}$. Then $\mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}}$ is an affine semigroup. Let K be a field. Then we set

$$K[\mathcal{P}] := K[\mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}}],$$

i.e., $K[\mathcal{P}]$ is an affine semigroup K-algebra associated with $\mathcal{P}$, and we call it a *toric ring* arising from $\mathcal{P}$.

For an integral convex polytope $\mathcal{P} \subset \mathbb{R}^N$, we say that $\mathcal{P}$ ($K[\mathcal{P}]$ or $\mathbb{R}_{\geq 0}\mathcal{A}_{\mathcal{P}}$) is *normal* if it satisfies

$$\mathbb{R}_{\geq 0}\mathcal{A}_{\mathcal{P}} \cap \mathbb{Z}\mathcal{A}_{\mathcal{P}} = \mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}}.$$

The detailed version of this paper will be submitted for publication elsewhere.

We say that $\mathcal{P}$ ($K[\mathcal{P}]$ or $\mathbb{R}_{\geq 0}\mathcal{A}_{\mathcal{P}}$) is *very ample* if the set

$$(\mathbb{R}_{\geq 0}\mathcal{A}_{\mathcal{P}} \cap \mathbb{Z}\mathcal{A}_{\mathcal{P}}) \setminus \mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}}$$

is finite. Thus, normal integral convex polytopes are always very ample.

Let, as before, d and n be positive integers with $n \geq d+1$. Given integers $\tau_1, \ldots, \tau_n$ with $\tau_1 < \cdots < \tau_n$, one of our goals is to classify the integers $\tau_1, \ldots, \tau_n$ with $\tau_1 < \cdots < \tau_n$ for which $C_d(\tau_1, \ldots, \tau_n)$ is normal. Even though to find such a complete classification seems to be difficult, many fascinating problems arise in the natural way. Our first main result is that if $\tau_{i+1} - \tau_i \geq d^2 - 1$ for $1 \leq i < n$, then $C_d(\tau_1, \ldots, \tau_n)$ is normal. Moreover, it is also shown that if $d \geq 4$ and $\tau_3 - \tau_2 = 1$ or $\tau_{n-1} - \tau_{n-2} = 1$, then $C_d(\tau_1, \ldots, \tau_n)$ is non-very ample.

Let $\mathcal{P}$ be an integral cyclic polytope. We will also consider the Cohen–Macaulayness and the Gorensteinness of the toric ring $K[\mathcal{P}]$. By proving that $K[\mathcal{P}]$ always satisfies Serre's condition (R_1), it follows that $K[\mathcal{P}]$ is Cohen–Macaulay if and only if $K[\mathcal{P}]$ is normal. Thus the characterization of Cohen–Macaulayness of integral cyclic polytopes is nothing but that of normality. Moreover, it turns out that $K[\mathcal{P}]$ is Gorenstein if and only if one has $d = 2, n = 3$ and $(\tau_2 - \tau_1, \tau_3 - \tau_2) = (2, 1)$ or $(1, 2)$.

In addition, we also define another toric rings arising from integral cyclic polytopes. Let

$$Q := Q_d(\tau_1, \ldots, \tau_n) := \mathbb{Z}_{\geq 0}\{(1, \tau_i, \tau_i^2, \ldots, \tau_i^d) \in \mathbb{Z}^{d+1} : i = 1, \ldots, n\}.$$

Then we write $K[Q]$ for the toric ring associated with the configuration Q. In other words, $K[Q]$ is a toric ring arising from the matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \tau_1 & \tau_2 & \cdots & \tau_n \\ \vdots & \vdots & \vdots & \vdots \\ \tau_1^d & \tau_2^d & \cdots & \tau_n^d \end{pmatrix}, \tag{1.1}$$

which is nothing but the Vandermonde matrix. We will show that if $d \geq 2$ and $n = d+2$, then $K[Q]$ is not normal.

2. Normal cyclic polytopes and non-very ample cyclic polytopes

The first main result of this draft is the following

Theorem 1 ([5, Theorem 2.1 and Theorem 3.1]). *Let d and n be positive integers with $n \geq d+1$ and $C_d(\tau_1, \ldots, \tau_n)$ an integral cyclic polytope, where $\tau_1 < \cdots < \tau_n$.*
(a) For each $1 \leq i \leq n-1$, if

$$\tau_{i+1} - \tau_i \geq d^2 - 1,$$

then $C_d(\tau_1, \ldots, \tau_n)$ is normal.
(b) Let $d \geq 4$. If

$$\textit{either} \quad \tau_3 - \tau_2 = 1 \quad \textit{or} \quad \tau_{n-1} - \tau_{n-2} = 1$$

is satisfied, then $C_d(\tau_1, \ldots, \tau_n)$ is not very ample.

Remark 2. Since each lattice length of an edge $\mathrm{conv}(\{(\tau_i, \tau_i^2, \ldots, \tau_i^d), (\tau_j, \tau_j^2, \ldots, \tau_j^d)\})$ of $\mathcal{P}$ coincides with $\tau_j - \tau_i$, where $i < j$, it follows immediately from [4, Theorem 1.3 (b)] that $\mathcal{P}$ is normal if $\tau_{i+1} - \tau_i \geq d(d+1)$ for $1 \leq i \leq n-1$. Thus, our constraint $\tau_{i+1} - \tau_i \geq d^2 - 1$ on integral cyclic polytopes is better than a general case, but this bound is still very rough. For example, $C_3(0,1,2,3)$ is normal, while we have $\tau_2 - \tau_1 = \tau_3 - \tau_2 = \tau_4 - \tau_3 = 1 < 8$. Similarly, $C_4(0,1,3,5,6)$ is also normal, although one has $\tau_2 - \tau_1 = \tau_5 - \tau_4 = 1$ and $\tau_3 - \tau_2 = \tau_4 - \tau_3 = 2$.

On the case where $d = 2$, it is well known that there exists a unimodular triangulation for every integral convex polytope of dimension 2. Therefore, integral convex polytopes of dimension 2 are always normal.

On the case where $d = 3$ and $d = 4$, exhaustive computational experiences lead us to give the following

Conjecture 3. *(a) All cyclic polytopes of dimension 3 are normal.*
(b) A cyclic polytope of dimension 4 is normal if and only if we have

$$\tau_3 - \tau_2 \geq 2 \quad \textit{and} \quad \tau_{n-1} - \tau_{n-2} \geq 2.$$

3. Cohen–Macaulay toric rings and Gorenstein toric rings arising from cyclic polytopes

Recall that a Noetherian ring R is said to satisfy (S_n) if

$$\mathrm{depth} R_{\mathfrak{p}} \geq \min\{n, \dim R_{\mathfrak{p}}\}$$

for all $\mathfrak{p} \in \mathrm{Spec}(R)$, and satisfy (R_n) if $R_{\mathfrak{p}}$ is a regular local ring for all $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\dim R_{\mathfrak{p}} \leq n$. The conditions (S_n) and (R_n) are called Serre's conditions.

The well-known criterion for normality of a Noetherian ring, Serre's Criterion (cf. [2, Theorem 2.2.22]), says that a Noetherian ring is normal if and only if it satisfies (R_1) and (S_2).

By using the combinatorial criterion of (R_1), which can be found in [1, Exercises 4.15 and 4.16], we can show

Proposition 4. *Let $\mathcal{P}$ be an integral cyclic polytope. Then $K[\mathcal{P}]$ always satisfies the condition (R_1).*

As a consequence of this proposition, we obtain

Theorem 5. *Let $\mathcal{P}$ be an integral cyclic polytope and $K[\mathcal{P}]$ its toric ring. Then the following conditions are equivalent:*

(1) *$K[\mathcal{P}]$ is normal;*
(2) *$K[\mathcal{P}]$ is Cohen–Macaulay;*
(3) *$K[\mathcal{P}]$ satisfies the condition (S_2).*

Remark 6. One can also prove that an integral cyclic polytope is normal if and only if it is seminormal. See [1, p. 66] for the definition and basic properties of seminormality. We use the notation from that book. Now, assume that $\mathcal{P}$ is not normal. Then there exists a point m in $\mathbb{R}_{\geq 0}\mathcal{A}_{\mathcal{P}} \cap \mathbb{Z}\mathcal{A}_{\mathcal{P}}$ which is not contained in $\mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}}$. This point m lies in the interior of a unique face $\mathcal{F}$ of $\mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}}$. But using the same construction as above, we can show that $\mathbb{Z}(\mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}} \cap \mathcal{F}) = \mathbb{Z}^{d+1} \cap \mathcal{H}$, where $\mathcal{H}$ is the linear subspace spanned by $\mathcal{F}$. Thus $m \in \mathbb{Z}(\mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}} \cap \mathcal{F})$ is an exceptional point, and therefore $(\mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}} \cap \mathcal{F})_*$ is not normal. Hence, $\mathcal{P}$ is not seminormal.

Moreover, we also obtain a complete characterization when $K[\mathcal{P}]$ is Gorenstein as follows.

Theorem 7. *Let $\mathcal{P}$ be an integral cyclic polytope and $K[\mathcal{P}]$ its toric ring. Then $K[\mathcal{P}]$ is Gorenstein if and only if*

$$d = 2,\ n = 3,\ (\tau_2 - \tau_1, \tau_3 - \tau_2) = (1, 2) \text{ or } (2, 1)$$

is satisfied.

4. The semigroup ring associated only with vertices of a cyclic polytope

Let Q denote the afiine semigroup $Q_d(\tau_1, \ldots, \tau_n)$ arising from the matrix (1.1). Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring over a field K and $K[Q]$ an affine semigroup K-algebra generated by the monomials $\{t_0 t_1^{\tau_i} \cdots t_d^{\tau_i^d} : i = 1, \ldots, n\}$, which is a subring of the Laurent polynomial ring $K[t_0, t_1^{\pm}, \ldots, t_d^{\pm}]$. Let I_Q be the kernel of the surjective ring homomorphism $S \to K[Q]$ which sends each x_i to $t_0 t_1^{\tau_i} \cdots t_d^{\tau_i^d}$. The ideal I_Q is just the toric ideal associated with the matrix (1.1). In particular, it is homogeneous with respect to the usual $\mathbb{Z}$-grading on S.

When $n = d + 1$, since the matrix (1.1) is nonsingular, $K[Q]$ is regular. In particular, it is normal. When $d = 1$, the matrix (1.1) can be transformed into

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & \tau_2 - \tau_1 & \cdots & \tau_n - \tau_1 \end{pmatrix}.$$

Since I_Q is preserved even if we divide a common divisor of $(\tau_2 - \tau_1), \ldots, (\tau_n - \tau_1)$ out of the second row, we may assume the greatest common divisor of $\tau_2 - \tau_1, \ldots, \tau_n - \tau_1$ is equal to 1. The ideal I_Q is a defining ideal of a projective monomial curve in $\mathbb{P}^{n-1}$, and it is well known (cf. [3]) that the corresponding curve is normal if and only if it is a rational normal curve of degree $n - 1$, that is, $\tau_i - \tau_1 = i - 1$ for all i with $2 \leq i \leq n$ (after the above transformation and re-setting each $\tau_i - \tau_1$). Consequently, in the case $d = 1$, the ring $K[Q]$ is normal if and only if $\tau_2 - \tau_1 = \tau_3 - \tau_2 = \cdots = \tau_n - \tau_{n-1}$. Hence we assume that $d \geq 2$ and $n \geq d + 1$.

Theorem 8. *Let Q be as above.*
(a) If $d \geq 2$ and $n = d + 2$, then $K[Q]$ is not normal.
(b) When $n \geq d + 3$, if $\prod_{k=1}^{d}(\tau_{d+1} - \tau_k) \nmid \prod_{k=1}^{d}(\tau_s - \tau_k)$ for some s with $d + 2 \leq s \leq n$, then $K[Q]$ is not normal.

Remark 9. When $n = d + 2$, since I_Q is principal, $K[Q]$ is Gorenstein. Hence, it is also Cohen–Macaulay.

Conjecture 10. *Let $K[Q]$ be as above. Then*
(a) $K[Q]$ is never normal if $d \geq 2$ and $d \geq n + 3$;
(b) $K[Q]$ is never Cohen–Macaulay if $d \geq 2$ and $d \geq n + 3$.

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CONSTRUCTIONS OF AUSLANDER-GORENSTEIN LOCAL RINGS

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Abstract. Generalizing the notion of crossed product, we provide systematic constructions of Auslander-Gorenstein local rings starting from an arbitrary Auslander-Gorenstein local ring.

1. Introduction

Auslander-Gorenstein rings (see Definition 2) appear in various areas of current research. For instance, regular 3-dimensional algebras of type A in the sense of Artin and Schelter, Weyl algebras over fields of characteristic zero, enveloping algebras of finite dimensional Lie algebras and Sklyanin algebras are Auslander-Gorenstein rings (see [2], [3], [4] and [12], respectively). However, little is known about constructions of Auslander-Gorenstein rings. It was shown in [7] that a left and right noetherian ring is an Auslander-Gorenstein ring if it admits an Auslander-Gorenstein resolution over another Auslander-Gorenstein ring. In this note, generalizing the notion of crossed product (see e.g. [8], [11] and so on), we will provide systematic constructions of Auslander-Gorenstein local rings starting from an arbitrary Auslander-Gorenstein local ring.

In order to provide the construction, we recall the notion of Frobenius extensions of rings due to Nakayama and Tsuzuku [9, 10], which we modify as follows. A ring A is said to be an extension of a ring R if A contains R as a subring, and the notation A/R is used to denote that A is an extension ring of R. A ring extension A/R is said to be Frobenius if the following conditions are satisfied: (F1) $A \in \text{Mod-}R$ and $A \in \text{Mod-}R^{\text{op}}$ are finitely generated projective; and (F2) $A \cong \text{Hom}_R(A, R)$ in Mod-A and $A \cong \text{Hom}_{R^{\text{op}}}(A, R)$ in Mod-A^{op} (see [1]). If R is a noetherian ring, a Frobenius extension A/R is a typical example of a noetherian ring A admitting Auslander-Gorenstein resolution over R, so that if R is an Auslander-Gorenstein ring then so is A with inj dim $A \leq$ inj dim R, where the equality holds whenever A/R is split, i.e., the inclusion $R \to A$ is a split monomorphism of R-R-bimodules (Proposition 7).

Generalizing the notion of crossed product, we will define new multiplications on the ring of full matrices and the group ring of finite cyclic groups. Let $n \geq 2$ be an integer and set $I(n) = \{1, \ldots, n\}$. We fix a cyclic permutation

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & 1 & \cdots & n-1 \end{pmatrix}$$

of $I(n)$. Then the law of composition $I(n) \times I(n) \to I(n), (i, j) \mapsto \pi^{-i}(j)$ makes $I(n)$ a cyclic group. We denote by $\Omega(n)$ the set of mappings $\omega : I(n) \times I(n) \to \mathbb{Z}$ satisfying the

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following conditions: (W1) $\omega(i,i)=0$ for all $i \in I(n)$; (W2) $\omega(i,j)+\omega(j,k) \geq \omega(i,k)$ for all $i,j,k \in I(n)$; (W3) $\omega(i,j)+\omega(j,i) \geq 1$ unless $i=j$; and (W4) $\omega(i,j)+\omega(j,\pi(i)) = \omega(i,\pi(i))$ for all $i,j \in I(n)$. We fix $\omega \in \Omega(n)$ and a ring R together with a pair (σ,c) of $\sigma \in \mathrm{Aut}(R)$ and $c \in R$ such that $\sigma(c)=c$ and $xc = c\sigma(x)$ for all $x \in R$. For instance, for any ring R and any $\sigma \in \mathrm{Aut}(R)$, a skew power series ring $R[[t;\sigma]]$ has such a pair (σ,t) (Example 8).

Denote by $\Omega_+(n)$ the subset of $\Omega(n)$ consisting of $\omega \in \Omega(n)$ such that $\omega(1,i)=\omega(i,n)=0$ for all $i \in I(n)$. Set $\chi(i)=\sum_{k=1}^{i}\omega(k,\pi(k))$ for $i \in I(n)$. Assume that $\omega \in \Omega_+(n)$ and that $\sigma^{\chi(n)} = \mathrm{id}_R$. Let A be a free right R-module with a basis $\{v_i\}_{i\in I(n)}$ and define a multiplication on A subject to the following axioms: (G1) $v_iv_j = v_{\pi^{-j}(i)}c^{\omega(\pi^{-j}(i),j)}$ for all $i,j \in I(n)$; and (G2) $xv_i = v_i\sigma^{-\chi(i)}(x)$ for all $x \in R$ and $i \in I(n)$. Then A is an associative ring with $1=v_n$ and R is considered as a subring of A via the injective ring homomorphism $R \to A, x \mapsto v_nx$; A/R is a split Frobenius extension; A is commutative if R is commutative and $\sigma^{\chi(i)} = \mathrm{id_R}$ for all $i \in I(n)$; and A is local if R is local and $c \in \mathrm{rad}(R)$ (Theorem 16).

2. Preliminaries

For a ring R we denote by $\mathrm{rad}(R)$ the Jacobson radical of R, by $R^\times$ the set of units in R, by $\mathrm{Z}(R)$ the center of R, by $\mathrm{Aut}(R)$ the group of ring automorphisms of R, for $\sigma \in \mathrm{Aut}(R)$ by R^σ the subring of R consisting of all $x \in R$ with $\sigma(x)=x$, and for $n \geq 2$ by $\mathrm{M}_n(R)$ the ring of $n \times n$ full matrices over R. We denote by Mod-R the category of right R-modules. Left R-modules are considered as right R^{op}-modules, where R^{op} denotes the opposite ring of R. In particular, we denote by inj dim R (resp., inj dim R^{op}) the injective dimension of R as a right (resp., left) R-module and by $\mathrm{Hom}_R(-,-)$ (resp., $\mathrm{Hom}_{R^{\mathrm{op}}}(-,-)$) the set of homomorphisms in Mod-R (resp., Mod-R^{op}).

We start by recalling the notion of Auslander-Gorenstein rings.

Proposition 1 (Auslander). *Let R be a left and right noetherian ring. Then for any $n \geq 0$ the following are equivalent.*

(1) *In a minimal injective resolution $I^\bullet$ of R in* Mod-R, flat dim $I^i \leq i$ *for all* $0 \leq i \leq n$.
(2) *In a minimal injective resolution $J^\bullet$ of R in* Mod-R^{op}, flat dim $J^i \leq i$ *for all* $0 \leq i \leq n$.
(3) *For any $1 \leq i \leq n+1$, any $M \in$* mod-R *and any submodule X of* $\mathrm{Ext}^i_R(M,R) \in$ mod-R^{op} *we have* $\mathrm{Ext}^j_{R^{\mathrm{op}}}(X,R)=0$ *for all* $0 \leq j < i$.
(4) *For any $1 \leq i \leq n+1$, any $X \in$* mod-R^{op} *and any submodule M of* $\mathrm{Ext}^i_{R^{\mathrm{op}}}(X,R) \in$ mod-R *we have* $\mathrm{Ext}^j_R(M,R)=0$ *for all* $0 \leq j < i$.

Definition 2 ([4]). For a left and right noetherian ring R we say that R satisfies the Auslander condition if it satisfies the equivalent conditions in Proposition 1 for all $n \geq 0$, and that R is an Auslander-Gorenstein ring if inj dim R = inj dim $R^{\mathrm{op}} < \infty$ and if it satisfies the Auslander condition.

Next, we recall the notion of Frobenius extensions of rings due to Nakayama and Tsuzuku [9, 10] which we modify as follows (see [1, Section 1]).

Definition 3 ([1])**.** A ring A is said to be an extension of a ring R if A contains a ring R as a subring, and the notation A/R is used to denote that A is an extension of a ring R. A ring extension A/R is said to be Frobenius if the following conditions are satisfied:
(F1) $A \in \text{Mod-}R$ and $A \in \text{Mod-}R^{\text{op}}$ are finitely generated projective; and
(F2) $A \cong \text{Hom}_R(A, R)$ in Mod-A and $A \cong \text{Hom}_{R^{\text{op}}}(A, R)$ in Mod-A^{op}.

It should be noted that if A/R is a Frobenius extension then so is $A^{\text{op}}/R^{\text{op}}$. The next proposition is well-known and easily verified.

Proposition 4. *Let A/R be a ring extension with $\phi : A \xrightarrow{\sim} \text{Hom}_R(A, R)$ in* Mod-A*. Then the following hold.*

1. *There exists a ring homomorphism $\theta : R \to A$ such that $x\phi(1) = \phi(1)\theta(x)$ for all $x \in R$. In particular, ϕ is an isomorphism of R-A-bimodules if and only if $\theta(x) = x$ for all $x \in R$.*
2. *If $A \in$ Mod-R is finitely generated projective then so is* $\text{Hom}_R(A, R) \in \text{Mod-}R^{\text{op}}$ *and $A \xrightarrow{\sim} \text{Hom}_{R^{\text{op}}}(\text{Hom}_R(A, R), R), a \mapsto (h \mapsto h(a))$, which is an isomorphism of A-R-bimodules.*
3. *If $A \in$ Mod-R is finitely generated projective, and if ϕ is an isomorphism of R-A-bimodules, then $A \in$ Mod-R^{op} is finitely generated projective and we have an isomorphism of A-R-bimodules $\psi : A \xrightarrow{\sim} \text{Hom}_{R^{\text{op}}}(A, R)$ with $\psi(a)(b) = \phi(b)(a)$ for all $a, b \in A$, so that A/R is a Frobenius extension.*

Definition 5. Let A/R be a Frobenius extension with $\phi : A \xrightarrow{\sim} \text{Hom}_R(A, R)$ in Mod-A and $\theta : R \to A$ a ring homomorphism such that $x\phi(1) = \phi(1)\theta(x)$ for all $x \in R$. In general, $\theta(R) \neq R$ ([1]). Following [9, 10], we say that A/R is a Frobenius extension of second kind if θ induces a ring automorphism of R and that A/R is a Frobenius extension of first kind if $\theta(x) = x$ for all $x \in R$, i.e., ϕ is an isomorphism of R-A-bimodules.

Definition 6 ([1])**.** A ring extension A/R is said to be split if the inclusion $R \to A$ is a split monomorphism of R-R-bimodules.

Proposition 7 ([1])**.** *For any Frobenius extension A/R the following hold.*

1. *If R is an Auslander-Gorenstein ring then so is A with* inj dim $A \leq$ inj dim R*.*
2. *Assume that A/R is split. If A is an Auslander-Gorenstein ring then so is R with* inj dim $R =$ inj dim A*.*

We end this section with recalling the notion of skew power series rings.

Example 8. Let R be a ring and $\sigma \in \text{Aut}(R)$. Let $R[t; \sigma]$ be a free right R-module with a basis $\{t^p\}_{p \geq 0}$ and define a multiplication on $R[t; \sigma]$ subject to the following axioms: (P1) $t^p t^q = t^{p+q}$ for all $p, q \geq 0$; and (P2) $xt^p = t^p \sigma^p(x)$ for all $x \in R$ and $p \geq 0$. Then $R[t; \sigma]$ is an associative ring with $1 = t^0$ and t^p is the pth power of $t = t^1$ for all $p \geq 2$. We consider R as a subring of $R[t; \sigma]$ via the injective ring homomorphism $R \to R[t; \sigma], x \mapsto t^0 x$.

Next, setting $(t^p) = \sum_{q \geq p} t^q R$ for $p \geq 1$, we have a descending chain of two-sided ideals $(t) \supset (t^2) \supset \cdots$ in $R[t; \sigma]$ and set $R[[t; \sigma]] = \varprojlim R[t; \sigma]/(t^p)$. Namely, $R[[t; \sigma]]$ is the ring of formal power series and contains $R[t; \sigma]$ as a subring. Also, every $\tau \in \text{Aut}(R)$ with $\tau\sigma = \sigma\tau$ is extended to a ring automorphism of $R[[t; \sigma]]$ such that $\sum_{p \geq 0} t^p x_p \mapsto$

$\sum_{p\geq 0} t^p\tau(x_p)$ which we denote again by τ. In particular, $\sigma \in \mathrm{Aut}(R[[t;\sigma]])$ with $\sigma(t) = t$ and $at = t\sigma(a)$ for all $a \in R[[t;\sigma]]$.

3. Structure system

Throughout the rest of this note, we set $I(n) = \{1, \dots, n\}$ with $n \geq 2$ and fix a cyclic permutation

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & 1 & \cdots & n-1 \end{pmatrix}$$

of $I(n)$. Then $\pi^{-i}(j) = \pi^{-j}(i)$ for all $i, j \in I(n)$ and the law of composition

$$I(n) \times I(n) \to I(n), (i, j) \mapsto \pi^{-i}(j)$$

makes $I(n)$ a cyclic group.

We denote by $\Omega(n)$ the set of mappings $\omega : I(n) \times I(n) \to \mathbb{Z}$ satisfying the following conditions:

(W1) $\omega(i, i) = 0$ for all $i \in I(n)$;
(W2) $\omega(i, j) + \omega(j, k) \geq \omega(i, k)$ for all $i, j, k \in I(n)$;
(W3) $\omega(i, j) + \omega(j, i) \geq 1$ unless $i = j$; and
(W4) $\omega(i, j) + \omega(j, \pi(i)) = \omega(i, \pi(i))$ for all $i, j \in I(n)$.

Example 9. Let $n = 4$. Then, setting

$$(\omega(i,j))_{1\leq i,j\leq 4} = \begin{pmatrix} 0 & 4 & 4 & 3 \\ 1 & 0 & 2 & -1 \\ -1 & 3 & 0 & -1 \\ 2 & 4 & 6 & 0 \end{pmatrix},$$

we have $\omega \in \Omega(4)$.

Lemma 10. *For any $\omega \in \Omega(n)$ the following hold.*

(1) $\omega(\pi(i), \pi(j)) = \omega(i, j) - \omega(i, \pi(i)) + \omega(j, \pi(j))$ *for all* $i, j \in I(n)$.
(2) $\omega(1, i) = 0$ *for all* $i \in I(n)$ *if and only if* $\omega(i, n) = 0$ *for all* $i \in I(n)$.

We denote by $\Omega_+(n)$ the subset of $\Omega(n)$ consisting of $\omega \in \Omega(n)$ such that $\omega(1, i) = \omega(i, n) = 0$ for all $i \in I(n)$ (cf. Lemma 10(2)).

Example 11. Let $n = 4$. Then, setting

$$(\omega(i,j))_{1\leq i,j\leq 4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 \\ 3 & 2 & 3 & 0 \end{pmatrix},$$

we have $\omega \in \Omega_+(4)$.

Lemma 12. *For any $\omega \in \Omega_+(n)$ the following hold.*

(1) $\omega(i, \pi(i)) = \omega(i, 1) = \omega(n, \pi(i)) \geq 1$ *unless* $i = 1$.
(2) $\omega(i, j) + \omega(\pi^j(i), k) = \omega(i, \pi^{-j}(k)) + \omega(\pi^{-j}(k), j)$ *for all* $i, j, k \in I(n)$.

We denote by $X_+(n)$ the set of mappings $\chi : I(n) \to \mathbb{Z}$ satisfying the following conditions:

(X1) $\chi(1) < \chi(2) < \cdots < \chi(n)$;
(X2) $\chi(i) + \chi(n-i+1) = \chi(n)$ for all $i \in I(n)$; and
(X3) $\chi(j-i) \leq \chi(j) - \chi(i) \leq \chi(j-i+1)$ for all $i, j \in I(n)$ with $i < j$.

Remark 13. For any $\chi : I(n) \to \mathbb{Z}$ satisfying the condition (X2) we have $\chi(1)+\chi(n) = \chi(n)$ and hence $\chi(1) = 0$.

Example 14. Let $n = 4$. Then, setting $\chi(1) = 0,\ \chi(2) = 3,\ \chi(3) = 5$ and $\chi(4) = 8$, we have $\chi \in X(4)$.

Proposition 15. *For any $\omega \in \Omega_+(n)$, setting $\chi(i) = \sum_{k=1}^{i} \omega(k, \pi(k))$ for $i \in I(n)$, we have $\chi \in X_+(n)$ and*

$$\omega(i,j) = \begin{cases} \chi(i) - \chi(j) + \chi(j-i+1) & \text{if } i \leq j, \\ \chi(i) - \chi(j) - \chi(i-j) & \text{if } i > j \end{cases}$$

for all $i, j \in I(n)$, so that we have a bijection $\Omega_+(n) \xrightarrow{\sim} X_+(n), \omega \mapsto \chi$.

4. Group rings

Throughout the rest of this note, we fix a ring R together with a pair (σ, c) of $\sigma \in \mathrm{Aut}(R)$ and $c \in R$ satisfying the following condition

$$(*) \quad \sigma(c) = c \quad \text{and} \quad xc = c\sigma(x) \text{ for all } x \in R.$$

Note that if $c \in R^\times$ then $\sigma(x) = c^{-1}xc$ for all $x \in R$, and that the condition (*) is satisfied if either $c = 0$ and σ is arbitrary, or $c \in \mathrm{Z}(R)$ and $\sigma = \mathrm{id}_R$. We refer to Example 8 for a non-trivial example. As usual, we require that $c^0 = 1$ even if $c = 0$. We fix $\omega \in \Omega_+(n)$ and, setting $\chi(i) = \sum_{k=1}^{i} \omega(k, \pi(k))$ for $i \in I(n)$, assume that $\sigma^{\chi(n)} = \mathrm{id}_R$.

Let A be a free right R-module with a basis $\{v_i\}_{i \in I(n)}$ and define a multiplication on A subject to the following axioms:

(G1) $v_i v_j = v_{\pi^{-j}(i)} c^{\omega(\pi^{-j}(i), j)}$ for all $i, j \in I(n)$; and
(G2) $xv_i = v_i \sigma^{-\chi(i)}(x)$ for all $x \in R$ and $i \in I(n)$.

Denoting by $\{\beta_i\}_{i \in I(n)}$ the dual basis of $\{v_i\}_{i \in I(n)}$ for the free left R-module $\mathrm{Hom}_R(A, R)$, we have $a = \sum_{i \in I(n)} v_i \beta_i(a)$ for all $a \in A$. It is not difficult to see that for any $a, b \in A$ and $i \in I(n)$ we have

$$\beta_i(ab) = \sum_{j \in I(n)} c^{\omega(i,j)} \sigma^{-\chi(j)}(\beta_{\pi^j(i)}(a)) \beta_j(b).$$

Theorem 16. *The following hold.*

1. *A is an associative ring with $1 = v_n$ and contains R as a subring via the injective ring homomorphism $R \to A, x \mapsto v_n x$.*
2. *A/R is a split Frobenius extension of first kind.*
3. *$v_i v_j = v_j v_i$ for all $i, j \in I(n)$. In particular, A is commutative if R is commutative and $\sigma^{\chi(i)} = \mathrm{id}_R$ for all $i \in I(n)$. Furthermore, for any $i \in I(n)$ with $i \neq n$ we have ${v_i}^r = c^s$ for some $2 \leq r \leq n$ and $s \geq 1$.*

(4) *There exists an injective ring homomorphism*

$$\rho : A \to \mathrm{M}_n(R), a \mapsto (c^{\omega(i,j)}\sigma^{-\chi(j)}(\beta_{\pi^j(i)}(a)))_{i,j\in I(n)}$$

such that $a \in A^\times$ *for all* $a \in A$ *with* $\rho(a) \in \mathrm{M}_n(R)^\times$.

(5) *If* $c \in \mathrm{rad}(R)$ *then* $\beta_n(a) \in R^\times$ *for all* $a \in A^\times$ *and* $R/\mathrm{rad}(R) \xrightarrow{\sim} A/\mathrm{rad}(A)$ *canonically, so that if* R *is local then so is* A.

Remark 17. Every $\tau \in \mathrm{Aut}(R)$ with $\tau\sigma = \sigma\tau$ and $\tau(c) = c$ is extended to a ring automorphism of A such that $\sum_{i\in I(n)} v_i x_i \mapsto \sum_{i\in I(n)} v_i \tau(x_i)$ which we denote again by τ. In particular, $\sigma \in \mathrm{Aut}(A)$ with $\sigma(c) = c$ and $ac = c\sigma(a)$ for all $a \in A$, so that for any $v \in \mathrm{Z}(A)^\sigma$ we can replace $(R; \sigma, c)$ by $(A; \sigma, vc)$ in the construction above.

In the following, we denote by $R[\omega; \sigma, c]$ the ring A constructed above.

Example 18. If $\chi(i) = (i-1)p$ with $p \geq 1$ for all $i \in I(n)$ then

$$R[t; \sigma^p]/(t^n - c^p) \xrightarrow{\sim} R[\omega; \sigma, c], t \mapsto v_{n-1},$$

where $(t^n - c^p) = (t^n - c^p)R[t; \sigma^p]$.

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τ-TILTING THEORY

OSAMU IYAMA

ABSTRACT. In this short note, we discuss background of τ-tilting theory which was introduced in [2].

Brenner-Butler [5] によって導入された**傾加群** (tilting module) の概念は, 今日では表現論において欠かせないものとなっている. 傾加群は, 森田理論の基本概念である**射影生成元** (progenerator) の一般化であり, また, Rickard による導来圏の森田理論 [14] における基本概念である**傾複体** (tilting complex) の特別な場合でもある. 今日では傾理論は, 群（有限群, 代数群）の表現論や代数幾何学, ミラー対称性予想をはじめとして, 様々な数学で用いられており, 環論の持つ普遍性を示す一例となっている.

傾加群に対して, 近年盛んに研究されている事柄の一つとして, **変異** (mutation) が挙げられる. 一般に変異とは, 特別な性質を持つ与えられた対象から, 同様の性質を持つ新しい対象を構成する操作のことである. **例外列** (exceptional sequence) の変異 [7] と**団傾対象** (cluster tilting object) の変異 [6, 10] の 2 種類が広く知られているが, いずれもある種の三角圏の構造を解析するものであり, 特に後者は高次元 Auslander-Reiten 理論 [9] や団代数 (cluster algebra) の圏論化 [12] にも応用される重要なものである.

傾加群に対する同様の操作である**傾変異** (tilting mutation) は, Riedtmann-Schofield [15], Happel-Unger [8] らによって研究されてきた[1]. 傾変異とは「基本的傾加群が与えられたときに, 一つの直既約な直和因子を入れ替えことによって, 新たな基本的傾加群を得る」操作であり, 傾変異理論とは, 加群圏の特徴的な部分（＝傾加群）を調べることによって, 加群圏全体の構造を理解しようとするものである. クイバーの**鏡映** (reflection) や Auslander-Platzeck-Reiten 傾加群, 有限群のモジュラー表現論における奥山, Rickard による傾複体などは, 全て傾変異の特別な場合である.

傾変異の注意点は,「直和因子の選び方によっては, 傾変異をすることが出来ない」点であり, これが他の変異操作と比較した場合に不十分な点である. これを解消するためには, 扱う対象の範囲を傾加群から少し広げることにより, 変異がいつでも可能となるようにすること（傾変異の「**完備化**」）が標準的であり, 以下の 3 種類が研究されている.

(a) **準傾複体** (silting complex)
(b) 団傾対象 (cluster tilting object)
(c) **台 τ 傾加群** (support τ-tilting module)

(a) は, 上で述べた Rickard の傾複体を一般化した概念であり, 導来圏の対象となっている. 詳細は相原氏との共著 [3] を参照されたい. (a) の欠点は, 導来圏はもとの加群圏よりもはるかに巨大であるため, 加群圏の構造解析のためには, 大部分の準傾複体は不要となる点である. より加群圏に近い圏の中で変異を行える方が, 完備化と呼ばれるに相応しい.

上でも述べた (b) は, この点を改善したものである. 団傾対象は, **団圏** (cluster category) と呼ばれる三角圏の対象であり, 団圏は加群圏を「少しだけ拡張して」構成されたもので

The detailed version of this paper has been submitted for publication elsewhere.

[1]彼らは変異という用語を用いていないのだが, 今日では変異と呼ぶ方が自然である.

あるため, 導来圏よりもはるかに加群圏に近い. 反面, 団圏を構成するためには DG 多元環が必要であるため, 取り扱いは必ずしも容易ではない. そのため, 傾変異のより扱いやすい拡張を与えることは, 重要な課題であった.

[2] で導入された (c) は, これらの要望に答えるものである. 台 τ 傾加群は, 特別な加群として定義されるものであり, 加群圏以外の圏を扱う必要が一切無い.

以下, 簡単に定義を与える. A を体 K 上の有限次元多元環とする. Auslander-Reiten 移動を τ で表わす. 有限生成 A 加群 M が τ **リジッド** (τ-rigid) であるとは, $\mathrm{Hom}_A(M, \tau M) = 0$ が成立することである. τ リジッド加群は Auslander-Smalø[4] によって, 80 年代に研究された概念であるが, 不思議なことに今日までほとんど忘れられており, 特別な呼称さえ与えられていなかった. 論文 [2] では, リジッド加群（$\mathrm{Ext}^1_A(M, M) = 0$ を満たす加群）の類似物である点に着目して, τ リジッド加群という名称を導入した.

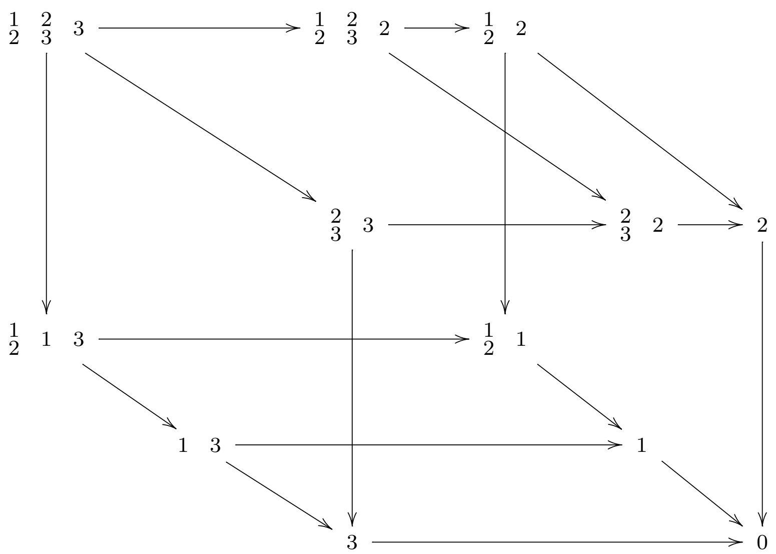
τ リジッド加群 M が τ **傾加群** (τ-tilting module) であるとは, 等式 $|M| = |A|$ が成立することである. ここで $|M|$ は, M の非同型な直既約直和因子の個数を表わす. 傾加群は τ 傾加群であるが, 一般に τ 傾加群は傾加群よりもはるかにたくさん存在する.

傾変異の完備化を与えるためには, 台 τ 傾加群の概念が必要となる. A のある巾等元 e に対する剰余環 $A/(e)$ 上の τ 傾加群を, **台 τ 傾加群** (support τ-tilting module) と呼ぶ. 以下, 台 τ 傾加群に関する諸性質を箇条書きする. 詳細は [2] を参照されたい.

- Bongartz 完備化の存在.
- 台 τ 傾加群に関する, 変異の一意的可能性.
- 台 τ 傾加群に関する, 変異クイバーと Hasse クイバーの一致.
- 台 τ 傾加群と, 関手的有限なねじれ部分圏の一対一対応.
- 台 τ 傾加群と, 2 項準傾複体の一対一対応.
- A が 2-Calabi-Yau 三角圏 C に付随する 2-Calabi-Yau 傾斜多元環の場合, 台 τ 傾加群と, C の団傾対象の一対一対応.

台 τ 傾加群に関する最近の結果は, [1, 11, 13, 16] 等を参照されたい.

最後に, A がクイバー $1 \xrightarrow{a} 2 \xrightarrow{b} 3$ と関係式 $ab = 0$ で与えられる場合, 台 τ 傾加群の Hasse クイバーを図示する.



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CLUSTER-TILTED ALGEBRAS OF CANONICAL TYPE AND QUIVERS WITH POTENTIAL

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Abstract. Let $\operatorname{coh}\mathbb{X}$ be the category of coherent sheaves over a weighted projective line and $\mathcal{C}_{\mathbb{X}}$ the classical cluster category associated with $\operatorname{coh}\mathbb{X}$. It is known that the morphism spaces in $\mathcal{C}_{\mathbb{X}}$ carry a natural $\mathbb{Z}/2\mathbb{Z}$-grading. Also, by results of Keller and Amiot, it is known that in this setting cluster-tilted algebras are Jacobian algebras of graded quivers with potential. We show that if T and T' are two cluster-tilting objects in $\mathcal{C}_{\mathbb{X}}$ which are related by mutation, then the corresponding cluster-tilted algebras are related by mutation of graded quivers with potential, thus enhancing Hübner's description of the quiver with relations of the corresponding tilted algebras.

Key Words: Cluster-tilted algebras, canonical algebras, quivers with potential, weighted projective lines.

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1. Introduction

The category $\operatorname{coh}\mathbb{X}$ of coherent sheaves over weighted projective lines was introduced in [7] as a geometric tool to study the representation theory of the so-called canonical algebras. It turns out that the category $\operatorname{coh}\mathbb{X}$ is an abelian hereditary category, hence it has associated cluster category $\mathcal{C}_{\mathbb{X}}$ in the sense of [4]. The category $\mathcal{C}_{\mathbb{X}}$ was studied in more detail in [3], where it is shown that the category $\mathcal{C}_{\mathbb{X}}$ can be obtained from $\operatorname{coh}\mathbb{X}$ by a suitable enlargment of the morphism spaces. Moreover, both categories $\operatorname{coh}\mathbb{X}$ and $\mathcal{C}_{\mathbb{X}}$ are equivepped with a "mutation operation", which acts on the isomorphism classes of a distinguished class of objects: basic tilting sheaves in $\operatorname{coh}\mathbb{X}$ and basic cluster-tilting objects in $\mathcal{C}_{\mathbb{X}}$. The aim of this notes is to describe the effect of this mutation operation on the endomorphism algebras of these objects, *c.f.* Theorem 4. We do so by incorporating the machinery of graded quivers with potential and their mutations introduced in [2] following the ungraded version of [5]. We note that a description of the Gabriel quiver of these endomorphism algebras was done in [8, Kor. 4.16] at the level of $\operatorname{coh}\mathbb{X}$. Thus, Theorem 4 although a minor enhancement of *loc. cit.*, provides a very convenient way to keep track of the changes on the relations both at the level of $\mathcal{C}_{\mathbb{X}}$ and $\operatorname{coh}\mathbb{X}$. This is illustrated by an example at the end of this notes.

In Section 2 we give a brief description of the category $\operatorname{coh}\mathbb{X}$, followed by a crash-course on the theory of graded quivers with potential and their mutations. At the end of the section we explain the connection between the topics discussed beforehand. In Section 3 we state the main theorem of this notes and give an example to illustrate the phenomenon described.

An expanded version of this paper will be submitted for publication elsewhere.

2. Preliminaries

In this section we collect the concepts and results that we need throughout this notes.

2.1. **Coherent sheaves over weighted projective lines.** Let k be an algebraically closed field and choose a tuple $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_t)$ of pairwise disctinct points of $\mathbb{P}^1_k$. Also choose a *parameter sequence* $\mathbf{p} = (p_1, \ldots, p_t)$ of positive integers with $p_i \geqslant 2$ for each $i \in \{1, \ldots, t\}$. We call the triple $\mathbb{X} = (\mathbb{P}^1_k, \boldsymbol{\lambda}, \mathbf{p})$ a *weighted projective line.* The category $\operatorname{coh}\mathbb{X}$ of *coherent sheaves* over $\mathbb{X}$ is defined as follows: consider the rank 1 abelian gruop with presentation

$$\mathbb{L} = \mathbb{L}(\mathbf{p}) = \langle \vec{x}_1, \ldots, \vec{x}_t \mid p_1\vec{x}_1 = \cdots = p_t\vec{x}_t =: \vec{c}\rangle$$

and the $\mathbb{L}$-graded algebra

$$S = S(\mathbf{p}, \boldsymbol{\lambda}) = k[x_1, \ldots, x_t]/\langle x_i^{p_i} - \lambda_i' x_2^{p_2} - \lambda_i'' x_1^{p_1} \mid i \in \{3, \ldots, t\}\rangle$$

where $\deg x_i = \vec{x}_i$ and $\lambda_i = [\lambda_i' : \lambda_i''] \in \mathbb{P}^1_k$ for each $i \in \{1, \ldots, t\}$. Note that the ideal which defines S is generated by homogeneous polynomials of degree $\vec{c}$. Then $\operatorname{coh}\mathbb{X}$ is the quotient of the category $\operatorname{mod}^{\mathbb{L}} S$ of finitely generated $\mathbb{L}$-graded S-modules by it's Serre subcategoy $\operatorname{mod}_0^{\mathbb{L}} S$ of finite lenght $\mathbb{L}$-graded S-modules. We refer the reader to [7] and [11] for basic results and properties of the category $\operatorname{coh}\mathbb{X}$.

The category $\operatorname{coh}\mathbb{X}$ enjoys several nice properties; it is an abelian, hereditary k-linear category with finite dimensional Hom and Ext spaces. Given a sheaf E, shifting the grading induces *twisted sheaves* $E(\vec{x})$ for each $\vec{x} \in \mathbb{L}$. In particular, twisting the grading by the dualizing element $\vec{\omega} := \sum_{i=1}^{t}(\vec{c} - \vec{x}_i) - 2\vec{c}$ gives the following version of *Serre's duality*:

$$\operatorname{Ext}^1_{\mathbb{X}}(E, F) \cong D\operatorname{Hom}_{\mathbb{X}}(F, E(\vec{\omega}))$$

for any E and F in $\operatorname{coh}\mathbb{X}$. This implies that $\operatorname{coh}\mathbb{X}$ has almost-split sequences and that the Auslander-Reiten translation is given by the auto-equivalence $\tau E = E(\vec{\omega})$. The free module S induces a structue sheaf in $\operatorname{coh}\mathbb{X}$ which we denote by $\mathcal{O}$. We recall that there are two group homomorphisms

$$\deg, \operatorname{rk} : K_0(\mathbb{X}) \to \mathbb{Z}$$

which, together with the function

$$\text{slope} = \frac{\deg}{\operatorname{rk}} : K_0(\mathbb{X}) \to \mathbb{Q} \cup \{\infty\},$$

play an important role in the theory We refer the reader to [7] for precise definitions.

Definition 1. A sheaf T is called a *tilting sheaf* if $\operatorname{Ext}^1_{\mathbb{X}}(T, T) = 0$ and it is maximal with this property or, equivalently, the number of pairwise non-isomorphic indecomposable direct summands of T equals $2 + \sum_{i=1}^{t}(p_i - 1)$, the rank of the Grothendieck group of $\operatorname{coh}\mathbb{X}$.

The connection between the category $\operatorname{coh}\mathbb{X}$ and canonical algebras is explained by the following proposition:

Proposition 2. [7, Prop. 4.1] *Let T be the following vector bundle:*

$$
\begin{tikzcd}
 & \mathcal{O}(\vec{x}_1) \arrow[r] & \cdots \arrow[r] & \mathcal{O}((p_1-1)\vec{x}_1) \arrow[ddr] & \\
 & \mathcal{O}(\vec{x}_2) \arrow[r] & \cdots \arrow[r] & \mathcal{O}((p_2-1)\vec{x}_2) \arrow[dr] & \\
\mathcal{O} \arrow[uur] \arrow[ur] \arrow[dr] & & \vdots & & \mathcal{O}(\vec{c}) \\
 & \mathcal{O}(\vec{x}_t) \arrow[r] & \cdots \arrow[r] & \mathcal{O}((p_t-1)\vec{x}_t) \arrow[ur] &
\end{tikzcd}
$$

Then T is a titling bundle *and* $\mathrm{End}_{\mathbb{X}}(T)$ *is the canonical algebra of parameter sequence* $\boldsymbol{\lambda}$ *and weight sequence* $\mathbf{p}$*. The tilting bundle T is called the* canonical configuration *in* $\operatorname{coh}\mathbb{X}$*.*

There is an involutive operation on the set of isomorphism classes of basic tilting sheaves called *mutation*, *c.f.* [8, Def. 2.9]. Let $T = T_1 \oplus \cdots \oplus T_n$ be a basic tilting sheaf and $k \in \{1, \dots, n\}$. The *mutation at k* of T is the basic tilting sheaf $\mu_k(T) = T'_k \oplus \bigoplus_{i \neq k} T_i$ where $T'_k = \ker\alpha \oplus \operatorname{coker}\alpha^*$ and

$$\alpha : \bigoplus_{i \to k} T_i \to T_k \quad \text{and} \quad \alpha^* : T_k \to \bigoplus_{k \to j} Tj.$$

Note that α is a monomorphism (resp. epimorphism) if and only if α^* is a monomorphism (resp. epimorphism), *c.f.* [8, Prop. 2.6, Prop. 2.8].

2.2. **Graded quivers with potential and their mutations.** Quivers with potentials and their Jacobian algebras where introduced in [5] as a tool to prove several of the conjectures of [6] about cluster algebras in a rather general setting. Their graded counter part, which is the one we are concerned with, was introduced in [2].

Let $Q = (Q_0, Q_1)$ be a finite quiver without loops or two cycles and $d : Q_1 \to \mathbb{Z}/2\mathbb{Z}$ a *degree function* on the set of arrows of Q. Thus, the complete path algebra $\widehat{kQ}$ has a natural $\mathbb{Z}/2\mathbb{Z}$-grading. A *potential* is a (possibly infinite) linear combination of cyclic paths in Q; we are only interested in potentials which are homogeneous as elements of $\widehat{kQ}$. For a cyclic path $a_1 \cdots a_d$ in Q and $a \in Q_1$, let

$$\partial_a(a_1 \cdots a_d) = \sum_{a_i = a} a_{i+1} \cdots a_d a_1 \cdots a_{i-i}$$

and extend it by lineary to an arbitrary potential. The maps ∂_a are called *cyclic derivatives.*

Definition 3. A graded quiver with potential (graded QP for short) is a quadruple (Q, W, d) where (Q, d) is a $\mathbb{Z}/2\mathbb{Z}$-graded finite quiver without loops and two cycles and W is a homogeneous potential for Q. The *graded Jacobian algebra* of (Q, W, d) is the graded algebra

$$\mathrm{Jac}(Q, W, d) \cong \frac{\widehat{kQ}}{\partial(W)}$$

where $\partial(W)$ is the closure in $\widehat{kQ}$ of the ideal generated by the set $\{\partial_a(W) \mid a \in \tilde{Q}_1\}$.

For each vertex of Q there is a pair of well defined involutive operations on the right equivalence-classes of graded QPs, [5, Def. 4.2], called *left and right mutations*. They differ of each other at the level of the grading only, and as their non-graded versions they consist of a mutation step and a reduction step.

Let (Q, W, d) be graded QP with W homogeneous of degree r and $k \in Q_0$. The *non-reduced left mutation at k* of (Q, W, d) is the graded QP $\tilde{\mu}_k^L(Q, W, d) = (Q', W', d')$ defined as follows:

(1) The quivers Q and Q' have the same vertex set.
(2) All arrows of Q which are not adjacent to k are also arrows of Q' and of the same degree.
(3) Each arrow $a : i \to k$ of Q is replaced in Q' by an arrow $a^* : k \to i$ of degree $d(a) + r$.
(4) Each arrow $b : k \to j$ of Q is replaced in Q' by an arrow $b^* : j \to k$ of degree $d(b)$.
(5) Each composition $i \xrightarrow{a} k \xrightarrow{b} j$ in Q is replaced in Q' by an arrow $[ba] : i \to j$ of degree $d(a) + d(b)$.
(6) The new potential is given by

$$W' = [W] + \sum_{i \xrightarrow{a} k \xrightarrow{b} j} [ba]a^*b^*$$

where $[W]$ is the potential obtained from W by replacing each composition $i \xrightarrow{a} k \xrightarrow{b} j$ which appears in W with the corresponding arrow $[ba]$ of Q'.

By [2, Thm. 4.6], there exist a graded QP (Q'_{red}, W'_{red}, d') which is right equivalent to (Q', W', d'), *c.f.* [5, Def. 4.2], and such that Q' has neither loops or two cycles. The *left mutation at k* of (Q', W', d') is then defined as

$$\mu_k^L(Q, W, d) := (Q'_{red}, W'_{red}, d').$$

Note that right equivalent quivers with potential have the same Jacobian algebras. The *right mutation at k* $\mu_k^R(Q, W, d)$ of (Q, W, d) is defined almost identically (reduction step included), just by replacing (iii) and (iv) above by

(iii') Each arrow $a : i \to k$ of Q is replaced in Q' by an arrow $a^* : k \to i$ of degree $d(a)$.
(iv') Each arrow $b : k \to j$ of Q is replaced in Q' by an arrow $b^* : j \to k$ of degree $d(b) + r$.

2.3. **Graded QPs and cluster-tilted algebras of canonical type.** Let $\mathcal{C} = \mathcal{C}_{\mathbb{X}}$ be the cluster-category of $\mathbb{X}$, *c.f.* [4]. It follows from [3, Prop. 2.3] that $\mathcal{C}$ can be taken as the category whose objects are precisely the objects of $\operatorname{coh}\mathbb{X}$, but whose morphism spaces are given by

$$\operatorname{Hom}_{\mathcal{C}}(X, Y) := \operatorname{Hom}_{\mathbb{X}}(X, Y) \oplus \operatorname{Ext}^1_{\mathbb{X}}(X, \tau^{-1}Y).$$

Moreover, isomorphism classes in $\operatorname{coh}\mathbb{X}$ and $\mathcal{C}_{\mathbb{X}}$, and tilting sheaves in $\operatorname{coh}\mathbb{X}$ are precisely the so-called *cluster-tilting objects* in $\mathcal{C}$, *i.e.* objects $T \in \mathcal{C}$ such that $\operatorname{Hom}_{\mathcal{C}}(T, T[1]) = 0$ and such that if $X \in \mathcal{C}$ is such that $\operatorname{Hom}_{\mathcal{C}}(T \oplus X, (T \oplus X)[1]) = 0$ then $X \in \operatorname{add} T$. For a detailed study of the combinatorics of cluster-tilting objects we refer the reader to [4] and [10] for their higher counterparts.

We recall from [9] that if Λ is a finite dimensional algebra of finite global dimension n, the $n+1$-preprojective algebra of Λ is the graded algebra

$$\Pi_{n+1}(\Lambda) := \bigoplus_{i=0}^{\infty} \mathrm{Ext}^n_\Lambda(D\Lambda, \Lambda).$$

Let $T \in \mathrm{coh}\,\mathbb{X}$ be a basic tilting sheaf. The endomorphism algebra $\mathrm{End}_\mathbb{X}(T)$ has global dimension less or equal than 2, thus, by [9, Thm. 6.11(a)], the 3-preprojective algebra of Λ can be realized as a graded Jacobian algebra using the following simple construction: Let Q be the Gabriel quiver of the basic algebra $\mathrm{End}_\mathbb{X}(T)$ so that

$$\mathrm{End}_\mathbb{X}(T) \cong \frac{kQ}{\langle r_1, \dots r_s\rangle}$$

where $\{r_1, \dots, r_s\}$ is a set of minimal relations. Consider the quiver

$$\tilde{Q} = Q \amalg \{r_i^* : t(r_i) \to s(r_i) \mid r_i : s(r_i) \dashrightarrow t(r_i)\},$$

i.e. $\tilde{Q}$ is obtained from Q by adding an arrow in the opposite direction for each relation defining $\mathrm{End}_\mathbb{X}(T)$. Thus we can define a homogeneous potential W in $\tilde{Q}$ of degree 1 by

$$W := \sum_{i=1}^{s} r_i r_i^*,$$

and there is an isomorphism of graded algebras

$$\mathrm{Jac}(\tilde{Q}, W, d) \cong \Pi_3(\mathrm{End}_\mathbb{X}(T)) \cong \mathrm{End}_\mathcal{C}(T).$$

3. Mutations of cluster-tilting objects and graded QPs

In this section we describe the effect of mutation on the endomorphism algebra of a cluster-tilting object in the cluster category $\mathcal{C}_\mathbb{X}$ using the machinery of graded quivers with potential. We must mention that this was partially done by T. Hübner in [8, Kor. 4.16] who described the effect of mutation of a tilting sheaf on it's endomorphism algebra. Since both cluster categories and (graded) quivers with potential were not available at that time and although Hübner's description of the quiver was equivalent to the one that we present, describing the relations would have beed rather complicated. Thus, even if Theorem 4 is a minor refinement of *loc. cit.*, it provides a simple algorithm to compute the relations of both the endomorphism algebra of the mutated tilting sheaf and of it's associated cluster-tilted algebra.

Theorem 4. *Let $\mathbb{X}$ be an arbitrary weighted projective line and $T = \bigoplus_{i=1}^n T_i$ a basic tilting sheaf over $\mathbb{X}$ such that $\mathrm{End}_\mathcal{C}(T) \cong \mathrm{Jac}(\tilde{Q}, W, d)$,* c.f. *Section 2.3. Let $k \in \{1, \dots, n\}$ and suppose that T_k is a formal sink of T. Then there is an isomorphism of graded algebras*

$$\mathrm{End}_\mathcal{C}(\mu_k(T)) \cong \mathrm{Jac}(\mu_k^R(\tilde{Q}, W, d)).$$

Analogously, if T_k is a formal source of T, then there is an isomorphism of graded algebras

$$\mathrm{End}_\mathcal{C}(\mu_k(T)) \cong \mathrm{Jac}(\mu_k^L(\tilde{Q}, W, d)).$$

We end this notes with an example illustrating Theorem 4, *c.f.* [3, Sec. 3].

Example 5. Let $\mathbf{p} = (3, 3, 3)$ so that

$$\mathbb{L} = \langle \vec{x}, \vec{y}, \vec{z} \mid 3\vec{x} = 3\vec{y} = 3\vec{z} =: \vec{c} \rangle$$

(we do not need to worry about $\boldsymbol{\lambda}$ in this particular case). Consider the canonical configuration T of $\operatorname{coh}\mathbb{X}$, *c.f.* Proposition 2. Then $\Lambda = \operatorname{End}_{\mathbb{X}}(T)$ is given by the quiver

$$\begin{tikzcd}
& \mathcal{O}(\vec{x}) \arrow[rrr, "x_2"] & & & \mathcal{O}(2\vec{x}) \arrow[ddr, "x_3"] & \\
& & \mathcal{O}(\vec{y}) \arrow[r, "y_2"'] & \mathcal{O}(2\vec{y}) \arrow[drr, "y_3"'] & & \\
\mathcal{O} \arrow[uur, "x_1"] \arrow[urr, "y_1"'] \arrow[ddr, "z_1"'] & & & & & \mathcal{O}(\vec{c}) \arrow[lllll, dashed, gray, bend left] \\
& & & & & \\
& \mathcal{O}(\vec{z}) \arrow[rrr, "z_2"'] & & & \mathcal{O}(2\vec{z}) \arrow[uur, "z_3"'] &
\end{tikzcd}$$

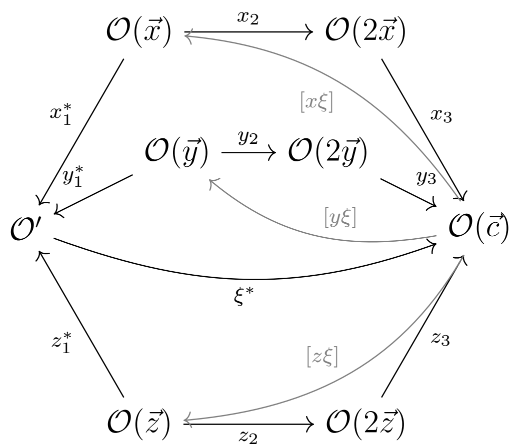
subject to the relation $x^3 + y^3 + z^3 = 0$. As explained in Section 2.3, the cluster-tilted algebra $\operatorname{End}_{\mathcal{C}}(T) \cong \Pi_3(\Lambda)$ is given by the Jacobian algebra of the graded quiver

$$\begin{tikzcd}
& \mathcal{O}(\vec{x}) \arrow[rrr, "x_2"] & & & \mathcal{O}(2\vec{x}) \arrow[ddr, "x_3"] & \\
& & \mathcal{O}(\vec{y}) \arrow[r, "y_2"'] & \mathcal{O}(2\vec{y}) \arrow[drr, "y_3"'] & & \\
\mathcal{O} \arrow[uur, "x_1"] \arrow[urr, "y_1"'] \arrow[ddr, "z_1"'] & & & & & \mathcal{O}(\vec{c}) \arrow[lllll, gray, bend left, "\xi"] \\
& & & & & \\
& \mathcal{O}(\vec{z}) \arrow[rrr, "z_2"'] & & & \mathcal{O}(2\vec{z}) \arrow[uur, "z_3"'] &
\end{tikzcd}$$

with potential

$$W = (x^3 + y^3 + z^3)\xi$$

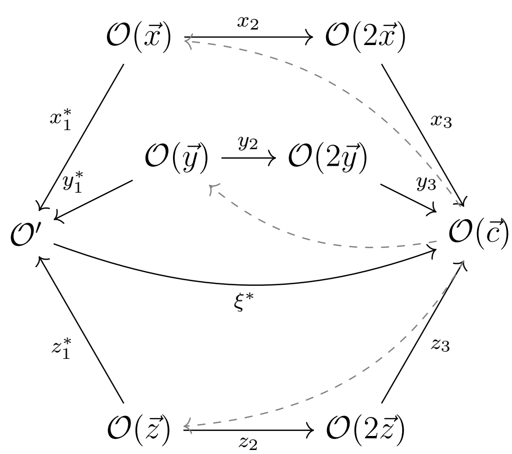
where the only arrow of degree 1 is colored gray. It is easy to see that $\mathcal{O}$ is a formal source of T since a relation in $\operatorname{End}_{\mathbb{X}}(T)$ begins at $\mathcal{O}$, *c.f.* [8, Kor. 3.5]. Then the algebra $\operatorname{End}_{\mathcal{C}}(\mu_{\mathcal{O}}T)$ is given by the Jacobian algebra of the graded quiver



with potential

$$W' = x^2[x\xi] + y^2[y\xi] + z^2[z\xi] + [x\xi]\xi^* x_1^* + [y\xi]\xi^* y_1^* + [z\xi]\xi^* z_1^*.$$

Note that we use the *left* mutation of graded quivers with potential. As explained in Section 2.3, by taking the degree zero part of $\mathrm{End}_{\mathbb{X}}(\mu_{\mathcal{O}}T)$ we obtain that $\mathrm{End}_{\mathbb{X}}(\mu_{\mathcal{O}}T)$ is isomorphic to the algebra given by the quiver



subject to the relations

$$x^2 + \xi^* x_1^* = 0$$
$$y^2 + \xi^* y_1^* = 0$$
$$z^2 + \xi^* z_1^* = 0.$$

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CLASSIFYING SERRE SUBCATEGORIES VIA ATOM SPECTRUM

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Abstract. We introduce the atom spectrum of an abelian category as a topological space consisting of all the equivalence classes of monoform objects. In terms of the atom spectrum, we give a classification of Serre subcategories of an arbitrary noetherian abelian category.

Key Words: Serre subcategory, Atom spectrum, Monoform object.

2010 *Mathematics Subject Classification*: Primary 18E10; Secondary 18E15, 16D90, 13C60.

1. Introduction

Classification of subcategories has been studied by a number of authors, for example, [2], [3], [4], [7], and [1]. Subcategories themselves are interesting objects. Moreover we expect that the structure of subcategories reflects some important properties of the whole category.

Throughout this report, we fix an abelian category $\mathcal{A}$. First of all, we recall the definition of a Serre subcategory.

Definition 1. A full subcategory $\mathcal{X}$ of $\mathcal{A}$ is called a *Serre subcategory* if it is closed under subobjects, quotient objects, and extensions.

Remark 2. This condition is equivalent to that for any short exact sequence

$$0 \to L \to M \to N \to 0$$

in $\mathcal{A}$, M belongs to $\mathcal{X}$ if and only if L and N belong to $\mathcal{X}$.

A prototype of classifications of subcategories is the following theorem shown by Gabriel [2]. For a ring R, denote by $\operatorname{Mod} R$ the category of all the R-modules and by $\operatorname{mod} R$ the category of finitely generated R-modules. We say that a subset Φ of $\operatorname{Spec} R$ is *closed under specialization* if for any $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec} R$, $\mathfrak{p} \subset \mathfrak{q}$ and $\mathfrak{p} \in \Phi$ imply $\mathfrak{q} \in \Phi$.

Theorem 3 (Gabriel [2])**.** *Let R be a commutative noetherian ring. Then we have the following bijection*

$$\begin{array}{ccc} \{\textit{Serre subcategories of } \operatorname{mod} R\} & \to & \{\Phi \subset \operatorname{Spec} R \mid \Phi \textit{ is closed under specialization}\} \\ \mathcal{X} & \mapsto & \displaystyle\bigcup_{M \in \mathcal{X}} \operatorname{Supp} M. \end{array}$$

In this report, we generalize this theorem to any abelian category with some noetherian property.

The detailed version of this paper has been submitted for publication elsewhere.

2. Monoform objects

The key notion of this report is that of monoform objects. We recall the definition of them.

Definition 4. A nonzero object H in $\mathcal{A}$ is called *monoform* if for any nonzero subobject L of H, there does not exist a nonzero subobject of H which is isomorphic to a subobject of H/L.

The following theorem states an important relationship between monoform objects and Serre subcategories.

Theorem 5. *Let M be an object in $\mathcal{A}$. M is monoform if and only if M does not belong to the smallest Serre subcategory containing all the objects of the form M/N where N is a nonzero subobject of M.*

Proposition 6. *Let H be a monoform object in $\mathcal{A}$. Then the following hold.*

(1) *Any nonzero subobject of H is also monoform.*
(2) *H is uniform, that is, for any nonzero subobjects L_1 and L_2 of H, $L_1 \cap L_2 \neq 0$.*

Definition 7. For monoform objects H and H' in $\mathcal{A}$, we say that H is *atom-equivalent* to H' if there exists a nonzero subobject of H which is isomorphic to a subobject of H'.

Remark 8. In fact, the relation of atom equivalence is an equivalence relation between monoform objects in $\mathcal{A}$ since any monoform object is uniform.

Now we define the notion of atoms, which was originally introduced by Storrer [6] in the case of module categories.

Definition 9. Denote by $\operatorname{ASpec}\mathcal{A}$ the quotient set (or quotient class) of the set of monoform objects in $\mathcal{A}$ by atom equivalence. We call it the *atom spectrum* of $\mathcal{A}$. Elements of $\operatorname{ASpec}\mathcal{A}$ are called *atoms* in $\mathcal{A}$. The equivalence class of a monoform object H in $\mathcal{A}$ is denoted by $\overline{H}$.

In section 4, we see that there exists a bijection between $\operatorname{ASpec}(\operatorname{Mod} R)$ and $\operatorname{Spec} R$. Hence the atom spectrum is a generalization of the prime spectrum in the commutative ring theory.

Definition 10. Let M be an object in $\mathcal{A}$.

(1) Define the *atom support* of M by
$$\operatorname{ASupp} M = \{\overline{H} \in \operatorname{ASpec}\mathcal{A} \mid H \text{ is a subquotient of } M\}.$$
(2) Define the set of *associated atoms* of M by
$$\operatorname{AAss} M = \{\overline{H} \in \operatorname{ASpec}\mathcal{A} \mid H \text{ is a subobject of } M\}.$$

The following proposition is a generalization of a proposition which is well-known in the commutative ring theory.

Proposition 11. *Let $0 \to L \to M \to N \to 0$ be a short exact sequence in $\mathcal{A}$. Then the following hold.*

(1) $\operatorname{ASupp} M = \operatorname{ASupp} L \cup \operatorname{ASupp} N$.
(2) $\operatorname{AAss} L \subset \operatorname{AAss} M \subset \operatorname{AAss} L \cup \operatorname{AAss} N$.

3. Main theorem

In order to generalize Gabriel's theorem (Theorem 3), we need to consider a generalized condition of "closed under specialization". This condition is given by the following topology.

Definition 12. Define a topology on $\operatorname{ASpec}\mathcal{A}$ as follows: we say that a subset (or subclass) Φ of $\operatorname{ASpec}\mathcal{A}$ is open if for any α, there exists $H \in \alpha$ such that $\operatorname{ASupp} H \subset \Phi$.

Proposition 13. *Open subsets of* $\operatorname{ASpec}\mathcal{A}$ *define a topology on* $\operatorname{ASpec}\mathcal{A}$ *which has an open basis* $\{\operatorname{ASupp} M \mid M \in \mathcal{A}\}$.

We recall the definition of noetherian abelian categories.

Definition 14.
1. An object M in $\mathcal{A}$ is called *noetherian* if for any ascending chain $L_0 \subset L_1 \subset \cdots$ of subobjects of M, there exists $n \geq 0$ such that $L_n = L_{n+1} = \cdots$.
2. An abelian category $\mathcal{A}$ is called *noetherian* if it is skeletally small (that is, the class of isomorphism classes forms a set), and any object in $\mathcal{A}$ is noetherian.

Remark 15. The skeletally smallness is just a set-theoretical assumption. It $\mathcal{A}$ ensures that $\operatorname{ASpec}\mathcal{A}$ is a set. We do not need to assume it if we allow $\operatorname{ASpec}\mathcal{A}$ to be a proper class.

Theorem 16 ([5])**.** *Let $\mathcal{A}$ be a noetherian abelian category. Then there exists a bijection*

$$\begin{array}{rcl} \{\textit{Serre subcategories of } \mathcal{A}\} & \to & \{\textit{open subsets of } \operatorname{ASpec}\mathcal{A}\} \\ \mathcal{X} & \mapsto & \displaystyle\bigcup_{M \in \mathcal{X}} \operatorname{ASupp}\mathcal{X}. \end{array}$$

The inverse map is given by $\Phi \mapsto \{M \in \mathcal{A} \mid \operatorname{ASupp} M \subset \Phi\}$.

4. In the case of module categories

In the case of module categories, the atom spectrum is described in terms of one-sided ideals.

Proposition 17. *Let R be a ring. Then any atom in* $\operatorname{Mod} R$ *is represented by a monoform object of the form $R/\mathfrak{p}$, where $\mathfrak{p}$ is a right ideal of R. Moreover if R is right noetherian, then* $\operatorname{ASpec}(\operatorname{mod} R)$ *is homeomorphic to* $\operatorname{ASpec}(\operatorname{Mod} R)$.

Proposition 18. *Let R be a commutative ring. Then the following hold.*

1. *For any ideal $\mathfrak{a}$ of R, $R/\mathfrak{a}$ is monoform in* $\operatorname{Mod} R$ *if and only if $\mathfrak{a}$ is a prime ideal of R.*
2. *For any prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ of R, $R/\mathfrak{p}$ is atom-equivalent to $R/\mathfrak{q}$ if and only if $\mathfrak{p} = \mathfrak{q}$. Therefore the correspondence $\mathfrak{p} \mapsto \overline{R/\mathfrak{p}}$ gives a bijection* $\operatorname{Spec} R \to \operatorname{ASpec}(\operatorname{Mod} R)$.
3. *For any R-module M,* $\operatorname{ASupp} M = \operatorname{Supp} M$, *and* $\operatorname{AAss} M = \operatorname{Ass} M$.
4. *For any subset Φ of* $\operatorname{Spec} R$, *Φ is open in the sense of* $\operatorname{ASpec}(\operatorname{Mod} R)$ *if and only if Φ is closed under specialization.*

Remark 19. In the case where R is noetherian, we can formulate these claims by using $\operatorname{ASpec}(\operatorname{mod} R)$ instead of $\operatorname{ASpec}(\operatorname{Mod} R)$. Then new claims which we obtain also hold.

In the case of artinian rings, monoformness is stated in terms of composition factors.

Proposition 20. *Let R be a right artinian ring. Then a finitely generated R-module M is monoform if and only if it has simple socle S such that there exists no other composition factor of M which is isomorphic to S.*

Proposition 21. *Let R be a right artinian ring and $\{S_1, \ldots, S_n\}$ be a maximal set of pairwise nonisomorphic simple modules. Then* $\operatorname{ASpec} R = \{\overline{S_1}, \ldots, \overline{S_n}\}$ *with the discrete topology.*

Example 22. Let R be the ring of lower triangular matrices over a field K, that is,

$$R = \begin{bmatrix} K & 0 \\ K & K \end{bmatrix}.$$

Then all the right ideals of R are

$$0, \begin{bmatrix} 0 & 0 \\ K & 0 \end{bmatrix}, \mathfrak{p}_a = K \begin{bmatrix} 1 & 0 \\ a & 0 \end{bmatrix} (a \in K), \mathfrak{m}_1 = \begin{bmatrix} 0 & 0 \\ K & K \end{bmatrix}, \mathfrak{m}_2 = \begin{bmatrix} K & 0 \\ K & 0 \end{bmatrix}, R.$$

All the comonoform right ideals of R are

$$\mathfrak{p}_a (a \in K), \mathfrak{m}_1, \mathfrak{m}_2.$$

Since

$$\frac{R}{\mathfrak{p}_a} \cong \begin{bmatrix} K & K \end{bmatrix}, \frac{R}{\mathfrak{m}_1} \cong \begin{bmatrix} K & 0 \end{bmatrix}, \frac{R}{\mathfrak{m}_2} \cong \frac{\begin{bmatrix} K & K \end{bmatrix}}{\begin{bmatrix} K & 0 \end{bmatrix}},$$

we have $\widetilde{\mathfrak{p}_a} = \widetilde{\mathfrak{m}_1} \neq \widetilde{\mathfrak{m}_2}$. Therefore all the Serre subcategories of $\operatorname{mod} R$ are $\{\text{zero objects}\}$, $\langle R/\mathfrak{m}_1 \rangle_{\text{Serre}}$, $\langle R/\mathfrak{m}_2 \rangle_{\text{Serre}}$, and $\operatorname{mod} R$, where $\langle R/\mathfrak{m}_i \rangle_{\text{Serre}}$ is the smallest Serre subcategory containing $R/\mathfrak{m}_i$.

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QUIVER VARIETIES AND QUANTUM CLUSTER ALGEBRAS

YOSHIYUKI KIMURA

ABSTRACT. Inspired by a previous work [Nak11] of Nakajima, we consider a class of (equivariant) perverse sheaves on acyclic graded quiver varieties and study the Fourier-Sato-Deligne transform from representation theoretical point of view. In particular, we get a monoidal categorification of quantum cluster algebra with specific coefficient. As a corollary, the strong positivity conjecture is verified. This is based on a talk in the 45th Symposium on Ring Theory and Representation Theory in Shinshu University and a preprint [KQ12].

1. INTRODUCTION

Cluster algebras were invented by Fomin and Zelevinsky in [FZ02] with an aim to provide concrete and combinatorial formalism for the study of Lusztig's dual canonical basis and total positivity. They are commutative algebras generated by certain combinatorially defined generators (*the cluster variables*). The quantum deformations were defined in[BZ05]. Fomin and Zelevinsky stated their original motivation as follows:

> "We conjecture that the above examples can be extensively generalized: for any simply-connected connected semisimple group G, the coordinate rings $\mathbb{C}[G]$ and $\mathbb{C}[G/N]$, as well as coordinate rings of many other interesting varieties related to G, have a natural structure of a cluster algebra. This structure should serve as an algebraic framework for the study of dual canonical bases in these coordinate rings and their q-deformations. In particular, we conjecture that all monomials in the variables of any given cluster (the *cluster monomials*) belong to this dual canonical basis."

However, despite the many successful applications of (quantum) cluster algebras to other areas (cf. the introductory survey by Keller [Kel12] and Geiss, Leclerc and Schröer [GLS12]), the link between (quantum) cluster monomials and the dual canonical basis of quantum groups remains largely elusive.

Also, the following positivity conjecture has attracted a lot of interest since the invention of cluster algebras.

Conjecture 1 (Laurent positivity conjecture)**.** *With respect to any given seed, each cluster variable expands into a Laurent polynomial with non-negative integer coefficients.*

This conjecture has been proved for cluster algebras arising from surfaces by Musiker, Schiffler, and Williams [MSW11], for cluster algebras containing a bipartite seed by Nakajima [Nak11], and the quantized version for quantum cluster algebras with respect to an acyclic initial seed by [Qin12a]. Recently, Efimov [Efi11]obtained further partial results

The detailed version of this paper [KQ12] has been submitted for publication elsewhere.

on this conjecture for quantum cluster algebras containing an acyclic seed using mixed Hodge modules.

In [HL10], Hernandez and Leclerc proposed monoidal categorification of cluster algebra.

Definition 2 (monoidal categorification)**.** Let $\mathcal{A}$ be a cluster algebra (of geometric type). Let $\mathcal{C}$ be a monoidal abelian category. We say that $\mathcal{C}$ is a monoidal categorification of $\mathcal{A}$ if the Grothendieck ring $K_0(\mathcal{C})$ of $\mathcal{C}$ is isomorphic to $\mathcal{A}$ as ring and the basis of $K_0(\mathcal{C})$ which consists of simple objects of $\mathcal{C}$ includes the set of cluster monomials[1].

The existence of monoidal categorification of cluster algebra yields the following consequence on cluster algebras (with geometric coefficients).

Conjecture 3 (strong positivity conjecture)**.** *Let $\mathcal{A}$ be a cluster algebra (with geometric coefficients $\mathbb{ZP}$). Then there exists a $\mathbb{Z}$-basis $\mathcal{B}$ of $\mathcal{A}$ which contains the set of cluster monomials and has non-negative structure constants.*

In [HL10] they gave a conjecture on the monoidal categorification of T-system cluster algebra (with level ℓ) using the tensor subcategory $\mathcal{C}_\ell$ of finite dimensional representations of (untwisted) quantum affine algebras and proved $\ell = 1$ case for A_n and D_4.In [Nak04, Nak11], Nakajima studied finite dimensional representation of quantum affine algebra via perverse sheaves on graded quiver varieties and gave a proof of $\ell = 1$ case for bipartite quiver.

In [KQ12, Qin12b], we studied graded quiver varieties which are associated with acyclic quiver and generalized the Nakajima's proof for $\ell = 1$ cases using the Nakayama functor on quiver representations.

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2. Quantum cluster algebras

2.1. **Quantum cluster algebras.** We briefly recall the definition of (quantum) cluster algebras. For more details, see [KQ12]. A quiver $Q = (Q_0, Q_1)$ is an oriented graph where Q_0 is a set of vertices and Q_1 a set of arrows. For each arrow α, we denote its outgoing vertex by $\mathrm{out}(h)$ and its incoming vertex by $\mathrm{in}(h)$. For a quiver Q, we associate doubled quiver H by adding opposite arrows $\overline{Q_1} := \{\overline{\alpha}\colon \mathrm{in}(\alpha) \to \mathrm{out}(\alpha) \mid \alpha \in Q_1\}$. We say $(Q_0, \overline{Q_1})$ as opposite quiver. Sometimes we also denote Q_0 and Q_1 by I and Ω respectively. We say that Q is p-acyclic if Q does not contain oriented cycles whose length are less than p and is acyclic if Q does not contain any oriented cycles. For 2-acyclic quiver Q (with frozen vertices), we can define cluster algebra $\mathcal{A}(Q)$ (with geometric coefficients).

[1] We remark that the correspondence between the set of isomorphism class of prime real simple objects in $\mathcal{C}$ and the set of cluster variables is required in [HL10]. Under the monoidal categorification, the correspondence can be shown in [GLS11c, Corollary 8.6]

Berenstein and Zelevinsky[BZ05] have introduced a quantum analogue of cluster algebra with geometric coefficients using quantum torus.

Let v be a formal parameter and we consider a ring $\mathbb{Z}[v^{\pm 1}]$ or $\mathbb{Q}[v^{\pm 1}]$. Let $m \geq n$ be be two positive integers. Let Λ be an $m \times m$ skew-symmetric integer matrix and $\tilde{B}$ an $m \times n$ integer matrix. The upper $n \times n$ submatrix of $\tilde{B}$, denoted by B, is called principal part of $\tilde{B}$.

Definition 4. The pair $(\Lambda, \widetilde{B})$ is called compatible if we have $\Lambda(-\widetilde{B}) = \begin{bmatrix} D \\ 0 \end{bmatrix}$ for some $n \times n$ diagonal matrix D whose diagonal entries are strictly positive integers. It is called a unitary compatible pair if moreover D is the identity matrix 1_n. The matrix Λ is called the Λ-matrix of $(\Lambda, \widetilde{B})$ and the matrix $\widetilde{B}$ is called the B-matrix of $(\Lambda, \widetilde{B})$.

We write $\Lambda(g, h)$ for $g^t \Lambda h, g, h \in \mathbb{Z}^m$, where g^t is the transpose of $g \in \mathbb{Z}^m$ as matrix.

Definition 5. The quantum torus $\mathcal{T} = \mathcal{T}(\Lambda)$ over $\mathbb{Z}[v^{\pm}]$ is the Laurent polynomial ring $\mathbb{Z}[v^{\pm 1}][x_1^{\pm 1}, ..., x_m^{\pm 1}]$, endowed with the following twisted product $*$ such that we have

$$x^g x^h = v^{\Lambda(g,h)} x^{g+h}$$

for any $g, h \in \mathbb{Z}^m$. Here for any $g = (g_i) \in \mathbb{Z}^m$, x^g denote the monomial $\prod_{1 \leq i \leq m} x_i^{g_i}$.

For $\epsilon \in \{\pm 1\}$, we define $m \times m$-matrix $E_\epsilon = (e_{ij})$ and $n \times n$-matrix $F_\epsilon = (f_{ij})$ as follows.

$$e_{ij} = \begin{cases} \delta_{ij} & \text{if } j \neq k \\ -1 & \text{if } i = j = k \\ \max(0, -\epsilon b_{ik}) & \text{if } i \neq k, j = k, \end{cases}$$

$$f_{ij} = \begin{cases} \delta_{ij} & \text{if } i \neq k \\ -1 & \text{if } i = j = k \\ \max(0, \epsilon b_{kj}) & \text{if } i = k, j \neq k. \end{cases}$$

Fix a compatible pair $(\Lambda, \widetilde{B})$ and the quantum torus $\mathcal{T} = \mathcal{T}(\Lambda)$. Since quantum torus is a Ore domain, we can consider its fraction skew-field $\mathcal{F}$ and $\mathcal{T}$ can be considered as a subalgebra of $\mathcal{F}$.

Definition 6. (1)A quantum seed is a tuple $(\Lambda, \widetilde{B}, (x_i)_{1 \leq i \leq m})$ where $\{x_i\}_{1 \leq i \leq m} \subset \mathcal{F}$ and $(\Lambda, \widetilde{B})$ a compatible pair.

(2) For a quantum seed and $1 \leq k \leq n$, we define quantum seed mutation $\mu_k(\Lambda, \widetilde{B}, (x_i)_{1 \leq i \leq m}) = (\Lambda', \widetilde{B'}, (x'_i)_{1 \leq i \leq m})$ as follows.

$$\Lambda' = E_\epsilon(B)^t \Lambda E_\epsilon(B),$$
$$\widetilde{B'} = E_\epsilon(B)^t \widetilde{B} F_\epsilon(B),$$
$$x'_i = \begin{cases} x_i & \text{if } i \neq k \\ x'_k & \text{if } i = k, \end{cases}$$

where x'_k is defined in the following equation.

$$x_k x'_k = v^{\Lambda(e_k, \sum_{1\le i\le m}[b_{ik}]_+ e_i)} \prod_{1\le i\le m} x_i^{[b_{ik}]_+} + v^{\Lambda(e_k, \sum_{1\le i\le m}[-b_{ik}]_+ e_i)} \prod_{1\le i\le m} x_i^{[-b_{ik}]_+}.$$

(3) Let $\mathbb{T}_n$ be the regular n-tree with distinct colors $\{1, \cdots, n\}$ at each vertices. Quantum cluster pattern is an assigment of quantum seed from $\mathbb{T}_n$ such that we have

$$(\Lambda(t'), \widetilde{B}(t'), (x_i(t'))_{1\le i\le m}) = \mu_k(\Lambda(t), \widetilde{B}(t), (x_i(t))_{1\le i\le m})$$

for each edge $t - t'$ which is colored by k.

(4) For a quantum cluster pattern, we set $\mathcal{X}_q = \bigcup_t \{x_i(t)\}_{1\le i\le m}$ and call by the set of quantum cluster variables. The quantum cluster algebra $\mathcal{A}_q$ is the $\mathbb{Z}[v^{\pm 1}]$-subalgebra which is generated by $\mathcal{X}_q$.

Quantum Laurent phenomena say that $\mathcal{A}_q$ is a subalgebra of the quantum torus $\mathcal{T}$ (cf. [BZ05]). For a quiver whose principal part is acyclic, it is known that quantum Laurent expansion at initial seed can be written as a generating function the Serre polynomial of the quiver Grassmannian associated with the correponding cluster tilting object. For more details, see [Qin12a]. For a Hodge-theoretic interpretation of quantum Laurent phoenomena, see [Efi11].

3. Quiver varieties

3.1. **Definition.** For a quiver Q, we consider a repetition quiver $\widehat{Q}$ as follows:

$\widehat{Q}_0 = Q_0 \times (1 + 2\mathbb{Z})$

$\widehat{Q}_1 = \{(\alpha, n)\colon (\mathrm{out}(\alpha), n) \to ()\}_{\alpha\in Q_1, n\in\mathbb{Z}} \cup \{\sigma(\alpha, n)\colon (\mathrm{in}(\alpha), n) \to (\mathrm{out}(\alpha), n-2)\}$

For an acyclic quiver Q, we consider a repetition quiver $\widehat{\Gamma} = (\widehat{\Gamma}_0, \widehat{\Gamma_1})$ with $\widehat{\Gamma}_0 = Q_0 \times \mathbb{Z}$ which contains $\widehat{Q}$ as a fullsubquiver on $Q_0 \times (1 + 2\mathbb{Z})$. We also add new arrows $\{\sigma(\alpha, n)\colon (\mathrm{in}(\alpha), 2n+1) \to (\mathrm{out}(\alpha), 2n-1)\}_{\alpha\in Q_1, n\in\mathbb{Z}}$ and $\{a(i, n)\colon (i, 2n+1) \to (i, 2n)\}_{i\in Q_0, n\in\mathbb{Z}} \cup \{b(i, n)\colon (i, 2n) \to (i, 2n-1)\}_{i\in Q_0, n\in\mathbb{Z}}$. Let $\mathcal{R}$ be a mesh category supported only on $Q_0 \times (1 + 2\mathbb{Z})$ and $\mathcal{S}$ be a fullsubcategory of $\mathcal{R}$ supported on $Q_0 \times 2\mathbb{Z}$.

We consider a finite dimensional $\widehat{\Gamma_0}$-graded vector space $V \oplus W$ where V is the $\widehat{Q_0}$-component and W is the $Q_0 \times 2\mathbb{Z}$-component.

Let $\mathrm{Rep}_{V\oplus W}(\mathcal{R})$ be a variety of representations of $\mathcal{R}$-module whose dimension vector is $V \oplus W$. A point $(B, \alpha, \beta) \in \mathrm{Rep}_{V\oplus W}(\mathcal{R})$ is said to be stable (resp. costable) if the following condition holds:

If a $\widehat{Q_0}$-graded subspace V' of V is B-invariant and contained in $\mathrm{Ker}(\beta)$ (resp. contains $\mathrm{Im}(\alpha)$), then $V' = 0$ (resp. $V' = V$).

We denote by $\mathrm{Rep}_{V\oplus W}(\mathcal{R})^{st}$ the (possibly empty) set of stable points.

Definition 7 (graded quiver varieties)**.** (1) The set-theoretical quotient $\mathcal{M}(V, W) = \mathrm{Rep}_{V\oplus W}(\mathcal{R})^{st}/G_V$ of the set of stable points with respect to the group action defined by base change of the product of general linear groups G_V is called smooth graded quiver variety.

(2) The affine algebraic-geometric quotient $\mathcal{M}_0(V, W) = \mathrm{Rep}_{V\oplus W}(\mathcal{R})//G_V$ is called affine graded quiver variety.

The smooth graded quiver variety can be defined as a homogenous spectrum of semi G_V invariants of $\mathrm{Rep}_{V\oplus W}(\mathcal{R})$ with respect to a character $\chi\colon G_V \to \mathbb{G}_m$. So there is a natural (G_W-equivariant) projective morphism $\pi\colon \mathcal{M}(V,W) \to \mathcal{M}_0(V,W)$ by general theory of geometric invariant theory. Since $\mathcal{M}_0(V,W)$ parametrizes semisimple representations, we can consider its union along all V. We denote it by $\mathcal{M}_0(W)$. The following gives a "description" of $\mathcal{M}_0(W)$ and is due to Leclerc-Plamondon [LP12] based on a result by Lusztig.

Theorem 8. *We have a natural G_W-equivariant isomorphism $\Phi_0\colon \mathcal{M}_0(W) \simeq \mathrm{Rep}_W(\mathcal{S})$.*

Let $\mathcal{P}_W$ be the set of isomorphism class of (G_W-equivariant) simple perverse sheaves on $\mathcal{M}_0(W)$ which appear in $\pi_!\mathbb{C}_{\mathcal{M}(V,W)}$ for some shifts and V and $\mathcal{Q}_W$ be the fullsubcategory of $D^b(\mathcal{M}_0(W))$ which is generated by $\mathcal{P}_W$ by shifts and direct sums. Let K_W be the quantum Grothendieck group of $\mathcal{Q}_W$ which is defined by shifts and direct sum and has a structure of $\mathbb{Z}[v^{\pm 1}]$-module.

We have a natural stratification on $\mathcal{M}_0(W)$ and the classification of $\mathcal{P}_W$ in terms of the stratification.

Let $\mathcal{M}_0^{reg}(V,W)$ be the (possible empty) open subsets of $\mathcal{M}_0(V,W)$ which consists of closed G_V-orbits whose stablizer is trivial and the dimension vector (V,W) is called dominant if $W - C_Q(z)V \geq 0$, where $C_Q(z)$ is the quantum Cartan matrix defined by

$$C_Q(z)_{ij} = \#\left\{h \in \Omega \,\middle|\, {}^{\mathrm{out}(h)=i}_{\mathrm{in}(h)=j}\right\} z - \left\{h \in \Omega \,\middle|\, {}^{\mathrm{out}(h)=j}_{\mathrm{in}(h)=i}\right\} z^{-1},$$

where $z\colon \mathbb{Z}^{\widehat{\Gamma_0}} \to \mathbb{Z}^{\widehat{\Gamma_0}}$ is the shift defined by $(zW)_i(n) = W_i(n-1)$.

Theorem 9. (1) *$\mathcal{M}_0(V,W)$ is not empty if and only if (V,W) is dominant. If (V,W) is dominant, $\mathcal{M}(V,W)$ is connected.*

(2) $\mathcal{M}_0(W) = \bigsqcup \mathcal{M}_0^{reg}(V,W)$

(3) $\mathcal{P}_W = \{\mathbf{IC}_W(V) \mid (V,W)$ *is dominant*$\}$, *where* $\mathbf{IC}_W(V) := \mathbf{IC}(\mathcal{M}_0(V,W),\mathbb{C})$ *is the intersection cohomology complex associated with the stratum $\mathcal{M}_0^{reg}(V,W)$.*

3.2. **Quantum Grothendieck ring.** Let $0 \subset W^2 \subset W$ be a $\mathcal{S}_0$-graded subspace and $W^1 = W/W^2$ and fix a splitting $W \simeq W^1 \oplus W^2$. Let $\lambda\colon \mathbb{G}_m \to G_W$ be the 1-parameter subgroup defined by $\lambda(t) = \mathrm{id}_{W^1} \oplus t\mathrm{id}_{W^2}$. Then $\mathbb{G}_m$ acts on $\mathcal{M}(V,W)$ and $\mathcal{M}_0(W)$. Let $\mathcal{T}_0(W^1,W^2)$ be the closed subvariety of $\mathcal{M}_0(W)$ which consists of points such that $\lim_{t\to 0} t\cdot[B,\alpha,\beta]$ exists. Then we have the following diagram:

$$\mathcal{M}_0(W^1)\times\mathcal{M}_0(W^2) \xleftarrow[\kappa_0]{} \mathcal{T}_0(W^1,W^2) \xrightarrow[\iota_0]{} \mathcal{M}_0(W)\ ,$$

where $\kappa_0\colon \mathcal{T}_0(W^1,W^2) \to \mathcal{M}_0(W^1)\times\mathcal{M}_0(W^2)$ be the morphism defined by taking limit $\lim_{t\to 0} t\cdot[B,\alpha,\beta]$ and $\iota_0\colon \mathcal{T}_0(W^1,W^2) \hookrightarrow \mathcal{M}_0(W)$ be the closed embedding. Let $\widetilde{\mathrm{Res}} := (\kappa_0)_!\iota_0^*\colon D^b(\mathcal{M}_0(W)) \to D^b(\mathcal{M}_0(W^1)\times\mathcal{M}_0(W^2))$ be the restriction functor defined by the above morphism. It can be shown that $\widetilde{\mathrm{Res}}(\mathcal{Q}_W) \subset \mathcal{Q}_{W^1}\boxtimes\mathcal{Q}_{W^2}$. Using the restriction functor (with some shifts), we get the following definition of quantum Grothendieck ring.

Definition 10. Let $\mathcal{R}_v$ be the subring of $\prod_W \mathrm{Hom}_{\mathbb{Z}[v^{\pm 1}]}(K_W,\mathbb{Z}[v^{\pm 1}])$ which consists the $\mathbb{Z}[v^{\pm 1}]$-linear module homomorphisms satisfy

$$\langle f_W, \mathrm{IC}_W(V)\rangle = \langle f_{W-C_Q(z)V}, \mathrm{IC}_{W-C_Q(z)V}(0)\rangle$$

for arbitrary dominant (V,W).

Let $\{L_W(V)\}$ be the dual basis of $\{\mathbf{IC}(\mathcal{M}(V,W))\}$ and $L=\{L_W\}$ be the basis of $\mathcal{R}_v$ which is determined by $\{L_W(0)\}$. It is known that L has positive structure constants and there is a embedding $\mathcal{R}$ into quantum torus using the generating function with respect to the pairing with $\{\pi_W(V)\}$, where $\pi_W(V)=\pi_!\mathbb{C}_{\mathcal{M}(V,W)}[\dim \mathcal{M}(V,W)]$.

We consider the support condition $(*)_\ell$ on $W=\bigoplus_{(i,n)\in\mathcal{S}_0} W_i(n)$:

$$W_i(n)=0 \text{ unless } n\in\{0,2,\cdots,2\ell\}.$$

Let $\mathcal{R}_{v,\ell}$ be the $\mathbb{Z}[v^{\pm1}]$-subalgebra which satisfies the support condition $(*)_\ell$ and $\mathcal{R}_\ell$ be the specialization at $v=1$. It can be shown that $L\mid_{v=1}\cap\mathcal{R}_\ell$ gives a basis of $\mathcal{R}_\ell$.

3.3. **T-system quiver.** For an acyclic quiver $Q=(Q_0,Q_1)$ and non-negative integer ℓ, we consider the following ice quiver $T_{Q,\ell}$. Let $(T_{Q,\ell})_0$ be the set $Q_0\times\{0,1,\cdots,\ell\}$ and $(T_{Q,\ell})_1=\{(\alpha,k)\colon(\mathrm{out}(\alpha),k)\to(\mathrm{in}(\alpha),k)\}_{\alpha\in Q_1,0\le k\le\ell-1}\cup\{\sigma(\alpha,k)\colon(\mathrm{in}(\alpha),k)\to(\mathrm{out}(\alpha),k-1)\}_{\alpha\in Q_1,1\le k\le\ell}\cup\{t_{i,k}\colon(i,k)\to(i,k+1)\}_{i\in Q_0,0\le k\le\ell-1}$. We call $T_{Q,\ell}$ by T-system quiver with level ℓ and we set $(T_{Q,\ell})_0^{\mathrm{fr}}=Q_0\times\{\ell\}$.

It is a special case of the quivers in [BFZ05] and [GLS11b, GLS11a] which are associated with the $\ell+1$ power $c_Q^{\ell+1}$ of the acyclic Coxeter element c_Q and the corresponding unipotent subgroup $N(c_Q^{\ell+1})$.

Conjecture 11. *There is a ring isomorphism $\Phi\colon\mathcal{A}(T_{Q,\ell})\simeq\mathcal{R}_\ell$ and the image of the cluster monomials is contained in the basis $L\mid_{v=1}\cap\mathcal{R}_\ell$.*

We remark that the subring $\mathcal{R}_\ell$ is an analogue of $K_0(\mathcal{C}_\ell)$, where $\mathcal{C}_\ell$ is the tensor subcategory in [HL10] and there is a natural quantum analogue between the quantum cluster algebra [GLS11a] and the (twisted) quantum Grothendieck ring. This should yields the quantization conjecture in [Kim12].

4. Level 1 case

We prove the above conjecture holds in $\ell=1$ case.

4.1. **Description of quiver varieties.** We consider the $\ell=1$ case. Let $W=\bigoplus_{(i,n)\in\mathcal{S}_0}W_i(n)$ be $\mathcal{S}_0$-graded vector space such that $W_i(n)=0$ unless $n\in\{0,2\}$. Since the full subquiver of $\mathcal{S}$ on $Q_0\times\{0,2\}$ does not contain oriented cycles and the mesh relations, $\mathrm{Rep}_W(\mathcal{S})$ is an affine space. Let S_i be a simple module of Q, I_i be an injective envelop of S_i and P_i be the projective cover of S_i.

Proposition 12. *For $W=W(0)\oplus W(2)$ be $\mathcal{S}_0$-graded vector space such that $W_i(n)=0$ unless $n\in\{0,2\}$. We set $P^{W(2)}=\bigoplus_{i\in Q_0}W_i(2)\otimes P_i$ and $I^{W(0)}=\bigoplus_{i\in Q_0}W_i(0)\otimes I_i$. Then we have an isomorphism:*

$$\Phi_0\colon\mathcal{M}_0(W)\simeq\mathbf{E}_W:=\mathrm{Hom}_Q(P^{W(2)},I^{W(0)}).$$

We also assume $\hat{Q}_0$-graded vector space V satisfies $V_i(n)=0$ unless $n\in\{1\}$. Let $\mathcal{F}(V,W)$ be the quiver Grassmann of $I^{W(0)}$ with dimension vector $V(1)$. Let $\widetilde{\mathcal{F}}(V,W)$ be the variety of pairs (z,S) with $z\in\mathrm{Hom}_Q(P^{W(2)},I^{W(0)})$ and $S\in\mathcal{F}(V,W)$ which satisfy $\mathrm{Im}(z)\subset S$. Let $\pi\colon\widetilde{\mathcal{F}}(V,W)\to\mathbf{E}_W$ be the first projection.

Proposition 13. *We have a G_W-equivariant isomorphism $\Phi\colon \mathcal{M}(V,W) \simeq \widetilde{\mathcal{F}}(V,W)$ which satisfies the following commutative diagram:*

$$\begin{tikzcd}
\mathcal{M}(V,W) \arrow[r, "\Phi"] \arrow[d, "\pi"] & \widetilde{\mathcal{F}}(V,W) \arrow[d, "\pi"] \\
\mathcal{M}_0(W) \arrow[r, "\Phi_0"'] & \mathbf{E}_W
\end{tikzcd}.$$

4.2. **Fourier-Deligne-Sato transform.** Since $\widetilde{\mathcal{F}}(V,W)$ is a vector subbundle over $\mathcal{F}(V,W)$ of the trivial bundle $\mathbf{E}_W \times \mathcal{F}(V,W)$, we consider its annihilator bundle $\widetilde{\mathcal{F}}^{\perp}(V,W) \subset \mathbf{E}^*{}_W \times \mathcal{F}(V,W)$. By the Nakayama duality, we have $\mathbf{E}_W^* \simeq \mathrm{Hom}_Q(I^{W(0)}, I^{W(2)})$ and

$$\widetilde{\mathcal{F}}^{\perp}(V,W) = \{(z^*, S) \in \mathbf{E}_W^* \times \mathcal{F}(V,W) | S \subset \mathrm{Ker}(z^*)\}.$$

Let $\pi^{\perp}\colon \widetilde{\mathcal{F}}^{\perp}(V,W) \to \mathbf{E}_W^*$ be the first projection. Then the fiber $(\pi^{\perp})^{-1}(z^*)$ is the quiver Grassmannian of $\mathrm{Ker}(z^*)$ with dimension vector $V(1)$. Let $\Psi\colon D^b(\mathbf{E}_W) \simeq D^b(\mathbf{E}_W^*)$ be the Fourier-Deligne-Sato transform and $\mathcal{L}_W$ be the subset of $\mathcal{P}_W$ which consists the Fourier transform $\Psi(\mathbf{IC}_W(V))$ has entire support $\mathbf{E}_W^*$. We note that $\mathbf{IC}_W(0) \in \mathcal{L}_W$.

We consider the following alternating sum of L_W:

$$\mathbb{L}_W = \sum_{\mathbf{IC}_W(V) \in \mathcal{L}_W} (-1)^{\dim \mathcal{M}(V,W)} \mathrm{rank}\Psi(\mathbf{IC}_W(V)) L_{W - C_Q(z)},$$

where $\mathrm{rank}\Psi(\mathbf{IC}_W(V))$ is the generic rank of $\Psi(\mathbf{IC}_W(V))$. It can be shown that $\mathbb{L}_W$ yields the quantum cluster character in [Qin12a].

Let $A_W = \mathrm{Aut}_Q(I^{W(0)}) \times \mathrm{Aut}_Q(I^{W(2)})$ be the automorphism group. We have natural projection of groups $A_W \to G_W$. By construction $\pi^{\perp}$ is equivariant with respect to the A_W-action, so the simple perverse sheaves which can be obtained by $\pi^{\perp}$ are A_W-equivariant perverse sheaves. By considering A_W-action, we get the following characterization.

Theorem 14. *If $\mathbf{E}_W^*$ has an open A_W-orbit, we have $\mathcal{L}_W = \{\mathbf{IC}_W(0)\}$.*

The sufficient condition for which $\mathbf{E}_W^*$ contains an open A_W-orbit can be characterized by the canonical decomposition of injective presentation by Derksen-Fei[DF09]. In particular, it can be shown that the set of quantum cluster monomials is contained in the "dual canonical basis" $\{L_W\}$. So we get the proof of the conjecture for $\ell = 1$ case.

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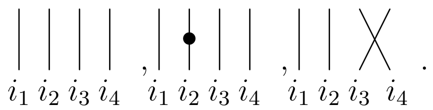
CYCLOTOMIC KLR ALGEBRAS OF CYCLIC QUIVERS

MASAHIDE KONISHI

Abstract. For cyclic quiver, cyclotomic KLR algebras are defined by fixing α and Γ, two weights on vertices. We fix α and Γ in a special (but essential) case, and then show that there are systematic changes of structures.

1. Introduction

Khovanov-Lauda-Rouquier algebra (KLR algebra for short) is defined by Khovanov and Lauda, and independently Rouquier in 2008. Generators and Relations are obtained from a quiver Γ and a weight α on its vertices. We can regard generators as concatenation of such diagrams :



An another weight Λ on vertices of Γ defines a cyclotomic ideal. We call a quotient of the KLR algebra by the cyclotomic ideal a cyclotomic KLR algebra. After here, we fix quiver Γ as its vertices are $\{0, 1, 2, \cdots, n-1\}$, and its arrows are from i to $i+1$ (also $n-1$ to 0), and set $\alpha = \sum_{i:vertex} \alpha_i$, $\Lambda = \Lambda_0$.

Our aim is to describe changes of structures of cyclotomic KLR algebras for n.

2. Preliminaries

After here, K is a field and I_n is a set consisting all of permutations of $(0, 1, \cdots, n-1)$.

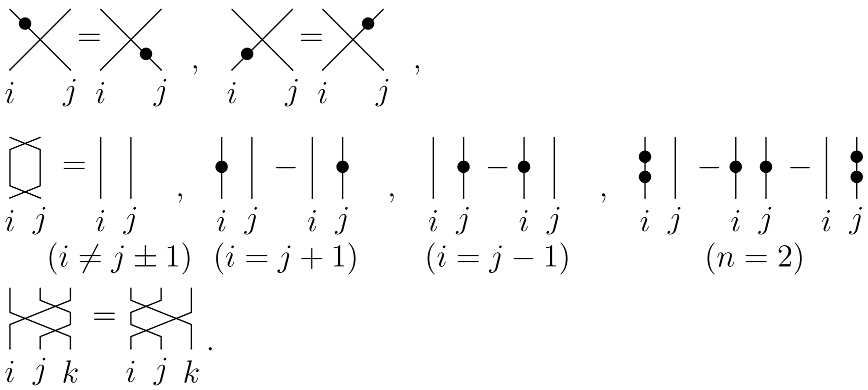
Definition 1. A KLR algebra $H_{\Gamma,\alpha}$ is an algebra obtained by following generators and relations.

- generators: $\{\mathbf{e}(\mathbf{i}) \,|\, \mathbf{i} \in I_n\} \cup \{y_1, \cdots, y_n\} \cup \{\psi_1, \cdots, \psi_{n-1}\}$
- relations:
  $\mathbf{e}(\mathbf{i})\mathbf{e}(\mathbf{j}) = \delta_{\mathbf{i},\mathbf{j}}$,
  $\sum_{\mathbf{i} \in \mathrm{Seq}(\alpha)} \mathbf{e}(\mathbf{i}) = 1$,
  $y_k \mathbf{e}(\mathbf{i}) = \mathbf{e}(\mathbf{i}) y_k$,
  $\psi_k \mathbf{e}(\mathbf{i}) = \mathbf{e}(\mathbf{s_k} \cdot \mathbf{i}) \psi_k$,
  $y_k y_l = y_l y_k$,
  $\psi_k y_l = y_l \psi_k \ (l \neq k, k+1)$,
  $\psi_k \psi_l = \psi_l \psi_k \ (|k-l| > 1)$,
  $\psi_k y_{k+1} \mathbf{e}(\mathbf{i}) = y_k \psi_k \mathbf{e}(\mathbf{i})$,

The detailed version of this paper will be submitted for publication elsewhere.

$$
\begin{aligned}
&y_{k+1}\psi_k \mathbf{e}(\mathbf{i}) = \psi_k y_k \mathbf{e}(\mathbf{i}),\\
&\psi_k^2 \mathbf{e}(\mathbf{i}) = \begin{cases} \mathbf{e}(\mathbf{i}) & (i_k \not\leftrightarrow i_{k+1}) \\ (y_{k+1} - y_k)\mathbf{e}(\mathbf{i}) & (i_k \to i_{k+1}) \\ (y_k - y_{k+1})\mathbf{e}(\mathbf{i}) & (i_k \leftarrow i_{k+1}) \\ (y_{k+1} - y_k)(y_k - y_{k+1})\mathbf{e}(\mathbf{i}) & (i_k \leftrightarrow i_{k+1}) \end{cases},\\
&\psi_k \psi_{k+1} \psi_k \mathbf{e}(\mathbf{i}) = \psi_{k+1}\psi_k\psi_{k+1}\mathbf{e}(\mathbf{i}).
\end{aligned}
$$

The three generators are respectively coresponding to the three diagrams in section 1. A multiplication of two generators are obtained as a concatenation of two diagrams (but if the colors of connecting part are different, it becomes 0). Each relations are also given by following diagrams :



A cyclotomic ideal and a cyclotomic KLR algebra are defined from Λ as follows.

Definition 2. Generators of cyclotomic ideal are as follows :

$$\{y_1 \mathbf{e}(\mathbf{i}) \,|\, \mathbf{i} \in I_n, i_1 = 0\} \cup \{\mathbf{e}(\mathbf{i}) \,|\, \mathbf{i} \in I_n, i_1 \neq 0\}.$$

Denote H_n for corresponding cyclotomic KLR algebra, a quotient of $H_{\Gamma,\alpha}$ by the ideal.

3. Properties

In this section, we describe four properties of H_n. We need some representation theoretical facts written in next section for proof.

Theorem 3. *The number of* $\mathbf{i} \in I_n$ *satisfying* $\mathbf{e}(\mathbf{i}) \neq 0$ *is exactly* 2^{n-2}. *Moreover, the set consisting all of such* $\mathbf{e}(\mathbf{i})$*s is complete set of primitive orthogonal idempotents.*

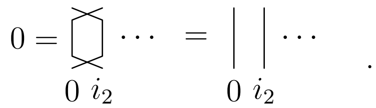
Proof. Fix n. We show there are at most 2^{n-2} $\mathbf{i}$s satisfying $\mathbf{e}(\mathbf{i}) \neq 0$ by constructing $\mathbf{i}$ from i_1 to i_b avoiding $\mathbf{e}(\mathbf{i}) = 0$. The rest part is proved in next section.

In the case of $n = 2$, there is only $(0, 1)$.

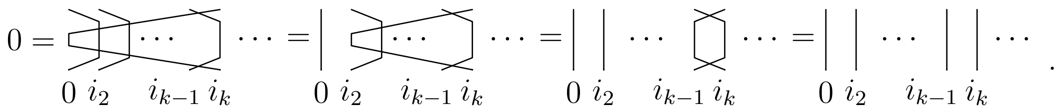
In the case of $n > 2$, at first i_1 must be 0 from the definition of the cyclotomic ideal. Next, i_2 must be 1 or $n - 1$ which are neighborhood of 0 in the quiver. If not, we obtain

$$
\begin{aligned}
\mathbf{e}((0, i_2, \cdots)) &= \psi_1^2 \mathbf{e}((0, i_2, \cdots)) \\
&= \psi_1 \mathbf{e}((i_2, 0, \cdots))\psi_1 \\
&= 0.
\end{aligned}
$$

We can write this equation by using diagrams as follows :



We must keep taking one of the two neighborhoods for $i_k (2 < k < n-1)$. If not, $\mathbf{e}(\mathbf{i}) = 0$ from following equation :



At last, we can set the rest number for i_n. Then we can obtain 2^{n-2} $\mathbf{i}$s constructed by using above method. $\square$

Proposition 4. *Let* $\mathbf{e}(\mathbf{i}) \neq 0$ *in* H_n. *Then these properties hold :*

(*a*) $y_k \mathbf{e}(\mathbf{i}) = 0$ $(1 \leq k < n)$,
(*b*) $y_n^2 \mathbf{e}(\mathbf{i}) = 0$,
(*c*) $y_n \mathbf{e}(\mathbf{i}) \neq 0$.

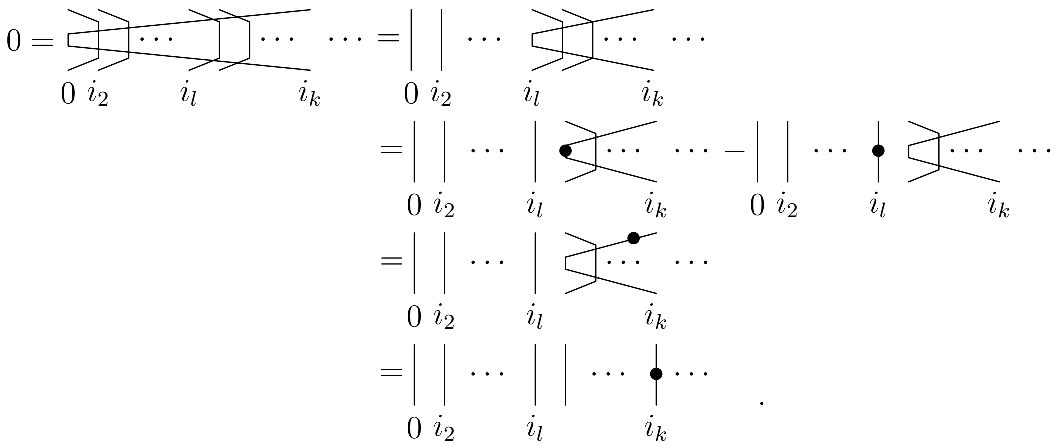
Proof. (*c*) will be proved in next section.

In the case of $n = 2$, (*a*) is by definition, (*b*) follows by expanding $\psi \mathbf{e}(0,1)\psi$.

In the case of $n > 2$, we prove (*a*) by induction for k.

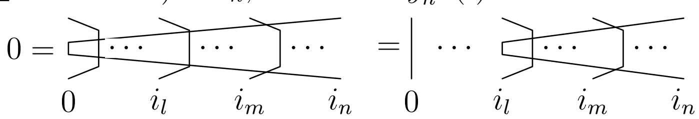
For $k = 1$, $y_k \mathbf{e}(\mathbf{i}) = 0$ from definition.

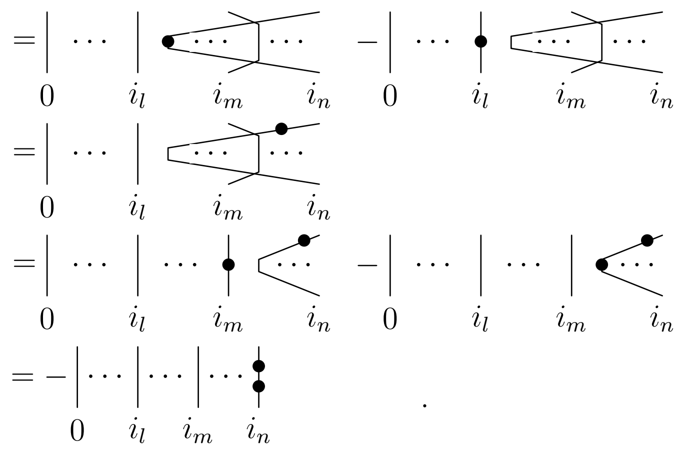
We show $y_k \mathbf{e}(\mathbf{i}) = 0$ for $k < n$. By Thm.3, there is unique $1 \leq l < k$ such that i_k and i_l are neighborhoods. Using $y_l \mathbf{e}(\mathbf{i}) = 0$ by assumption of induction, we obtain $y_k \mathbf{e}(\mathbf{i}) = 0$ from following equation :



We assume $i_l \to i_k$ in this equation, but if $i_l \leftarrow i_k$ the difference is only signs. Therefore (*a*) follows.

In the same way, since $y_k \mathbf{e}(\mathbf{i}) = 0$ for $k < n$ and there are two neighborhoods i_l,i_m $(1 \leq l < m < n)$ of i_n, we obtain $y_n^2 \mathbf{e}(\mathbf{i}) = 0$ as follows :





Also we assume there $i_l \to i_n \to i_m$, but the difference with the case $i_l \leftarrow i_n \leftarrow i_m$ is only signs. Therefore (b) follows. $\square$

For H_n, set two subsets I_n^e, I_n^1 of I_n as follows :

$$\begin{aligned} I_n^e &= \{\mathbf{i} \in I_n \,|\, \mathbf{e}(\mathbf{i}) \neq 0\} \\ I_n^1 &= \{\mathbf{i} \in I_n^e \,|\, i_2 = 1\} \end{aligned}$$

And set an idempotent $\mathbf{e}$ of H_n as follows :

$$\mathbf{e} = \sum_{\mathbf{i} \in I_n^1} \mathbf{e}(\mathbf{i})$$

At last, set two maps $\hat{\ }: I_{n-1}^e(\alpha) \to I_n^1(\alpha), \bar{\ }: I_n^1(\alpha) \to I_{n-1}^e(\alpha)$ as follows :

$$\begin{aligned} \hat{\mathbf{i}} &= (0, 1, i_2 + 1, \cdots, i_{n-1} + 1) \quad \textit{for} \quad \mathbf{i} = (0, i_2, \cdots, i_{n-1}), \\ \bar{\mathbf{i}} &= (0, i_3 - 1, \cdots, i_n - 1) \quad \textit{for} \quad \mathbf{i} = (0, 1, i_3, \cdots, i_n). \end{aligned}$$

In other word, $\hat{\ }$ increments i_k except i_1 and inserts 1 at second, $\bar{\ }$ decrements i_k except i_1 and remove i_2. Both maps are bijection and inversion of the other.

Proposition 5. *For each $n > 2$, an isomorphism of algebras*

$$H_{n-1} \xrightarrow{\sim} \mathbf{e}H_n\mathbf{e}$$

is obtained as follows :

$$\mathbf{e}(\mathbf{i}) \mapsto \mathbf{e}(\hat{\mathbf{i}}) \ , \ y_{n-1} \mapsto y_n \ , \ \psi_k \mapsto \psi_{k+1} \ .$$

Proof. For $\mathbf{e}(\mathbf{i})$, $\mathbf{e}(\mathbf{i}) = 0$ and $\mathbf{e}(\hat{\mathbf{i}}) = 0$ are equivalent. For y_k, what we check is only $y_{n-1} \in H_{n-1}$ and $y_n \in H_n$ by Prop.4. It is easy to check each relations is preserved. Since elements in $\mathbf{e}H_n\mathbf{e}$ can be presented without ψ_1, we can make the inversion map $\mathbf{e}H_n\mathbf{e} \to H_{n-1}$ as follows :

$$\mathbf{e}(\mathbf{i}) \mapsto \mathbf{e}(\bar{\mathbf{i}}) \ , \ y_n \mapsto y_{n-1} \ , \ \psi_k \mapsto \psi_{k-1} \ .$$

$\square$

Proposition 6. *For each H_n, the two indecomposable projective modules corresponding to two primitive idempotents $\mathbf{e}(\mathbf{i})$ and $\mathbf{e}(\mathbf{j})$ are isomorphic if and only if $i_n = j_n$.*

In particular, the isomorphic class of indecomposable projective modules has $(n-1)$ elements.

4. Appendix : Representation Theoretical Facts

Using isomorphism given in [BK], each H_n is replaced by well-known object in representation theory. Using the facts in it, we complete the proofs of previous section.

Theorem 7 (Brundan-Kleshchev,Rouquier)**.**

(a) $\displaystyle\bigoplus_{|\alpha|=n} H_{\Gamma,\alpha,\Lambda} \cong \mathcal{H}_q^{\Lambda}(n)$

The right side is Ariki-Koike algebra determined by Λ and $n, q = \sqrt[n]{1} \in \mathbb{C}$.

(b) *H_{C_n}, α, Λ is a block. That is, an indecomposable two-sided ideal.*

We set $\Lambda = \Lambda_0$. In this case, Ariki-Koike algebra is Hecke algebra $H_q(\mathcal{S}_r)$ of type A. The following theorem holds. For notations in the theorem, see Mathas([4] p.50 Ex.18).

Theorem 8 (Dipper-James)**.** *Let λ be a partition of r.*
There exists $H_q(\mathcal{S}_r)$-module S^{λ} with following properties :
Let n be minimum integer satisfying $1 + q + q^2 + \cdots + q^{n-1} = 0$.

(a) *If λ is n-regular (the same number doesn't continue n times), then* top *of S^{λ} is uniquely determined. In this case, we denote D^{λ} for* topS^{λ}*.*

(b) *$\{D^{\lambda} \mid \lambda : n\text{-regular}\}$ is complete list of simple $H_q(\mathcal{S}_r)$-modules.*

The following lemma holds in general.

Lemma 9. *Let P^{λ} a indecomposable projective module corresponding to D^{λ}.*
As a left module,

$$H_q(\mathcal{S}_r) \cong \bigoplus_{\lambda} (\dim D^{\lambda}) P^{\lambda}$$

The following property holds in this time [5].

Theorem 10. *As an element of Grothendieck group,*

- $\left[D^{(n)}\right] = \left[S^{(n)}\right]$
- $\left[D^{(n-k,1^k)}\right] = -\left[D^{(n-k+1,1^{k-1})}\right] + \left[S^{(n-k,1^k)}\right]$

By using hook length formula, the following property holds.

Proposition 11.

$$\dim S^{(n-k,1^k)} = \binom{n-1}{k}$$

Proof. The Young diagram corresponding to $(n-k, 1^k)$ is as follows :

$$\begin{array}{|c|c|c|c|c|}\hline n & n-k-1 & \cdots & 2 & 1 \\ \hline k \\ \cline{1-1} \cdots \\ \cline{1-1} 1 \\ \cline{1-1} \end{array}$$

$$\begin{aligned} \dim S^{(n-k,1^k)} &= \frac{n!}{n \cdot k!(n-k-1)!} \\ &= \frac{(n-1)!}{((n-1)-k)!k!} \\ &= \binom{n-1}{k} \end{aligned}$$

□

By using Thm.10 and Prop.11, the following property holds.

Proposition 12. *For* $0 \leq k \leq n-1$*, denote* $\lambda_k = (n-k, 1^k)$*.*

$$\sum_{k=0}^{n-1} \mathrm{dim} D^{\lambda_k} = 2^{n-2}$$

Proof. Since $\mathrm{dim} D^{\lambda_k} = -\mathrm{dim} D^{\lambda_{k-1}} + \mathrm{dim} S^{\lambda_k}$,
we obtain $\mathrm{dim} D^{\lambda_k} + \mathrm{dim} D^{\lambda_{k-1}} = \mathrm{dim} S^{\lambda_k} = \binom{n-1}{k}$.
Therefore if n is odd,

$$\sum_{k=0}^{n-1} \mathrm{dim} D^{\lambda_k} = 1 + \binom{n-1}{2} + \binom{n-1}{4} + \cdots + \binom{n-1}{n-1} = 2^{n-2}$$

if even,

$$\sum_{k=0}^{n-1} \mathrm{dim} D^{\lambda_k} = \binom{n-1}{1} + \binom{n-1}{3} + \cdots + \binom{n-1}{n-1} = 2^{n-2}$$

□

Therefore we obtain the following corollary.

Corollary 13. *Every* 2^{n-2} $\mathbf{e}(\mathbf{i})$*s obtained in Thm.3 is primitive idempotent.*

The folloing preperty holds.

Proposition 14. *If* $\mathbf{e}(\mathbf{i}) \neq 0$ *then* $y_n \mathbf{e}(\mathbf{i}) \neq 0$*.*

Proof. There are no elements except for $y_n \mathbf{e}(\mathbf{i})$ in $\mathbf{e}(\mathbf{i}) H_n \mathbf{e}(\mathbf{i})$ such that linearly independent to $\mathbf{e}(\mathbf{i})$. On the other hand, since there are no indecomposable simple projective modules by Thm.10, $\mathrm{dim}(\mathrm{End}(\mathbf{e}(\mathbf{i}) H_n)) \geq 2$. Hence $y_n \mathbf{e}(\mathbf{i}) \neq 0$ from $\mathrm{End}(\mathbf{e}(\mathbf{i}) H_n) \cong \mathbf{e}(\mathbf{i}) H_n \mathbf{e}(\mathbf{i})$. □

About Prop.6, if part follows from [1] and only if part follows from the fact ;
H_n is Morita equivalent to Brauer tree algebra of A_n type.

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GOLDIE EXTENDING MODULES

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ABSTRACT. Let R be a ring. A right R-module M is said to be *Goldie extending* (*u-Goldie extending*) if, for any (uniform) submodule X of M, there exist an essential submodule Y of X and a direct summand N of M such that Y is essential in N. A Goldie extending module is introduced by Akalan-Birkenmeier-Tercan [1]. Note that Goldie extending modules are dual to H-supplemented modules (cf. [7]).

In this paper, we show some characterizations of Goldie extending and consider generalizations of relative injectivity. And we apply them to the study of the open problems "When is a direct sum of Goldie extending (uniform) modules Goldie extending ?" and "Is the property Goldie extending inherited by direct summands ?" in Akalan-Birkenmeier-Tercan [1].

Key Words: (Goldie) extending modules, Internal exchange property.

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1. Introduction

Throughout this paper R is a ring with identity and all modules considered are unitary right R-modules. A submodule X of a module M is said to be *essential* in M or an *essential submodule* of M, if $X \cap Y \neq 0$ for any non-zero submodule Y of M and we write $X \subseteq_e M$ in this case. Y is called a *closed* in M or a *closed submodule* of M if Y has no proper essential extensions inside M. Let $A \subseteq B \subseteq M$. B is said to be *closure* of A in M if B is closed in M and $A \subseteq_e B$. $K <_\oplus N$ means that K is a direct summand of N.

Let $M = M_1 \oplus M_2$ and let $\varphi : M_1 \to M_2$ be a homomorphism. Put $\langle M_1 \xrightarrow{\varphi} M_2 \rangle = \{m_1 - \varphi(m_1) \mid m_1 \in M_1\}$. Then this is a submodule of M which is called *the graph* with respect to $M_1 \xrightarrow{\varphi} M_2$. Note that $M = M_1 \oplus M_2 = \langle M_1 \xrightarrow{\varphi} M_2 \rangle \oplus M_2$.

Let $\{M_i \mid i \in I\}$ be a family of modules. The direct sum decomposition $M = \oplus_I M_i$ is said to be *exchangeable* if, for any direct summand X of M, there exists $\overline{M_i} \subseteq M_i$ $(i \in I)$ such that $M = X \oplus (\oplus_I \overline{M_i})$. A module M is said to have the (*finite*) *internal exchange property* if, any (finite) direct sum decomposition $M = \oplus_I M_i$ is exchangeable.

A module M is said to be *extending* (*u-extending*) if, for any (uniform) submodule X of M, there exists a direct summand N of M such that X is essential in N. An indecomposable extending module is called *uniform*. A module M is said to be *semi-continuous* if M is extending with the finite internal exchange property. A module M is said to be *quasi-continuous* if M is extending with the following condition (C_3):

(C_3) If A and B are direct summands of M such that $A \cap B = 0$, then $A \oplus B$ is a direct summand of M.

The detailed version of this paper will be submitted for publication elsewhere.

A module M is said to be *G-extending* or *Goldie extending* (*u-G-extending* or *u-Goldie extending*) if, for any (uniform) submodule X of M, there exist an essential submodule Y of X and a direct summand N of M such that Y is essential in N. A module M is said to be *G^+-extending* if any direct summand of M is G-extending (cf. [1]). Let $\{M_i \mid i \in I\}$ be a family of modules and put $M = \oplus_I M_i$. Then M is said to be *(u-)G-extending for the decomposition $M = \oplus_I M_i$* if, for any (uniform) submodule X of M, there exist an essential submodule Y of X, a direct summand N of M and a submodule M_i' of M_i ($i \in I$) such that $M = N \oplus (\oplus_I M_i')$ and Y is essential in N.

We see that the following implications hold:

quasi-continuous $\Rightarrow$ semi-continuous $\Rightarrow$ extending $\Rightarrow$ G^+-extending.

In general, the converse is not ture. For example, $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ is semi-continuous but not quasi-continuous. $\mathbb{Z} \oplus \mathbb{Z}$ is extending but not semi-continuous. And $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$ is G^+-extending but not extending.

A module A is said to be *B-ejective* if, for any submodule X of B and any homomorphism $f : X \to A$, there exist an essential submodule X' of X and a homomorphism $g : B \to A$ such that $g|_{X'} = f|_{X'}$ (cf. [1]).

For undefined terminologies, the reader is referred to [2], [3], [7] and [9].

2. G-extending modules and generalizations of relative injectivities

Firstly, we show a connection between extending modules and G-extending modules.

Proposition 1. *Let M be a module and consider the following conditions:*

(1) *M is G-extending and B is essentially A-injective for any decomposition $M = A \oplus B$,*

(2) *M is extending.*

Then (1) $\Rightarrow$ (2) holds. In particular, if M has the finite internal exchange property, then the converse holds.

Proposition 2. *Let A and B be modules. Then A is B-injective if and only if A is B-ejective and essentially B-injective.*

Let M be a module with the decomposition $M = A \oplus B$. If M is G-extending for the decomposition $M = A \oplus B$, then A is G-extending. Thus we obtain the following:

Theorem 3. *Let M be a module with the finite internal exchange property. Then M is G^+-extending if and only if M is G-extending.*

A module A is said to be *weakly (weakly mono-)B-ojective* if, for any submodule X of B and any homomorphism (monomorphism) $f : X \to A$, there exist an essential submodule X' of X, decompositions $A = A_1 \oplus A_2$, $B = B_1 \oplus B_2$, a homomorphism (monomorphism) $g_1 : B_1 \to A_1$ and a monomorphism $g_2 : A_2 \to B_2$ satisfying the following condition $(*)$:

$(*)$ For any $x' \in X'$, we express x' and $f(x')$ in $B = B_1 \oplus B_2$ and $A = A_1 \oplus A_2$ as $x' = b_1 + b_2$ and $f(x') = a_1 + a_2$, respectively. Then $g_1(b_1) = a_1$ and $g_2(a_2) = b_2$ (cf. [4], [6]). Now we consider some properties of weakly ojectivities.

Proposition 4. *Let A be a module and let B be a extending module with the finite internal exchange property. Then*

(1) *If A is weakly B-ojective, then A is weakly B'-ojective for any $B' <_{\oplus} B$.*
(2) *If A is weakly B-ojective, then A is weakly mono-B-ojective.*

By a quite similar proof of [8, Theorem 2.1], we get the following:

Proposition 5. *Let A be an extending module with the finite internal exchange property and let B be a G^+-extending module. If A is weakly B-ojective, then A' is weakly B-ojective for any $A' <_{\oplus} A$.*

Theorem 6. *Let M_1 and M_2 be G^+-extending modules and put $M = M_1 \oplus M_2$. If M_1' is weakly mono-M_2'-ojective for any $M_i' <_{\oplus} M_i$ $(i = 1, 2)$, then M is G-extending for the decomposition $M = M_1 \oplus M_2$.*

The following is a main result in this section:

Theorem 7. *Let M_1 and M_2 be G-extending modules with the finite internal exchange property and put $M = M_1 \oplus M_2$. Then the following conditions are equivalent:*
(1) *M is G-extending for $M = M_1 \oplus M_2$,*
(2) *$N = M_1' \oplus M_2'$ is G-extending for $N = M_1' \oplus M_2'$, for any $M_i' <_{\oplus} M_i$ $(i = 1, 2)$,*
(3) *M_1' is weakly M_2'-ojective for any $M_i' <_{\oplus} M_i$ $(i = 1, 2)$.*

Let A and B be modules and let $f : A \to B$ be a monomorphism. f is called a *proper monomorphism* if f is not an isomorphism. If there exists a proper monomorphism from A to B, we write $A \prec B$ or $A \overset{f}{\prec} B$. If there is no proper monomorphism from A to B, we write $A \not\prec B$. By Theorem 7, we obtain the following:

Theorem 8. *Let M_1 and M_2 be G^+-extending and put $M = M_1 \oplus M_2$. Suppose that $M \not\prec M$. Then the following conditions are equivalent:*
(1) *M is G^+-extending and the decomposition $M = M_1 \oplus M_2$ is exchangeable,*
(2) *M is G^+-extending for $M = M_1 \oplus M_2$,*
(3) *M_1' is weakly mono-M_2'-ojective for any $M_i' <_{\oplus} M_i$ $(i = 1, 2)$.*

3. Direct sums of uniform modules

In this section, we consider the problem "When is a direct sum of uniform modules (G-)extending ?". Firstly we show the following:

Proposition 9. *Let $\{U_i \mid i \in I\}$ be a family of uniform modules and put $M = \oplus_I U_i$. Then the following conditions are equivalent:*
(1) *M is u-G-extending for $M = \oplus_I U_i$,*
(2) *For any $J \subseteq I$, $N = \oplus_J U_j$ is u-G-extending for $N = \oplus_J U_j$,*
(3) *U_i is weakly mono-U_j-ojective for any $i \neq j$.*

The following theorem is obtained by a quite similar proof of [5, Theorem 2.3].

Theorem 10. *(cf.* [5, Theorem 2.3]*) Let $\{U_i \mid i \in I\}$ be a family of uniform modules and put $M = \oplus_I U_i$. We consider the following condition:*
(1) *U_i is weakly mono-U_j-ojective for any $i \neq j \in I$,*
(2) *There is no infinite sequence f_1, f_2, f_3, f_4, $\cdots$ of proper monomorphisms f_k : $U_{i_k} \to U_{i_{k+1}}$ with all $i_k \in I$ distinct.*

If M satisfies the conditions (a) and (b), then M is G-extending for $M = \oplus_I U_i$.

Let $\{U_i \mid i \in I\}$ be a family of uniform modules and put $M = \oplus_I U_i$. If M is G-extending for $M = \oplus_I U_i$ and U_i is essentially U_j-injective $(i \neq j)$, then the condition (b) in Theorem 10 holds. Thus we obtain the following result:

Theorem 11. *Let $\{U_i \mid i \in I\}$ be a family of uniform modules and put $M = \oplus_I U_i$. Then the following conditions are equivalent:*

(1) *M is extending with the (finite) internal exchange property,*
(2) *M is extending and the decomposition $M = \oplus_I U_i$ is exchangeable,*
(3) (a) *M is u-extending for the decomposition $M = \oplus_I U_i$,*
  (b) *M satisfies the condition (b) in Theorem 10,*
(4) (a) *M is G-extending for the decomposition $M = \oplus_I U_i$,*
  (b) *U_i is essentially U_j-injective for any $i \neq j$,*
  (c) *(A_2') holds for all U_i and $\{U_j \mid i \neq j \in I\}$,*
(5) (a) *M is u-G-extending for the decomposition $M = \oplus_I U_i$,*
  (b) *U_i is essentially U_j-injective for any $i \neq j$,*
  (c) *(A_2') holds for all U_i and $\{U_j \mid i \neq j \in I\}$,*
  (d) *M satisfies the condition (b) in Theorem 10,*
(6) (a) *U_i is essentially U_j-injective and weakly mono-U_j-ojective for any $i \neq j$,*
  (b) *(A_2') holds for all U_i and $\{U_j \mid i \neq j \in I\}$,*
  (c) *M satisfies the condition (b) in Theorem 10.*

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REALIZING CLUSTER CATEGORIES OF DYNKIN TYPE A_n AS STABLE CATEGORIES OF LATTICES

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Abstract. Cluster tilting objects of the cluster category $\mathcal{C}$ of Dynkin type A_{n-3} are known to be indexed by triangulations of a regular polygon P with n vertices. Given a triangulation of P, we associate a quiver with potential with frozen vertices such that the frozen part of the associated Jacobian algebra has the structure of a $K[\![x]\!]$-order denoted as Λ_n. Then we show that $\mathcal{C}$ is equivalent to the stable category of the category of Λ_n-lattices.

Let $n \geq 3$ be an integer, K be a field and $R = K[\![x]\!]$ be the formal power series ring in one variable over K.

1. The Order

Definition 1. Let Λ be an *R-order*, i.e. an R-algebra which is a finitely generated free R-module. A left Λ-module L is called a *Λ-lattice* if it is finitely generated free as an R-module. We denote by $\mathrm{CM}(\Lambda)$ the category of Λ-lattices.

The order we use to study cluster categories of type A_{n-3} is Λ_n:

$$\Lambda_n = \begin{bmatrix} R & R & R & \cdots & R & (x^{-1}) \\ (x) & R & R & \cdots & R & R \\ (x^2) & (x) & R & \cdots & R & R \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (x^2) & (x^2) & (x^2) & \cdots & R & R \\ (x^2) & (x^2) & (x^2) & \cdots & (x) & R \end{bmatrix}_{n\times n}.$$

The detailed version of this paper will be submitted for publication elsewhere.

The category $\mathrm{CM}(\Lambda_n)$ has Auslander-Reiten sequences. We draw the Auslander-Reiten quiver of Λ_n for n even: it is similar for n odd.

$$
\begin{tikzcd}[column sep=small]
 & L_{1,1,n-2} \arrow[dr] \arrow[rrr, dotted, no head] & & & L_{\frac{n}{2},1,\frac{n}{2}-1} \arrow[dr] & \\
L_{2,n-2,0} \arrow[ur] \arrow[dr] & & L_{1,2,n-3} \arrow[r, dotted, no head] & L_{\frac{n}{2}-1,2,\frac{n}{2}-1} \arrow[ur] \arrow[dr] & & L_{\frac{n}{2},2,\frac{n}{2}-2} \\
 & L_{3,n-3,0} \arrow[ur] \arrow[rrr, dotted, no head] \arrow[dd, dotted, no head] & & & L_{\frac{n}{2}-1,3,\frac{n}{2}-2} \arrow[ur] \arrow[dd, dotted, no head] & \\
 & & & & & \\
 & L_{\frac{n}{2},3,\frac{n}{2}-3} \arrow[dr] \arrow[rrr, dotted, no head] & & & L_{2,n-3,1} \arrow[dr] & \\
L_{\frac{n}{2},2,\frac{n}{2}-2} \arrow[ur] \arrow[dr] & & L_{\frac{n}{2}+1,2,\frac{n}{2}-3} \arrow[r, dotted] & L_{1,n-2,1} \arrow[ur] \arrow[dr] & & L_{2,n-2,0} \\
 & L_{\frac{n}{2}+1,1,\frac{n}{2}-2} \arrow[ur] \arrow[rrr, dotted, no head] & & & L_{1,n-1,0} \arrow[ur] &
\end{tikzcd}
$$

where

$$
L_{m_1,m_2,m_3} = \begin{bmatrix} R \\ \vdots \\ R \\ (x) \\ \vdots \\ (x) \\ (x^2) \\ \vdots \\ (x^2) \end{bmatrix},
$$

with R appearing m_1 times, (x) appearing m_2 times and (x^2) appearing m_3 times.

This is a Mobius strip, both the first and last row of which consist of $\frac{n}{2}$ projective-injective Λ_n-lattices, with $n-3$ τ-orbits between them.

2. The Jacobian Algebra

Let Q be a finite quiver. The complete path algebra $\widehat{KQ}$ is the completion of the path algebra with respect to the $\mathcal{J}$-adic topology for $\mathcal{J}$ the ideal generated by all arrows of Q. A quiver with potential (QP for short) is a finite quiver with a linear combination of cycles of the quiver.

Quivers of triangulations of surfaces are defined in [2] and [3] before. And QPs arising from such triangulations are defined in [4]. We extend their definitions in the case of regular polygons.

Let us fix a postive integer $n \geq 3$ and a triangulation $\triangle$ of a regular polygon with n vertices (n-gon for short).

Definition 2. The quiver $Q_\triangle$ of the triangulation $\triangle$ is the quiver the vertices of which are the (internal and external) edges of the triangulation. Whenever two edges a and b share a joint vertex, the quiver $Q_\triangle$ contains a normal arrow $a \to b$ if a is a predecessor of b with respect to clockwise orientation inside a triangle at the joint vertex of a and b. Moreover, for every vertex of the polygon with at least one internal incident edge in the triangulation, there is a dashed arrow $a \dashrightarrow b$ where a and b are its two incident external edges, a being a predecessor of b with respect to clockwise orientation.

In the following we denote $Q = Q_\triangle$. A cycle in Q is called a *cyclic triangle* if it consists of three normal arrows, and a minimal cycle in Q is called a *big cycle* if it contains exactly one dashed arrow.

Definition 3. We define the set of frozen vertices F of Q as the subset of Q_0 consisting of the n external edges of the n-gon, and the potential as

$$W = \sum \textit{cyclic triangles} - \sum \textit{big cycles}.$$

According to [1], the associated Jacobian algebra is defined by

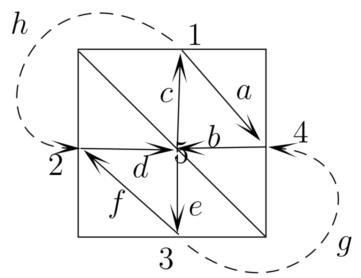
$$\mathcal{P}(Q, W, F) = \widehat{KQ}/\mathcal{J}(W, F),$$

where $\mathcal{J}(W, F)$ is the closure

$$\mathcal{J}(W, F) = \overline{\langle \partial_a W \mid a \in Q_1,\ s(a) \notin F \quad \text{or} \quad e(a) \notin F \rangle}$$

with respect to the $\mathcal{J}_{\widehat{KQ}}$-adic topology and $s(a)$ (resp. $e(a)$) is the starting vertex (resp. ending vertex) of the arrow a. Notice that cyclic derivatives associated with arrows between frozen vertices are excluded.

Example 4. We illustrate the construction of $Q_\triangle$ and W when $\triangle$ is a triangulation of a square:



In this case $W = abc + def - beg - dch$, $F = \{1, 2, 3, 4\}$ and

$$\mathcal{J}(W, F) = \overline{\langle ca - eg, ab - hd, fd - gb, ef - ch \rangle}.$$

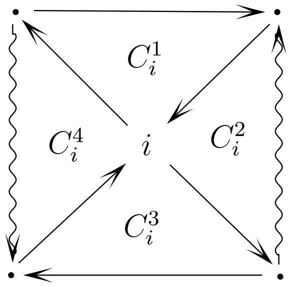
Remark 5. Since each arrow which is not between frozen vertices is shared by a big cycle and a cyclic triangle, it follows that all relations in $\mathcal{P}(Q,W,F)$ are commutativity relations.

3. A basis

Let $\triangle$ be a triangulation of the n-gon, (Q,W,F) be the associated QP with frozen vertices and $\mathcal{P}(Q,W,F)$ be its Jacobian algebra.

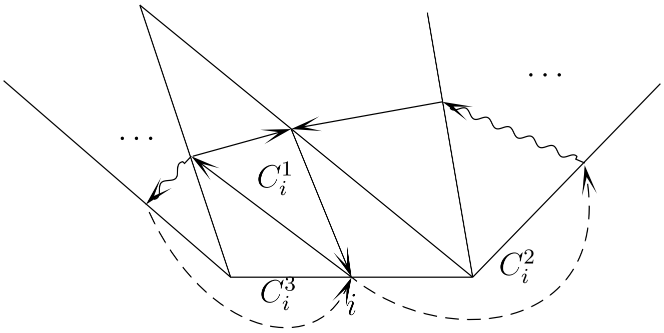
Let $i \in Q_0$. We consider all minimal cycles $C_i^1, \dots, C_i^k$ passing through i. It is easy to check that in general there are only three cases:

1: i is one of the internal edges of the triangulation, the only case is the following:



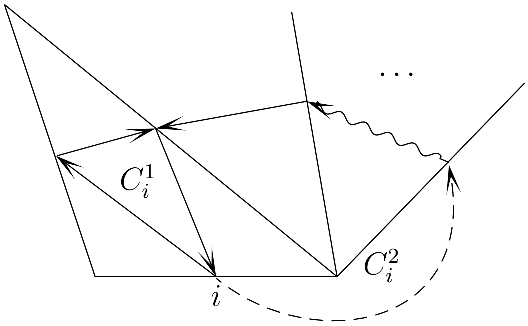
$$C_i := C_i^1 = C_i^2 = C_i^3 = C_i^4 \text{ holds in } \mathcal{P}(Q,W,F).$$

2: i is an external edge with four adjacent edges,



$$C_i := C_i^1 = C_i^2 = C_i^3 \text{ holds in } \mathcal{P}(Q,W,F).$$

3: i is an external edge with three adjacent edges,



$$C_i := C_i^1 = C_i^2 \text{ holds in } \mathcal{P}(Q, W, F).$$

By a countable basis of a complete topological vector space, we mean a linearly independent set of elements which spans a dense vector subspace. It is known that $\widehat{KQ}$ has a countable basis P_Q which is the set of all paths on Q. We say that two paths w_1 and w_2 are *equivalent* ($w_1 \sim w_2$) if $w_1 = w_2$ in $\mathcal{P}(Q, W, F)$. This gives an equivalence relation on P_Q and $P_Q/\sim$ is a countable basis of the Jacoabian algebra $\mathcal{P}(Q, W, F)$.

Consider the element $C := \sum_{i \in Q_0} C_i$ in the associated Jacobian algebra $\mathcal{P}(Q, W, F)$, then C is in the center of $\mathcal{P}(Q, W, F)$.

Definition 6. For any vertices $i, j \in Q_0$, a path w from i to j is called *C-free* if there is no path w' from i to j satisfying $w \sim w'C$.

Considering all the C-free paths, we have the following proposition.

Proposition 7. For any vertices $i, j \in Q_0$,

(1) there exists a unique C-free path w_0 from i to j up to $\sim$.
(2) $\{w_0, w_0C, w_0C^2, w_0C^3, \ldots\}$ is a countable basis of $e_i\mathcal{P}(Q, W, F)e_j$.

The Jacobian algebra $\mathcal{P}(Q, W, F)$ is an R-algebra through $x \mapsto C$. According to this proposition, it is an R-order whose set of generators consists of C-free paths.

4. Main Results

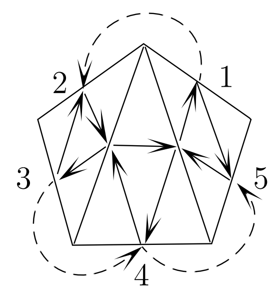
Let $\triangle$ be a triangulation of the n-gon ($n \geq 3$), (Q, W, F) be the associated QP with frozen vertices and $\mathcal{P}(Q, W, F)$ be its Jacobian algebra. Let Λ_n be the order defined in Section 1.

Theorem 8. *The cluster category $\mathcal{C}_{A_{n-3}}$ of type A_{n-3} is equivalent to the stable category of the category $\mathrm{CM}(\Lambda_n)$ of Λ_n-lattices.*

Theorem 9. *Let e_F be the sum of the idempotents at frozen vertices. Then*

(1) *$e_F\mathcal{P}(Q, W, F)e_F$ is isomorphic to Λ_n as an R-order.*
(2) *the Λ_n-module $e_F\mathcal{P}(Q, W, F)$ is a cluster tilting object of $\mathrm{CM}(\Lambda_n)$, i.e. a Λ_n-lattice X satisfies $\mathrm{Ext}^1(e_F\mathcal{P}(Q, W, F), X) = 0$ precisely when it is a direct summand of direct sum of finite copies of $e_F\mathcal{P}(Q, W, F)$.*
(3) *$\mathrm{End}_{\Lambda_n}(e_F\mathcal{P}(Q, W, F))$ is isomorphic to $\mathcal{P}(Q, W, F)$ as an R-order.*

Example 10. The Jacobian algebra associated with the following triangulation of the pentagon:



is isomorphic to the following R-order:

$$\mathcal{P}(Q,W,F) \cong \begin{bmatrix} R & R & R & R & (x^{-1}) & R & R \\ (x) & R & R & R & R & R & R \\ (x^2) & (x) & R & R & R & (x) & (x) \\ (x^2) & (x^2) & (x) & R & R & (x) & (x) \\ (x^2) & (x^2) & (x^2) & (x) & R & (x^2) & (x) \\ (x) & (x) & R & R & R & R & R \\ (x) & (x) & (x) & R & R & (x) & R \end{bmatrix}$$

It is clear that $e_F\mathcal{P}(Q,W,F)e_F \cong \Lambda_5$ holds in this case. The Auslander-Reiten quiver of Λ_5 is the following:

$$\begin{tikzcd}[column sep=small, row sep=small]
\begin{bmatrix} R \\ (x) \\ (x^2) \\ (x^2) \\ (x^2) \end{bmatrix} \arrow[dr] & & \begin{bmatrix} R \\ R \\ (x) \\ (x^2) \\ (x^2) \end{bmatrix} \arrow[dr] & & \begin{bmatrix} R \\ R \\ R \\ (x) \\ (x^2) \end{bmatrix} \arrow[dr] & \\
& \begin{bmatrix} R \\ (x) \\ (x) \\ (x^2) \\ (x^2) \end{bmatrix} \arrow[ur] \arrow[dr] & & \begin{bmatrix} R \\ R \\ (x) \\ (x) \\ (x^2) \end{bmatrix} \arrow[ur] \arrow[dr] & & \begin{bmatrix} R \\ R \\ R \\ (x) \\ (x) \end{bmatrix} \\
\begin{bmatrix} R \\ R \\ R \\ (x) \\ (x) \end{bmatrix} \arrow[ur] \arrow[dr] & & \begin{bmatrix} R \\ (x) \\ (x) \\ (x) \\ (x^2) \end{bmatrix} \arrow[ur] \arrow[dr] & & \begin{bmatrix} R \\ R \\ (x) \\ (x) \\ (x) \end{bmatrix} \arrow[ur] \arrow[dr] & \\
& \begin{bmatrix} R \\ R \\ R \\ R \\ (x) \end{bmatrix} \arrow[ur] & & \begin{bmatrix} R \\ (x) \\ (x) \\ (x) \\ (x) \end{bmatrix} \arrow[ur] & & \begin{bmatrix} R \\ (x) \\ (x^2) \\ (x^2) \\ (x^2) \end{bmatrix}
\end{tikzcd}$$

As a Λ_5-module, $e_F\mathcal{P}(Q,W,F)$ is isomorphic to

$$\begin{bmatrix} R \\ (x) \\ (x^2) \\ (x^2) \\ (x^2) \end{bmatrix} \oplus \begin{bmatrix} R \\ R \\ (x) \\ (x^2) \\ (x^2) \end{bmatrix} \oplus \begin{bmatrix} R \\ R \\ R \\ (x) \\ (x^2) \end{bmatrix} \oplus \begin{bmatrix} R \\ R \\ R \\ R \\ (x) \end{bmatrix} \oplus \begin{bmatrix} R \\ (x) \\ (x) \\ (x) \\ (x) \end{bmatrix} \oplus \begin{bmatrix} R \\ R \\ (x) \\ (x) \\ (x) \end{bmatrix} \oplus \begin{bmatrix} R \\ R \\ (x) \\ (x) \\ (x^2) \end{bmatrix}$$

which is cluster tilting in $\mathrm{CM}(\Lambda_5)$. As expected, its summands correspond bijectively to the (internal and external) edges of the triangulation, or equivalently to the vertices of the corresponding quiver.

The stable category $\underline{\mathrm{CM}}(\Lambda_5)$ is equivalent to $\mathcal{C}_{A_2}$ and has a cluster tilting object

$$\begin{bmatrix} R \\ R \\ (x) \\ (x) \\ (x) \end{bmatrix} \oplus \begin{bmatrix} R \\ R \\ (x) \\ (x) \\ (x^2) \end{bmatrix}.$$

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CHARACTERIZATION OF GORENSTEIN STRONGLY KOSZUL HIBI RINGS BY F-INVARIANTS

KAZUNORI MATSUDA

Abstract. Hibi rings are a kind of graded toric ring on a finite distributive lattice $D = J(P)$, where P is a partially ordered set. In this article, we compute diagonal F-thresholds and F-pure thresholds of Hibi rings and give a characterization of Hibi rings which satisfy the equality between these invariants in terms of its trivialness in the sense of Herzog-Hibi-Restuccia.

1. Introduction

This is a partially joint work with T. Chiba.

Firstly, we recall the definition of Hibi rings(see[Hib]).

Let $P = \{p_1, p_2, \ldots, p_N\}$ be a finite partially ordered set(poset for short), and let$J(P)$ be the set of all poset ideals of P, where a poset ideal of P is a subset I of P such that if $x \in I$, $y \in P$ and $y \leq x$ then $y \in I$.

A *chain* X of P is a totally ordered subset of P. The *length* of a chain X of P is $\#X - 1$, where $\#X$ is the cardinality of X. The *rank* of P, denoted by rank$P$, is the maximum of the lengths of chains in P. A poset is called *pure* if its all maximal chains have the same length. For $x, y \in P$, we say that y *covers* x, denoted by $x \lessdot y$, if $x < y$ and there is no $z \in P$ such that $x < z < y$.

Definition 1. ([Hib]) Let the notation be as above. Let φ be the following map:

$$\varphi : J(P) \longrightarrow k[T, X_1, \ldots, X_N], \qquad I \longmapsto T \prod_{p_i \in I} X_i$$

Then the *Hibi ring* $R(P)$ is defined as follows:

$$R(P) = k[\varphi(I) \mid I \in J(P)].$$

Remark 2. (1) ([Hib]) Hibi rings are graded toric rings.
(2) $\dim R(P) = \#P + 1$.
(3) ([Hib]) $R(P)$ is Gorenstein if and only if P is pure.

Finally, we define $\mathrm{rank}^* P$ and $\mathrm{rank}_* P$ for a poset P in order to state our main theorem.

A sequence $C = (q_1, \ldots, q_t)$ is called a *path* of P if C satisfies the following conditions:

(1) $q_1, \ldots, q_t$ are distinct elements of P,
(2) q_1 is a minimal element of P and $q_{t-1} \lessdot q_t$,
(3) $q_i \lessdot q_{i+1}$ or $q_{i+1} \lessdot q_i$.

The detailed version of this paper will be submitted for publication elsewhere.

In short, we regard the Hasse diagram of P as a graph, and consider paths on it. In particular, if q_t is a maximal element of P, then we call C *maximal* path. For a path $C = (q_1, \ldots, q_t)$, we denote $C = q_1 \to q_t$.

For a path $C = (q_1, \ldots, q_t)$, q_i is said to be a *locally maximal element* of C if $q_{i-1} \lessdot q_i$ and $q_{i+1} \lessdot q_i$, and a *locally minimal element* of C if $q_i \lessdot q_{i-1}$ and $q_i \lessdot q_{i+1}$. For convenience, we consider that q_1 is a locally minimal element and q_t is a locally maximal element of C.

For a path $C = (q_1, \ldots, q_t)$, if $q_1 \leq \cdots \leq q_t$ then we call C an *ascending chain* and if $q_1 \geq \cdots \geq q_t$ then we call C a *descending chain*. We denote a ascending chain by a symbol A and a descending chain by a symbol D. For a ascending chain $A = (q_1, \ldots, q_t)$, we put $t(A) = q_t$ and $< A >= \{q \in P \mid q \leq t(A)\}$. Since $< A >$ is a poset ideal of P generated by A, we note that $< A > \in J(P)$.

Let $C = (q_1, \ldots, q_t)$ be a path and $V(C)$ the vertices of C. We now introduce the notion of the *decomposition* of C. We decompose $V(C)$ as follows:

$$V(C) = V(A_1) \coprod V(D_1) \coprod V(A_2) \coprod \cdots \coprod V(D_{n-1}) \coprod V(A_n)$$

such that

$$\begin{aligned} V(A_1) &= \{q_1, \ldots, q_{a(1)}\}, \\ V(D_1) &= \{q'_1, \ldots, q'_{d(1)}\}, \\ V(A_2) &= \{q_{a(1)+1}, \ldots, q_{a(2)}\}, \\ &\vdots \\ V(D_{n-1}) &= \{q'_{d(n-2)+1}, \ldots, q'_{d(n-1)}\}, \\ V(A_n) &= \{q_{a(n-1)+1}, \ldots, q_{a(n)} = q_t\}, \end{aligned}$$

where $\{q_{a(1)}, \ldots, q_{a(n)}\}$ is the set of locally maximal elements and $\{q_1, q'_{d(1)}, \ldots, q'_{d(n-1)}\}$ is the set of locally minimal elements of C. Then A_i are ascending chains and D_j are descending chains. This decomposition is denoted by $C = A_1 + D_1 + A_2 + \cdots + D_{n-1} + A_n$.

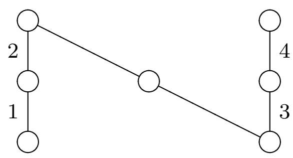
For a path $C = (q_1, \ldots, q_t)$, we define *the upper length* by

$$\text{length}^* C = \#\{(q_i, q_{i+1}) \in E(C) \mid q_i \lessdot q_{i+1}\},$$

where $E(C)$ is the set of edges of C.

Example 3. (1) If C is a chain, then $\text{length}^* C = \text{length} C$.

(2) Consider the following path C:

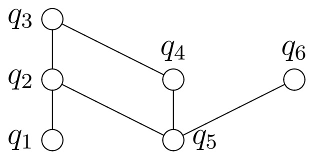


Then $\text{length}^* C = 4$.

Next, we introduce the condition (*).

Definition 4. For a path $C = (q_1, \ldots, q_t)$, we say that C *satisfies a condition* (*) if C satisfies the following conditions: for all q_r which is locally maximal element or locally minimal element of C, $q_{s'} \not\leq q_s$ for all $s' > r$ and $r > s$.

Example 5. Consider the following poset P:



Then, $C_1 = (q_1, q_2, q_5, q_6)$ satisfies the condition (*), but $C_2 = (q_1, q_2, q_3, q_4, q_5, q_6)$ does not satisfy the condition (*) because $q_2 \geq q_5$.

Remark 6. (1) For a path $C = (q_1, \ldots, q_t)$ such that C satisfies a condition (*) and q_t is a locally maximal element, we can extend C to a path $\tilde{C} = (q_1, \ldots, q_t, \ldots, q_{t'})$ such that $\tilde{C}$ is a maximal path which satisfies a condition (*). Indeed, if q_t is not a maximal element of P, then there exists q_{t+1} such that $q_t \lessdot q_{t+1}$. We decompose $C = A_1 + D_1 + \ldots + D_{n-1} + A_n$. If $q_{t+1} \in < A_i >$ for some i, then so is q_t. This means that C does not satisfy a condition (*), a contradiction. Hence a path $C' = (q_1, \ldots, q_t, q_{t+1})$ also satisfies a condition (*). Therefore, by repeating this operation, we can extend C to a path $\tilde{C} = (q_1, \ldots, q_t, \ldots, q_{t'})$ such that $\tilde{C}$ is a maximal path which satisfies a condition (*).

(2) Let $C = (q_1, \ldots, q_t)$ be a path of P. If C is a unique path such that its starting point is q_1 and its end point is q_t, then C satisfies a condition (*). Indeed, if C does not so, there exists a locally maximal(or minimal) element q_r such that $q_{s'} \leq q_s$ for some $s < r < s'$. Then, $C' = (q_1, \ldots, q_s, q_{s'}, \ldots, q_t)$ is also a path, but this is a contradiction.

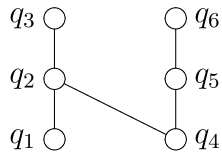
Now, we can define the upper rank $\mathrm{rank}^* P$ and the lower rank $\mathrm{rank}_* P$ for a poset P.

Definition 7. For a poset P, we define

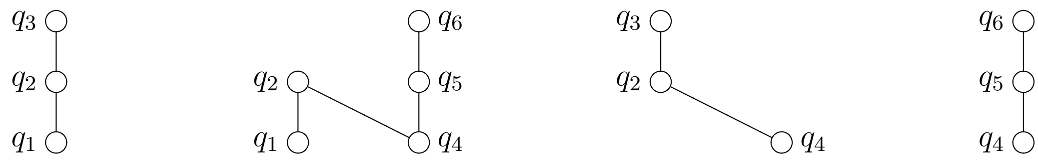
$$\mathrm{rank}^* P = \max\{\mathrm{length}^* C \mid C \text{ is a maximal path which satisfies a condition}(*)\},$$
$$\mathrm{rank}_* P = \min\{\mathrm{length}^* C \mid C \text{ is a maximal path which satisfies a condition}(*)\}.$$

We call $\mathrm{rank}^* P$ *upper rank* and $\mathrm{rank}_* P$ *lower rank* of P. We note that $\#P - 1 \geq \mathrm{rank}^* P \geq \mathrm{rank} P \geq \mathrm{rank}_* P$.

Example 8. Consider the following poset P:



Then, the following paths satisfy the condition (*):



Hence we have $\mathrm{rank}^* P = 3$ and $\mathrm{rank}_* P = \mathrm{rank} P = 2$.

2. Diagonal F-thresholds of Hibi rings

In this section, we recall the definition and several basic results of F-threshold and give a formula of the F-thresholds of Hibi rings.

2.1. **Definition and basic properties.** Let R be a Noetherian ring of characteristic $p > 0$ with $\dim R = d \geq 1$. Let $\mathfrak{m}$ be a maximal ideal of R. Suppose that $\mathfrak{a}$ and J are $\mathfrak{m}$-primary ideals of R such that $\mathfrak{a} \subseteq \sqrt{J}$ and $\mathfrak{a} \cap R^\circ \neq \emptyset$, where R° is the set of elements of R that are not contained in any minimal prime ideal of R.

Definition 9 (see [HMTW])**.** Let $R, \mathfrak{a}, J$ be as above. For each nonnegative integer e, put $\nu_{\mathfrak{a}}^J(p^e) = \max\{r \in \mathbb{N} \mid \mathfrak{a}^r \not\subseteq J^{[p^e]}\}$, where $J^{[p^e]} = (a^{p^e} \mid a \in J)$. Then we define

$$c^J(\mathfrak{a}) = \lim_{e \to \infty} \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e}$$

if it exists, and call it the *F-threshold* of the pair $(R, \mathfrak{a})$ with respect to J. Moreover, we call $c^{\mathfrak{a}}(\mathfrak{a})$ the *diagonal F-threshold* of R with respect to $\mathfrak{a}$.

About basic properties and examples of F-thresholds, see [HMTW]. In this section, we summarize basic properties of the diagonal F-thresholds $c^{\mathfrak{m}}(\mathfrak{m})$.

Example 10.
(1) Let $(R, \mathfrak{m})$ be a regular local ring of positive characteristic. Then $c^{\mathfrak{m}}(\mathfrak{m}) = \dim R$.
(2) Let $k[X_1, \ldots, X_d]^{(r)}$ be the r-th Veronese subring of a polynomial ring $S = k[X_1, \ldots, X_d]$. Put $\mathfrak{m} = (X_1, \ldots, X_d)^r R$. Then $c^{\mathfrak{m}}(\mathfrak{m}) = \frac{r+d-1}{r}$.
(3) ([MOY, Corollary 2.4]) If $(R, \mathfrak{m})$ is a local ring with $\dim R = 1$, then $c^{\mathfrak{m}}(\mathfrak{m}) = 1$.

Example 11. ([MOY, Theorem 2]) Let $S = k[X_1, \ldots, X_m, Y_1, \ldots, Y_n]$ be a polynomial ring over k in $m + n$ variables, and put $\mathfrak{n} = (X_1, \ldots, X_m, Y_1, \ldots, Y_n)S$. Take a binomial $f = X_1^{a_1} \cdots X_m^{a_m} - Y_1^{b_1} \cdots Y_n^{b_n} \in S$, where $a_1 \geq \cdots \geq a_m, b_1 \geq \cdots \geq b_n$. Let $R = S_{\mathfrak{n}}/(f)$ be a binomial hypersurface local ring with the unique maximal ideal $\mathfrak{m}$. Then

$$c^{\mathfrak{m}}(\mathfrak{m}) = m + n - 2 + \frac{\max\{a_1 + b_1 - \min\{\sum_{i=1}^m a_i, \sum_{j=1}^n b_j\}, 0\}}{\max\{a_1, b_1\}}.$$

In [CM], we gave a formula of $c^{\mathfrak{m}}(\mathfrak{m})$ of Hibi rings.

Theorem 12 (see [CM])**.** *Let P be a finite poset, and $R = R(P)$ the Hibi ring made from P. Let $\mathfrak{m} = R_+$ be the graded maximal ideal of R. Then*

$$c^{\mathfrak{m}}(\mathfrak{m}) = \mathrm{rank}^* P + 2.$$

3. F-pure thresholds of Hibi rings

In this section, we recall the definition of the F-pure threshold and give a formula of the F-pure thresholds of Hibi rings. This formula is given by Chiba.

The F-pure threshold, which was introduced by [TW], is an invariant of an ideal of an F-finite F-pure ring. F-pure threshold can be calculated by computing generalized test ideals (see [HY]), and [Bl] showed how to compute generalized test ideals in the case of toric rings and its monomial ideals. Since Hibi rings are toric rings, we can compute F-pure thresholds of the homogeneous maximal ideal of arbitrary Hibi rings, and will be described in terms of poset.

Definition 13 (see [TW]). Let R be an F-finite F-pure ring of characteristic $p > 0$, $\mathfrak{a}$ a nonzero ideal of R, and t a non-negative real number. The pair $(R, \mathfrak{a}^t)$ is said to be F-pure if for all large $q = p^e$, there exists an element $d \in \mathfrak{a}^{\lceil t(q-1) \rceil}$ such that the map $R \longrightarrow R^{1/q}$ $(1 \mapsto d^{1/q})$ splits as an R-linear map. Then the F-pure threshold $\mathrm{fpt}(\mathfrak{a})$ is defined as follows:

$$\mathrm{fpt}(\mathfrak{a}) = \sup\{t \in \mathbb{R}_{\geq 0} \mid (R, \mathfrak{a}^t) \text{ is } F\text{-pure}\}.$$

Hara and Yoshida [HY] introduced the generalized test ideal $\tau(\mathfrak{a}^t)$ (t is a non negative real number). Then $\mathrm{fpt}(\mathfrak{a})$ can be calculated as the minimum jumping number of $\tau(\mathfrak{a}^c)$, that is,

$$\mathrm{fpt}(\mathfrak{a}) = \sup\{t \in R_{\geq 0} \mid \tau(\mathfrak{a}^t) = R\}.$$

Chiba gave a formula of $\mathrm{fpt}(\mathfrak{m})$ of Hibi ring $R = R(P)$.

Theorem 14 (see [CM]). *Let P be a finite poset, and $R = R(P)$ the Hibi ring made from P. Let $\mathfrak{m} = R_+$ be the graded maximal ideal of R. Then*

$$\mathrm{fpt}(\mathfrak{m}) = \mathrm{rank}_* P + 2.$$

4. $-a(R)$ of Hibi rings and Characterization of Hibi rings which satisfy $c^{\mathfrak{m}}(\mathfrak{m}) = -a(R) = \mathrm{fpt}(\mathfrak{m})$

The first main theorem of this article is the following:

Theorem 15 (see [CM], [BH]). *Let P be a poset, and $R = R(P)$ the Hibi ring made from P. Let $\mathfrak{m} = R_+$ the unique graded maximal ideal of R. Then*

$$\begin{aligned} c^{\mathfrak{m}}(\mathfrak{m}) &= \mathrm{rank}^* P + 2, \\ -a(R) &= \mathrm{rank} P + 2, \\ \mathrm{fpt}(\mathfrak{m}) &= \mathrm{rank}_* P + 2, \end{aligned}$$

where $a(R)$ is a-invariant of R(see [GW]*). In particular, $c^{\mathfrak{m}}(\mathfrak{m}) \geq -a(R) \geq \mathrm{fpt}(\mathfrak{m})$.*

In this section, we give a characterization of Hibi rings which satisfy $c^{\mathfrak{m}}(\mathfrak{m}) = -a(R) = \mathrm{fpt}(\mathfrak{m})$, that is, we consider the following question:

Question: When does $c^{\mathfrak{m}}(\mathfrak{m}) = -a(R) = \mathrm{fpt}(\mathfrak{m})$ hold for Hibi rings?

Hirose, Watanabe and Yoshida [HWY] showed that for any homogeneous affine toric ring R with the unique graded maximal ideal $\mathfrak{m}$, R is Gorenstein if and only if $\mathrm{fpt}(\mathfrak{m}) = -a(R)$. Hence we need to study Hibi rings which satisfy $c^{\mathfrak{m}}(\mathfrak{m}) = -a(R)$.

Let P_1, P_2 be posets and let $R_1 = R(P_1)$, $R_2 = R(P_2)$ be Hibi rings made from P_1, P_2 respectively. In order to give an answer of the above question, we observe the tensor products and Segre products of R_1 and R_2(see [Hib], [HeHiR]).

Firstly, we define some notions.

Definition 16. A ring R is *trivial* if R can be made by the following operations : starting from polynomial rings, repeated applications of tensor products and Segre products.

Definition 17. (see [HeHiR]) A poset P is *simple* if there is no element of P which is comparable with any other element of P.

Tensor Products:

Let P be a *not* simple poset. Then there exists $p \in P$ such that p is comparable with any other element of P. Put $P_1 = \{q \in P \mid q < p\}$ and $P_2 = \{q \in P \mid q > p\}$. Then

$$R(P) \simeq R_1 \otimes R_2$$

holds. Moreover, it is easy to see that

$$\begin{aligned} \mathrm{rank}^* P &= \mathrm{rank}^* P_1 + \mathrm{rank}^* P_2 + 2, \\ \mathrm{rank} P &= \mathrm{rank} P_1 + \mathrm{rank} P_2 + 2, \\ \mathrm{rank}_* P &= \mathrm{rank}_* P_1 + \mathrm{rank}_* P_2 + 2. \end{aligned}$$

Hence we have

$$\mathrm{rank}^* P = \mathrm{rank} P = \mathrm{rank}_* P$$

$$\Updownarrow$$

$$\mathrm{rank}^* P_1 = \mathrm{rank} P_1 = \mathrm{rank}_* P_1 \quad \text{and} \quad \mathrm{rank}^* P_2 = \mathrm{rank} P_2 = \mathrm{rank}_* P_2.$$

Segre Products:

Let P be a *not* connected (that is, its Hasse diagram is not connected) poset. Then there exist two non-empty subposets P_1 and P_2 of P such that the elements of P_1 and P_2 are incomparable. Then

$$R(P) \simeq R_1 \# R_2$$

holds. Moreover, it is easy to see that

$$\begin{aligned} \mathrm{rank}^* P &= \max\{\mathrm{rank}^* P_1, \mathrm{rank}^* P_2\}, \\ \mathrm{rank} P &= \max\{\mathrm{rank} P_1, \mathrm{rank} P_2\}, \\ \mathrm{rank}_* P &= \min\{\mathrm{rank}_* P_1, \mathrm{rank}_* P_2\}. \end{aligned}$$

Hence we have

$$\mathrm{rank}^* P_1 = \mathrm{rank} P_1 \ \text{ and } \ \mathrm{rank}^* P_2 = \mathrm{rank} P_2 \quad \Rightarrow \quad \mathrm{rank}^* P = \mathrm{rank} P$$

and

$$\mathrm{rank} P = \mathrm{rank}_* P \quad \Rightarrow \quad \mathrm{rank} P_1 = \mathrm{rank}_* P_1 \ \text{ and } \ \mathrm{rank} P_2 = \mathrm{rank}_* P_2$$

holds. If P is pure, then the converses of the above assertion are also true, that is

$$\mathrm{rank}^* P = \mathrm{rank} P = \mathrm{rank}_* P$$

$$\Updownarrow$$

$$\mathrm{rank}^* P_1 = \mathrm{rank} P_1 = \mathrm{rank}_* P_1 \quad \text{and} \quad \mathrm{rank}^* P_2 = \mathrm{rank} P_2 = \mathrm{rank}_* P_2$$

holds since $\mathrm{rank} P = \mathrm{rank} P_1 = \mathrm{rank} P_2$.

By using these observation, we prove the following proposition.

Proposition 18. *Let P be a finite poset, and $R = R(P)$ the Hibi ring made from P. Let $\mathfrak{m} = R_+$ be the graded maximal ideal of R. Then if R is trivial, then* $\mathrm{rank}^* P = \mathrm{rank} P$, *that is, $c^{\mathfrak{m}}(\mathfrak{m}) = -a(R)$. Moreover, if P is pure, the converse is also true.*

Proof. The first assertion is clear from the above observation and the fact that $c^{\mathfrak{m}}(\mathfrak{m}) = -a(R)$ if R is a polynomial ring.

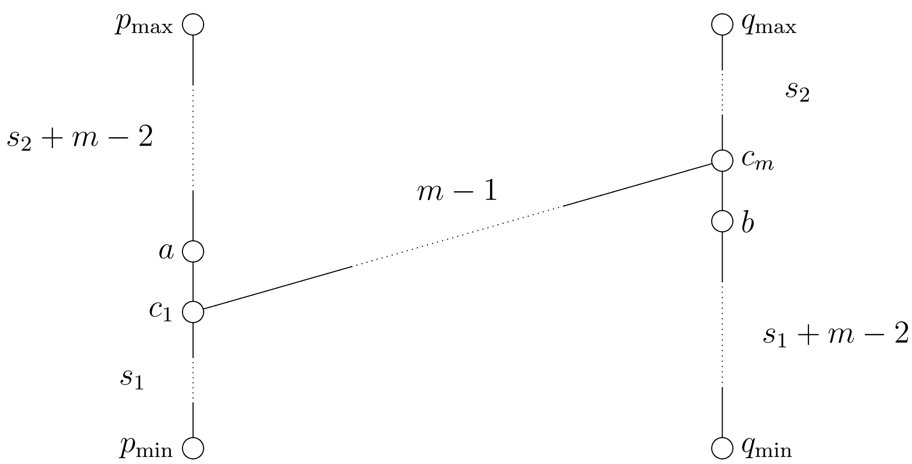
We prove that the converse is true if P is pure. Assume that R is *not* trivial. From the above observation, we may assume that P is simple and connected.

Firstly, we refer the following lemma.

Lemma 19. *([HeHiR, Lemma 3.5]) Every simple and connected poset P possesses a saturated ascending chain $A = c_1 \to c_m$ $(m \geq 2)$ together with a, $b \in P$ satisfying the following condition : (i) $c_m \gtrdot b$; (ii) $a \gtrdot c_1$; (iii) $c_1 \not\leq b$; (iv) $a \not\leq c_m$.*

Hence, it is enough to show that $\mathrm{rank}^* P > \mathrm{rank} P$ under the situation as in Lemma 3.5.

We consider three paths $C_1 = p_{\min} \to p_{\max}$, $C_2 = p_{\min} \to q_{\max}$ and $C_3 = q_{\min} \to q_{\max}$ as the following:



We put $\mathrm{length}(p_{\min} \to c_1) = s_1$ and $\mathrm{length}(c_m \to q_{\max}) = s_2$. Since P is pure,

$$\mathrm{rank} P = \mathrm{length} C_1 = \mathrm{length} C_2 = \mathrm{length} C_3 = s_1 + s_2 + m - 1.$$

Hence we have

$$\mathrm{length}(a \to p_{\max}) = s_2 + m - 2, \quad \mathrm{length}(q_{\min} \to b) = s_1 + m - 2.$$

Let $C = q_{\min} \to c_m \to c_1 \to p_{\max}$ be a path. Then it is easy to show that C satisfies a condition (*). Moreover,

$$\begin{aligned} \mathrm{length}^* C &= (s_2 + m - 1) + (s_1 + m - 1) \\ &= s_1 + s_2 + 2m - 2 \\ &> s_1 + s_2 + m - 1 \\ &= \mathrm{rank} P \end{aligned}$$

since $m \geq 2$. Therefore we have $\mathrm{rank}^* P > \mathrm{rank} P$. □

In [HeHiR], Herzog, Hibi and Restuccia introduced the notion of strongly Koszulness for homogeneous k-algebra, and they proved that a Hibi ring is strongly Koszul if and only if it is trivial(see [HeHiR, Theorem 3.2]). Moreover, from [HWY], we can see that for any Hibi ring $R = R(P)$ with the unique graded maximal ideal $\mathfrak{m}$, $\mathrm{rank} P = \mathrm{rank}_* P$ if and only if P is pure. Therefore, we get the following theorem:

Theorem 20 (see [CM], [HeHiR])**.** *Let P be a finite poset, and $R = R(P)$ the Hibi ring made from P. Let $\mathfrak{m} = R_+$ be the graded maximal ideal of R. The the following assertions are equivalent:*

(1) *R is trivial and Gorenstein.*
(2) *R is strongly Koszul and Gorenstein.*
(3) *R satisfies $c^{\mathfrak{m}}(\mathfrak{m}) = -a(R) = \mathrm{fpt}(\mathfrak{m})$.*
(4) *P satisfies $\mathrm{rank}^* P = \mathrm{rank} P = \mathrm{rank}_* P$.*

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DERIVED GABRIEL TOPOLOGY, LOCALIZATION AND COMPLETION OF DG-ALGEBRAS

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Abstract. Gabriel topology is a special class of linear topology on rings, which plays an important role in the theory of localization of (not necessary commutative) rings []. Several evidences have suggested that there should be a corresponding notion for dg-algebras. In my talk I introduced a notion of Gabriel topology on dg-algebras, derived Gabriel topology, and showed its basic properties.

In the same way as the definition of derived Gabriel topology on a dg-algebra, we gave the definition of topological dg-modules over a dg-algebra equipped with derived Gabriel topology. An important example of topology on dg-modules is the finite topology on the bi-dual module $M^{\circledast\circledast}$ of a dg-module M by another dg-module J.

We show that a derived bi-duality dg-module is quasi-isomorphic to the homotopy limit of a certain tautological functor. This is a simple observation, which seems to be true in wider context. From the view point of derived Gabriel topology, this is a derived version of results of J. Lambek about localization and completion of ordinary rings. However the important point is that we can obtain a simple formula for the bi-duality modules only when we come to the derived world from the abelian world.

We give applications. 1. we give a generalization and an intuitive proof of Efimov-Dwyer-Greenlees-Iyenger Theorem which asserts that the completion of commutative ring satisfying some conditions is obtained as a derived bi-commutator. (We can also prove Koszul duality for dg-algebras with Adams grading satisfying mild conditions.) 2. We prove that every smashing localization of dg-category is obtained as a derived bi-commutator of some pure injective module. This is a derived version of the classical results in localization theory of ordinary rings.

These applications show that our formula together with the viewpoint that a derived bi-commutator is a completion in some sense, provide us a fundamental understanding of a derived bi-duality module.

Key Words: Derived bi-duality, homotopy limit, dg-algebras, completion, localization, Koszul duality, Lambek Theorem.

1. Introduction

The following situation and its variants are ubiquitous in Algebras and Representation theory:

Let R be a ring, J an R-module and $E := \mathrm{End}_R(J)^{\mathrm{op}}$ the opposite ring of the endomorphism ring of J over R. Then we have the duality

$$(-)^* := \mathrm{Hom}_R(-, J) : \mathrm{Mod}R \rightleftarrows (\mathrm{Mod}E)^{\mathrm{op}} : \mathrm{Hom}_E(-, J) =: (-)^*$$

and the unite map $\epsilon_M : M \to M^{**}$ is given by the evaluation map:

$$\epsilon_M(m) : \mathrm{Hom}_R(M, J) \to J,\ f \mapsto f(m) \text{ for } m \in M.$$

The detailed version of this paper will has been submitted for publication elsewhere.

The bi-dual R^{**} of R is called the *bi-commutator* (or the *double centralizer*) and denoted by $\mathrm{Bic}_R(J)$. The following is more popular expression (or the usual definition) of the bi-commutator

$$\mathrm{Bic}_R(J) := \mathrm{End}_E(J)^{\mathrm{op}}.$$

The bi-commutator has a ring structure and the evaluation map $\epsilon_R : R \to \mathrm{Bic}_R(J)$ become a ring homomorphism. In particular, the case where the canonical algebra homomorphism $R \to \mathrm{Bic}_R(J)$ become an isomorphism, the module J is said to have the *double centralizer property.* Dualities together with evaluation maps, bi-commutators and double centralizer properties are one of the central topics in Algebras and Representation theory. (See e.g. [5, 8, 10, 11, 17, 26])

Recently the concern with the *derived bi-commutators* (or the *derived double centralizers*) has been growing:

Let R a ring (or more generally dg-algebra) J an (dg-)R-module and $\mathcal{E} := \mathbb{R}\mathrm{End}_R(J)^{\mathrm{op}}$ the opposite dg-algebra of the endomorphism dg-algebra of J. Then the *derived bi-commutator* is defined by

$$\mathbb{B}\mathrm{ic}_R(J) := \mathbb{R}\mathrm{End}_{\mathcal{E}}(J)^{\mathrm{op}}.$$

There also exists a canonical algebra homomorphism $R \to \mathbb{B}\mathrm{ic}_R(J)$. In particular, the case where the canonical algebra homomorphism $R \to \mathbb{B}\mathrm{ic}_R(J)$ become an isomorphism, the module J is said to have the *derived double centralizer property.* Derived double centralizer property for special modules has been extensively studied as a part of Koszul duality. (See e.g. [12, 22].)

In [2, Section 4.16], Dwyer-Greenlees-Iyenger call a pair (R, J) *dc-complete*, in the case where J has derived double centralizer property. They proved the following surprising and impressive theorem, which we will refer as *completion theorem.*

Theorem 1 ([2],[3]). *Let R be a commutative Noetherian ring and $\mathfrak{a}$ an ideal such that the residue ring $R/\mathfrak{a}$ is of finite global dimension. We denote by $\widehat{R}$ the $\mathfrak{a}$-adic completion. Then we have a quasi-isomorphism*

$$\widehat{R} \simeq \mathbb{B}\mathrm{ic}_R(R/\mathfrak{a})$$

where $\mathbb{B}\mathrm{ic}_R(R/\mathfrak{a})$ is the derived bi-commutator of $R/\mathfrak{a}$ over R.

From the view point of Derived-Categorical Algebraic Geometry (DCAG), all important procedure in Algebraic Geometry should have derived-categorical interpretation. In [7] Kontsevich claimed that formal completion for a scheme is obtained as a derived bi-commutator. Following this idea, Efimov [3] introduced the derived bi-commutator of subcategory $\rfloor\mathcal{J} \subset \mathcal{D}(R)$ and proved a scheme version of completion theorem. Since formal completion plays an important role in Algebraic Geometry, completion theorem and its scheme version are expected to become important in DCAG. Therefore it is desirable to obtain better understanding of this theorem.

In the proof of completion theorem, Grothedieck vanishing theorem for local cohomology is used. Since it is special theorem for commutative Noetherian rings, it is preferable to obtain more categorical proof. Recently Porta, Shaul and Yekutieli [21] generalized completion theorem for a commutative ring R and a weakly proregular ideal $\mathfrak{a}$ based on their work [20] about the derived functors of the completion functors and the torsion functors. However it is still remain unclear that to what extent we can obtain a transcendental outcome by a homological operation with finite input. In this paper we establish a simple description of the derived bi-commutator, which enable us to give a more intuitive proof of completion theorem. Actually the description is given by a certain tautological homotopy limit, and hence seems to state that every derived bi-commutator is completion in some sense. (We can make this precise by introducing the notion of derived Gabriel topology.)

For this purpose, we study derived bi-duality:

$$(-)^{\circledast} := \mathbb{R}\mathrm{Hom}_R(-, J) : \mathcal{D}(R) \rightleftarrows \mathcal{D}(\mathcal{E})^{\mathrm{op}} : \mathbb{R}\mathrm{Hom}_{\mathcal{E}}(-, J) =: (-)^{\circledast}.$$

For a special class of modules J, derived bi-duality is already studied in the context of Gorenstein dg-algebras [4, 6, 13]. We consider general dg-modules J and establish a simple description of the derived bi-dual module $M^{\circledast\circledast}$ via a certain tautological homotopy limit. This is the main result of this paper. As an application other than completion theorem, we discuss smashing localization of dg-categories.

As mentioned above, derived bi-dualities, derived bi-commutator and derived double centralizer property are expected to play prominent roles in Algebras, Representation theory, Derived-Categorical Algebraic Geometry. Our main theorem together with the view point that derived bi-commutators are completion in some sense, would have many applications. Moreover since the main theorem is proved in a formal argument, the same formula should hold in more wider context. Bi-duality is a basic operation which is ubiquitous in mathematics. So it can be expect that our main theorem become an indispensable tool in many area of mathematics.

Below we give an outline, in which the readers see that if we omit homotopy theoretical details, things become very simple. However, we will see that it is inevitable to work with homotopy theory.

2. Derived bi-duality via homotopy limit

Let $\mathcal{A}$ be a dg-algebra and J a dg $\mathcal{A}$-module. We denote $\mathcal{E} := (\mathbb{R}\mathrm{End}_{\mathcal{A}}(J))^{\mathrm{op}}$ be the opposite dg-algebra of the endomorphism dg-algebra. Then J has a natural dg $\mathcal{E}$-module structure. We obtain the dualities

$$(-)^{\circledast} := \mathbb{R}\mathrm{Hom}_{\mathcal{A}}(-, J) : \mathcal{D}(\mathcal{A}) \rightleftarrows \mathcal{D}(\mathcal{E})^{\mathrm{op}} : \mathbb{R}\mathrm{Hom}_{\mathcal{E}}(-, J) =: (-)^{\circledast}.$$

There are natural transformations $\epsilon : 1_{\mathcal{D}(\mathcal{A})} \to (-)^{\circledast\circledast}$ induced from evaluation morphisms.

We denote by $\langle J \rangle$ the smallest thick subcategory containing J. Namely $\langle J \rangle$ is the full triangulated subcategory of $\mathcal{D}(\mathcal{A})$ consisting those objects which constructed from J by taking cones, shifts, and direct summands finitely many times.

Let M be a dg $\mathcal{A}$-module. We denote by $\langle J \rangle_{M/}$ the under category. Namely, the objects of $\langle J \rangle_{M/}$ are morphisms $k : M \to K$ with $K \in \langle J \rangle$ and the morphisms from $k : M \to K$ to $\ell : M \to L$ are the morphisms $\psi : K \to L$ in $\langle J \rangle$ such that $\ell = \psi \circ k$. This category $\langle J \rangle_{M/}$ comes naturally equipped with the co-domain functor $\Gamma : \langle J \rangle_{M/} \to \mathcal{D}(\mathcal{A})$ which sends an object $k : M \to K$ to its co-domain K.

$$\Gamma : \langle J \rangle_{M/} \to \mathcal{D}(\mathcal{A}), \quad [k : M \to K] \mapsto K.$$

The following simple formula is the main theorem.

Theorem 2. *We have the following quasi-isomorphism*

$$M^{\circledast\circledast} \simeq \operatorname*{holim}_{\langle J\rangle_{M/}} \Gamma$$

Remark 3. In the above Theorem 2, Theorem 4 and Corollary 5, we omit homotopy theoretical details. For the rigorous statements see [14].

To explain an idea of a proof, we give the following heuristic arguments. First we claim that if K belongs to $\langle J\rangle$, then the evaluation map $\epsilon_K : K \to K^{\circledast\circledast}$ is an isomorphism. Indeed the case $K = J$ is clear. Since the bi-dual $(-)^{\circledast\circledast}$ is an exact functor, we can check the claim for general $K \in \langle J\rangle$.

It follows from the above claim that every morphism $k : M \to K$ with $K \in \langle J\rangle$ factors though $\epsilon_M : M \to M^{\circledast\circledast}$.

$$\begin{tikzcd} M \arrow[r, "\epsilon_M"] \arrow[d, "k"] & M^{\circledast\circledast} \arrow[d, "k^{\circledast\circledast}"] \\ K & K^{\circledast\circledast} \arrow[l, "\epsilon_K^{-1}", "\cong"'] \end{tikzcd}$$

It seems that the derived bi-dual module $M^{\circledast\circledast}$ satisfies one of the two conditions of the limit of the family $M \to K$ of morphisms. In the following way, we can catch a glimpse of the other condition that we can reach from $K \in \langle J\rangle$ to $M^{\circledast\circledast}$:

It is well-known that a dg-module is obtained as a filtered homotopy colimit perfect modules. Hence the dg $\mathcal{E}$-module $M^{\circledast}$ is quasi-isomorphic to the homotopy colimit of some family $\{P_\lambda\}_\Lambda$ of perfect $\mathcal{E}$-modules.

$$M^{\circledast} \simeq \operatorname*{hocolim}_{\Lambda} P_\lambda \tag{2.1}$$

Applying the dual functor $(-)^{\circledast}$ to this quasi-isomorphism, we obtain the following (quasi-)isomorphisms

$$M^{\circledast\circledast} \simeq (\operatorname*{hocolim}_{\Lambda} P_\lambda)^{\circledast} \simeq \operatorname*{holim}_{\Lambda}(P_\lambda^{\circledast}).$$

It is clear that $\mathcal{E}^{\circledast} \simeq J$. Therefore, since P_λ is a perfect $\mathcal{E}$-module, the dual $P_\lambda^{\circledast}$ belongs to $\langle J\rangle$. This shows that we can reach from $K \in \langle J\rangle$ to $M^{\circledast\circledast}$. Actually the following Theorem 4 which is a version of the quasi-isomorphism (2.1) is a key of the proof of the main theorem.

Theorem 4. *Let X be a dg $\mathcal{E}$-module. We denote by $\operatorname{Perf}\mathcal{E}$ the category of perfect $\mathcal{E}$-modules. Then the over category $(\operatorname{Perf}\mathcal{E})_{/X}$ comes naturally equipped with the domain functor*

$$\Upsilon : \operatorname{Perf}\mathcal{E} \to \mathcal{D}(\mathcal{E}),\ [p : P \to X] \mapsto P.$$

Then the canonical morphism

$$\operatorname*{hocolim}_{(\operatorname{Perf}\mathcal{E})_{/X}} \Upsilon \to X$$

is a quasi-isomorphism.

Since the bi-dual $\mathcal{A}^{\circledast\circledast}$ of $\mathcal{A}$ is naturally isomorphic to the derived bi-commutator $\mathbb{B}\mathrm{ic}_{\mathcal{A}}(J)$,

$$\mathcal{A}^{\circledast\circledast} = \mathbb{R}\mathrm{Hom}_{\mathcal{E}}(\mathbb{R}\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, J), J) \cong \mathbb{R}\mathrm{Hom}_{\mathcal{E}}(J, J) = \mathbb{B}\mathrm{ic}_{\mathcal{A}}(J)$$

in particular, we have the following corollary.

Corollary 5.

$$\mathbb{B}\mathrm{ic}_{\mathcal{A}}(J) \simeq \operatorname*{holim}_{\langle J\rangle_{\mathcal{A}/}} \Gamma.$$

These theorem and corollary provide us a fundamental understanding of derived bi-duality functors.

3. Completion via derived bi-commutator

As the first application, we generalize the completion theorem and give an intuitive proof.

Let R be a ring and $\mathfrak{a}$ a two-sided ideal. An (right) R-module M is called $\mathfrak{a}$-*torsion* if for any $m \in M$ there exists $n \in \mathbb{Z}_{\geq 1}$ such that $m\mathfrak{a}^n = 0$. We denote by $\mathfrak{a}$-tor the full subcategory of Mod$R$ consisting of $\mathfrak{a}$-torsion modules. We denote by $\mathcal{D}_{\mathfrak{a}\text{-tor}}(R)$ the full subcategory of $\mathcal{D}(R)$ consisting of complexes with $\mathfrak{a}$-torsion cohomology groups. We denote by $\mathcal{D}(\mathfrak{a}\text{-tor})$ the full subcategory of $\mathcal{D}(R)$ consisting of complexes each term of which is $\mathfrak{a}$-torsion module.

Theorem 6. *Assume that the canonical inclusion functor $\mathcal{D}(\mathfrak{a}\text{-tor}) \to \mathcal{D}_{\mathfrak{a}\text{-tor}}(R)$ gives an equivalence and that $R/\mathfrak{a}^n$ belongs to $\langle R/\mathfrak{a}\rangle$ for $n \geq 0$. We denote by $\widehat{R}$ the $\mathfrak{a}$-adic completion. Then we have a quasi-isomorphism*

$$\mathbb{B}\mathrm{ic}_R(R/\mathfrak{a}) \simeq \widehat{R}.$$

"*Proof*".

Assumption. In this "Proof" we assume that holim = lim. We identify quasi-isomorphisms with isomorphisms.

We denote by $\mathcal{I}$ the (non-full) subcategory of $\langle R/\mathfrak{a}\rangle_{R/}$ which consists of objects $\pi^n : R \to R/\mathfrak{a}^n$ for $n \geq 1$ and of morphisms $\pi^m \to \pi^n$ induced from the canonical projections $\varphi^{m,n} : R/\mathfrak{a}^m \to R/\mathfrak{a}^n$ for $m \geq n$. In other words, $\mathcal{I}$ is the image of the functor $(\mathbb{Z}_{\geq 1})^{\mathrm{op}} \to \mathcal{D}(\mathcal{A})$ which sends an object n to π^n and a morphism $m \to n$ to $\pi^m \to \pi^n$ where we consider the ordered set $\mathbb{Z}_{\geq 1}$ as a category in the standard way. Therefore we have

$$\lim_{\mathcal{I}} \Gamma|_{\mathcal{I}} \cong \lim_{n\to\infty} R/\mathfrak{a}^n \cong \widehat{R}.$$

Thanks to Corollary 5 the problem is reduced to show that $\lim_{\mathcal{I}} \Gamma|_{\mathcal{I}} \cong \lim_{\langle R/\mathfrak{a}\rangle_{R/}} \Gamma$. Therefore it is enough to prove that $\mathcal{I}$ is a left cofinal subcategory of $\langle R/\mathfrak{a}\rangle_{R/}$. Namely only we have to show that the over category $\mathcal{I}_{/k}$ is non-empty and connected for each $k \in \langle R/\mathfrak{a}\rangle_{R/}$.

Let $k : R \to K$ be an object of $\langle R/\mathfrak{a}\rangle_{R/}$. It is clear that $\langle R/\mathfrak{a}\rangle$ contained in $\mathcal{D}_{\mathfrak{a}\text{-tor}}(R)$. Since we assume that $\mathcal{D}(\mathfrak{a}\text{-tor}) \xrightarrow{\sim} \mathcal{D}_{\mathfrak{a}\text{-tor}}(R)$, K belongs to $\mathcal{D}(\mathfrak{a}\text{-tor})$. It follows that K is (quasi-)isomorphic to a complex each term of which is an $\mathfrak{a}$-torsion modules. Therefore a morphism $k : R \to K$ canonically factors through some cyclic $\mathfrak{a}$-torsion module $R/\mathfrak{a}^n$.

$$\begin{tikzcd} R \arrow[d, "\pi^n"'] \arrow[dr, "k"] & \\ R/\mathfrak{a}^n \arrow[r, "\psi"'] & K \end{tikzcd}$$

In other words, there exists a morphism $\psi : \pi^n \to k$ in $\langle R/\mathfrak{a}\rangle_{R/}$. This proves the non-emptiness of $\mathcal{I}_{/k}$. Since the factorization $k = \psi \circ \pi^n$ is canonical, we see that $\mathcal{I}_{/k}$ is connected. This shows that $\mathcal{I}$ is left co-final in $\langle R/\mathfrak{a}\rangle_{R/}$ and completes the "proof". "□"

In [21] Porta, Shaul and Yekutieli generalized completion theorem (Theorem 1) by using a compact generator of $\mathcal{D}_{\mathfrak{a}\text{-tor}}(R)$.

Theorem 7 ([21, Theorem 4.2])**.** *Let R be a commutative ring and $\mathfrak{a}$ a weakly pro-regular ideal. Let K be a compact generator of $\mathcal{D}_{\mathfrak{a}\text{-tor}}(R)$. Then we have the following quasi-isomorphism of dg-algebras under R.*

$$\mathbb{B}\mathrm{ic}_R(K) \simeq \widehat{R}.$$

By our method, we give a generalization of this theorem.

Theorem 8. *Let R be a ring and $\mathfrak{a}$ an two-sided ideal such that the canonical functor $\mathcal{D}(\mathfrak{a}\text{-tor}) \to \mathcal{D}_{\mathfrak{a}\text{-tor}}(R)$ gives an equivalence. Let K be a compact generator of $\mathcal{D}_{\mathfrak{a}\text{-tor}}(R)$. Then we have a quasi-isomorphism*

$$\mathbb{B}\mathrm{ic}_R(K) \simeq \widehat{R}.$$

It is proved by [20, Corollary 3.31] that if a ring R is commutative and an ideal $\mathfrak{a}$ is weakly pro-regular, then the canonical functor $\mathcal{D}(\mathfrak{a}\text{-tor}) \to \mathcal{D}_{\mathfrak{a}\text{-tor}}(R)$ gives an equivalence. Therefore Theorem 8 implies Theorem 7.

Remark 9. The conditions on Theorem 6 and Theorem 8 are not practical. The reason why we put these artificial conditions is not to obtain generality but to clarify to what extent the derived bi-commutator gives the completion.

The condition that the canonical functor $\mathbf{can}_{\mathfrak{a}} : \mathcal{D}(\mathfrak{a}\text{-tor}) \to \mathcal{D}_{\mathfrak{a}\text{-tor}}(R)$ gives an equivalence is satisfied if the subcategory $\mathfrak{a}$-tor is closed under taking injective hull. This condition is satisfied if a right ideal $\mathfrak{a}$ has the Artin-Rees property. In particular, in the case where a ring R is commutative Noetherian, for any ideal $\mathfrak{a}$ the functor $\mathbf{can}_{\mathfrak{a}}$ is an equivalence. As we mentioned before, if a ring R is commutative and an ideal $\mathfrak{a}$ is weakly pro-regular, then the canonical functor $\mathbf{can}_{\mathfrak{a}}$ is an equivalence. It should be noted that if R is commutative Noetherian, any ideal $\mathfrak{a}$ is weakly pro-regular (See [1, 20, 23]). It is showed in [20, Example 3.35] that a weakly pro-regular ideal in non-Noetherian ring naturally appears.

The following question arises: find a necessary and sufficient condition on rings R and ideals $\mathfrak{a}$ such that the canonical functor $\mathbf{can}_{\mathfrak{a}}$ is an equivalence.

4. Smashing localization via derived bi-commutator

First we recall the following classical fact.

Theorem 10 ([10, Corollary 3.4.1], [17, Theorem 7.1])**.** *Let $f : R \to S$ be a (right) Gabriel localization of a ring R, that is, f is an epimorphism in the category of rings and S is left flat over R. Let J be a co-generator of the torsion theory which corresponds to the Gabriel localization f. If we take a product $J' := J^{\kappa}$ of copies of J over large enough cardinal κ, then we have an isomorphism*

$$\mathrm{Bic}_R(J') \cong S.$$

In this section we prove a derived version. A morphisms $f : \mathcal{A} \to \mathcal{B}$ of dg-algebras is called *smashing localization* (or *homological epimorphism*) if the restriction functor $f_* : \mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{A})$ is fully faithful. Recall that a ring homomorphism $R \to S$ is an epimorphism in the category of rings if and only if the restriction functor $f_* : \mathrm{Mod}S \to \mathrm{Mod}R$ is fully faithful. Therefore smashing localization can be considered as a dg-version of epimorphisms of rings.

Theorem 11. *Let $\mathcal{A} \to \mathcal{B}$ be a smashing localization of dg-algebras and J be a pure injective co-generator of $\mathcal{D}(\mathcal{B})$. Then we have a quasi-isomorphism over $\mathcal{A}$*

$$\mathbb{B}\mathrm{ic}_{\mathcal{A}}(f_* J') \simeq \mathcal{B}.$$

where $J' = J^{\Pi\kappa}$ is a large enough product of J.

The notion of pure injective co-generator which is introduced by Krause [9] is a dg-version of injective co-generator for the module category $\mathrm{Mod}R$ of an ordinary ring R.

Remark 12. Nicolás and Saorin [18] proved that for any smashing localization $F : \mathcal{D}(\mathcal{A}) \to \mathcal{S}$, there exists a subcategory $\mathcal{I} \subset \mathcal{D}(\mathcal{A})$ such that the functor $\mathbb{L}\iota_{\mathcal{I}}^* : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathbb{B}\mathrm{ic}_{\mathcal{A}}(\mathcal{I}))$ induced from the canonical morphism $\iota_{\mathcal{I}} : \mathcal{A} \to \mathbb{B}\mathrm{ic}_{\mathcal{A}}(\mathcal{I})$ is equivalent to F.

In our way of the proof, an essential point is the following theorem.

Theorem 13. *Let J a pure injective co-generator of $\mathcal{D}(\mathcal{A})$ and M a dg $\mathcal{A}$-module. If we take a product $J' = J^{\Pi\kappa}$ of copies of J over large enough cardinal κ, then the evaluation morphism is a quasi-isomorphism*

$$\epsilon_M : M \xrightarrow{\sim} M^{\circledast\circledast}$$

where the bi-dual is taken over J'.

From the view point that a derived bi-commutator is a completion, we can give an intuitive proof of Theorem 13 by using Theorem 2. (In the case where $\mathcal{A}$ is an ordinary ring and M is a module, the same results is already proved by Shamir [24] in a different way.) In the rest of this subsection, we use the same assumption with that of “Proof” of Theorem 6.

For the sake of simplicity we deal with the case where $\mathcal{A}$ is an ordinary ring, M is an $\mathcal{A}$-module and J is an injective co-generator of $\mathrm{Mod}\mathcal{A}$. Then the module M has an injective resolution by the products of J

$$0 \to M \to J^{\Pi\kappa_0} \to J^{\Pi\kappa_1} \to J^{\Pi\kappa_2} \to \cdots .$$

We can reduce the problem to the following theorem by setting $\kappa := \sup\{\kappa_i \mid i \in \mathbb{Z}\}$.

Theorem 14. *Let $M \xrightarrow{\sim} J^\bullet$ be an injective resolution of M.*

$$0 \to M \to J^0 \to J^1 \to J^2 \to \cdots . \tag{4.1}$$

Assume that J^i is a direct summand of J. Then the evaluation map $\epsilon_M : M \to M^{\circledast\circledast}$ is a quasi-isomorphism.

We denote by I^n the totalization of the n-th truncated resolution.

$$I^n := \mathrm{tot}[J^0 \to J^1 \to \cdots \to J^n].$$

Then by assumption the complex I^n belongs to the thick subcategory $\langle J \rangle$ generated by J. Therefore the canonical morphism $\pi^n : M \to I^n$ belongs to the under category $\langle J \rangle_{M/}$. Moreover we have a canonical morphism $\varphi^{n+1,n} : I^{n+1} \to I^n$ for $n \geq 0$ which is compatible with π^n.

$$\begin{tikzcd}
 & M \arrow[d, "\pi^n"'] \arrow[dr, "\pi^{n-1}"] & & \\
\cdots \arrow[r, "\varphi^{n+1,n}"'] & I^n \arrow[r, "\varphi^{n,n-1}"'] & I^{n-1} \arrow[r, "\varphi^{n-1,n-2}"'] & \cdots
\end{tikzcd} .$$

Note that since the limit $\lim_{n\to\infty} I^n$ is the totalization of the injective resolution (4.1), the morphisms $\{\pi^n\}$ induces a (quasi-)isomorphism $M \to \lim_{n\to\infty} I^n$. We will see that the family $\{\pi^n : M \to I^n\}_{n\geq 0}$ is an "approximation" for the morphisms $k : M \to K$ with $K \in \langle J \rangle$.

We denote by $\mathcal{I}$ the subcategory of $\langle J \rangle_{M/}$ consisting of objects $\pi^n : M \to I^n$ and of morphisms $\phi^{m,n} : \pi^m \to \pi^n$ so that $\mathcal{I}$ is isomorphic to $(\mathbb{Z}_{\geq 0})^{\mathrm{op}}$. Then it is clear that

$$\lim_{\mathcal{I}} \Gamma|_{\mathcal{I}} \cong \lim_{n\to\infty} I^n \simeq M.$$

Therefore by "Theorem" 2 it is enough to prove that the subcategory $\mathcal{I} \subset \langle J \rangle_{M/}$ is left co-final. Namely for each $k \in \langle J \rangle_{M/}$ the over category $\mathcal{I}_{/k}$ is non-empty and connected.

We recall the following elementary fact from Homological algebra: Let M' be another $\mathcal{A}$-module and $M \xrightarrow{\sim} J'^{\bullet}$ an injective resolution. Assume that an $\mathcal{A}$-homomorphism $f : M \to M'$ is given. Then (1) there exists a morphism $\psi : J^{\bullet} \to J'^{\bullet}$ of complexes which completes the commutative diagram

$$\begin{tikzcd}
M \arrow[r] \arrow[d, "f"'] & J^{\bullet} \arrow[d, "\psi"] \\
M' \arrow[r] & J'^{\bullet}.
\end{tikzcd}$$

(2) This morphism ψ is not uniquely determined. (3) However it is uniquely determined up to homotopy.

Using the same methods of the proof of (1), we can check that $\mathcal{I}_{/k}$ is non-empty. By the same reason with (2), the category $\mathcal{I}_{/k}$ is *not* connected. However in the same way of the proof of (3), we can verify that $\mathcal{I}_{/k}$ is "*homotopically connected*". We explain detail in the special case where the co-domain K of $k : M \to K$ is an injective module:

Since the canonical morphism $\pi^0 : M \to I^0 = J^0$ is injective, there exists an extension $\psi : I^0 \to K$ of π^0. This shows that $\mathcal{I}_{/k} \neq \emptyset$. However there is no canonical choice of an extension. Moreover since the degree 0-part of the canonical morphism $\varphi^{n,0} : I^n \to I^0$ is the identity map $1_{J^0} : J^0 \to J^0$, two extensions ψ and ψ' are not connected to each other in $\mathcal{I}_{/k}$, unless $\psi = \psi'$. Nevertheless we can see that for any pair (ψ, ψ') of extensions, there exists a homotopy commutative diagram

$$\begin{tikzcd}
\pi^1 \arrow[r, "\varphi^{1,0}"] \arrow[d, "\varphi^{1,0}"'] & \pi^0 \arrow[d, "\psi"] \\
\pi^0 \arrow[r, "\psi'"'] & k.
\end{tikzcd}$$

Hence the objects ψ and ψ' of $\mathcal{I}_{/k}$ is homotopically connected to each other in $\mathcal{I}_{/k}$. This shows that it is inevitable to work with homotopy theory.

5. Koszul duality for Adams graded dg-algebras (a part of joint work with A. Takahashi)

The following theorem will be proved and applied in [16].

Theorem 15. *Let $\mathcal{A} := \mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \cdots$ be an $\mathbb{N}$-Adams graded dg-algebra. If the $\mathcal{A}_0$-modules $\mathcal{A}_n$ satisfies a mild condition. Then we have a quasi-isomorphism*

$$\mathbb{B}\mathrm{ic}_{\mathcal{A}}(\mathcal{A}/\mathcal{A}_{\geq 1}) \simeq \mathcal{A}$$

The proof is given in the same way of the proof of Theorem 6. Here we consider the Adams grading as a "linear topology" on $\mathcal{A}$. The condition is that $\mathcal{A}$ is "complete" with respect to this topology.

6. From the view point of Derived Gabriel topology

Gabriel topology is a special class of linear topology on rings, which plays an important role in the theory of localization of rings [25]. The notion of derived Gabriel topology, which is a derived version of Gabriel topology, is introduced in [15]. From the view point of derived Gabriel topology, Theorem 2 says that the derived bi-dual $M^{\circledast\circledast}$ equipped with "the finite topology" is the "J-adic completion" of M. In this sense Theorem 2 is inspired by the following results of J. Lambek.

Theorem 16 ([10, Theorem 4.2],(See also [11, Theorem 3.7]))**.** *Let R be a ring and J an injective R-module. For an R-module M, we denote by $Q(M)$ the module of quotients with respect to J. Assume that every torsionfree factor module of $Q(M)$ is J-divisible. Then the (ordinary) bi-duality $\mathrm{Hom}_{\mathrm{End}_R(J)}(\mathrm{Hom}_R(M,J),J)$ equipped with the finite topology is the J-adic completion of $Q(M)$.*

Recently many results in ring theory have been becoming to have their derived analogue ([19, 27]). However it can be said that the statements of these derived versions are parallel to that of the original versions. Contrary to this, our derived version of Lambek theorem is definitely improved from the original version. The assumptions and conditions in the original version is removed in the derived version. So the point is that we can obtain a simple formula for the bi-duality modules only when we come to the derived world from the abelian world.

At the first sight, three theorems below concerning on derived bi-dualities

- Completion theorem
- Localization theorem

- Koszul duality

seem to be theorems of different kind. However in the present paper we will see that these are consequences of a simple formula, which is the main theorem 2. From the view point of derived Gabriel topology, these theorems are consequences of completeness of each algebras with respect to appropriate topologies.

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SELFINJECTIVE ALGEBRAS AND QUIVERS WITH POTENTIALS

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Abstract. We study silting mutations (Okuyama-Rickard complexes) for selfinjective algebras given by quivers with potential (QPs). We show that silting mutation is compatible with QP mutation. As an application, we get a family of derived equivalences of Jacobian algebras.

1. Introduction

Derived categories are nowadays considered as an essential tool in the study of many areas of mathematics. In the representation theory of algebras, derived equivalences of algebras have been one of the central themes and extensively investigated. It is well-known that endomorphism algebras of tilting complexes are derived equivalent to the original algebra [20]. Therefore it is an important problem to give concrete methods to calculate endomorphism algebras of tilting complexes. In this note, we focus on one of the fundamental tilting complexes over selfinjective algebras, known as Okuyama-Rickard complexes, which play an important role in the study of Broué's abelian defect group conjecture. From a categorical viewpoint, they are nowadays interpreted as a special case of silting mutation [3]. We provide a method to determine the quivers with relations of the endomorphism algebras of Okuyama-Rickard complexes when selfinjective algebras are given by quivers with potential (QPs for short).

The notion of QPs was introduced by [7], which plays a significant role in the study of cluster algebras (we refer to [13]). Recently it has been discovered that mutations of QPs (Definition 2) give rise to derived equivalences [5, 15, 18, 22]. The aim of this note is to give a similar (but different) type of derived equivalences by comparing QP mutation and silting mutation (Definition 4).

Conventions. Let K be an algebraically closed field and $D := \mathrm{Hom}_K(-, K)$. All modules are left modules. For a finite dimensional algebra Λ, we denote by $\mathrm{mod}\Lambda$ the category of finitely generated Λ-modules and by $\mathrm{add}M$ the subcategory of $\mathrm{mod}\Lambda$ consisting of direct summands of finite direct sums of copies of $M \in \mathrm{mod}\Lambda$. The composition fg means first f, then g. For a quiver Q, we denote by Q_0 vertices and Q_1 arrows of Q and by $a : s(a) \to e(a)$ the start and end vertices of an arrow or path a.

2. Preliminaries

2.1. **Quivers with potential.** We recall the definition of quivers with potential. We follow [7].

The detailed version of this paper will be submitted for publication elsewhere.

• Let Q be a finite connected quiver without loops. We denote by KQ_i the K-vector space with basis consisting of paths of length i in Q, and by $KQ_{i,cyc}$ the subspace of KQ_i spanned by all cycles. We denote the *complete path algebra* by

$$\widehat{KQ} = \prod_{i\geq 0} KQ_i$$

and by $J_{\widehat{KQ}}$ the Jacobson radical of $\widehat{KQ}$. A *quiver with potential* (QP) is a pair (Q, W) consisting of a finite connected quiver Q without loops and an element $W \in \prod_{i\geq 2} KQ_{i,\mathrm{cyc}}$, called a *potential.* For each arrow a in Q, the *cyclic derivative* $\partial_a : \widehat{KQ}_{cyc} \to \widehat{KQ}$ is defined as the continuous linear map satisfying $\partial_a(a_1 \cdots a_d) = \sum_{a_i=a} a_{i+1} \cdots a_d a_1 \cdots a_{i-1}$ for a cycle $a_1 \cdots a_d$. For a QP (Q, W), we define the *Jacobian algebra* by

$$\mathcal{P}(Q, W) = \widehat{KQ}/\mathcal{J}(W),$$

where $\mathcal{J}(W) = \overline{\langle \partial_a W \mid a \in Q_1 \rangle}$ is the closure of the ideal generated by $\partial_a W$ with respect to the $J_{\widehat{KQ}}$-adic topology.

• A QP (Q, W) is called *trivial* if W is a linear combination of cycles of length 2 and $\mathcal{P}(Q, W)$ is isomorphic to the semisimple algebra $\widehat{KQ_0}$. It is called *reduced* if $W \in \prod_{i\geq 3} KQ_{i,\mathrm{cyc}}$.

Following [9], we use this terminology.

Definition 1. We call a QP (Q, W) *selfinjective* if $\mathcal{P}(Q, W)$ is a finite dimensional self-injective algebra.

Next we recall the definition of mutation of QPs.

Definition 2. For each vertex k in Q not lying on a 2-cycle, we define a new QP $\widetilde{\mu}_k(Q, W) := (Q', W')$ as follows.

(a) Q' is a quiver obtained from Q by the following changes.
- Replace each arrow $a : k \to v$ in Q by a new arrow $a^* : v \to k$.
- Replace each arrow $b : u \to k$ in Q by a new arrow $b^* : k \to u$.
- For each pair of arrows $u \xrightarrow{b} k \xrightarrow{a} v$, add a new arrow $[ba] : u \to v$

(b) $W' = [W] + \Delta$ is defined as follows.
- $[W]$ is obtained from the potential W by replacing all compositions ba by the new arrows $[ba]$ for each pair of arrows $u \xrightarrow{b} k \xrightarrow{a} v$.
- $\Delta = \sum_{\substack{a,b\in Q_1 \\ e(b)=k=s(a)}} [ba]a^*b^*.$

Then *mutation* $\mu_k(Q, W)$ is defined as a *reduced part* of $\widetilde{\mu}_k(Q, W)$ (we refer to [7]).

2.2. **Silting mutation.** The notion of silting objects was introduced by [14], which is a generalization of tilting objects. Recently its theory has been rapidly developed and many connections have been discovered, for example [6, 3, 8, 16]. In this subsection, we briefly recall their definitions and properties.

Now let Λ be a finite dimensional algebra and $\mathcal{T} := \mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)$ be the homotopy category of bounded complexes of finitely generated projective Λ-modules.

Definition 3. Let T be an object of $\mathcal{T}$. We call T *silting* (respectively, *tilting*) if $\mathrm{Hom}_{\mathcal{T}}(T, T[i]) = 0$ for any positive integer $i > 0$ (for any integer $i \neq 0$) and satisfies $\mathcal{T} = \mathsf{thick}T$, where $\mathsf{thick}T$ denote by the smallest thick subcategory of $\mathcal{T}$ containing T.

We call a morphism $f : X \to Y$ *left minimal* if any morphism $g : Y \to Y$ satisfying $fg = f$ is an isomorphism. For an object $M \in \mathcal{T}$, we call a morphism $f : X \to M'$ *left* $(\mathrm{add}M)$*-approximation* of X if M' belongs to $\mathrm{add}M$ and $\mathrm{Hom}_{\mathcal{T}}(f, M'')$ is surjective for any object M'' in $\mathrm{add}M$. Dually we define a *right minimal* morphism and a *right* $(\mathrm{add}M)$*-approximation*.

Definition 4. Let T be a basic silting object in $\mathcal{T}$ and take an arbitrary decomposition $T = X \oplus M$. We take a minimal left $(\mathrm{add}M)$-approximation $f : X \to M'$ of X and a triangle

$$X \xrightarrow{f} M' \longrightarrow Y \longrightarrow X[1].$$

We put $\mu_X(T) := Y \oplus M$ and call it a *left silting mutation* of T with respect to X. Dually we define a *right silting mutation*.

We recall an important result of silting mutation.

Theorem 5. [3, Theorem 2.31] *Any mutation of a silting object is again a silting object.*

Next we give some notations for our setting.

Let Q be a finite connected quiver and $\Lambda := KQ/(R)$ be a finite dimensional algebra. We denote by $\{e_k \mid k \in Q_0\}$ a complete set of primitive orthogonal idempotents of Λ. Take a set of vertices $I := \{k_1, \dots, k_n\} \subset Q_0$ and we denote by $e_I := e_{k_1} + \cdots + e_{k_n}$ and $\mu_I(\Lambda) := \mu_{\Lambda e_I}(\Lambda)$. We remark that an Okuyama-Rickard complex is nothing but a silting object of $\mathcal{T}$ [3, Theorem 2.50].

By Theorem 5, $\mu_I(\Lambda)$ is always a silting object of $\mathcal{T}$, but it is not necessarily a tilting object. However, for selfinjective algebras, it is a tilting object if it satisfies a condition given by Nakayama permutations.

Definition 6. Let Λ be a selfinjective algebra above. Then there exists a permutation $\sigma : Q_0 \to Q_0$ satisfying $D(e_k\Lambda) \cong \Lambda e_{\sigma(k)}$ for any $k \in Q_0$, where $\nu := D\mathrm{Hom}_\Lambda(-, \Lambda) : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda$ is the Nakayama functor. We call σ the *Nakayama permutation* of Λ.

Note that $\Lambda e_I \cong \nu(\Lambda e_I)$ if and only if $I = \sigma I$. The following result is useful. We refer to [1, 3] for the proof.

Proposition 7. *Let Λ be a selfinjective algebra above. Then $\mu_I(\Lambda)$ is a tilting object in $\mathcal{T}$ if and only if $I = \sigma I$.*

3. Main results

For a set of vertices $I := \{k_1, \dots, k_n\} \subset Q_0$, we assume the following conditions.

(a1) Any vertex in I is not contained in 2-cycles in Q.
(a2) There are no arrows between vertices in I.

In this case, since the mutation is independent of the choice of order of mutations, we can define the successive mutation

$$\mu_I(Q,W) := \mu_{k_1} \circ \cdots \circ \mu_{k_n}(Q,W).$$

Then our main result is the following.

Theorem 8. *Let (Q,W) be a selfinjective QP and $\Lambda := \mathcal{P}(Q,W)$. Let I be a set of vertices of Q_0 satisfying the conditions* (a1) *and* (a2)*. Then we have a K-algebra isomorphism*

$$\mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)}(\mu_I(\Lambda)) \cong \mathcal{P}(\mu_I(Q,W)).$$

We will give the proof in the next section. Combining with Theorem 7, we have the following result.

Corollary 9. *Let I be a set of vertices of Q_0 satisfying $\sigma I = I$ and the conditions* (a1) *and* (a2)*. Then $\mathcal{P}(Q,W)$ and $\mathcal{P}(\mu_I(Q,W))$ are derived equivalent.*

Proof. By Theorem 7, $\mu_I(\Lambda)$ is a tilting object of $\mathcal{T}$. Then Λ and $\mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)}(\mu_I(\Lambda))$ are derive equivalent [20] and the result follows from Theorem 8 $\square$

Moreover, since selfinjectivity is preserved by derived equivalence [4], we have the following result, which is given in [9, Theorem 4.2].

Corollary 10. *Let I be a set of vertices of Q_0 satisfying $\sigma I = I$ and the conditions* (a1) *and* (a2)*. Then $\mu_I(Q,W)$ is a selfinjective QP.*

We note that the Nakayama permutation of $\mu_I(Q,W)$ is again given by the same permutation [9, Proposition 4.4.(b)]. By this corollary, we can apply Corollary 9 to new QPs repeatedly and, consequently, obtain a lot of derived equivalences.

Example 11. Let (Q,W) be the QP given as follows

$$\begin{tikzcd}
 & & 1 \arrow[dl, "a_1"'] & & \\
 & 2 \arrow[dl, "a_2"'] & & 6 \arrow[ul, "a_6"'] & \\
3 \arrow[rr, "a_3"] & & 4 \arrow[rr, "a_4"] & & 5 \arrow[ul, "a_5"']
\end{tikzcd}, \qquad W = a_1a_2a_3a_4a_5a_6.$$

Then (Q,W) is a selfinjective QP with a Nakayama permutation $(153)(264)$. Let $\Lambda := \mathcal{P}(Q,W)$ and $\mathcal{T} := \mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)$ and take a silting object in $\mathcal{T}$

$$\mu_1(\Lambda) = \left\{ \begin{array}{ccc} \overset{-1}{\Lambda e_1} & \xrightarrow{a_1} & \overset{0}{\Lambda e_2} \\ & & \oplus \\ & & \Lambda(1-e_1). \end{array} \right.$$

Then by Theorem 8, we have an isomorphism

$$\mathrm{End}_{\mathcal{T}}(\mu_1(\Lambda)) \cong \mathcal{P}(\mu_1(Q,W)),$$

where $\mu_1(Q,W)$ is the QP given as follows

$$\begin{tikzcd}
 & 1 \arrow[dr, "a_6^*"] & \\
2 \arrow[ur, "a_1^*"] \arrow[d, "a_2"'] & & 6 \arrow[ll, "{[a_6a_1]}"] \\
3 \arrow[r, "a_3"] & 4 \arrow[r, "a_4"] & 5, \arrow[u, "a_5"']
\end{tikzcd}
\qquad [a_6a_1]a_1^*a_6^* + [a_6a_1]a_2a_3a_4a_5.$$

Next we consider the σ-orbit of the vertex 1 and let $I = \{1, 3, 5\}$. Then we have a tilting object

$$\mu_I(\Lambda) = \left\{ \begin{array}{ccc} \Lambda e_1 \oplus \overset{-1}{\Lambda e_3} \oplus \Lambda e_5 & \xrightarrow{\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_3 & 0 \\ 0 & 0 & a_5 \end{pmatrix}} & \Lambda e_2 \oplus \overset{0}{\Lambda e_4} \oplus \Lambda e_6 \\ & & \oplus \\ & & \Lambda(1 - e_I). \end{array} \right.$$

Then we have an isomorphism

$$\mathrm{End}_{\mathcal{T}}(\mu_I(\Lambda)) \cong \mathcal{P}(\mu_I(Q, W)),$$

where $\mu_I(Q, W)$ is the QP given as follows

$$\begin{tikzcd}
 & & 1 \arrow[dr, "a_6^*"] & & \\
 & 2 \arrow[ur, "a_1^*"] \arrow[dr, "{[a_2a_3]}"'] & & 6 \arrow[ll, "{[a_6a_1]}"'] \arrow[dr, "a_5^*"] & \\
3 \arrow[ur, "a_2^*"] & & 4 \arrow[ll, "a_3^*"] \arrow[ur, "{[a_4a_5]}"'] & & 5, \arrow[ll, "a_4^*"]
\end{tikzcd}$$

$[a_6a_1]a_1^*a_6^* + [a_2a_3]a_3^*a_2^* + [a_4a_5]a_5^*a_4^* + [a_6a_1][a_2a_3][a_4a_5]$.

We note that, although $\mathcal{P}(\mu_I(Q, W))$ is selfinjective and derived equivalent to $\mathcal{P}(Q, W)$, $\mathcal{P}(\mu_1(Q, W))$ is neither selfinjective nor derived equivalent to $\mathcal{P}(Q, W)$.

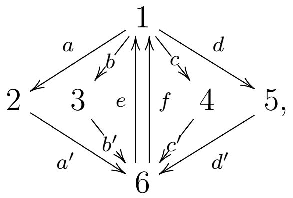
Example 12. Let (Q, W) be the QP given as follows

$$\begin{tikzcd}
1 \arrow[r] & 2 \arrow[d] & 3 \arrow[l] \\
4 \arrow[u] \arrow[d] & 5 \arrow[l] \arrow[r] & 6 \arrow[u] \arrow[d] \\
7 \arrow[r] & 8 \arrow[u] & 9, \arrow[l]
\end{tikzcd}$$

where the potential is the sum of each small squares. Then (Q, W) is a selfinjective QP with a Nakayama permutation $(19)(28)(37)(46)(5)$. For σ-orbits $I^1 := \{1, 9\}$ and $I^3 := \{3, 7\}$, we have selfinjective QPs $\mu_{I^1}(Q, W)$ and $\mu_{I^3} \circ \mu_{I^1}(Q, W)$ and their Jacobian algebras are derived equivalent to $\mathcal{P}(Q, W)$.

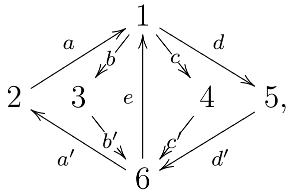
$$\xrightarrow{\mu_{I^1}(Q,W)}
\begin{tikzcd}
1 \arrow[d] & 2 \arrow[l] \arrow[d] & 3 \arrow[l] \\
4 \arrow[ur] \arrow[d] & 5 \arrow[l] \arrow[r] & 6 \arrow[u] \arrow[dl] \\
7 \arrow[r] & 8 \arrow[u] \arrow[r] & 9, \arrow[u]
\end{tikzcd}
\qquad
\xrightarrow{\mu_{I^3} \circ \mu_{I^1}(Q,W)}
\begin{tikzcd}
1 \arrow[d] & 2 \arrow[l] \arrow[r] \arrow[d] & 3 \arrow[d] \\
4 \arrow[ur] \arrow[dr] & 5 \arrow[l] \arrow[r] & 6 \arrow[ul] \arrow[dl] \\
7 \arrow[u] & 8 \arrow[l] \arrow[u] \arrow[r] & 9. \arrow[u]
\end{tikzcd}$$

Example 13. Let (Q, W) be the QP associated with tubular algebra of type $(2, 2, 2, 2)$



$$W = aa'e + bb'e + cc'e + aa'f + \lambda bb'f + dd'f, \lambda \in K \setminus \{0, 1\}.$$

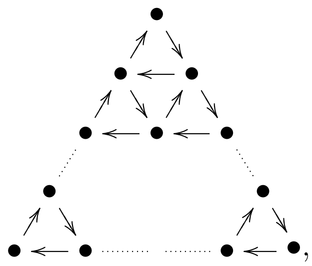
Then (Q, W) is a selfinjective QP [9] and the Nakayama permutation is the identity. Thus mutation of the QP at any vertex admits a derived equivalence in this case. For example, $\mu_2(Q, W)$ is the following QP with $\lambda' = \frac{\lambda}{\lambda - 1}$



$$bb'e + cc'e + dd'e + \lambda' bb'a'a + dd'a'a.$$

Thus $\mu_2(Q, W)$ is a selfinjective QP and $\mathcal{P}(\mu_2(Q, W))$ is derived equivalent to $\mathcal{P}(Q, W)$.

Example 14. Let (Q, W) be the QP given as follows



where the potential is the sums of each small triangles. Then (Q, W) is a selfinjective QP and one can easily get a lot of derived equivalence classes of algebras by the same procedures. See [9, Figure 4] for one of the concrete description. We refer to [12], which enables one to compute quiver mutations immediately.

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POWER RESIDUES

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Dedicated to professors Kazuo Kishimoto, Takasi Nagahara and Hisao Tominaga for their support

ABSTRACT. We present improved reports about the Feit Thompson conjecture until now and some new results for a prime 5.

Key Words: Feit Thompson conjecture, power residue symbol, Eisenstein reciprocity law, common index divisors.
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Let $p < q$ be primes and we set

$$f := \frac{q^p - 1}{q - 1} \text{ and } t := \frac{p^q - 1}{p - 1}.$$

Feit and Thompson [5] conjectured that f never divides t. If it would be proved, the proof of their odd order theorem [6] would be greatly simplified (see [1] and [7]).
The inequality $f < t$ may be trivial but here we confirm this as follows: It is easy for $p = 2$ from $2^q > q + 2$ by $q \geq 3$. Noting $\dfrac{x}{\log x}$ is strict increasing for $x \geq 3$, we have $\dfrac{q}{\log q} > \dfrac{p}{\log p}$ and hence $p^q > q^p$ by $q > p \geq 3$. Thus we have

$$\frac{p^q - 1}{p - 1} > \frac{p^q - 1}{q - 1} > \frac{q^p - 1}{q - 1} \text{ for } q > p \geq 3.$$

If $q \equiv 1 \bmod p$, in particular $p = 2$, then f never divides t. In fact, $f\ell = t$ implies a contradiction as follows:

$$1 \equiv t = f\ell = (q^{p-1} + \cdots + 1)\ell \equiv p\ell \equiv 0 \bmod p.$$

Contrary to the simple proof, this is important and fundamental in our discussions and it shall be freely used without previous notices. In this paper, small Latin letters represent integers in case no proviso and we use very often the notation $s \overset{p}{=} t$ in stead of $s \equiv t \bmod p$.

1. COMMON PRIME DIVISORS OF f AND t

Using computer and Proposition 1,(2), Stephans [15] found that f and t have a greatest common (prime) divisor $112643 = 2pq + 1$ for primes $p = 17$ and $q = 3313$. This example is so far of the only one with a common divisor $(f, t) > 1$. In case $p = 2$, $(f, t) = 1$. In fact, if r is a common prime divisor of $f = q + 1$ and $t = 2^q - 1$, then r is odd and q is the order of 2 mod r. Hence $r \equiv 1 \bmod q$ by Fermat little theorem. This implies a contradiction $r \leqq q + 1 < r$ since r is odd.

The paper is in a final form and no version of it will be submitted for publication elsewhere.

The next Proposition 1 follows in the range of rational integers.

Proposition 1 ([15], [4] and [11]). *Assume r is a common prime divisor of f and t. Then we have*

(1) *p is the order of $q \bmod r$ and q is the order of $p \bmod r$*
(2) $r \equiv 1 \bmod 2pq$.
(3) *If $p \equiv 3 \bmod 4$ or $q \equiv 3 \bmod 4$, then $r \equiv 1 \bmod 4$.*
(4) *If $p \equiv 3 \bmod 4$ and $q \equiv 1 \bmod 4$, then f never divides t.*

Proof. (1): It follows from the assumption that $q^p \equiv 1 \bmod r$ and $p^q \equiv 1 \bmod r$. If $q \equiv 1 \bmod r$, then $0 \equiv f = q^{p-1} + \cdots + 1 \equiv p \bmod r$ and so $r = p$, which implies a contradiction $0 \equiv t \equiv 1 \bmod p$. Similarly, we have $p \not\equiv 1 \bmod r$.
(2): Since p is odd, f and r are odd. Thus (2) follows from (1) and Fermat little theorem.
(3): Let λ_p be the Legendre symbol by p. Since $\lambda_r(p) = 1$ by $p^q \equiv 1 \bmod r$ and $\lambda_p(r) = 1$ by (2), the quadratic reciprocity $1 = \lambda_r(p)\lambda_p(r) = (-1)^{\frac{p-1}{2}\frac{r-1}{2}} = (-1)^{\frac{r-1}{2}}$ shows our result for p and similarly for q.
(4): Using (3), we have a contradiction $1 \equiv f = q^{p-1} + \cdots + q + 1 \equiv p \equiv 3 \bmod 4$. □

2. Results using Eisenstein reciprocity law

We set $\zeta = e^{\frac{2\pi i}{p}}$ for odd prime p and $\eta := \zeta^c(\zeta - q)$ where $c(q-1) \overset{p}{=} 1$. Then η is primary prime (see [9, p.206]) and $f = \prod_{\sigma \in G} \eta^\sigma = N(\eta)$ where G is the Galois group of $\mathbb{Q}(\zeta)$ over $\mathbb{Q}$.
We consider an integer $g := \sum_{a=1}^{p-1} \lambda_p(a) q^a$ for the Gauss sum $g(\lambda_p) = \sum_{a=1}^{p-1} \lambda_p(a)\zeta_p^a$ where $\lambda_p(a)$ is the Legendre symbol by p. Then we have $g \overset{\eta}{=} g(\lambda_p)$. More strongly, $g^2 \overset{f}{=} (-1)^{\frac{p-1}{2}} p$ by a computation using $q^p \equiv 1 \bmod f$ as that of $g(\lambda_p)^2$. The next is easy from the definition of p-th power residue symbol (see [9, p.205]).

Lemma 2. *Let χ_A be the p-th power residue symbol by an integral ideal $A \not\ni p$ of $\mathbb{Q}(\zeta)$.*

(a) $\chi_A(-1) = 1$.
(b) $\chi_\alpha(\beta) = 1$ *where α, β are real and non unit elements in $\mathbb{Q}(\zeta)$.*
(c) $\chi_A(\zeta) = \zeta^{\frac{N(A)-1}{p}}$.

Proof. (a): It follows from $\chi_A(-1) = \chi_A((-1)^p) = \chi_A(-1)^p = 1$.
(b): $\chi_\alpha(\beta)$ is real by $\overline{\chi_\alpha(\beta)} = \chi_{\bar{\alpha}}(\bar{\beta}) = \chi_\alpha(\beta)$, where $^-$ is a complex conjugate. 1 is the only real root of $x^p = 1$ for odd p.
(c): If $a \equiv 1$ and $b \equiv 1 \bmod p$, then it follows from $(a-1)(b-1) \equiv 0 \bmod p^2$ that

$$\frac{ab-1}{p} \equiv \frac{a-1}{p} + \frac{b-1}{p} \bmod p.$$

Thus if $\chi_B(\zeta) = \zeta^{\frac{N(B)-1}{p}}$ and $\chi_C(\zeta) = \zeta^{\frac{N(C)-1}{p}}$, then $\chi_{BC}(\zeta) = \zeta^{\frac{N(BC)-1}{p}}$ by $N(BC) = N(B)N(C)$. In case A is prime, (c) is clear by $A \not\supset (p) = (1-\zeta)^{p-1}$ and in general case, it follows from the above. □

The Eisenstein reciprocity law (see [9, p.207]) is used freely in this section.

Theorem 3 (Eisenstein). *$\chi_\alpha(b) = \chi_b(\alpha)$ for a primary $\alpha \in \mathbb{Q}(\zeta)$ and $b \in \mathbb{Z}$ such that p, α and b are relatively prime to each other.*

For $p = 3$, we have the next results.

Proposition 4. *Assume $p = 3$ and f divides t.*
(1) $f = q^2 + q + 1$ *is prime.*
(2) $\chi_\eta(g) = 1$.
(3) $f \overset{4}{=} 1$.
(4) $q \equiv -1 \bmod 72$.

Proof. (1) : If f is composite, then we have a contradiction $(q+1)^2 < f = q^2 + q + 1$ using Proposition 1,(2) (see [4] and [11]).
(2): Since $\chi_\eta(-1) = 1$ by Lemma 2,(a) and $\chi_\eta(3)^q = \chi_\eta(3^q) = 1$, we have the next by $q \equiv -1 \bmod 3$.

$$\chi_\eta(g)^2 = \chi_\eta(g(\lambda_3)^2) = \chi_\eta(-1)\chi_\eta(3) = 1.$$

(3): Since $g^2 \overset{f}{=} -3$ and $\lambda_f(3) = \lambda_f(3)^q = \lambda_f(3^q) = 1$, we have

$$1 = \lambda_f(g^2) = \lambda_f(-1)\lambda_f(3) = (-1)^{\frac{f-1}{2}} \text{ (see [4] and [11]).}$$

(4): $f = q^2+q+1$ is prime by (1) and $(3, f) = 1$ by Proposition 1, (2). Thus $(f, g) = 1$ since $g^2 \overset{f}{=} -3$ and so using the quadratic reciprocity on Jacobi symbols and $g = q - q^2 \overset{f}{=} 2q+1$, we have the next from $q \overset{12}{=} -1$ by (3) that

$$\begin{aligned}
\lambda_f(g) &= \lambda_f(2q+1) = (-1)^{\frac{q^2(q+1)}{2}}\lambda_{2q+1}(f) \\
&= \lambda_{2q+1}(4f) = \lambda_{2q+1}((2q+1)^2 + 3) \\
&= \lambda_{2q+1}(3) = (-1)^q\lambda_3(2q+1) = -\lambda_3(-1) \\
&= 1.
\end{aligned}$$

Thus $g \equiv a^2 \bmod f$ for some $a \in \mathbb{Z}$ and $(a, f) = 1$. Hence $-3 \equiv g^2 \equiv a^4 \bmod f$ and

$$1 \equiv a^{f-1} \equiv (-3)^{\frac{f-1}{4}} = (-3^q)^{\frac{q+1}{4}} \equiv (-1)^{\frac{q+1}{4}} \bmod f.$$

Therefore $q \equiv -1 \bmod 8$ (see [4], [3], [8] and [16] in this order).
Using cubic reciprocity or Eisenstein reciprocity law and Lemma 2, we have the next by (2).

$$\begin{aligned}
1 &= \chi_\eta(g)^2 = \chi_\eta(2q+1)^2 = \chi_{2q+1}(\eta)^2 \\
&= \chi_{2q+1}(\omega)^2 \cdot \chi_{2q+1}((\omega + 1/2)^2) \\
&= \omega^{2((2q+1)^2-1)/3} \cdot \chi_{2q+1}(-3/4) = \omega^{2((2q+1)^2-1)/3}
\end{aligned}$$

where $\omega = e^{\frac{2\pi i}{3}}$. Hence $8q(q+1) \equiv 0 \bmod 9$ (see [12]). □

For $p = 5$, we have new results.

Proposition 5. *If $p = 5$ and f divides t, then $q \overset{25}{=} -1$ or $q \overset{25}{=} 2$ or $q \overset{25}{=} 1/2$.*

Proof. It follows from $g = q(q-1)^2(q+1)$ that

$$1 = \chi_\eta(g)^{2(q-1)} = \{\chi_\eta(q)\chi_\eta(q-1)^2\chi_\eta(q+1)\}^{2(q-1)}$$

and using freely Eisenstein reciprocity law (Theorem 3) and Lemma 2, the equations (2.1), (2.2), (2.3) follow from each computation in the last of the proof.

$$\chi_\eta(q)^{2(q-1)} = \zeta^{2q\cdot\frac{q^4-1}{5}} \tag{2.1}$$

$$\chi_\eta(q-1)^{4(q-1)} = \zeta^{2(q+1)\cdot\frac{(q-1)^4-1}{5}} \tag{2.2}$$

$$\chi_\eta(q+1)^{2(q-1)} = \begin{cases} 1 & \text{if } q \overset{5}{=} -1 \\ \zeta^{(q+1)\cdot\frac{(q+1)^4-1}{5}} & \text{if } q \overset{5}{\neq} -1 \end{cases} \tag{2.3}$$

In case $q \overset{5}{=} -1$, since values are 1 in (2.2) and (2.3), we obtain $2q(q^4-1) \overset{25}{=} 0$ by the power of ζ in (2.1) and so $q \overset{25}{=} -1$ by $2q(q-1)(q^2+1) \overset{5}{\neq} 0$ using $q \overset{5}{=} -1$.
In case $q \overset{5}{\neq} -1$, considering the power of ζ,

$$2q(q^4-1) + 2(q+1)((q-1)^4-1) + (q+1)((q+1)^4-1) \overset{25}{=} 0.$$

It follows from the above and $q(q+1) \overset{5}{\neq} 0$ that

$$5q^3 - 6q^2 - 5q - 6 \overset{25}{=} 0.$$

This has solutions $q \overset{25}{=} 2$ or $q \overset{25}{=} 1/2$.
The computation of (2.1).

$$\chi_\eta(q)^{2(q-1)} = \chi_q(\eta)^{2(q-1)} = \chi_q(\zeta^{c+1})^{2(q-1)} = \zeta^{2q\cdot\frac{q^4-1}{5}}.$$

The computation of (2.2).

$$\begin{aligned} \chi_\eta(q-1)^{4(q-1)} &= \chi_{q-1}(\eta)^{4(q-1)} \\ &= \chi_{q-1}(\zeta^c)^{4(q-1)}\chi_{q-1}(\zeta-1)^{4(q-1)} \\ &= \chi_{q-1}(\zeta^4)\chi_{q-1}(\zeta-1)^{4(q-1)} \\ &= \chi_{q-1}(\zeta^{2(q+1)})\chi_{q-1}(\zeta-2+\zeta^{-1})^{2(q-1)} \\ &= \zeta^{2(q+1)\cdot\frac{(q-1)^4-1}{5}}. \end{aligned}$$

The computation of (2.3). In case $q \overset{5}{=} -1$, setting s by $q+1 = 5^e s$ and $(s,5)=1$, we have

$$\begin{aligned} \chi_\eta(q+1)^{2(q-1)} &= \chi_s(\eta)^{2(q-1)} \\ &= \chi_s(\zeta^c)^{2(q-1)}\chi_s(\zeta+1)^{2(q-1)} \\ &= \chi_s(\zeta^2)\chi_s(\zeta+1)^{2(q-1)} \\ &= \chi_s(\zeta)^{q+1}\chi_s(\zeta+2+\zeta^{-1})^{q-1} = 1. \end{aligned}$$

In case $q \overset{5}{\neq} -1$,

$$\begin{aligned}
\chi_\eta(q+1)^{2(q-1)} &= \chi_{q+1}(\zeta^c)^{2(q-1)}\chi_{q+1}(\zeta+1)^{2(q-1)} \\
&= \chi_{q+1}(\zeta^2)\chi_{q+1}(\zeta+1)^{2(q-1)} \\
&= \chi_{q+1}(\zeta^{q+1})\chi_{q+1}(\zeta+2+\zeta^{-1})^{(q-1)} \\
&= \zeta^{(q+1)\cdot\frac{(q+1)^4-1}{5}}. \qquad \square
\end{aligned}$$

3. Common index divisors

Let $F = \mathbb{Q}(\mu)$ be a number field of dimension m over $\mathbb{Q}$ and let D_F be the integer ring of F. We set $\Delta(\alpha_1, \alpha_2, \cdots, \alpha_m) := |\alpha_k^{(\ell)}|$ where $\alpha_k^{(\ell)}$ $(0 \leqq \ell \leqq m-1)$ are conjugates of $\alpha_k \in F$ $(1 \leqq k \leqq m)$.
For an integral basis $\eta_1, \eta_2, \cdots, \eta_m$ of D_F, $d(F) := \Delta(\eta_1, \eta_2, \cdots, \eta_m)^2$ is called the discriminant of F. For $\alpha \in F$, $d(\alpha) := \Delta(1, \alpha, \alpha^2, \cdots, \alpha^{m-1})^2$ is also called the discriminant of α. It is easy to see $d(\alpha) = I(\alpha)^2 d(F)$ where $I(\alpha) \in \mathbb{Z}$.
A prime number p is called a common index divisor of F if p divides $I(\gamma)$ for all $\gamma \in D_F$.

Example 6. (1) (Dedekind) :
$h(x) = x^3 + x^2 - 2x + 8$ is irreducible over $\mathbb{Q}$. Let α be a root. Then $d(\alpha) = -2^2 \cdot 503$, $d(\mathbb{Q}(\alpha)) = -503$, $I(\alpha) = 2$, and 2 is a common index divisor of $\mathbb{Q}(\alpha)$. The Galois group of $h(x)$ is the symmetric group S_3 of degree 3.
(2) (Stephans): Both 17 and 3313 are common index divisors in some subfields of $\mathbb{Q}(\zeta_r)$ where $r = 112643$ and $\zeta_r = e^{\frac{2\pi i}{r}}$.

In general, $n > p$ for a prime p if and only if there exists a number field K of degree n such that a prime p is a common index divisor of K (see [17] for 'if' part and [2] for 'only if' part).

4. Reviews from Ireland and Rosen [9]

Using the same notations in section 1, we note that $(f, p-1) = 1$. In fact, if ℓ is a prime common divisor of f and $p-1$, then $q^p \overset{\ell}{=} 1$ and $\ell < p$. We obtain p is the order of $q \mod \ell$ since $q \overset{\ell}{=} 1$ implies a contradiction $0 \overset{\ell}{=} f = q^{p-1} + \cdots + q + 1 \overset{\ell}{=} p \overset{\ell}{=} 1$. Thus $\ell \overset{p}{=} 1$ contradicts to $\ell < p$. Hence $f \mid t$ if and only if $p^q \overset{f}{=} 1$.
From this, we remember the next well known assertion. In the text books on the elementary number theory, we can usually see that for an odd prime r and a divisor n of $r-1$, an equation $x^n \overset{r}{=} a$ is solvable if and only if $a^{\frac{r-1}{n}} \overset{r}{=} 1$. This assertion is just the Euler's criterion for $n = 2$ and the existence of primitive roots is essential in this proof.
In this section, we shall observe [9, p.197, Corollary] is a generalization of this and an improvement of [13, Theorem] by Artin map (see [14]).
Considering in general $a^n \overset{m}{=} 1$, we may assume without loss generality n is the order of $a \bmod m$ and a is a prime by Dirichlet theorem, since there exist infinite many prime numbers p with $p \overset{m}{=} a$ because a and m are relatively prime. Thus we consider here the congruence $p^n \overset{m}{=} 1$ where p is a prime and n is the order of $p \bmod m$.

This section is almost all rewrite of [9, p.196-197] with a slight improvement. Here we set p is prime, D is the integer ring of $K = \mathbb{Q}(\zeta_m)$ where $\zeta_m = e^{\frac{2\pi i}{m}}$, and P is a prime ideal of D containing p.

The following Lemma is essential in this section. Lemma 7 and Corollary 8 were stated in [9, p.196].

Lemma 7. *If p does not divide m, then $D \equiv \mathbb{Z}[\zeta_m] \bmod p$.*

Proof. We set $\zeta = \zeta_m$ Since $\{1, \zeta, \ldots, \zeta^{\varphi(m)-1}\}$ is a basis of K over $\mathbb{Q}$, we obtain $D \ni \alpha = \sum r_k \zeta^k$ where $r_k \in \mathbb{Q}$. Thus $\mathrm{Tr}(\alpha\zeta^\ell) = \sum r_k \mathrm{Tr}(\zeta^k\zeta^\ell)$, where Tr is the trace from K to $\mathbb{Q}$. Solving this linear equations about r_k, we have $dr_k \in \mathbb{Z}$, namely, $dD \subset \mathbb{Z}[\zeta]$ where $d = |\mathrm{Tr}(\zeta^k\zeta^\ell)|$ is the discriminant of a cyclotomic polynomial $\Phi_m(x)$ of order m. If $d \overset{p}{=} 0$, then $\Phi_m(x)$ has a multiple root α in D/P and hence $\Phi_m(\alpha) = 0$ and $\Phi'_m(\alpha) = 0$. Substituting α in the differential $mx^{m-1} = \Phi_m(x)'g(x) + \Phi_m(x)g(x)'$ of $x^m - 1 = \Phi_m(x)g(x)$, we have $m\alpha^{m-1} = 0$ and $\alpha = 0$ by the condition, which yields a contradiction $0 = \Phi_m(\alpha) = \Phi_m(0) = \pm 1$. Thus we have $d \underset{p}{\neq} 0$ and $D \equiv \mathbb{Z}[\zeta] \mod p$. □

It is easy to see for $(a, m) = 1,\ \sigma_a : \zeta_m \to \zeta_m^a$ are automorphisms of K and $G = \{\sigma_a \mid 1 \leqq a < m,\ (a, m) = 1\}$ is the Galois group of K over $\mathbb{Q}$.

Corollary 8. (1) *$\alpha^{\sigma_p} \overset{p}{=} \alpha^p$ for $\alpha \in D$.*
(2) *$P^{\sigma_p} = P$.*
(3) *p is unramified in D.*

Proof. We set $\zeta = \zeta_m$. There exists $\beta \in D$ with $\alpha = p\beta + \sum a_k\zeta^k$ by Lemma 7.
(1) follows from

$$\alpha^{\sigma_p} = p\beta^{\sigma_p} + \sum_k a_k\zeta^{pk} \overset{p}{=} \sum_k a_k^p\zeta^{pk} \overset{p}{=} \alpha^p.$$

(2): For $\mu \in P$, $\mu^{\sigma_p} \overset{p}{=} \mu^p \overset{P}{=} 0$ and so $\mu^{\sigma_p} \in P$. This implies $P^{\sigma_p} \subset P$ and hence $P^{\sigma_p^{-1}} = P^{\sigma_p^{n-1}} \subset P$ where n is the order of σ_p.
(3): Let P be a prime ideal with $p \in P^2$ and let $\nu \in P$ but $\nu \notin P^2$. Then for the order n of σ_p, $\nu = \nu^{\sigma_p^n} \overset{p}{=} \nu^{p^n} \overset{P^2}{=} 0$ by (1) and $p^n \geqq 2$. Hence we have a contradiction $\nu \in P^2$ from $p \in P^2$. □

The next Lemma 9,(1) is restated of [9, p.182].

Lemma 9. (1) *G is transitive on the set Ω of distinct prime ideals of D containing p.*
(2) *$p^{|G_P|}$ is the order of D/P, namely, $|G_P|$ is a degree of P where G_P is the stabilizer of P.*

Proof. (1): Assume there exists $Q \in \Omega$ with $Q \neq P^\sigma$ for all $\sigma \in G$. Then there exists an element α satisfying $\alpha \equiv 0 \bmod Q$ and $\alpha \equiv 1 \bmod P^\sigma$ for all $\sigma \in G$. $N(\alpha) := \prod_{\sigma \in G} \alpha^\sigma \in \mathbb{Z} \cap Q = p\mathbb{Z} \subset P$ and so a contradiction $\alpha^\tau \in P$ for some τ, namely $\alpha \in P^{\tau^{-1}}$.
(2): We set d is the degree of P and $c = |\Omega|$. Then $d = |G_P|$ follows from $cd = \varphi(m) = |G| = |G : G_P||G_P| = c|G_P|$ since p is unramified by Corollary 8,(3).

We set L is the fixed subfield of K by σ_p. The next is just [9, p.197, Corollary] and contains [13, Theorem] which follows from Artin map (see [14, p.96]).

Theorem 10. *$G_P = \langle \sigma_p \rangle$.*

Proof. We set that n is the order of σ_p, $d = |G_P|$ and $\langle \nu \rangle = (D/P)^{\times}$. Then n is divisor of d since $\langle \sigma_p \rangle \subset G_P$ by Corollary 8,(2). On the other hand $p^d - 1$ is the order of ν by lemma 9,(2) and so $p^d - 1$ is a divisor of $p^n - 1$ since $\nu = \nu^{\sigma_p^n} = \nu^{p^n}$ by Corollary 8,(1) and hence $\nu^{p^n - 1} = 1$. It is false for $n < d$ and so $n = d$. $\square$

Theorem 10 is an extension of the next familiar theorem in elementary number theory. *If r is prime and n is a divisor of $r-1$, then $p^n \overset{r}{=} 1$ if and only if $p \overset{r}{=} x^{\frac{r-1}{n}}$ is solvable.* In fact, Assume $p^n \overset{r}{=} 1$. Then we may assume n is the order of σ_p and $\langle \sigma_p \rangle = \langle \sigma_c^{\frac{r-1}{n}} \rangle$ since the subgroup of order n is unique in the cyclic $\langle \sigma_c \rangle$ where c is a primitive root of r. Hence $p \overset{r}{=} x^{\frac{r-1}{n}}$ is solvable by $\sigma_p = \sigma_c^{\frac{(r-1)}{n}k}$ for some k. The other side is trivial.

Let D_M be the integer ring of a subfield M of K and Let P_M be prime ideal of D_M containing p.

Corollary 11. *$D/P = \mathbb{F}_{p^{|G_P|}}$ and $D_M/P_M = \mathbb{F}_p$ for any subfield M of L.*

Proof. First assertion is clear from Theorem 10. Second assertion follows from

$$\alpha^p \overset{p}{=} \alpha^{\sigma_p} = \alpha \text{ for } \alpha \in D_M \text{ and so } \alpha^p \overset{P_M}{=} \alpha. \qquad \square$$

We note that $D/P = \mathbb{F}_p$ if and only if p splits completely in D by Corollary 8,(3). The next is an extension of [13, Theorem].

Corollary 12. *Assume $p^n \overset{m}{=} 1$ and set $s = [L : \mathbb{Q}]$. Then in case $s > p$, p is a common index divisor of L and in case $p = s$, $h_\theta(x) \overset{p}{\neq} x^p - x$ has a multiple root in $\mathbb{F}_p = D_L/P_L$ where $L = \mathbb{Q}(\theta)$ and $h_\theta(x)$ is the minimal polynomial of θ over $\mathbb{Q}$.*

Proof. If there exists an element of $\mu \in D_L$ such that p does not divide $I(\mu) \in \mathbb{Z}$ where $d(\mu) = I(\mu)^2 d(L)$ for the discriminants $d(\mu)$ and $d(L)$ of μ and L, respectively. Noting that p does not divide $d(L)$ by Dedekind's theorem on discriminant (see [14, p.88, Remark 2.15]) since p is unramified in K and so in L, we have $d(\mu) \overset{p}{\neq} 0$ and so the minimal polynomial $g_\mu(x)$ of μ over $\mathbb{Q}$ has distinct roots in $\mathbb{F}_p$. Thus $s = \deg g_\mu(x) \leqq p$. In particular case $s = p$, $g_\mu(x) \overset{p}{=} x^p - x$. $\square$

We prove again Proposition 1,(3) (see [11] and [13]).

Corollary 13. *If r is a common prime divisor of f and t, then $p \equiv 1 \bmod 4$ or $r \equiv 1 \bmod 4$.*

Proof. We set $m = r$ and consider Guss sum $g(\lambda) = \sum_{k=1}^{r-1} \lambda(k)\zeta_r^k$ where λ is a quadratic character by r and $\zeta_r = e^{\frac{2\pi i}{r}}$. It is well known that $g(\lambda)^2 = (-1)^{\frac{r-1}{2}} r \overset{p}{=} (-1)^{\frac{r-1}{2}}$ and $g(\lambda) = \theta - \theta_1 = 2\theta + 1$ by $\theta + \theta_1 = -1$ where $\theta = \sum_{\lambda(a)=1} \zeta_r^a$ and $\theta_1 = \sum_{\lambda(b)=-1} \zeta_r^b$. $M = \mathbb{Q}(\theta) = \mathbb{Q}(g)$ is a quadratic subfield of L by $r \overset{2pq}{=} 1$ (see Proposition 1,(2)).

Since $\theta \overset{P_M}{\equiv} b$ for $b \in \mathbb{Z}$ by Corollary 11,

$$(-1)^{\frac{r-1}{2}\frac{p-1}{2}} \overset{p}{\equiv} g(\lambda)^{p-1} = (2\theta+1)^{p-1} \overset{P_M}{\equiv} (2b+1)^{p-1} \overset{p}{\equiv} 1.$$

Noting $2b+1 \overset{p}{\not\equiv} 0$ by above equations except the last equivalence, we can complete these from Fermat little theorem. □

We prove again the part $q \overset{9}{\equiv} -1$ of Proposition 4,(4) (see [12] and [13]).

Corollary 14. *If f divides t for a prime $p = 3$, then $q \overset{9}{\equiv} -1$.*

Proof. The assumption implies $q \overset{3}{\equiv} -1$ and $f = q^2 + q + 1$ is prime by Proposition 4. Let c be a primitive root of f and set $\zeta = e^{\frac{2\pi i}{f}}$. Then $\sigma : \zeta \to \zeta^c$ is a generator of the Galois group G of $K = \mathbb{Q}(\zeta)$ over $\mathbb{Q}$, let L_3 be the correspond subfield to $H = \langle \sigma^3 \rangle$. and let $G = \bigcup_{s=0}^{2} H\sigma^s$ be a coset decomposition by H. We set also $\theta = \sum_{\tau \in H} \zeta^\tau$ and $\theta_s = \theta^{\sigma^s}$ for $s = 0, 1, 2$. We can see $[L_3 : \mathbb{Q}] = 3$ and $L_3 = \mathbb{Q}(\theta)$ by [14, p.61, Theorem 2.6].
Let $g = g(\chi)$ be a cubic Gauss sum for the cubic residue character χ by a primary prime divisor $\eta = \omega(\omega - q)$ of $f = \eta\bar{\eta}$ in $\mathbb{Z}[\omega]$, where $\omega = e^{\frac{2\pi i}{3}}$. Namely, we set $g_s = g(\chi^s) = \sum_{t=0}^{f-1} \chi^s(t)\zeta^t$ which are rewritten as follow

$$g_s = \sum_{t=0}^{2} \sum_{k=0}^{\frac{f-4}{3}} \chi^s(c^{3k+t})\zeta^{c^{3k+t}} = \sum_{t=0}^{2} \chi(c)^{st} (\sum_{\tau \in H} \zeta^\tau)^{\sigma^t} = \sum_{t=0}^{2} \omega^{st}\theta_t.$$

These equations are also solved about θ_s as $3\theta_s = \sum_{t=0}^{2} \bar{\omega}^{st} g_t$. We can set the minimal polynomial $h_\theta = x^3 + x^2 + a_2 x + a_3$ of θ over $\mathbb{Z}$ by $\sum_{s=0}^{2} \theta_s = -1$.
We shall show $a_3 \overset{3}{\equiv} -a_2$. Noting [9, p.92, Proposition 8.2.2] and $\bar{\theta}_s = \theta_s$ since the complex conjugate $^-$ is the element of order 2 in H,

$$f = |g(\chi)|^2 = g(\chi)\overline{g(\chi)} = \theta_0^2 + \theta_1^2 + \theta_2^2 + (\omega + \omega^2)a_2 = 1 - 3a_2.$$

Hence we have

$$a_2 = (1-f)/3 = -q \cdot (q+1)/3 \overset{3}{\equiv} (q+1)/3.$$

It follows from equations $3\theta_s = \sum_{t=0}^{2} \bar{\omega}^{st} g_t$ that

$$-3^3 a_3 = (3\theta_0)(3\theta_1)(3\theta_2) = \prod_{s=0}^{2} (\sum_{t=0}^{2} \bar{\omega}^{st} g_t) = g_0^3 + g_1^3 + g_2^3 - 3g_0g_1g_2.$$

Using Stickelberger relation $g_1^3 = f\eta$ ([9, p.115, Corollary]), we can see the next from $g_0 = -1,\ g_2 = \bar{g}_1$ and $\eta + \bar{\eta} = q - 1$.

$$-a_3 = (-1 + f(\eta + \bar{\eta}) + 3f)/3^3 = ((q+1)/3)^3 \overset{3}{\equiv} (q+1)/3 \overset{3}{\equiv} a_2.$$

Thus we have $a_3 \overset{3}{\equiv} -a_2$.
Since $h_\theta \neq x^3 - x$ has a multiple root b in $\mathbb{F}_3$ by Corollary 12, we have $h'_\theta(b) \overset{3}{\equiv} 0$, namely, $b \overset{3}{\equiv} a_2$, where $h'_\theta(x)$ is a derivative of $h_\theta(x)$. Thus $0 \overset{3}{\equiv} h_\theta(a_2) \overset{3}{\equiv} a_2 - a_2^2 + a_3 \overset{3}{\equiv} -a_2^2$ and $0 \overset{3}{\equiv} a_2 \overset{3}{\equiv} (q+1)/3$. □

Example 15. If m has a primitive root, namely, $m = 2, 4, r^e$ and $2r^e$ where r is odd primes (see [9, p.44]), then G is cyclic and L_s is the unique subfield with $[L_s : \mathbb{Q}] = s$. Thus we have next results from Corollary 12.

(1) If $\ell^{r-1} \overset{r^2}{=} 1$ for primes ℓ, r with $\ell < r$, then ℓ is a common index divisor of a subfield L_r of $\mathbb{Q}(\zeta_{r^2})$.

(2) If $p^q \overset{r}{=} 1$ for primes p, r with $qq' = r-1$ and $p < q'$, then p is a common index divisor of a subfield $L_{q'}$ of $\mathbb{Q}(\zeta_r)$ (see [13, Theorem]).

Question. *If f divides t, then is f square free ?*

This question follows from the next observations: If f divides t, then we can see $p^q \equiv 1$ and $q^p \equiv 1 \bmod f$. Thus if f is divided by a prime square r^2, we have $p^{r-1} \equiv 1$, $q^{r-1} \equiv 1 \bmod r^2$ by $r \equiv 1 \bmod 2pq$ (see Proposition 1,(2)). It is well known from computation by using computer that there are rare primes r satisfying $a^{r-1} \equiv 1 \bmod r^2$ for a fixed $a > 1$. Further, in this case $p \not\equiv q \bmod r$ for fixed numbers p, q.

5. Integral normal basis

Let K be a Galois extension over $\mathbb{Q}$ with the Galois group G and let D be the integer ring of K. If there exists an element $\mu \in D$ such that $D = \sum_{\sigma \in G} \mu^\sigma \mathbb{Z}$, then we call $\{\mu^\sigma \mid \sigma \in G\}$ a normal basis and μ a normal basis element.

Here we set D_m is the integer ring of the cyclotomic field $K = \mathbb{Q}(\zeta_m)$ with the Galois group G, where $\zeta_m = e^{\frac{2\pi i}{m}}$. We set also D_θ is the integer ring of a proper subfield $\mathbb{Q}(\theta)$ of K and G_α is the stabilizer of $\alpha \in K$. In the text book [14, p.73-74], it was proved that the integer rings of subfields in $\mathbb{Q}(\zeta_r)$ for a prime r have normal bases and this plays an important role in [13]. Moreover, the integer rings of quadratic fields $\mathbb{Q}(\sqrt{n})$ have normal bases if and only if $n \equiv 1 \bmod 4$.

In the last of this paper, we shall show the following. It seems to be closely rated to the above Question.

Proposition 16. *D_m has a normal basis if and only if m is square free.*

Proof. Assume m is square free. In case m is a prime, D_m has a normal basis by [14, p.74, Remark 2.10] and so our result holds by the method in the proof of [10, p.68, Proposition 17 and p.75, Theorem 4].

Conversely, we assume D_m has a normal basis and m is divided by the square r^2 of a prime r. Then using [14, p.74, Theorem 2.12], we may assume $m = r^2$ and D_θ with $[\mathbb{Q}(\theta) : \mathbb{Q}] = r$ has a normal basis element μ. Thus we can show that $D_{r^2} = \sum_{\rho \in G_\omega} \mu^\rho D_\omega$ where $\omega = \zeta_{r^2}^r$. In fact, $[\mathbb{Q}(\omega) : \mathbb{Q}] = r-1$ yields $G = G_\theta \times G_\omega$ and $K = \mathbb{Q}(\theta) \cdot \mathbb{Q}(\omega) = \mathbb{Q}(\theta)[\omega] = \mathbb{Q}[\theta, \omega]$. Noting $d = \pm 1$ if α/d is an algebraic integer for an algebraic integer α and $d \in \mathbb{Z}$ with $(\alpha, d) = 1$, we obtain

$$D_{r^2} = D_\theta D_\omega = (\sum_{\nu \in G/G_\theta} \mu^\nu \mathbb{Z}) D_\omega = \sum_{\rho \in G_\omega} \mu^\rho D_\omega$$

Since D_{r^2} has a basis $\{1, \zeta, \dots, \zeta^{\ell-1}\}$ with $\zeta = \zeta_{r^2}$ and $\ell = r^2 - r$ by $D_{r^2} = \mathbb{Z}[\zeta]$ (see [10, p.75, Theorem 3]), we have

$$\mu = \sum_{k=0}^{\ell-1} a_k \zeta^k = \sum_{t=0}^{r-1} \sum_{s=0}^{r-2} a_{rs+t} \zeta^{rs+t} = \sum_{t=0}^{r-1} \alpha_t \zeta^t \text{ where } \alpha_t = \sum_{s=0}^{r-2} a_{rs+t} \omega^s \in D_\omega.$$

We set $\tau = \sigma_b$ with $b = c^{r-1}$ where c is a primitive root for r^2. Noting $G_\omega = \langle \tau \rangle$ is the Galois group of $\mathbb{Q}(\zeta)$ over $\mathbb{Q}(\omega)$, we can see from this equation that

$$\mu^{\tau^s} = \sum_{k=0}^{r-1} \alpha_k \zeta^{k\tau^s} = \sum_{k=0}^{r-1} \alpha_k \omega^{k\frac{b^s-1}{r}} \zeta^k.$$

This is equivalent to

$$(\mu, \mu^\tau, \mu^{\tau^2}, \dots, \mu^{\tau^{r-1}}) = (1, \zeta, \dots, \zeta^{r-1})A, \text{ where } A := (\alpha_k \omega^{k\frac{b^s-1}{r}})_{k,s}.$$

The next calculation implies a contradiction such that a unit $|A|$ is contained in rD_ω. Since r is the order of $b = c^{r-1} \bmod r^2$, we have for $r > k > 0$,

$$k\frac{b^s-1}{r} \equiv k\frac{b^t-1}{r} \bmod r, \text{ i.e., } b^s \equiv b^t \bmod r^2 \text{ if and only if } s \equiv t \bmod r.$$

Thus for any $k > 0$, we obtain

$$\sum_{s=0}^{r-1} \omega^{k\frac{b^s-1}{r}} = \sum_{t=0}^{r-1} \omega^t = \frac{\omega^r - 1}{\omega - 1} = 0.$$

This equation shows that we can change the first column of $|A|$ is equal to $(r\alpha_0,\ 0,\ \cdots,\ 0)^t$ and so we have a contradiction such that a unit $|A|$ is contained in rD_ω. $\square$

We confirm Proposition 16 for $r = 2$ and Kronecker-Weber theorem for quadratic fields (see [10, p.210, Corollary 3] or [14, p.133]).

Confirmation. *The quadratic field $\mathbb{Q}(\sqrt{n})$ with the discriminant d is a subfield of $\mathbb{Q}(\zeta_d)$.* In fact, ℓ represent primes and we set $s = {}^\#\{\ell \mid \ell \equiv -1 \bmod 4, \ell | n\}$. Using $g_\ell^2 = (-1)^{\frac{\ell-1}{2}} \ell$ in any case, where g_ℓ is a quadratic Gauss sum by ℓ, we can see our assertion.
In case $n \equiv 1 \bmod 4$, noting s is even,

$$\mathbb{Q}(\sqrt{n}) \subset \prod_{\ell | n} \mathbb{Q}(\zeta_\ell) = \mathbb{Q}(\zeta_n).$$

In case $n \equiv -1 \bmod 4$, noting s is odd and $\mathbb{Q}(\sqrt{-1}) = \mathbb{Q}(\zeta_4)$,

$$\mathbb{Q}(\sqrt{n}) \subset \mathbb{Q}(\zeta_4) \prod_{\ell | n} \mathbb{Q}(\zeta_\ell) = \mathbb{Q}(\zeta_{4n}).$$

In case $n \equiv 2 \bmod 4$, we set $n = 2n_0$ where n_0 is odd. Noting the above two cases and $\mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\zeta_8)$ by $\zeta_8 + \zeta_8^{-1} = \sqrt{2}$,

$$\mathbb{Q}(\sqrt{n}) \subset \mathbb{Q}(\sqrt{2})\mathbb{Q}(\sqrt{n_0}) \subset \mathbb{Q}(\zeta_8) \prod_{\ell | n_0} \mathbb{Q}(\zeta_\ell) = \mathbb{Q}(\zeta_{4n}).$$

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ON HOCHSCHILD COHOMOLOGY OF A CLASS OF WEAKLY SYMMETRIC ALGEBRAS WITH RADICAL CUBE ZERO

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Abstract. This paper is based on my talk given at the Symposium on Ring Theory and Representation Theory held at Shinsyu University, Japan, 7–9 September 2012. In this paper, we provide an explicit minimal projective bimodule resolution for some weakly symmetric algebras with radical cube zero. Then by using this resolution we compute the dimension of its Hochschild cohomology groups and determine the Hochschild cohomology ring modulo nilpotence.

1. Introduction

We consider the bound quiver algebra $A = k\Gamma/I$ where Γ is the quiver with m vertices and $2m$ arrows as follows:

$$\begin{tikzcd} e_0 \arrow[loop left, "a_0"] \arrow[r, shift left, "a_1"] & e_1 \arrow[l, shift left, "\overline{a}_1"] \arrow[r, shift left, "a_2"] & \cdots \arrow[l, shift left, "\overline{a}_2"] \arrow[r, shift left, "a_{m-1}"] & e_{m-1} \arrow[l, shift left, "\overline{a}_{m-1}"] \arrow[loop right, "a_m"] \end{tikzcd}$$

for an integer $m \geq 3$, and I is the ideal of $k\Gamma$ generated by the following elements:

$$a_1\overline{a}_1 - a_0^2, \quad a_m^2 - \overline{a}_{m-1}a_{m-1}, \quad \overline{a}_1 a_0, \quad a_m\overline{a}_{m-1},$$
$$a_i\overline{a}_i - \overline{a}_{i-1}a_{i-1}, \quad a_j a_{j+1}, \quad \overline{a}_{l+1}\overline{a}_l,$$

for $2 \leq i \leq m-1$, $0 \leq j \leq m-1$ and $1 \leq l \leq m-2$. Then, the following elements form a k-basis of A.

$$e_i,\ a_j,\ \overline{a}_l,\ a_r\overline{a}_r,\ a_m^2$$

for $0 \leq i \leq m-1$, $0 \leq j \leq m$ and $1 \leq l, r \leq m-1$. It is known that A is a Koszul weakly symmetric algebra with radical cube zero.

We denote by A^e the enveloping algebra $A \otimes_k A^{op}$ of A, so that left A^e-modules correspond to A-bimodules. The Hochschild cohomology ring is given by $\mathrm{HH}^*(A) = \mathrm{Ext}^*_{A^e}(A, A) = \oplus_{n\geq 0}\mathrm{Ext}^n_{A^e}(A, A)$ with Yoneda product. It is well-known that $\mathrm{HH}^*(A)$ is a graded commutative ring. Let $\mathcal{N}$ denote the ideal of $\mathrm{HH}^*(A)$ which is generated by all homogeneous nilpotent elements. Then $\mathcal{N}$ is contained in every maximal ideal of $\mathrm{HH}^*(A)$, so that the maximal ideals of $\mathrm{HH}^*(A)$ are in 1-1 correspondence with those in the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(A)/\mathcal{N}$. In this paper, we describe the ring structure of $\mathrm{HH}^*(A)/\mathcal{N}$.

In [8], Snashall and Solberg defined the support varieties for finitely generated modules over a finite dimensional algebra by using the Hochschild cohomology ring modulo nilpotence. Furthermore, in [2], Erdmann, Holloway, Snashall, Solberg and Taillefer introduced some reasonable "finiteness conditions," denoted by (Fg), for any finite dimensional algebra, and they showed that if a finite dimensional algebra satisfies (Fg), then the support varieties have a lot of analogous properties of support varieties for finite group algebras.

The detailed version of this paper has been submitted for publication elsewhere.

Recently, in [3], Erdman and Solberg gave necessary and sufficient conditions for any Koszul algebra to satisfy (Fg). Consequently, they showed that A satisfies (Fg). So the Hochschild cohomology ring of A is finitely generated as an algebra. On the other hand, in the case where $m = 2$ and $\operatorname{char} k \neq 2$, A is precisely the principal block of the tame Hecke algebra $H_q(S_5)$ for $q = -1$. In this case, a k-basis of the Hochschild cohomology groups of A was described by Schroll and Snashall in [7]. They proved independently that A satisfies (Fg), and gave some properties of the support varieties for modules over A.

In this paper, we provide an explicit minimal projective bimodule resolution of A for $m \geq 3$, and then determine the ring structure of the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(A)/\mathcal{N}$.

The contents of this paper are organized as follows. In Section 2, we determine sets $\mathcal{G}^n$ ($n \geq 0$), introduced in [6], for the right A-module $A/\operatorname{rad} A$. Then, using $\mathcal{G}^n$, we construct a minimal projective resolution $(P_\bullet, \partial_\bullet)$ of A as an A^e-module (Theorem 1). In Section 3, we first determine the dimension of the Hochschild cohomology groups for $m \geq 3$ (Theorem 4), and then we give an explicit k-basis of the Hochschild cohomology groups (Propositions 2, 3) and determine the Hochschild cohomology ring modulo nilpotence (Theorem 6).

Throughout this paper, for any arrow a in Γ, we denote the origin of a by $o(a)$ and the terminus by $t(a)$. We write $\otimes_k$ as $\otimes$ for simplicity,

2. A projective bimodule resolution

In this section, we give an explicit minimal projective bimodule resolution

$$(P_\bullet, \partial_\bullet): \quad \cdots \xrightarrow{\partial_4} P_3 \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} A \to 0$$

of $A = k\Gamma/I$ for $m \geq 3$ by using the argument in [5].

Let $B = kQ/I'$ with a finite quiver Q and an admissible ideal I' in kQ. In [6], Green, Solberg and Zacharia introduced the following subsets $\mathcal{G}^n$ ($n \geq 0$) of kQ, and used the subsets to give a minimal projective resolution of the right B-module $B/\operatorname{rad} B$.

Let $\mathcal{G}^0$ the set of all vertices of Q, $\mathcal{G}^1$ the set of all arrows of Q and $\mathcal{G}^2$ a minimal set of generators of I. In [6], the authors proved that for each $n \geq 3$ there is a subset $\mathcal{G}^n$ of kQ satisfying the following two conditions:

(a) Each of the elements x of $\mathcal{G}^n$ is a uniform element satisfying

$$x = \sum_{y \in \mathcal{G}^{n-1}} y r_y = \sum_{z \in \mathcal{G}^{n-2}} z s_z \quad \text{for unique } r_y, s_z \in kQ.$$

(b) There is a minimal projective B-resolution of $B/\operatorname{rad} B$

$$(R_\bullet, \delta_\bullet): \quad \cdots \xrightarrow{\delta_4} R_3 \xrightarrow{\delta_3} R_2 \xrightarrow{\delta_2} R_1 \xrightarrow{\delta_1} R_0 \xrightarrow{\delta_0} B/J \to 0,$$

satisfying the following conditions:

(i) For each $j \geq 0$, $R_j = \bigoplus_{x \in \mathcal{G}^j} t(x)B$.

(ii) For each $j \geq 1$, the differential $\delta_j : R_j \to R_{j-1}$ is defined by

$$t(x)\lambda \longmapsto \sum_{y \in \mathcal{G}^{j-1}} r_y t(x)\lambda \quad \text{for } x \in \mathcal{G}^j \text{ and } \lambda \in B,$$

where r_y are elements in the expression (a).

In [5], Green, Hartman, Marcos and Solberg used the subsets $\mathcal{G}^n$ $(n \geq 0)$ of kQ to give a minimal projective bimodule resolution for any finite dimensional Koszul algebra. This set also appears in the papers [3], [7] and [9] in constructing minimal projective bimodule resolutions.

In order to give sets $\mathcal{G}^n$ $(n \geq 0)$ for $A = k\Gamma/I$, we first define the following quiver Δ and morphisms of quivers $\phi^i = (\phi^i_0, \phi^i_1) : \Delta \to \Gamma$ for $i = 0, 1, \ldots, m-1$.

Let Δ be the following locally finite quiver with vertices (x, y) and arrows $b^{(x,y)}$: $(x, y) \to (x+1, y)$ and $c^{(x,y)} : (x, y) \to (x, y+1)$ for integers $x, y \geq 0$ as follows:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
{\scriptstyle c^{(0,2)}}\Big\uparrow & & {\scriptstyle c^{(1,2)}}\Big\uparrow & & {\scriptstyle c^{(2,2)}}\Big\uparrow & & \\
(0,2) & \xrightarrow{b^{(0,2)}} & (1,2) & \xrightarrow{b^{(1,2)}} & (2,2) & \xrightarrow{b^{(2,2)}} & \cdots \\
{\scriptstyle c^{(0,1)}}\Big\uparrow & & {\scriptstyle c^{(1,1)}}\Big\uparrow & & {\scriptstyle c^{(2,1)}}\Big\uparrow & & \\
(0,1) & \xrightarrow{b^{(0,1)}} & (1,1) & \xrightarrow{b^{(1,1)}} & (2,1) & \xrightarrow{b^{(2,1)}} & \cdots \\
{\scriptstyle c^{(0,0)}}\Big\uparrow & & {\scriptstyle c^{(1,0)}}\Big\uparrow & & {\scriptstyle c^{(2,0)}}\Big\uparrow & & \\
(0,0) & \xrightarrow{b^{(0,0)}} & (1,0) & \xrightarrow{b^{(1,0)}} & (2,0) & \xrightarrow{b^{(2,0)}} & \cdots
\end{array}$$

For any integer z, let $Q(z)$ be the quotient and $\overline{z}$ the remainder when we divide z by m. Then we have $0 \leq \overline{z} \leq m-1$. We denote the sets of vertices of Δ and Γ by Δ_0 and Γ_0, respectively. Also, we denote the sets of arrows of Δ and Γ by Δ_1 and Γ_1, respectively. For each $i = 0, 1, \ldots, m-1$, we define the maps $\phi^i_0 : \Delta_0 \to \Gamma_0$ and $\phi^i_1 : \Delta_1 \to \Gamma_1$ by

(1) For $(x, y) \in \Delta_0$

$$\phi^i_0(x, y) := \begin{cases} e_{\overline{x-y+i}} & \text{if } Q(x-y+i) \in 2\mathbb{Z}, \\ e_{m-1-\overline{x-y+i}} & \text{if } Q(x-y+i) \notin 2\mathbb{Z}. \end{cases}$$

(2) For $b^{(x,y)}, c^{(x,y)} \in \Delta_1$

$$\phi^i_1(b^{(x,y)}) := \begin{cases} a_{\overline{x-y+i}+1} & \text{if } Q(x-y+i) \in 2\mathbb{Z}, \\ \overline{a}_{m-1-\overline{x-y+i}} & \text{if } Q(x-y+i) \notin 2\mathbb{Z}, \end{cases}$$

$$\phi^i_1(c^{(x,y)}) := \begin{cases} \overline{a}_{\overline{x-y+i}} & \text{if } Q(x-y+i) \in 2\mathbb{Z}, \\ a_{m-\overline{x-y+i}} & \text{if } Q(x-y+i) \notin 2\mathbb{Z}. \end{cases}$$

where we put $\overline{a}_0 := a_0$ for our convenience.

Then, for all $i = 0, 1, \ldots, m-1$ and arrows $b^{(x,y)}$ and $c^{(x,y)}$ in Δ, we have

$$\begin{aligned} o(\phi^i_1(b^{(x,y)})) &= o(\phi^i_1(c^{(x,y)})) = \phi^i_0(x, y), \\ t(\phi^i_1(b^{(x,y)})) &= \phi^i_0(x+1, y), \\ t(\phi^i_1(c^{(x,y)})) &= \phi^i_0(x, y+1). \end{aligned}$$

Thus ϕ_1^i is a morphism of quivers. Note that ϕ_1^i naturally induces the map between the set of paths of Δ and that of Γ as follows:

$$\phi_1^i(p_1 \cdots p_r) = \phi_1^i(p_1) \cdots \phi_1^i(p_r),$$

for a path $p_1 \cdots p_r (r \geq 1)$ of Δ where p_j is an arrow for $1 \leq j \leq r$.

Now, we can define the sets $\mathcal{G}^n$ $(n \geq 0)$ for A in the similar way in [9]. For integers $n \geq 0$, $x, y \geq 0$ with $x + y = n$ and $i = 0, 1, \dots, m-1$, we define the element $g_{x,y,i}^n$ in $k\Gamma$ by

$$g_{x,y,i}^n := \sum_p (-1)^{s_p} \phi_1^i(p),$$

where

- p ranges over all paths in Δ starting at $(0,0)$ and ending with (x,y); and
- s_p is an integer determined as follows: If we write $p = p_1 p_2 \dots p_n$ with p_j arrows in Δ for $1 \leq j \leq n$, then $s_p = \sum_{p_j = c^{(x',y')}} j$ where x' and y' are positive integers with $x' + y' = j - 1$.

For each $n \geq 0$, we put

$$\mathcal{G}^n := \{g_{x,n-x,i}^n | \, 0 \leq x \leq n \text{ and } 0 \leq i \leq m-1\}.$$

Then, for $n = 0, 1, 2$, $\mathcal{G}^n$ can be described as follows:

$$\begin{aligned}
&\mathcal{G}^0 = \{e_0, e_1, \dots, e_{m-1}\},\\
&\mathcal{G}^1 = \{a_1, \dots, , a_m, -a_0 - \overline{a}_1, -\overline{a}_2, \dots, -\overline{a}_{m-1}\},\\
&\mathcal{G}^2 =\\
&\{-\phi_1^i(c^{(0,0)}c^{(0,1)}), \phi_1^i(b^{(0,0)}c^{(1,0)}) - \phi_1^i(c^{(0,0)}b^{(0,1)}), \phi_1^i(b^{(0,0)}b^{(1,0)}) \mid 0 \leq i \leq m-1\}\\
&\ = \{-a_0a_1, -\overline{a}_1 a_0, -\overline{a}_i\overline{a}_{i-1}, a_1\overline{a}_1 - a_0^2, a_{j+1}\overline{a}_{j+1} - \overline{a}_j a_j, a_m^2 - \overline{a}_{m-1}a_{m-1},\\
&\quad a_{l+1}a_{l+2}, a_m\overline{a}_{m-1} \mid 2 \leq i \leq m-1, 1 \leq j \leq m-2 \text{ and } 0 \leq l \leq m-2\}.
\end{aligned}$$

And it is easily seen that $\mathcal{G}^n$ satisfies the conditions (a) and (b) for $m \geq 3$ in the beginning of this section.

Now, for any integer $n \geq 0$, we define a left A^e-module

$$P_n := \coprod_{g \in \mathcal{G}^n} Ao(g) \otimes t(g)A.$$

Using the argument of [5], we have the following minimal projective resolution of A.

Theorem 1. [4, Theorem 2.3] *The following sequence is a minimal projective resolution of the left A^e-module A.*

$$(P_\bullet, \partial_\bullet) : \cdots \to P_n \xrightarrow{\partial_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\pi} A \to 0,$$

where π is the multiplication map and left A^e-homomorphisms ∂_n are defined by

(1) *In the case where $i=0$,*

$$\partial_n(o(g^n_{x,n-x,0})\otimes t(g^n_{x,n-x,0}))=$$
$$\begin{cases} (-1)^n o(g^{n-1}_{0,n-1,0})\otimes\phi^0_1(c^{(0,n-1)}) \\ \quad +\begin{cases}\phi^0_1(c^{(0,0)})\otimes t(g^{n-1}_{n-1,0,0}) & \text{if } n\equiv 0,1 (\mathrm{mod}\, 4),\\ -\phi^0_1(c^{(0,0)})\otimes t(g^{n-1}_{n-1,0,0}) & \text{if } n\equiv 2,3 (\mathrm{mod}\, 4),\end{cases} & \text{if } x=0,\\ o(g^{n-1}_{x-1,n-x,0})\otimes\phi^0_1(b^{(x-1,n-x)})+(-1)^n o(g^{n-1}_{x,n-1-x,0})\otimes\phi^0_1(c^{(x,n-1-x)}) \\ \quad +(-1)^x\phi^0_1(b^{(0,0)})\otimes t(g^{n-1}_{x-1,n-x,1}) \\ \qquad +\begin{cases}(-1)^x\phi^0_1(c^{(0,0)})\otimes t(g^{n-1}_{n-1-x,x,0}) & \text{if } n\equiv 0,1 (\mathrm{mod}\, 4),\\ -(-1)^x\phi^0_1(c^{(0,0)})\otimes t(g^{n-1}_{n-1-x,x,0}) & \text{if } n\equiv 2,3 (\mathrm{mod}\, 4),\end{cases} \\ \qquad \text{if } 1\le x\le n-1, \\ o(g^{n-1}_{n-1,0,0})\otimes\phi^0_1(b^{(n-1,0)})+(-1)^n\phi^0_1(b^{(0,0)})\otimes t(g^{n-1}_{n-1,0,1}) & \text{if } x=n.\end{cases}$$

(2) *In the case where $1\le i\le m-2$,*

$$\partial_n(o(g^n_{x,n-x,i})\otimes t(g^n_{x,n-x,i}))=$$
$$\begin{cases} (-1)^n o(g^{n-1}_{0,n-1,i})\otimes\phi^i_1(c^{(0,n-1)})+\phi^i_1(c^{(0,0)})\otimes t(g^{n-1}_{0,n-1,i-1}) & \text{if } x=0,\\ o(g^{n-1}_{x-1,n-x,i})\otimes\phi^i_1(b^{(x-1,n-x)})+(-1)^n o(g^{n-1}_{x,n-1-x,i})\otimes\phi^i_1(c^{(x,n-1-x)}) \\ \quad +(-1)^x\phi^i_1(b^{(0,0)})\otimes t(g^{n-1}_{x-1,n-x,i+1}) \\ \qquad +(-1)^x\phi^i_1(c^{(0,0)})\otimes t(g^{n-1}_{x,n-1-x,i-1}) & \text{if } 1\le x\le n-1,\\ o(g^{n-1}_{n-1,0,i})\otimes\phi^i_1(b^{(n-1,0)})+(-1)^n\phi^i_1(b^{(0,0)})\otimes t(g^{n-1}_{n-1,0,i+1}) & \text{if } x=n.\end{cases}$$

(3) *In the case where $i=m-1$,*

$$\partial_n(o(g^n_{x,n-x,m-1})\otimes t(g^n_{x,n-x,m-1}))=$$
$$\begin{cases} (-1)^n o(g^{n-1}_{0,n-1,m-1})\otimes\phi^{m-1}_1(c^{(0,n-1)})+\phi^{m-1}_1(c^{(0,0)})\otimes t(g^{n-1}_{0,n-1,m-2}) \\ \quad \text{if } x=0,\\ o(g^{n-1}_{x-1,n-x,m-1})\otimes\phi^{m-1}_1(b^{(x-1,n-x)}) \\ \quad +(-1)^n o(g^{n-1}_{x,n-1-x,m-1})\otimes\phi^{m-1}_1(c^{(x,n-1-x)}) \\ \qquad +\begin{cases}(-1)^x\phi^{m-1}_1(b^{(0,0)})\otimes t(g^{n-1}_{n-x,x-1,m-1}) & \text{if } n\equiv 0,1 (\mathrm{mod}\, 4),\\ -(-1)^x\phi^{m-1}_1(b^{(0,0)})\otimes t(g^{n-1}_{n-x,x-1,m-1}) & \text{if } n\equiv 2,3 (\mathrm{mod}\, 4),\end{cases} \\ \qquad +(-1)^x\phi^{m-1}_1(c^{(0,0)})\otimes t(g^{n-1}_{x,n-1-x,m-2}), \\ \qquad\quad \text{if } 1\le x\le n-1,\\ o(g^{n-1}_{n-1,0,m-1})\otimes\phi^{m-1}_1(b^{(n-1,0)}) \\ \quad +\begin{cases}(-1)^n\phi^{m-1}_1(b^{(0,0)})\otimes t(g^{n-1}_{0,n-1,m-1}) & \text{if } n\equiv 0,1 (\mathrm{mod}\, 4),\\ -(-1)^n\phi^{m-1}_1(b^{(0,0)})\otimes t(g^{n-1}_{0,n-1,m-1}) & \text{if } n\equiv 2,3 (\mathrm{mod}\, 4),\end{cases} \\ \qquad \text{if } x=n.\end{cases}$$

3. Hochschild cohomology of A

In this section, we give a k-basis of the Hochschild cohomology groups of A and determine the ring structure of the Hochschild cohomology ring modulo nilpotence by using the minimal projective A^e-resolution given in Theorem 1.

By setting $P_n^* := \mathrm{Hom}_{A^e}(P_n, A)$ and $\partial_n^* = \mathrm{Hom}_{A^e}(\partial_n, A)$ for $n \geq 0$, we get the following complex.

$$(P_\bullet^*, \partial_\bullet^*): \quad 0 \to P_0^* \xrightarrow{\partial_1^*} P_1^* \xrightarrow{\partial_2^*} \cdots \xrightarrow{\partial_{n-1}^*} P_{n-1}^* \xrightarrow{\partial_n^*} P_n^* \xrightarrow{\partial_{n+1}^*} \cdots .$$

Then, for $n \geq 0$, the n-th Hochschild cohomology group $\mathrm{HH}^n(A)$ of A is given by $\mathrm{HH}^n(A) := \mathrm{Ext}^n_{A^e}(A, A) = \mathrm{Ker}\,\partial_{n+1}^*/\mathrm{Im}\,\partial_n^*$.

In the rest of the paper, for an integer $n \geq 0$, we set $p := Q(n)$ and $t := \overline{n}$, that is, p and t are unique integers such that $n = pm + t$ with $p \geq 0$ and $0 \leq t \leq m-1$.

Using the complex $(P_\bullet^*, \partial_\bullet^*)$, we compute a k-basis of $\mathrm{HH}^n(A)$ for $n \geq 0$. Now we consider the case where m is even. In the case where m is odd, we have the similar results.

Proposition 2. [4, Proposition 3.7] *Suppose that $m \geq 3$. Then the following elements form a k-basis of the center $Z(A) = \mathrm{HH}^0(A) = \mathrm{Ker}\,\partial_1^*$ of A.*

$$\sum_{i=0}^{m-1} e_i,\ a_0,\ a_m,\ a_j\overline{a}_j \quad \text{for } 1 \leq j \leq m.$$

Proposition 3. [4, Proposition 3.8] *Suppose $m \geq 3$ and m is even. For each $n = pm+t \geq 1$, the following elements form a k-basis of $\mathrm{HH}^{pm+t}(A)$.*

(1) *In the case where p and t are even, we have a k-basis of $\mathrm{HH}^{pm+t}(A)$ as follows:*

  (a) *If $x_1 = (p-\alpha)m + t/2$, $x_2 = \alpha m + t/2$,*

$$\chi_{n,\alpha}: \begin{cases} e_i \otimes \phi_0^i(x_1, n-x_1) \mapsto \begin{cases} e_i & \text{if } i \text{ is even,} \\ (-1)^{t/2} e_i & \text{if } i \text{ is odd,} \end{cases} \\ e_i \otimes \phi_0^i(x_2, n-x_2) \mapsto \begin{cases} e_i & \text{if } i \text{ is even,} \\ (-1)^{t/2} e_i & \text{if } i \text{ is odd,} \end{cases} \end{cases}$$

  for $0 \leq i \leq m-1$, $0 \leq \alpha \leq p/2$.

  (b) *If $x = pm/2 + t/2$, $\pi_{n,1}: e_0 \otimes \phi_0^0(x, n-x) \mapsto a_0$.*

  (c) *If $x = pm/2 + t/2$, $\pi_{n,2}: e_{m-1} \otimes \phi_0^{m-1}(x, n-x) \mapsto a_m$.*

  (d) *If $x = (p-\alpha)m + t/2$,*

  $F_{n,\alpha}: e_0 \otimes \phi_0^0(x, n-x) \mapsto a_1\overline{a}_1$ *for $0 \leq \alpha \leq p/2 - 1$.*

  (e) *If $x = pm/2 + t/2$, $\mathrm{char}\,k = 2$, $F_{n,p/2}: e_0 \otimes \phi_0^0(x, n-x) \mapsto a_1\overline{a}_1$.*

(2) *In the case where p is even and t is odd, we have a k-basis of $\mathrm{HH}^{pm+t}(A)$ as follows:*

(a) *If* $x_1 = (p-\alpha)m + (t-1)/2$ *and* $x_2 = \alpha m + (t-1)/2$,

$$\mu_{n,\alpha} : \begin{cases} e_i \otimes \phi_0^i(x_1, n-x_1) \mapsto \begin{cases} \overline{a}_i & \text{if } i \text{ is even}, \\ (-1)^{(t-1)/2}\overline{a}_i & \text{if } i \text{ is odd}, \end{cases} \\ e_i \otimes \phi_0^i(x_2, n-x_2) \mapsto \begin{cases} \overline{a}_i & \text{if } i \text{ is even}(\neq 0), \\ (-1)^{(t-1)/2}\overline{a}_i & \text{if } i \text{ is odd}, \end{cases} \\ e_{m-1} \otimes \phi_0^{m-1}(x_2+1, n-x_2-1) \mapsto (-1)^{(t+1)/2} a_m, \end{cases}$$

for $0 \le i \le m-1$, $0 \le \alpha \le p/2 - 1$.

(b) *If* $x = pm/2 + (t-1)/2$ *and* $\operatorname{char} k \neq 2$,

$$\mu_{n,p/2} : \begin{cases} e_i \otimes \phi_0^i(x, n-x) \mapsto \begin{cases} \overline{a}_i & \text{if } i \text{ is even}, \\ (-1)^{(t-1)/2}\overline{a}_i & \text{if } i \text{ is odd}, \end{cases} \\ e_i \otimes \phi_0^i(x+1, n-1-x) \mapsto \begin{cases} -a_{i+1} & \text{if } i \text{ is even}, \\ (-1)^{(t+1)/2} a_{i+1} & \text{if } i \text{ is odd}, \end{cases} \end{cases}$$

for $0 \le i \le m-1$.

(c) *If* $x = pm/2 + (t-1)/2$ *and* $\operatorname{char} k = 2$, $\mu_{n,p/2} : e_0 \otimes \phi_0^0(x, n-x) \mapsto a_0$.

(d) *If* $x = pm/2 + (t+1)/2$ *and* $\operatorname{char} k = 2$,

$\mu'_{n,p/2} : e_{m-1} \otimes \phi_0^{m-1}(x, n-x) \mapsto a_m$.

(e) *If* $x = (p-\alpha)m + (t-1)/2$,

$$\nu_{n,\alpha} : \begin{cases} e_0 \otimes \phi_0^0(x+1, n-1-x) \mapsto a_1, \\ e_1 \otimes \phi_0^1(x, n-x) \mapsto (-1)^{(t-1)/2}\overline{a}_1, \end{cases}$$

for $0 \le \alpha \le p/2 - 1$.

(f) *If* $x = pm/2 + (t-1)/2$, $E_{n,1} : e_0 \otimes \phi_0^0(x, n-x) \mapsto a_1\overline{a}_1$.

(g) *If* $x = pm/2 + (t+1)/2$, $E_{n,2} : e_{m-1} \otimes \phi_0^{m-1}(x, n-x) \mapsto a_m^2$.

(3) *In the case where* p *is odd and* t *is even, we have a* k*-basis of* $\mathrm{HH}^{pm+t}(A)$ *as follows*:

(a) *If* $x_1 = (p-\alpha-1)m + (m+t)/2$ *and* $x_2 = \alpha m + (m+t)/2$,

$$\chi_{n,\alpha} : \begin{cases} e_i \otimes \phi_0^i(x_1, n-x_1) \mapsto \begin{cases} e_i & \text{if } i \text{ is even}, \\ (-1)^{(m+t)/2} e_i & \text{if } i \text{ is odd}, \end{cases} \\ e_i \otimes \phi_0^i(x_2, n-x_2) \mapsto \begin{cases} e_i & \text{if } i \text{ is even}, \\ (-1)^{(m+t)/2} e_i & \text{if } i \text{ is odd}, \end{cases} \end{cases}$$

for $0 \le i \le m-1$, $0 \le \alpha \le (p-1)/2$.

(b) *If* $x = (p-1)m/2 + (m+t)/2$, $\pi_{n,1} : e_0 \otimes \phi_0^0(x, n-x) \mapsto a_0$.

(c) *If* $x = (p-1)m/2 + (m+t)/2$, $\pi_{n,2} : e_{m-1} \otimes \phi_0^{m-1}(x, n-x) \mapsto a_m$.

(d) *If* $x = (p-\alpha-1)m + (m+t)/2$,

$F_{n,\alpha} : e_0 \otimes \phi_0^0(x, n-x) \mapsto a_1\overline{a}_1$ *for* $0 \le \alpha \le (p-1)/2 - 1$.

(e) *If* $x = (p-1)m/2 + (m+t)/2$ *and* $\operatorname{char} k = 2$,

$F_{n,(p-1)/2} : e_0 \otimes \phi_0^0(x, n-x) \mapsto a_1\overline{a}_1$.

(4) *In the case where* p *and* t *are odd, we have a* k*-basis of* $\mathrm{HH}^{pm+t}(A)$ *as follows*:

(a) *If* $x_1=(p-\alpha-1)m+(m+t-1)/2$ *and* $x_2=\alpha m+(m+t-1)/2$,

$$\mu_{n,\alpha}:\begin{cases} e_i\otimes\phi_0^i(x_1,n-x_1)\mapsto\begin{cases}\overline{a}_i & \text{if } i \text{ is even,}\\ (-1)^{(m+t-1)/2}\overline{a}_i & \text{if } i \text{ is odd,}\end{cases}\\ e_i\otimes\phi_0^i(x_2,n-x_2)\mapsto\begin{cases}\overline{a}_i & \text{if } i \text{ is even}(\neq 0),\\ (-1)^{(m+t-1)/2}\overline{a}_i & \text{if } i \text{ is odd,}\end{cases}\\ e_{m-1}\otimes\phi_0^{m-1}(x_2+1,n-x_2-1)\mapsto(-1)^{(m+t+1)/2}a_m,\end{cases}$$

for $0\le i\le m-1$, $0\le\alpha\le(p-1)/2-1$.

(b) *If* $x=(p-1)m/2+(m+t-1)/2$ *and* $\operatorname{char}k\neq 2$,

$$\mu_{n,(p-1)/2}:\begin{cases} e_i\otimes\phi_0^i(x,n-x)\mapsto\begin{cases}\overline{a}_i & \text{if } i \text{ is even,}\\ (-1)^{(m+t-1)/2}\overline{a}_i & \text{if } i \text{ is odd,}\end{cases}\\ e_i\otimes\phi_0^i(x+1,n-1-x)\mapsto\\ \quad\begin{cases}-a_{i+1} & \text{if } i \text{ is even,}\\ (-1)^{(m+t+1)/2}a_{i+1} & \text{if } i \text{ is odd,}\end{cases}\end{cases}$$

for $0\le i\le m-1$.

(c) *If* $x=(p-1)m/2+(m+t-1)/2$ *and* $\operatorname{char}k=2$,

$\mu_{n,(p-1)/2}: e_0\otimes\phi_0^0(x,n-x)\mapsto a_0$.

(d) *If* $x=(p-1)m/2+(m+t+1)/2$ *and* $\operatorname{char}k=2$,

$\mu'_{n,(p-1)/2}: e_{m-1}\otimes\phi_0^{m-1}(x,n-x)\mapsto a_m$.

(e) *If* $x_1=pm+m-1$, $x_2=0$, $t=m-1$,

$$\psi_n:\begin{cases} e_i\otimes\phi_0^i(pm+m-1,0)\mapsto(-1)^i\overline{a}_i,\\ e_i\otimes\phi_0^i(0,pm+m-1)\mapsto(-1)^{i+1}a_{i+1},\end{cases}$$

for $0\le i\le m-1$.

(f) *If* $x=(p-\alpha-1)m+(m+t-1)/2$,

$$\nu_{n,\alpha}:\begin{cases} e_0\otimes\phi_0^0(x+1,n-1-x)\mapsto a_1,\\ e_1\otimes\phi_0^1(x,n-x)\mapsto(-1)^{(m+t-1)/2}\overline{a}_1,\end{cases}$$

for $0\le\alpha\le(p-1)/2-1$.

(g) *If* $x=(p-1)m/2+(m+t-1)/2$, $E_{n,1}: e_0\otimes\phi_0^0(x,n-x)\mapsto a_1\overline{a}_1$.

(h) *If* $x=(p-1)m/2+(m+t+1)/2$, $E_{n,2}: e_{m-1}\otimes\phi_0^{m-1}(x,n-x)\mapsto a_m^2$.

By Propositions 2 and 3, we have the dimension of $\mathrm{HH}^n(A)$.

Theorem 4. [4, Theorem 3.5] *In the case* $m\ge 3$, *we have* $\dim_k\mathrm{HH}^0(A)=m+3$ *and, for* $pm+t\ge 1$,

$$\dim_k\mathrm{HH}^{pm+t}(A)=p+\begin{cases}3 & \text{if } p \text{ is even and } \operatorname{char}k\neq 2,\\ 2 & \text{if } p \text{ is odd, } t\neq m-1 \text{ and } \operatorname{char}k\neq 2,\\ 3 & \text{if } p \text{ is odd, } t=m-1 \text{ and } \operatorname{char}k\neq 2,\\ 4 & \text{if } p \text{ is even and } \operatorname{char}k=2,\\ 3 & \text{if } p \text{ is odd, } t\neq m-1 \text{ and } \operatorname{char}k=2,\\ 4 & \text{if } p \text{ is odd, } t=m-1 \text{ and } \operatorname{char}k=2.\end{cases}$$

Remark 5. In the case $m = 2$, by Theorem 4, we have the dimension of the Hochschild cohomology groups of A given in [7].

By corresponding Yoneda product of the basis elements of $\mathrm{HH}^*(A)$ given in Propositions 2 and 3, we have the generators of $\mathrm{HH}^*(A)$ and the following results.

Theorem 6. *In the case where m is even with $m \geq 3$, and* $\mathrm{char}\, k \neq 2$, *The Hochschild cohomology ring modulo nilpotence* $\mathrm{HH}^*(A)/\mathcal{N}$ *of A is isomorphic to the polynomial ring of two variables $k[\chi_{2,0}, \chi_{2m,0}]$.*

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A GENERALIZATION OF GOLDIE TORSION THEORY

YASUHIKO TAKEHANA

Abstract. Throughout this paper R is a ring with a unit element, every right R-module is unital and Mod-R is the category of right R-modules. Let $\mathcal{C}$ be a subclass of Mod-R. A torsion theory for $\mathcal{C}$ is a pair of $(\mathcal{T},\mathcal{F})$ of classes of objects of $\mathcal{C}$ such that (i) $\mathrm{Hom}_R(T,F)=0$ for all $T\in\mathcal{T}$, $F\in\mathcal{F}$. (ii) If $\mathrm{Hom}_R(M,F)=0$ for all $F\in\mathcal{F}$, then $M\in\mathcal{T}$. (iii) If $\mathrm{Hom}_R(T,N)=0$ for all $T\in\mathcal{T}$, then $N\in\mathcal{F}$. Let $\mathcal{B}$ be a subclass of Mod-R, $\mathcal{F}=\{M\in \text{Mod-}R|\mathrm{Hom}_R(B,M)=0$ for any $B\in\mathcal{B}\}$ and $\mathcal{T}=\{M\in \text{Mod-}R|\mathrm{Hom}_R(M,P)=0$ for any $P\in\mathcal{T}\}$. Then $(\mathcal{T},\mathcal{F})$ is called to be a torsion theory generated by $\mathcal{B}$. If $\mathcal{B}$ is the class of all modules M/N such that N is essential in M, a torsion theory generated by $\mathcal{B}$ is called the Goldie torsion theory. In this paper we generalize Goldie torsion theory by using left exact radical σ and study the dualization of this.

1. Introduction

For a subclass $\mathcal{E}$ of Mod-R and for a short exact sequence $0\to A\to B\to C\to 0$, it is said that $\mathcal{E}$ is closed under taking extensions if $A,C\in\mathcal{E}$ then $B\in\mathcal{E}$. It is well known that if $\mathcal{B}$ is closed under taking factor modules, direct sums and extensions then $(\mathcal{B},\mathcal{F})$ is a torsion theory. A torsion theory cogenerated by a subclass of Mod-R is defined dually, as follows. Let $\mathcal{B}$ be a subclass of Mod-R, $\mathcal{T}=\{M\in \text{Mod-}R|\mathrm{Hom}_R(M,B)=0$ for any $B\in\mathcal{B}\}$ and $\mathcal{F}=\{M\in \text{Mod-}R|\mathrm{Hom}_R(P,M)=0$ for any $P\in\mathcal{T}\}$. Then $(\mathcal{T},\mathcal{F})$ is called to be a torsion theory cogenerated by $\mathcal{B}$. It is well known that $\mathcal{B}$ is closed under taking submodules, direct products and extensions then $(\mathcal{T},\mathcal{B})$ is a torsion theory. A torsion theory $(\mathcal{T},\mathcal{F})$ is called to be hereditary if $\mathcal{T}$ is closed under taking submodules. It is well known that $(\mathcal{T},\mathcal{F})$ is hereditary if and only if $\mathcal{F}$ is closed under taking injective hulls.

A subfunctor of the identity functor of Mod-R is called a preradical. For preradical σ, $\mathcal{T}_\sigma:=\{M\in \text{Mod-}R|\sigma(M)=M\}$ is the class of σ-torsion right R-modules, and $\mathcal{F}_\sigma:=\{M\in \text{Mod-}R|\sigma(M)=0\}$ is the class of σ-torsion free right R-modules. A preradical t is called to be idempotent(a radical) if $t(t(M))=t(M)(t(M/t(M))=0)$. It is well known that $(\mathcal{T}_\sigma,\mathcal{F}_\sigma)$ is a torsion theory for an idempotent radical σ. For a torsion theory $(\mathcal{T},\mathcal{F})$ and a module M we put $t(M)=\sum N(N\in T)$(or equivalently $t(M)=\cap N(M/N\in\mathcal{F})$), then $\mathcal{T}=\mathcal{T}_t$, and $\mathcal{F}=\mathcal{F}_t$, and t is called an associated idempotent radical for $(\mathcal{T},\mathcal{F})$.

A preradical t is called to be left exact if $t(N)=N\cap t(M)$ holds for any module M and its submodule N. For a preradical σ and a module M and its submodule N, N is called to be σ-dense submodule of M if $M/N\in\mathcal{T}_\sigma$. For a preradical σ, t is called σ-left exact preradical if $t(N)=N\cap t(M)$ holds for any module M and its σ-dense submodule N. If N is an essential and σ-dense submodule of M, then N is called to be a σ-essential submodule of M(M is a σ-essential extension of N). For an idempotent radical σ a

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module M is called to be σ-injective if the functor $\mathrm{Hom}_R(\_, M)$ preserves the exactness for any exact sequence $0 \to A \to B \to C \to 0$ with $C \in \mathcal{T}_\sigma$.

We denote $E(M)$ the injective hull of a module M. For an idempotent radical σ, $E_\sigma(M)$ is called the σ-injective hull of a module M, where $E_\sigma(M)$ is defined by $E_\sigma(M)/M := \sigma(E(M)/M)$. Then even if σ is not left exact, $E_\sigma(M)$ is σ-injective and a σ-essential extension of M, is a maximal σ-essential extension of M and is a minimal σ-injective extension of M.

2. σ-HEREDITARY TORSION THEORIES AND σ-STABLE TORSION THEORIES

Let σ be an idempotent radical. We call a torsion theory $(\mathcal{T}_t, \mathcal{F}_t)$ σ-hereditary if $\mathcal{T}_t$ is closed under taking σ-dense submodules. A σ-hereditary torsion theory is characterized in [1], as follows. A torsion theory $(\mathcal{T}_t, \mathcal{F}_t)$ is σ-hereditary if and only if t is σ-left exact, moreover if σ is left exact then $(\mathcal{T}_t, \mathcal{F}_t)$ is σ-hereditary if and only if $\mathcal{F}$ is closed under taking σ-injective hulls. A torsion theory $(\mathcal{T}, \mathcal{F})$ is called to be stable if $\mathcal{T}$ is closed under taking injective hulls. For a preradical σ, we call a torsion theory $(\mathcal{T}, \mathcal{F})$ σ-stable if $\mathcal{T}$ is closed under taking σ-injective hulls. σ-stable torsion theory is characterized in [3] as follows. A torsion theory $(\mathcal{T}_t, \mathcal{F}_t)$ is σ-stable if and only if for any σ-injective module E, $t(E)$ is also σ-injective(if and only if, for any module M it holds that $E_\sigma(t(M)) \subseteq t(E_\sigma(M))$).

It is well known that a torsion theory generated by a class of Mod-R closed under taking submodules and quotient modules is hereditary. We generalize this as follows.

Proposition 1. *Let σ be a left exact radical and $\mathcal{B}$ a class of modules closed under taking σ-dense submodules and quotient modules. Then a torsion theory generated by $\mathcal{B}$ is σ-hereditary.*

Proof. Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory generated by $\mathcal{B}$. We show that $\mathcal{F}$ is closed under taking σ-injective hulls. Let $M \in \mathcal{F}$. We will show that $E_\sigma(M) \in \mathcal{F}$. Suppose that $E_\sigma(M) \notin \mathcal{F}$, then there exists some $B \in \mathcal{B}$ such that $\mathrm{Hom}_R(B, E_\sigma(M)) \neq 0$, and so there exists $0 \neq f : B \to E_\sigma(M)$, and so $f(B) \neq 0$. As M is essential in $E_\sigma(M)$, it follows that $M \cap f(B) \neq 0$. Since $\mathcal{B}$ is closed under taking factor modules, $f(B) \in \mathcal{B}$. Since $f(B)/(M \cap f(B)) \cong (M + f(B))/M \subseteq E_\sigma(M)/M \in \mathcal{T}_\sigma$, $M \cap f(B)$ is a σ-dense in $f(B)$. Thus by the assumption of $\mathcal{B}$ it follows that $M \cap f(B) \in \mathcal{B}$. Since $M \in \mathcal{F}$, $M \cap f(B) \in \mathcal{F}$. Thus $M \cap f(B) \in \mathcal{B} \cap \mathcal{F} \subseteq \mathcal{T} \cap \mathcal{F} = \{0\}$. This is a contradiction to $M \cap f(B) \neq 0$, and so $E_\sigma(M) \in \mathcal{F}$. $\square$

Proposition 2. *Let E be a σ-injective module, $\mathcal{B} =: \{M | Hom_R(M, E) = 0\}$ and $\mathcal{F} =: \{M \in$ Mod-$R | Hom_R(B, M) = 0$ for any $B \in \mathcal{B}\}$. Then $(\mathcal{B}, \mathcal{F})$ is a σ-hereditary torsion theory.*

Proof. It is easily verified that $\mathcal{B}$ is closed under taking quotient modules, direct sums and extensions. Then $(\mathcal{B}, \mathcal{F})$ is a torsion theory. We only show that $\mathcal{B}$ is closed under taking σ-dense submodules. Let $M \in \mathcal{B}$ and N be a σ-dense submodule of M. Suppose that $N \notin \mathcal{B}$, then there exists a nonzero $f \in \mathrm{Hom}_R(N, E)$. By the σ-injectivity of E, f extends $f' \in \mathrm{Hom}_R(M, E)$, but this is a contradiction to the fact that $M \in \mathcal{B}$. $\square$

Proposition 3. *Let σ be a preradical and E be a σ-torsionfree module. We put $\mathcal{T} = \{M \in$ Mod-$R | Hom_R(M, E) = 0\}$ and $\mathcal{F} = \{M \in$ Mod-$R | Hom_R(T, M) = 0$ for any $T \in \mathcal{T}\}$. Then $(\mathcal{T}, \mathcal{F})$ is σ-stable torsion theory.*

Proof. Since σ is a preradical, $\mathcal{T}_\sigma$ is closed under taking factor modules and $\mathcal{F}_\sigma$ is closed under taking submodules. Since it is easily verified that $\mathcal{T}$ is closed under taking factor modules, direct sums and extensions, it holds that ($\mathcal{T}$,$\mathcal{F}$) is a torsion theory. We only show that $\mathcal{T}$ is closed under taking σ-injective hulls. Suppose that $M \in \mathcal{T}$ and $E_\sigma(M) \notin \mathcal{T}$. Then it holds that $\mathrm{Hom}_R(M, E) = 0$ and $\mathrm{Hom}_R(E_\sigma(M), E) \neq 0$. Thus there exists $f \in \mathrm{Hom}_R(E_\sigma(M), E)$ such that $\mathrm{Im}\, f \neq 0$. Since $f|_M \in \mathrm{Hom}_R(M, E) = 0$, it folows that $\ker f \supseteq M$. Since $\mathcal{T}_\sigma \ni E_\sigma(M)/M \twoheadrightarrow E_\sigma(M)/\ker f \cong \mathrm{Im}\, f \subseteq E \in \mathcal{F}_\sigma$. This is a contradiction to the fact that $\mathrm{Im}\, f \neq 0$. Thus it follows that $E_\sigma(M) \in \mathcal{T}$. □

3. A generalization of Goldie torsion theory

A torsion theory($\mathcal{T}, \mathcal{F}$) generated by $\{M/N | N$ is essential in $M\}$ is called to be Goldie torsion theory. Goldie torsion theory is hereditary and stable. Let $Z(M)$ denote the singular submodule of a module M. For a module M, $Z_2(M)$ is defined by $Z_2(M)/Z(M) := Z(M/Z(M))$. It is well known that $\mathcal{T} = \mathcal{T}_{Z_2}$ and $\mathcal{F} = \mathcal{F}_{Z_2}$.

For a left exact radical σ we call a torsion theory generated by $\{M/N | N$ is σ-essential in $M\}$ σ-Goldie torsion theory.

Theorem 4. *For a left exact radical σ, σ-Goldie torsion theory is hereditary and σ-stable.*

Proof. Let $\mathcal{B}$ be $\{M/N | N$ is an σ-essential submodule of a module $M\}$. It is easily verified that $\mathcal{B}$ is closed under taking submodules and factor modules. A torsion theory ($\mathcal{T}$,$\mathcal{F}$) generated by $\mathcal{B}$ is hereditary by Proposition 1($\sigma = 1$). We show that $\mathcal{T}$ is closed under taking σ-injective hulls. Let M be in $\mathcal{T}$. Suppose that $E_\sigma(M) \notin \mathcal{T}$. Then there exists a module F in $\mathcal{F}$ such that $\mathrm{Hom}_R(E_\sigma(M), F) \neq 0$. Since $E_\sigma(M)/M \in \mathcal{B}$, $\mathrm{Hom}_R(E_\sigma(M)/M, F) = 0$. Since $M \in \mathcal{T}$ and $F \in \mathcal{F}$, $\mathrm{Hom}_R(M, F) = 0$. Then there exists a short exact sequence $0 \to \mathrm{Hom}_R(E_\sigma(M)/M, F) \to \mathrm{Hom}_R(E_\sigma(M), F) \to \mathrm{Hom}_R(M, F)$. This is a contradiction, and so it follows that $E_\sigma(M) \in \mathcal{T}$. □

For a module M, we denote $Z_\sigma(M) := \{m \in M | mI = 0$ for some σ-essential right ideal I of $R\}$. If $m \in Z_\sigma(M)$, there exists some σ-essential right ideal I of R such that $mI = 0$, and then $(0 : m) \supseteq I$. Since $mR \cong R/(0 : m) \twoheadleftarrow R/I \in \mathcal{T}_\sigma \cap \mathcal{T}_Z$, $mR \in \mathcal{T}_\sigma \cap \mathcal{T}_Z$, and so $mR \subseteq \sigma(M) \cap Z(M)$. Thus $Z_\sigma(M) \subseteq \sum mR(_{m \in Z_\sigma(M)}) \subseteq Z(M) \cap \sigma(M)$. $Z_\sigma(M) \subseteq Z(M) \cap \sigma(M)$. Since Z and σ are left exact, $\sigma(Z(M)) = Z(M) \cap \sigma(M) = Z(\sigma(M))$, and so $Z_\sigma(M) \subseteq Z(\sigma(M)) = \sigma(Z(M))$. Conversely if $m \in Z(\sigma(M)) = \sigma(Z(M))$, then $R/(0 : m) \cong mR \in \mathcal{T}_Z \cap \mathcal{T}_\sigma$, and so $(0 : m)$ is σ-essential in R. Thus $m \in Z_\sigma(M)$, and so $Z(\sigma(M)) = \sigma(Z(M)) \subseteq Z_\sigma(M)$. Therefore $Z_\sigma(M) = Z(\sigma(M)) = \sigma(Z(M))$.

For a preradical r and a module M, put $r_1(M) := r(M)$. If β is not a limit ordinal, $r_\beta(M)/r_{\beta-1}(M) := r(M/r_{\beta-1}(M))$. If β is a limit ordinal, $r_\beta(M) := \sum_{\beta > \alpha} r_\alpha(M)$. This gives rise to an increasing sequence of preradicals. We put $\overline{r}(M) = \sum_\beta r_\beta(M)$, then $\overline{r}$ is a smallest radical larger than r. If r is idempotent, then $\overline{r}$ is also idempotent. Thus for an idempotent preradical r, a torsion theory ($\mathcal{T}$,$\mathcal{F}$) generated by $\mathcal{T}_r$ is given that $\mathcal{T} = \mathcal{T}_{\overline{r}}$ and $\mathcal{F} = \mathcal{F}_{\overline{r}}$.

We define $(Z_\sigma)_2(M)$ by $(Z_\sigma)_2(M)/Z_\sigma(M) := Z_\sigma(M/Z_\sigma(M))$.

Lemma 5. *Let σ be a radical and $\sigma(M) \supseteq N$ for a module M and its submodule N. Then it holds that $\sigma(M/N) = \sigma(M)/N$.*

Theorem 6. *For a left exact radical σ, it holds that $\mathbf{Z}_2\sigma = (Z_\sigma)_2 = \overline{Z_\sigma}$.*

Proof. By Lemma 5, $\sigma(M/Z(\sigma(M))) = \sigma(M)/Z(\sigma(M))$.

Thus $(Z_\sigma)_2(M)/Z_\sigma(M) = Z_\sigma(M/Z_\sigma(M)) = Z\{\sigma(M/Z(\sigma(M)))\}$

$= Z(\sigma(M)/Z(\sigma(M))) = Z_2(\sigma(M))/Z(\sigma(M))$. Thus $\mathbf{Z}_2\sigma = (Z_\sigma)_2$. Since $\mathbf{Z}_2$ and σ are left exact radicals, $(Z_\sigma)_2$ is a left exact radical. Since $\overline{Z_\sigma}$ is the smallest radical containing Z_σ, $(Z_\sigma)_2 \supseteq \overline{Z_\sigma}$. By construction of $\overline{Z_\sigma}$, it holds that $(Z_\sigma)_2 \subseteq \overline{Z_\sigma}$, and so $(Z_\sigma)_2 = \overline{Z_\sigma}$, as desired. □

Let G be a Goldie torsion functor. The followings are well known. (1) $G(M) = M$ if and only if $Z(M)$ is essential in M. (2) $G(M) = 0$ if and only if $Z(M) = 0$. (3) If $Z(R) = 0$, then $G = Z$. We can generalize this as follows.

Corollary 7. *Let G_σ be a σ-Goldie torsion functor. Then the following facts hold.*

(1) *$G_\sigma(M) = M$ if and only if $\sigma(M) = M$ and $Z(M)$ is essential in M.*
(2) *$G_\sigma(M) = 0$ if and only if $Z(\sigma(M)) = 0$.*
(3) *If $Z(\sigma(R)) = 0$, then $G_\sigma = Z_\sigma$*

4. σ-costable torsion theory and σ-cohereditary torsion theory

From now on, assume that R be a right perfect ring. A right R-module M is called σ-projective if the functor $\mathrm{Hom}_R(M, \quad)$ preserves the exactness for any exact sequence $0 \to A \to B \to C \to 0$ with $A \in \mathcal{F}_\sigma$. For an idempotent radical σ, a short exact sequence $[0 \to K_\sigma(M) \to P_\sigma(M) \overset{\pi_M^\sigma}{\to} M \to 0]$ is called σ-projective cover of a module M when $P_\sigma(M)$ is σ-projective, $K_\sigma(M)$ is σ-torsion free and $K_\sigma(M)$ is small in $P_\sigma(M)$. Since R is a right perfect ring and σ is an idempotent radical, σ-projective cover of a module always exists. A torsion theory $(\mathcal{T},\mathcal{F})$ is called costable if $\mathcal{F}$ is closed under taking projective covers. We generalize this in [4]. We call a torsion theory $(\mathcal{T},\mathcal{F})$ σ-costable if $\mathcal{F}$ is closed under taking σ-projective covers.

Proposition 8. *Let E be a module such that for any module X and any $f \in Hom_R(E, X)$, $f(E)$ is σ-torsionfree and not small in X. We denote $\mathcal{F} = \{M | Hom_R(E, M) = 0\}$ and $\mathcal{T} = \{M | Hom_R(M, F) = 0$ for any $F \in \mathcal{F}\}$. Then $(\mathcal{T}, \mathcal{F})$ is a σ-costable torsion theory.*

Proof. Let M be in $\mathcal{F}$. Then $\mathrm{Hom}_R(E, M) = 0$. Suppose that $P_\sigma(M) \notin \mathcal{F}$. Then $\mathrm{Hom}_R(E, P_\sigma(M)) \neq 0$. Thus there exists an $f \in \mathrm{Hom}_R(E, P_\sigma(M))$ such that $f(E) \neq 0$. Since $\pi_M^\sigma f \in \mathrm{Hom}_R(E, M) = 0$, $f(E) \subseteq K_\sigma(M)$ is small in $P_\sigma(M)$. Thus $f(E)$ is σ-torsionfree and small in $P_\sigma(M)$. This is a contradiction. Thus $\mathcal{F}$ is closed under taking σ-projective covers. □

Next we state the dual of Proposition 1. A preradical σ is called epi-preserving if $\sigma(M/N) = (\sigma(M) + N)/N$ holds for any module M and any submodule N of M. If σ is an epipreserving idempotent radical, $\mathcal{F}_\sigma$ is closed under taking factor modules and then $(\mathcal{T},\mathcal{F})$ is called cohereditary. We say that a subclass $\mathcal{C}$ of Mod-R is closed under taking σ-factor modules if : if $M \in \mathcal{C}$ and N is a σ-torsionfree submodule of M then $M/N \in \mathcal{C}$. We call a torsion theory $(\mathcal{T},\mathcal{F})$ σ-cohereditary if $\mathcal{F}$ is closed under taking σ-factor modules.

Proposition 9. *Let σ be an epi-preserving idempotent radical. Let $\mathcal{B}$ be a class of modules closed under taking σ-factor modules and submodules. Then the torsion theory cogenerated by $\mathcal{B}$ is σ-cohereditary.*

Proof. Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory cogenerated by $\mathcal{B}$. We show that $\mathcal{T}$ is closed under taking σ-projective covers. Let $M \in \mathcal{T}$. We show that $P_\sigma(M) \in \mathcal{T}$. Suppose that $P_\sigma(M) \notin \mathcal{T}$. Then there exists $B \in \mathcal{B}$ such that $\mathrm{Hom}_R(P_\sigma(M), \mathcal{B}) \neq 0$. Then there exists a nonzero homomorphism $\alpha : P_\sigma(M) \to B$. $\mathcal{B} \ni B \supseteq \alpha(P_\sigma(M)) \supseteq \alpha(K_\sigma(M)) \in \mathcal{F}_\sigma$. Since $\mathcal{B}$ is closed under taking σ-factor modules and submodules, $\alpha(P_\sigma(M))/\alpha(K_\sigma(M)) \in \mathcal{B}$. Then α induces $\widetilde{\alpha} : M \simeq P_\sigma(M)/K_\sigma(M) \twoheadrightarrow \alpha(P_\sigma(M))/\alpha(K_\sigma(M)) \in \mathcal{B}$. Since $M \in \mathcal{T}$, it follows that $\alpha(P_\sigma(M))/\alpha(K_\sigma(M)) = 0$, and so $\alpha(P_\sigma(M)) = \alpha(K_\sigma(M))$. Therefore it follows that $\alpha^{-1}\{\alpha(P_\sigma(M))\} = \alpha^{-1}\{\alpha(K_\sigma(M))\}$, and so $P_\sigma(M) = \alpha^{-1}(0) + K_\sigma(M)$. Since $K_\sigma(M)$ is small in $P_\sigma(M)$, it follows that $P_\sigma(M) = \alpha^{-1}(0)$. But this is a contradiction to the fact that $\alpha \neq 0$. Thus $P_\sigma(M) \in \mathcal{T}$. □

Proposition 10. *Let P be a σ-projective module and $\mathcal{B}$ be $\{M | Hom_R(P, M) = 0\}$. Then a torsion theory generated by $\mathcal{B}$ is σ-cohereditary.*

Proof. It is easily verified that $\mathcal{B}$ is closed under taking submodules, direct products and extensions. We only show that $\mathcal{B}$ is closed under taking σ-factor modules. Let $M \in \mathcal{B}$ and $N \in \mathcal{F}_\sigma$ be a submodule of M. Suppose that $M/N \notin \mathcal{B}$, then there exists a nonzero $f \in \mathrm{Hom}_R(P, M/N)$. Then f extends $f' \in \mathrm{Hom}_R(P, M)$ such $gf' = f$, where g is a canonical epimorphism from M to M/N. This is a contradiction to the fact that $M \in \mathcal{B}$. Thus a torsion theory $(\mathcal{T},\mathcal{B})$ generated by $\mathcal{B}$ is σ-cohereditary by Proposition 9. □

5. Dualization of σ-Goldie torsion theory

A module N is called small if N is a small submodule of some module. It is well known that N is small if and only if N is small in $E(M)$. Now we consider dualizations of σ-Goldie torsion theory.

Theorem 11. *Let σ be an epi-preserving idempotent radical. We denote $\mathcal{B} = \{N \in \mathcal{F}_\sigma | N$ is small in some module $M\}(= \{M \in \mathcal{F}_\sigma | M$ is small in $E(M)\})$. Then a torsion theory $(\mathcal{T},\mathcal{F})$ cogenerated by $\mathcal{B}$ is cohereditary and σ-costable.*

Proof. It is easily verified that $\mathcal{B}$ is closed under taking direct sums, factor modules and submodules. Thus by Proposition 9($\sigma = 1$), $\mathcal{F}$ is closed under taking factor modules, and so $(\mathcal{T},\mathcal{F})$ is a cohereditary torsion theory. Next we show that $\mathcal{F}$ is closed under taking σ-projective covers. Let $M \in \mathcal{F}$. Suppose that $P_\sigma(M) \notin \mathcal{F}$, then there exists a module X in $\mathcal{T}$ and $\mathrm{Hom}_R(X, P_\sigma(M)) \neq 0$. Consider the following exact sequence. $0 \to \mathrm{Hom}_R(X, K_\sigma(M)) \to \mathrm{Hom}_R(X, P_\sigma(M)) \to \mathrm{Hom}_R(X, M)$. Since $X \in \mathcal{T}$ and $M \in \mathcal{F}$, $\mathrm{Hom}_R(X,M) = 0$. Since $K_\sigma(M)$ is σ-torsion free small submodule of $P_\sigma(M)$, $K_\sigma(M)$ is in $\mathcal{B}$. Thus $\mathrm{Hom}_R(X, K_\sigma(M)) = 0$, and so $\mathrm{Hom}_R(X, P_\sigma(M)) = 0$. But this is a contradiction, and so it follows that $P_\sigma(M) \in \mathcal{F}$. □

A module of $\mathcal{B}$ in Theorem 11 is a generalization of small module. Last we give another extension of small modules. We call a module N a σ-small module if there exists a module L and a small σ-dense submodule K of L such that N is isomorphic to K.

Proposition 12. *A module N is a σ-small module if and only if N is small in $E_\sigma(N)$.*

Proof. Let N be a σ-small module. Then there exists a module L and its σ-dense small submodule K such that $N \cong K$. Consider the following diagram.

$$\begin{array}{ccc} N & \overset{h|_N}{\to} & K \subseteq^\circ L \twoheadrightarrow L/K \in \mathcal{T}_\sigma \\ \cap & & \cap \\ E_\sigma(N) & \underset{h}{\to} & E_\sigma(K), \end{array}$$

where h is an isomorphism and $h|_N$ is an isomorphism and an restriction of h to N. Then there exists a $g : L \to E_\sigma(K)$ such that $g|_K = 1_K$. Then $K = g(K) \subseteq^\circ g(L) \subseteq E_\sigma(K)$, and so K is small in $E_\sigma(K)$. Thus $h^{-1}(K)$ is small in $h^{-1}(E_\sigma(K))$, and so N is small in $E_\sigma(N)$. The converse is clear. $\square$

Proposition 13. *Let $\mathcal{B}$ be the class of all σ-small modules. Then a torsion theory generated by $\mathcal{B}$ is a σ-hereditary torsion theory. A torsion theory cogenerated by $\mathcal{B}$ is a cohereditary torsion theory.*

Proof. It is easiy verified that $\mathcal{B}$ is closed under taking factor modules and σ-dense submodules, then a torsion theory generated by $\mathcal{B}$ is a σ-hereditary torsion theory by Proposition 1 and Proposition12. A torsion theory cogenerated by $\mathcal{B}$ is a cohereditary torsion theory by Proposition 9. $\square$

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ON THE RELATION OF THE UPPER BOUND OF GLOBAL DIMENSION AND THE LENGTH OF SERIAL ALGEBRA WHICH HAS FINITE GLOBAL DIMENSION

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Abstract. The aim of this note is to study the relationship between the global dimension and the Loewy length of serial algebras that have finite global dimension. To compute global dimension, we define the associative quiver of an admissible sequence $(a_1, \cdots, a_n)$ of a serial algebra A. This note concludes the following result. For positive integer k with $k < n/2$, if the Loewy length $L(A)$ of A is minimal positive integer which greater than n/k, then the global dimension is less than or equal to $2n - 2k - 1$.

Key Words: serial algebra, global dimension

Let A be a finite dimensional basic connected serial algebra over an algebraically closed field, and n is the number of the non isomorphic simple left modules of A. If the global dimension gl.dim$A$ of A is finite, then gl.dim$A \leq 2n - 2$ and the Loewy length $L(A)$ of A is less than or equal to $2n - 1$[3]. In this note we consider the relationship $L(A)$ and gl.dim$A$.

1. Notation

The quiver of A is one of the following.

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n \arrow[lll, bend left] \end{tikzcd}$$

The algebra A whose quiver is the first one called chain type and the other called cyclic type. Let $P_i, S_i\,(1 \leq i \leq n)$ be the indecomposable left projective module and the simple left module of A corresponding to the vertex i. Then P_{i+1} is a projective cover of radP_i for $i = 1, 2, \cdots, n-1$, and P_1 is a projective cover of radP_n in case of cyclic type. The sequence of positive integers $(a_1, \cdots, a_n)$ where $a_i = L(P_i)\,(1 \leq i \leq n)$ is called the admissible sequence of A and has the property that $a_{i+1} \geq a_i - 1 \geq 1$ for all $i = 1, 2, \cdots, n-1$ and $a_1 \geq a_n - 1$. Conversely, for any sequence $(a_1, \cdots, a_n)$ of positive integers with this property, there is a serial algebra with this sequence as its admissible sequence.

Now, let A be a serial algebra with admissible sequence $(a_1, \cdots, a_n)$. Then P_i has unique composition series of following shape. Where i is the corresponding simple module of vertex i, and $[k]$ denotes the least positive residue of k modulo n for any positive integer k.

$$P_i = \begin{pmatrix} i \\ [i+1] \\ \vdots \\ [i + a_i - 1] \end{pmatrix}$$

The detailed version of this paper will be submitted for publication elsewhere.

2. Regular points of admissible sequence

In the paper [3], Gustafson introduced the notion of f-regular points and computed the global dimension of serial rings.

Definition 1. The function f on $\{1, 2, \cdots, n\}$ for admissible sequence $(a_1, \cdots, a_n)$ is defined by $f(i) = [i + a_i]$. The point $i \in \{1, \cdots, n\}$ is f-regular if $f^t(i) = i$ for some positive integer t.

Since $f^{n-1}(i)$ is f-regular for any i , the set of f-regular points is not empty.

Definition 2. For $i \in \{1, \cdots, n\}$, the distance $h(i)$ of i from f-regular points defined by $h(i) = 0$ for f-regular point i, and $h(i) = t$ if $f^{t-1}(i)$ is not f-regular but $f^t(i)$ is f-regular for positive integer t. The maximal distance d from f-regular points defined by $d = \max\{h(i) | i = 1, \cdots, n\}$.

The minimal projective resolution of S_i is following.

$$\cdots \to P_{f^2(i+1)} \to P_{f^2(i)} \to P_{f(i+1)} \to P_{f(i)} \to P_{i+1} \to P_i \to S_i \to 0$$

If the projective dimension proj.dimS_i of S_i is finite, then its left end is one of the following shape.

$$0 \to P_{f^k(i)} \to P_{f^{k-1}(i+1)} \to \cdots$$
$$0 \to P_{f^k(i+1)} \to P_{f^k(i)} \to \cdots$$

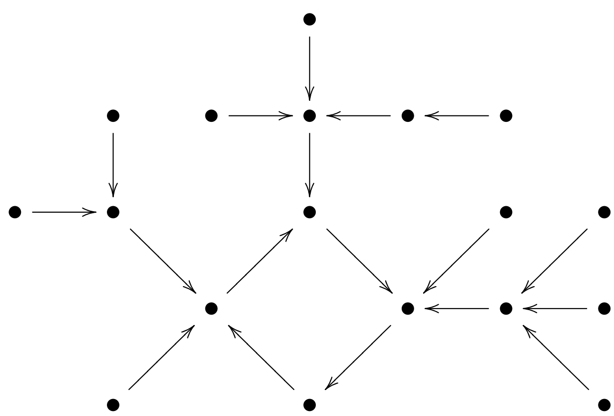
So $f^{k+1}(i) = f^k(i+1)$ or $f^{k+1}(i) = f^{k+1}(i+1)$ and this is f-regular. It follows that proj.dim$S_i \leq 2d$. Then we have following lemma.

Lemma 3 (Gustafson). *Let d be the maximal distance from f-regular points, then* gl.dim$A \leq 2d$.

3. Associative quiver

We define the associative quiver Q_A of A by $\{1, \cdots, n\}$ is set of vertices and an arrow i to j if $f(i) = j$. An associative quiver is a disjoint union of left serial quivers which are defined below.

Definition 4. A quiver called left serial if it has unique oriented cycle and when removing all arrows of this cycle, the remaining is a disjoint union of trees with unique sink which is a vertex of the cycle.



It follows that a vertex i is f-regular if and only if i belongs to the cycle of the associative quiver. We call "f-regular vertex" instead of "f-regular point" when we treat the point as the vertex of the associative quiver.

Example 5. Let A be a serial algebra with admissible sequence $(3,3,3,2,2)$. Its indecomposable left projective modules are following.

$$\begin{pmatrix}1\\2\\3\end{pmatrix}\begin{pmatrix}2\\3\\4\end{pmatrix}\begin{pmatrix}3\\4\\5\end{pmatrix}\begin{pmatrix}4\\5\end{pmatrix}\begin{pmatrix}5\\1\end{pmatrix}$$

The associative quiver of A is following. The vertices 1, 4, 2 and 5 are f-regular.

$$\begin{tikzcd} 3 \arrow[r] & 1 \arrow[r, bend left] & 4 \arrow[l, bend left] & 2 \arrow[r, bend left] & 5 \arrow[l, bend left] \end{tikzcd}$$

An associative quiver is a disjoint union of left serial quivers. But if gl.dim$A<\infty$, it is only one component. For, if gl.dim$A<\infty$, any morphism between indecomposable projective modules that corresponding to f-regular vertices is an isomorphism or 0. Because if there is a non isomorphic and non zero morphism $g: P_i \to P_j$, then its cokernel K has infinite chain of projective resolution.

$$\cdots \to P_{f^2(i)} \to P_{f^2(j)} \to P_{f(i)} \to P_{f(j)} \to P_i \to P_j \to K \to 0$$

If there exist two distinct cycles in the associative quiver, then it follows that there exist a non isomorphic and non zero morphism between the corresponding indecomposable projective modules on these cycles.

Proposition 6. *If* gl.dim$A<\infty$, *then* Q_A *is left serial (only one component).*

4. Serial algebra of chain type

Let A be a serial algebra of chain type with admissible sequence $(a_1,\cdots,a_n)$. It is well known that $L(A)\le n$ and gl.dim$A \le n-1$. Next theorem is generalization of this.

Theorem 7. *Suppose that A is a serial algebra of chain type with admissible sequence $(a_1,\cdots,a_n)$ and $l=L(A)$.*
Then gl.dim$A \le n-l+1$.

The inequality of global dimension of above theorem is sharp.

Example 8. Let A be a serial algebra with admissible sequence $(l, l-1, \cdots, 3, 2, 2, \cdots, 2, 1)$. gl.dim$A = n-l+1$. Indeed, proj.dim$S_{l-2} = n-l+1$ and this is the maximal.

$$0 \to P_n \to P_{n-1} \to \cdots \to P_{l-2} \to S_{l-2} \to 0$$

5. Serial algebra of cyclic type

Let A be a serial algebra of cyclic type with admissible sequence $(a_1,\cdots,a_n)$. If gl.dim$A<\infty$, then $L(A)\le 2n-1$[3]. It is well known that if $L(A)=2$ then gl.dim$A=\infty$. So, fixed $l\,(2<l<2n)$, we calculate upper bound of gl.dim$A$.

Definition 9. For $1\le i\le n-1$, the vertex i is called a step vertex if $a_{i+1}=a_i-1$ and vertex n is called a step vertex if $a_1=a_n-1$.

If i is a step vertex, then the associative quiver contains following graph as subquiver.

$$\begin{tikzcd} i \arrow[dr] & \\ & f(i) \\ i+1 \arrow[ur] & \end{tikzcd}$$

We don't avoid the case $i = f(i)$ and $i + 1 = f(i)$.

Lemma 10. *If Q_A has r f-regular vertices and s step vertices and if* gl.dim$A < \infty$*, then the maximal distance d from f-regular vertices is less than or equal to $n - r - s + 1$.*

Example 11. Let A be a serial algebra with admissible sequence $(4, 4, 4, 4, 3, 2, 2)$. The associative quiver is following. 1 and 5 are f-regular and 4 and 5 are step vertices. $d = h(3) = 4 = 7 - 2 - 2 + 1$ and gl.dim$A$ = proj.dim$S_2 = 7 \leq 8 = 2 \cdot d = 2 \cdot 4$.

$$\begin{tikzcd} 3 \arrow[r] & 7 \arrow[r] & 2 \arrow[r] & 6 \arrow[r] & 1 \arrow[r, bend left] & 5 \arrow[l, bend left] \\ & & & 4 \arrow[ur] & & \end{tikzcd}$$

If Q_A has only one step point, then d is maximal among the algebras which have fixed r regular points. In this case Q_A is following shape. There are r points that belong to the cycle and d is $n - r$. Lemma 3 shows that gl.dim$A \leq 2n - 2r$. In case of $n < l < 2n$, this bound is sharp. But the case $l \leq n$ is not.

$$\begin{tikzcd} & & & & \bullet \arrow[dr] & \\ \bullet \arrow[r] & \bullet \arrow[r] & \cdots \arrow[r] & \bullet \arrow[ur] & & \bullet \arrow[dl] \\ & & & & \cdots \arrow[ul] & \end{tikzcd}$$

For any positive real number x, let $\lceil x \rceil$ be the minimum positive integer that greater than x.

Theorem 12. *Let A be a a serial algebra of cyclic type with admissible sequence $(a_1, \cdots, a_n)$ which has finite global dimension and $l = L(A)$.*

(1) *If $n < l < 2n$, then* gl.dim$A \leq 4n - 2l$.
(2) *If $l = n$, then* gl.dim$A \leq 2n - 3$.
(3) *For any positive integer k with $n \geq 2k + 3$ and $\lceil \frac{n}{k+1} \rceil < \lceil \frac{n}{k} \rceil$,*
if $\lceil \frac{n}{k+1} \rceil \leq l < \lceil \frac{n}{k} \rceil$, then gl.dim$A \leq 2n - 2k - 3$.

These inequality of global dimension are sharp.

Example 13. For positive integer $t \, (1 \leq t < n)$, let A be a serial algebra with admissible sequence $(n + t, n + t - 1, \cdots, n + 1, n + 1, \cdots, n + 1, n)$.
This is the case (1) of the theorem which the equality holds. $h(i) = n - t \, (1 \leq i \leq$

$t)$, $h(t+j) = n-t-j\,(1 \le j \le n-t-1)$, and $h(n) = 0$. So $d = n-t$, and gl.dim$A = 2n-2t$. Q_A is following.

$$\begin{tikzcd}
1 \arrow[dr] & & & & & \\
2 \arrow[r] & t+1 \arrow[r] & t+2 \arrow[r] & \cdots \arrow[r] & n-1 \arrow[r] & n \arrow[loop right] \\
\ddots \arrow[ur] & t \arrow[u] & & & &
\end{tikzcd}$$

Above example is generalization of Gustafson's example.

Example 14 (Gustafson)**.** In example 13, the case of $t = 1$ is following.
Let A be a serial algebra with admissible sequence $(n+1, n+1, \cdots, n+1, n)$.
gl.dim$A = 2n-2$. Q_A is following.

$$\begin{tikzcd}
1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n-1 \arrow[r] & n \arrow[loop right]
\end{tikzcd}$$

Example 15. Let A be a serial algebra with admissible sequence $(n, n-1, \cdots, n-1, n-1)$. This is the case (2) of the theorem which the equality holds. gl.dim$A = 2n-3$. Q_A is following.

$$\begin{tikzcd}
n \arrow[r] & n-1 \arrow[r] & \cdots \arrow[r] & 2 \arrow[r] & 1 \arrow[loop right]
\end{tikzcd}$$

In the case (3) of the theorem, the equality of gl.dim$A$ holds when Q_A has $k+1$ regular vertices and one step vertex. In this case $d = n-k-1$, and gl.dim$A = 2d-1 = 2n-2k-3$.

Example 16. Let A be a serial algebra with admissible sequence $(3, 3, 3, 3, 2, 2)$. In this case, $n = 6, l = \lceil \frac{6}{2} \rceil = 3 < \lceil \frac{6}{1} \rceil, k = 1$, and gl.dim$A = 7 = 2 \cdot 6 - 2 \cdot 1 - 3$. Q_A is following.

$$\begin{tikzcd}
3 \arrow[r] & 6 \arrow[r] & 2 \arrow[r] & 5 \arrow[r] & 1 \arrow[r, bend left] & 4 \arrow[l, bend left]
\end{tikzcd}$$

Example 17. Let A be a serial algebra with admissible sequence
$(4, 4, 4, 4, 4, 3, 3, 3, 3, 3, 3, 3, 3, 3)$. In this case, $n = 14, l = \lceil \frac{14}{4} \rceil = 4 < \lceil \frac{14}{3} \rceil, k = 3$, and gl.dim$A = 19 = 2 \cdot 14 - 2 \cdot 3 - 3$. Q_A is following.

$$\begin{tikzcd}
 & & & & & & & & & & & 12 \arrow[dr] & \\
4 \arrow[r] & 8 \arrow[r] & 11 \arrow[r] & 14 \arrow[r] & 3 \arrow[r] & 7 \arrow[r] & 10 \arrow[r] & 13 \arrow[r] & 2 \arrow[r] & 6 \arrow[r] & 9 \arrow[ur] & & 1 \arrow[dl] \\
 & & & & & & & & & & & 5 \arrow[ul] &
\end{tikzcd}$$

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NONCOMMUTATIVE GRADED GORENSTEIN ISOLATED SINGULARITIES

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Abstract. Gorenstein isolated singularities play an essential role in representation theory of Cohen-Macaulay modules. In this article, we define a notion of noncommutative graded isolated singularity and study AS-Gorenstein isolated singularities. For an AS-Gorenstein algebra A of dimension $d \geq 2$, we show that A is a graded isolated singularity if and only if the stable category of graded maximal Cohen-Macaulay modules over A has the Serre functor. Using this result, we also show the existence of cluster tilting modules over certain fixed subalgebras of AS-regular algebras.

Key Words: graded isolated singularity, graded maximal Cohen-Macaulay module, AS-Gorenstein algebra, Serre functor, cluster tilting.

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1. Introduction

Throughout this paper, k is an algebraically closed field of characteristic 0. In representation theory of orders, which generalize both finite dimensional algebras and Cohen-Macaulay rings, studying the categories of Cohen-Macaulay modules is active (see [5] for details). In particular, the following results play key roles in the theory (we present graded versions due to [6, Corollary 2.5, Theorem 3.2, Theorem 4.2]).

Theorem 1. *Let R be a noetherian commutative graded local Gorenstein ring of dimension d and of Gorenstein parameter ℓ. Assume that R is an isolated singularity. Then the stable category of graded maximal Cohen-Macaulay modules has the Serre functor $(-\ell)[d-1]$.*

Theorem 2. *Let $S = k[x_1, \dots, x_d]$ be a polynomial ring generated in degree 1, G a finite subgroup of $\mathrm{SL}_d(k)$, and S^G the fixed subring of S.*

1. *Then the skew group algebra $S * G$ is isomorphic to $\underline{\mathrm{End}}_{S^G}(S)$ as graded algebras.*
2. *Assume that S^G is an isolated singularity. Then S is a $(d-1)$-cluster tilting module in the categories of graded maximal Cohen-Macaulay modules over S^G.*

The proofs of these results rely on commutative ring theory. This paper tries to give a noncommutative (not necessarily order) version of them.

One of the noncommutative analogues of polynomial rings (resp. Gorenstein local rings) is AS-regular algebras (resp. AS-Gorenstein algebras). In this paper, we define a notion of noncommutative graded isolated singularity by the smoothness of the noncommutative projective scheme (see also [8]), and we focus on studying AS-Gorenstein

The detailed version of this paper will be submitted for publication elsewhere.

isolated singularities. In particular, a noncommutative version of Theorem 1 will be given in Theorem 7, and a partial generalization of Theorem 2 for some fixed subalgebras of AS-regular algebras will be given in Theorem 11.

2. Preliminaries

Let A be a connected graded algebra and $\mathfrak{m} = \bigoplus_{i>0} A_i$ the maximal homogeneous two-sided ideal of A. The trivial A-module $A/\mathfrak{m}$ is denoted by k. We denote by $\mathsf{GrMod}\, A$ the category of graded right A-modules with degree zero A-module homomorphisms, and by $\mathsf{grmod}\, A$ the full subcategory consisting of finitely generated graded right A-modules. The group of graded k-algebra automorphisms of A is denoted by $\mathrm{GrAut}\, A$. Let M be a graded right A-module. For an integer $n \in \mathbb{Z}$, we define the truncation $M_{\geq n} := \bigoplus_{i \geq n} M_i \in \mathsf{GrMod}\, A$ and the shift $M(n) \in \mathsf{GrMod}\, A$ by $M(n)_i := M_{n+i}$ for $i \in \mathbb{Z}$. We write

$$\underline{\mathrm{Ext}}^i_A(M, N) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Ext}^i_{\mathsf{GrMod}\, A}(M, N(n)).$$

For a graded algebra automorphism $\sigma \in \mathrm{GrAut}\, A$, we define a new graded right A-module $M_\sigma \in \mathsf{GrMod}\, A$ by $M_\sigma = M$ as graded vector spaces with the new right action $m * a = m\sigma(a)$ for $m \in M$ and $a \in A$. We denote by $(-)^* = \underline{\mathrm{Hom}}_k(-, k)$ the graded Matlis duality. If M is locally finite, then $M^{**} \cong M$ as graded A-modules. We define the functor $\underline{\Gamma}_{\mathfrak{m}} : \mathsf{GrMod}\, A \to \mathsf{GrMod}\, A$ by $\underline{\Gamma}_{\mathfrak{m}}(-) = \lim_{n \to \infty} \underline{\mathrm{Hom}}_A(A/A_{\geq n}, -)$. The derived functor of $\underline{\Gamma}_{\mathfrak{m}}$ is denoted by $\mathrm{R}\underline{\Gamma}_{\mathfrak{m}}(-)$, and its cohomologies are denoted by $\underline{\mathrm{H}}^i_{\mathfrak{m}}(-) = h^i(\mathrm{R}\underline{\Gamma}_{\mathfrak{m}}(-))$.

Definition 3. A connected graded algebra A is called a d-dimensional AS-Gorenstein algebra (resp. AS-regular algebra) of Gorenstein parameter ℓ if

- A is noetherian,
- $\mathrm{id}_A A = \mathrm{id}_{A^{\mathrm{op}}} A = d < \infty$ (resp. $\mathrm{gldim}\, A = d < \infty$) and
- $\underline{\mathrm{Ext}}^i_A(k, A) \cong \underline{\mathrm{Ext}}^i_{A^{\mathrm{op}}}(k, A) \cong \begin{cases} k(\ell) & \text{if } i = d, \\ 0 & \text{if } i \neq d. \end{cases}$

If A is a d-dimensional AS-Gorenstein algebra of Gorenstein parameter ℓ, then i-th local cohomology $\underline{\mathrm{H}}^i_{\mathfrak{m}}(A)$ of A is zero for all $i \neq d$. The graded A-A bimodule $\omega_A := \underline{\mathrm{H}}^d_{\mathfrak{m}}(A)^*$ is called the canonical module of A. It is known that there exists a graded algebra automorphism $\nu \in \mathrm{GrAut}\, A$ such that $\omega_A \cong A_\nu(-\ell)$ as graded A-A bimodules (cf. [7, Theorem 1.2]). We call this graded algebra automorphism $\nu \in \mathrm{GrAut}\, A$ the generalized Nakayama automorphism of A.

We denote by $\mathsf{tors}\, A$ the full subcategory of $\mathsf{grmod}\, A$ consisting of finite dimensional modules over k, and

$$\mathsf{tails}\, A := \mathsf{grmod}\, A/\mathsf{tors}\, A$$

the quotient category, which is called the noncommutative projective scheme associated to A in [1]. If A is a commutative graded algebra finitely generated in degree 1 over k, then $\mathsf{tails}\, A$ is equivalent to the category of coherent sheaves on $\mathrm{Proj}\, A$ by Serre, justifying the terminology. We usually denote by $\mathcal{M} \in \mathsf{tails}\, A$ the image of $M \in \mathsf{grmod}\, A$. If $M, N \in \mathsf{grmod}\, A$, then $\mathcal{M} \cong \mathcal{N}$ in $\mathsf{tails}\, A$ if and only if $M_{\geq n} \cong N_{\geq n}$ in $\mathsf{grmod}\, A$ for some n, explaining the word of "tails".

We define a notion of noncommutative graded isolated singularity by the smoothness of the noncommutative projective scheme. Recall that the global dimension of $\mathsf{tails}\, A$ is defined by

$$\operatorname{gldim}(\mathsf{tails}\, A) := \sup\{i \mid \operatorname{Ext}^i_{\mathsf{tails}\, A}(\mathcal{M}, \mathcal{N}) \neq 0 \text{ for some } \mathcal{M}, \mathcal{N} \in \mathsf{tails}\, A\}.$$

Definition 4. A noetherian connected graded algebra A is called a graded isolated singularity if $\mathsf{tails}\, A$ has finite global dimension.

If A is a graded quotient of a polynomial ring generated in degree 1, then A is a graded isolated singularity (in the above sense) if and only if $A_{(\mathfrak{p})}$ is regular for any homogeneous prime ideal $\mathfrak{p} \neq \mathfrak{m}$, justifying the definition. It is easy to see that if A has finite global dimension, then $\mathsf{tails}\, A$ has finite global dimension, so A is a graded isolated singularity. The purpose of this paper is to study AS-Gorenstein isolated singularities.

For the rest of this section, we recall that Jørgensen and Zhang [9] gave a noncommutative graded version of Watanabe's theorem. Let A be a graded algebra and let $\sigma \in \operatorname{GrAut} A$, $M, N \in \mathsf{GrMod}\, A$. A k-linear graded map $f : M \to N$ is called σ-linear if $f : M \to N_\sigma$ is a graded A-module homomorphism. If A is AS-Gorenstein, then by [9, Lemma 2.2], $\sigma : A \to A$ induces a σ-linear map $\underline{\mathrm{H}}^d_{\mathfrak{m}}(\sigma) : \underline{\mathrm{H}}^d_{\mathfrak{m}}(A) \to \underline{\mathrm{H}}^d_{\mathfrak{m}}(A)$. Moreover, there exists a constant $c \in k^\times$ such that $\underline{\mathrm{H}}^d_{\mathfrak{m}}(\sigma) : \underline{\mathrm{H}}^d_{\mathfrak{m}}(A) \to \underline{\mathrm{H}}^d_{\mathfrak{m}}(A)$ is equal to $c(\sigma^{-1})^* : A^*(\ell) \to A^*(\ell)$. The constant c^{-1} is called the homological determinant of σ, and we denote $\operatorname{hdet} \sigma = c^{-1}$ (see [9, Definition 2.3]).

Theorem 5. [9, Theorem 3.3] *If A is AS-Gorenstein of dimension d, and G is a finite subgroup of* $\operatorname{GrAut} A$ *such that* $\operatorname{hdet} \sigma = 1$ *for all $\sigma \in G$, then the fixed subalgebra A^G is AS-Gorenstein of dimension d.*

3. Serre Functors

Definition 6. Let C be a k-linear category such that $\dim_k \operatorname{Hom}_{\mathsf{C}}(\mathcal{M}, \mathcal{N}) < \infty$ for all $\mathcal{M}, \mathcal{N} \in \mathsf{C}$. An autoequivalence $S : \mathsf{C} \to \mathsf{C}$ is called the Serre functor for C if we have a functorial isomorphism

$$\operatorname{Hom}_{\mathsf{C}}(\mathcal{M}, \mathcal{N}) \cong \operatorname{Hom}_{\mathsf{C}}(\mathcal{N}, S(\mathcal{M}))^*$$

for all $\mathcal{M}, \mathcal{N} \in \mathsf{C}$.

Note that the Serre functor is unique if it exists. Let A be an AS-Gorenstein algebra of dimension d. We say that $M \in \mathsf{grmod}\, A$ is graded maximal Cohen-Macaulay if $\underline{\operatorname{Ext}}^i_A(M, A) = 0$ for any $i > 0$. We denote by $\mathsf{CM}^{\mathsf{gr}}(A)$ the full subcategory of $\mathsf{grmod}\, A$ consisting of graded maximal Cohen-Macaulay modules, and by $\underline{\mathsf{CM}}^{\mathsf{gr}}(A)$ the stable category of $\mathsf{CM}^{\mathsf{gr}}(A)$. Thus $\underline{\mathsf{CM}}^{\mathsf{gr}}(A)$ has the same objects as $\mathsf{CM}^{\mathsf{gr}}(A)$ and the morphism set is given by

$$\operatorname{Hom}_{\underline{\mathsf{CM}}^{\mathsf{gr}}(A)}(M, N) = \operatorname{Hom}_{\mathsf{GrMod}\, A}(M, N)/P(M, N)$$

for any $M, N \in \mathsf{CM}^{\mathsf{gr}}(A)$, where $P(M, N)$ consists of the degree zero A-module homomorphisms that factor through a projective module in $\mathsf{GrMod}\, A$. The syzygy gives a functor $\Omega : \underline{\mathsf{CM}}^{\mathsf{gr}}(A) \to \underline{\mathsf{CM}}^{\mathsf{gr}}(A)$. By [2], we see that $\underline{\mathsf{CM}}^{\mathsf{gr}}(A)$ is a triangulated category with respect to the translation functor $M[-1] = \Omega M$.

We have the following main result in this section.

Theorem 7. *Let A be an AS-Gorenstein algebra of dimension $d \geq 2$. Then the following are equivalent.*

(1) *A is a graded isolated singularity.*
(2) *$\underline{\mathsf{CM}}^{\mathsf{gr}}(A)$ has the Serre functor $- \otimes_A \omega_A[d-1]$, that is, there exists a functorial isomorphism*

$$\operatorname{Hom}_{\underline{\mathsf{CM}}^{\mathsf{gr}}(A)}(M, N) \cong \operatorname{Hom}_{\underline{\mathsf{CM}}^{\mathsf{gr}}(A)}(N, M \otimes_A \omega_A[d-1])^*$$

for any $M, N \in \mathsf{CM}^{\mathsf{gr}}(A)$.

In order to give an example of this result, we prepare a noncommutative graded version of a classical result by Auslander. Let A be an AS-Gorenstein algebra. We call A CM-representation-finite if there exist finitely many indecomposable graded maximal Cohen-Macaulay modules $X_1, \ldots, X_n$ so that, up to isomorphism, the indecomposable graded maximal Cohen-Macaulay modules in $\mathsf{grmod}\, A$ are precisely the degree shifts $X_i(s)$ for $1 \leq i \leq n$ and $s \in \mathbb{Z}$.

Proposition 8. *Let A be an AS-regular algebra of dimension 2, and let G be a finite subgroup of* $\operatorname{GrAut} A$ *such that* $\operatorname{hdet} \sigma = 1$ *for all $\sigma \in G$. Then A^G is CM-representation-finite. In fact, the indecomposable maximal Cohen-Macaulay modules over A^G are precisely the indecomposable summands of $A(s)$. Moreover, A^G is an AS-Gorenstein isolated singularity.*

Example 9. Let

$$A = k\langle x, y\rangle/(xy - \alpha yx) \quad 0 \neq \alpha \in k, \quad \deg x = \deg y = 1.$$

Then A is an AS-regular algebra of dimension 2 and of Gorenstein parameter 2. We define a graded algebra automorphism $\sigma \in \operatorname{GrAut} A$ by $\sigma(x) = \xi x, \sigma(y) = \xi^2 y$ where ξ is a primitive 3-rd root of unity. One can check $\operatorname{hdet} \sigma = 1$. Let $G = \langle \sigma \rangle \leq \operatorname{GrAut} A$. Then A^G is AS-Gorenstein of dimension 2 and

$$H_{A^G}(t) = \frac{1 - t + t^2}{(1-t)^2(1+t+t^2)}.$$

It follows from Proposition 8 that A^G is CM-representation-finite and a graded isolated singularity. But A^G is not AS-regular because $H_{A^G}(t)^{-1} \notin \mathbb{Z}[t]$. Theorem 7 shows that $\underline{\mathsf{CM}}^{\mathsf{gr}}(A^G)$ has the Serre functor.

4. n-cluster tilting modules

The notion of n-cluster tilting subcategories plays an important role from the viewpoint of higher analogue of Auslander-Reiten theory [3], [4]. It can be regarded as a natural generalization of the classical notion of CM-representation-finiteness.

Definition 10. Let A be a balanced Cohen-Macaulay algebra. A graded maximal Cohen-Macaulay module $X \in \mathsf{CM}^{\mathsf{gr}}(A)$ is called an n-cluster tilting module if

$$\begin{aligned} \operatorname{add}_A\{X(s) \mid s \in \mathbb{Z}\} &= \{M \in \mathsf{CM}^{\mathsf{gr}}(A) \mid \underline{\operatorname{Ext}}^i_A(M, X) = 0\ (0 < i < n)\} \\ &= \{M \in \mathsf{CM}^{\mathsf{gr}}(A) \mid \underline{\operatorname{Ext}}^i_A(X, M) = 0\ (0 < i < n)\}. \end{aligned}$$

Note that A is CM-representation-finite if and only if A has a 1-cluster tilting module. In fact if A and A^G are as in Proposition 8, then A^G has a 1-cluster tilting module $A \in \mathsf{CM}^{\mathsf{gr}}(A^G)$.

Let A be a connected graded algebra and G a finite subgroup of $\operatorname{GrAut} A$. Then the skew group algebra $A * G$ is an $\mathbb{N}$-graded algebra defined by $A * G = \bigoplus_{i \in \mathbb{N}} (A_i \otimes_k kG)$ as a graded vector space with the multiplication

$$(a \otimes \sigma)(a' \otimes \sigma') = a\sigma(a') \otimes \sigma\sigma'$$

for any $a, a' \in A$ and $\sigma, \sigma' \in G$. We have the following main result in this section.

Theorem 11. *Let A be a AS-regular domain of dimension $d \geq 2$ and of Gorenstein parameter ℓ generated in degree 1. Take $r \in \mathbb{N}^+$ such that $r \mid \ell$. We define a graded algebra automorphism σ_r of A by $\sigma_r(a) = \xi^{\deg a} a$ where ξ is a primitive r-th root of unity, and write $G = \langle \sigma_r \rangle$ for the finite cyclic subgroup of* $\operatorname{GrAut}(A)$ *generated by σ_r. Then*

(1) *the skew group algebra $A * G$ is isomorphic to $\underline{\operatorname{End}}_{A^G}(A)$ as graded algebras.*
(2) *A^G is a graded isolated singularity, and $A \in \mathsf{CM}^{\mathsf{gr}}(A^G)$ is a $(d-1)$-cluster tilting module.*

Moreover, it follows from the study of skew group algebras [10, Lemma 13] that $\underline{\operatorname{End}}_{A^G}(A)$ in the above theorem is a generalized AS-regular algebra of dimension d (ie, $\underline{\operatorname{End}}_{A^G}(A)$ has global dimension d and satisfies generalized Gorenstein condition).

Theorem 11 is a partial generalization of Theorem 2. Thanks to this result, we can obtain examples of $(d-1)$-cluster tilting modules over non-orders.

Example 12. Let

$$A = k\langle x, y\rangle/(\alpha xy^2 + \beta yxy + \alpha y^2 x + \gamma x^3, \alpha yx^2 + \beta xyx + \alpha x^2 y + \gamma y^3), \deg x = \deg y = 1$$

where $\alpha, \beta, \gamma \in k$ are generic scalars. Then A is an AS-regular algebra of dimension 3 and Gorenstein parameter 4. Let

$$G = \langle \sigma_4 \rangle = \left\langle \begin{pmatrix} \xi & 0 \\ 0 & \xi \end{pmatrix} \right\rangle \leq \operatorname{GrAut} A$$

where ξ is a primitive 4-th root of unity. Then A^G is an AS-Gorenstein isolated singularity, and $A \in \mathsf{CM}^{\mathsf{gr}}(A^G)$ is a 2-cluster tilting module. Moreover, we see that $\underline{\operatorname{End}}_{A^G}(A)$ is a generalized AS-regular algebra of dimension 3.

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ULRICH MODULES AND SPECIAL MODULES OVER 2-DIMENSIONAL RATIONAL SINGULARITIES

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Abstract. In this talk, we study Ulrich ideals and Ulrich modules with respect to some ideals. As the main result, we classify all Ulrich ideals and Ulrich modules of 2-dimensional Gorenstein rational singularities. Moreover, we prove that the notion of Ulrich modules and that of special modules in 2-dimensional Gorenstein rational singularities.

Key Words: Cohen-Macaulay module, Gorenstein ring, Ulrich ideal, good ideal, Ulrich module, special module, rational singularity, Riemann-Roch theorem, McKay correspondence

2000 *Mathematics Subject Classification*: Primary 13H10, 13H15; Secondary 13A30, 14B05, 14C40, 16G50.

本講演は, 講演者の他, 後藤四郎氏 (明治大学), 高橋亮氏 (名古屋大学), 大関一秀氏 (山口大学) との共同研究の報告である ([4], [5]).

1. 背景

この講演を通じて, 特に断らない限り, $(A, \mathfrak{m}, k)$ は可換なネーター局所整域とし, 単位元 1_A を持つとする. また, $\mathfrak{m}$ は A のただ 1 つの極大イデアル, $k = A/\mathfrak{m}$ を剰余体とする. $d = \dim A$ を A の**クルル次元** (Krull dimension) を表すものとする.

さらに, 講演を通じて, M は有限生成 A 加群を表すものとし, $\ell_A(M)$ は M の**長さ** (length), $\mu_A(M) = \ell_A(M/\mathfrak{m}M)$ を M の**極小生成系の個数** (minimal number of generators) $\mathrm{rank}_A(M) = \dim_{Q(A)}(M \otimes_A Q(A))$ を M の**階数** (rank), $e_A(M) = e^0_{\mathfrak{m}}(M)$ を M の極大イデアルに関する**重複度** (multiplicity) を表すものとする. ここで, $Q(A)$ は A の**商体**を表す. 一般に, A の $\mathfrak{m}$-準素イデアル I に対して, 十分大きな整数 n を取れば, $\ell_A(A/I^{n+1})$ は n についての多項式となり, 次のように表すことができる：

$$\ell_A(A/I^{n+1}) = e_0\binom{n+d}{d} - e_1\binom{n+d-1}{d-1} + \cdots + (-1)^d e_d.$$

このとき, $e_0 = e(I)$ を I に関する**重複度**と言う. さらに, A が整域のときには,

$$e^0_I(M) = e(I) \cdot \mathrm{rank}_A(M)$$

により, I に関する M の重複度が計算できる.

本講演の主役である極大 Cohen-Macaulay 加群 (以下, 極大 Cohen-Macaulay 加群と呼ぶ) と Ulrich Cohen-Macaulay 加群の定義から始めよう.

The detailed version of this paper will be submitted for publication elsewhere.

Definition 1. $\mathrm{Ext}_A^i(A/\mathfrak{m}, M) = 0$ $(0 \le \forall i < d = \dim A)$ が成立するとき, M は**極大 Cohen-Macaulay** A **加群**であるという。明確なときは, 極大 Cohen-Macaulay 加群と呼ぶ。

極大 Cohen-Macaulay 加群について, 次の不等式は基本的である.

Lemma 2. *M が極大 Cohen-Macaulay 加群ならば,*

$$\mathrm{rank}_A M \le \mu_A(M) \le e_A(M)$$

が成り立つ.

Definition 3 (Brennan-Herzog-Ulrich [1]). M が極大 Cohen-Macaulay 加群で, 等号 $\mu_A(M) = e_A(M)$ が成立するとき, M は **Ulrich Cohen-Macaulay** A **加群** であるという. 以下では, **Ulrich 加群**と呼ぶ。

次に, special Cohen-Macaulay 加群の定義を思い出そう. その前に有理特異点について少し説明しよう. $(A, \mathfrak{m}, k)$ を２次元の完備局所整閉整域とし, 剰余体 k は標数 0 の代数的閉体と仮定する. ある特異点解消 $\pi\colon X \to \mathrm{Spec}\, A$ が存在して, $H^1(X, \mathcal{O}_X) = 0$ が成り立つとき, A は**有理特異点**であると言う.

有理特異点の代表的な例は商特異点である. 例えば, 次の環は代表的な例である.

$$k[[s^4, st, y^4]] \cong k[[x, y, z]]/(x^2 + y^4 + z^2).$$
$$k[[s^4, s^3t, s^2t^2, st^3, t^4]].$$

特に, **２次元の Gorenstein 有理特異点**は, **有理二重点** (rational double point) とも呼ばれ, 次のいずれかの方程式 f で定義される超曲面 $A = k[[x, y, z]]/(f)$ であることが知られている (cf. [9], [10]):

$$\begin{array}{ll}
(A_n) & z^2 + x^2 + y^{n+1}\ (n \ge 1) \\
(D_n) & z^2 + x^2y + y^{n-1}\ (n \ge 4) \\
(E_6) & z^2 + x^3 + y^4 \\
(E_7) & z^2 + x^3 + xy^3 \\
(E_8) & z^2 + x^3 + y^5.
\end{array}$$

さて, ２次元有理特異点上の special Cohen-Macaulay 加群の定義を与えておこう (注意：より一般に定義されているが, ここでは有理特異点に限定する).

Definition 4. A を２次元の有理特異点とし, M を極大 Cohen-Macaulay 加群とする. このとき, M が **special Cohen-Macaulay** A **加群**であるとは, $M^* := \mathrm{Hom}_A(M, A) \cong \mathrm{Syz}_A^1(M)$ が成り立つことと定める. 明らかな場合は, **special 加群**と呼ぶ。

Wunram [14] は２次元巡回商特異点上のすべての special 加群を分類した. また, Iyama-Wemyss [7] はその結果を拡張し, ２次元の商特異点における special 加群を完全に分類し, その Auslander-Reiten グラフを記述した.

先に与えた有理 (商) 特異点の例に対して, 直既約な Ulrich 加群と special 加群を明確にしておこう.

Example 5. $A = k[[s^4, st, t^4]] \cong k[[x, y, z]]/(x^2 + y^4 + z^2)$ の極小な特異点解消の双対グラフは $E_1 - E_2 - E_3$ である. ここで, $E_1^2 = E_2^2 = E_3^2 = -2$, $E_1E_2 = E_2E_3 = 1$, 及び $E_1E_3 = 0$ が成り立つ. McKay 対応により, 我々は E_i $(i = 1, 2, 3)$ に対応する直既約な極大 Cohen-Macaulay 加群 M_i を見出すことができる.

	生成系	階数	$\mu_A(M)$	$e(M)$	
M_0	A	1	1	2	free
M_1	$As + At^3$	1	2	2	Ulrich, special
M_2	$As^2 + At^2$	1	2	2	Ulrich, special
M_3	$As^3 + At$	1	2	2	Ulrich, special

Example 6. $A = k[[s^4, s^3t, s^2t^2, st^3, t^4]]$ とおく．極小特異点解消 $\pi\colon X \to \operatorname{Spec} A$ の双対グラフは, E_1 のみである．ここで, $E_1^2 = -4$ である．

Special McKay 対応 (cf. [14]) により, E_1 に対応する直既約な極大 Cohen-Macaulay A 加群 M_1 が存在する．

	生成系	階数	$\mu_A(M)$	$e(M)$	
M_0	A	1	1	4	free
M_1	$As + At$	1	2	4	special
M_2	$As^2 + Ast + At^2$	1	3	4	
M_3	$As^3 + As^2t + Ast^2 + At^3$	1	4	4	Ulrich

イデアル I に関する Ulrich 加群の概念と Ulrich イデアルの概念を定義しよう．

Definition 7. A を Cohen-Macaulay 局所整域とし, I を $\mathfrak{m}$ 準素イデアルとする．極大 Cohen-Macaulay A 加群 M が $\ell_A(M/IM) = e_I^0(M)$ と, $M/$ が A/I 自由加群であることをみたすとき, I **に関する Ulrich Cohen-Macaulay** A **加群**, 略して, I **に関する Ulrich 加群**であるという．

Definition 8. パラメーター系で生成されないイデアルが, I のある**極小還元** (minimal reduction) Q に対して $I^2 = QI$b をみたし, I/I^2 が A/I-自由加群ならば, **Ulrich イデアル**と言う．

Ulrich イデアルの代表的な例を与えておこう．

Example 9. (1) ([11])A が Cohen-Macaulay 局所整域で, **極小重複度** (minimal multiplicity) を持つとき, $\mathfrak{m}$ は Ulrich イデアルになる．

(2) $A = k[[x_1, \ldots, x_n]]/(x_1^{a_1} + \cdots + x_n^{a_n})$ のとき，もし $a_1 = 2b$ が偶数ならば, $I_b = (x_1^b, x_2, \ldots, x_n)$ は Ulrich イデアルになる．$k[[x, y, z]]/(x^3+y^3+z^3)$ も Ulrich イデアルを持つが, この形にはなっていない (後述未解決問題参照 Watanabe-Yoshida:No.1).

Ulrich イデアル上の Ulrich 加群の理論は, 重複度 2 の超曲面の (極大イデアルに関する) Ulrich 加群の理論を拡張したものと考えることができる．その視点でみたとき, 次の２つの結果はごく自然なものである ([1] 参照).

Theorem 10. $(A, \mathfrak{m}, k)$ を *Cohen-Macaulay* 局所整域とする．$i \geq d$ に対して, I が *Ulrich* イデアルならば, $\operatorname{Syz}_A^i(A/I)$ は I に関する *Ulrich* 加群である．$d = \dim A \geq 1$ のときは, 逆も正しい.

Theorem 11. M は I に関する *Ulrich* 加群と仮定する．このとき, $M^\vee = \operatorname{Hom}_A(M, K_A)$ が *Ulrich* 加群であることと, A/I が *Gorenstein* であることとは同値である.

Gorenstein 局所環において, Ulrich イデアルは興味深い特徴付けを持つ．イデアル I が **good イデアル**であるとは, $I^2 = QI$ をみたす極小還元 Q が存在して, $I^2 = QI$ であり, $2 \cdot \ell_A(A/I) = e_I^0(A)$ が成り立つときを言う．

Proposition 12. A は *Gorenstein* 局所環とする. このとき, 次は同値である.

(1) I は *Ulrich* イデアルである.

(2) I *is good* イデアルで, かつ, A/I は *Gorenstein* である.

(3) I *is* $\mu_A(I) = d+1$ をみたす *good* イデアルである.

本講演において焦点となっている 2 次元有理二重点において, good イデアルの概念は次に述べるように幾何的な特徴付けを持つ.

Theorem 13 (**Goto-Iai-Watanabe** [3] **).** $(A, \mathfrak{m}, k)$ を 2 次元有理二重点とするとき, $\mathfrak{m}$-準素イデアル I に対して, 次の条件は同値である :

(1) I は *good* イデアルである.

(2) I は整閉イデアルで, 極小特異点解消 $\pi\colon X \to \operatorname{Spec} A$ 上で表現される. すなわち, X 上の反ネフ因子 Z が存在して, $I\mathcal{O}_X = \mathcal{O}_X(-Z)$ は可逆で, $I = H^0(X, \mathcal{O}_X(-Z))$ が成り立つ.

代表的な例は極大イデアルである. 2 次元有理特異点において, 極大イデアルは good イデアルであるが, 実際, 極大イデアルは基本因子 $Z_0 = \sum_i n_i E_i$ で表現される整閉イデアルである.

Example 14. $A = k[[x,y,z]]/(x^2+y^4+z^2)$ を (A_3) 型の有理二重点とする. このとき, 基本因子は $Z_0 = E_1 + E_2 + E_3$ である. $Z = a_1E_1 + a_2E_2 + a_3E_3$ が反ネフ因子であるための条件を計算してみよう. $E_1^2 = E_2^2 = E_3^2 = -2$, $E_1E_2 = E_2E_3 = 1$ 及び $E_1E_3 = 0$ を用いて計算すると,

$$\begin{cases} ZE_1 &=& -2a_1 + a_2 & \leq & 0 \\ ZE_2 &=& a_1 - 2a_2 + a_3 & \leq & 0 \\ ZE_3 &=& a_2 - 2a_3 & \leq & 0 \end{cases}$$

を得る. ゆえに, ベクトル $[a_1, a_2, a_3]$ と $a_1E_1 + a_2E_2 + a_3E_3$ を同一視するとき, 反ネフ因子は $[1,1,1]$, $[1,2,1]$, $[1,2,2]$, $[1,2,3]$, $[2,2,1]$ 及び $[3,2,1]$ で張られる錘の格子点に対応する. なお, 後で見るように最初の 2 つのベクトルが Ulrich イデアルに対応する.

次に, 2 次元有理特異点上の special Cohen-Macaulay 加群の概念を一般化しよう. 以下, しばらく A は 2 次元の有理特異点とする.

Definition 15. A を 2 次元の有理特異点, I を $\mathfrak{m}$-準素イデアルとする. 極大 Cohen-Macaulay 加群 M が, special であり, M/IM が A/I 上自由であるとき, M は **I に関して special Cohen-Macaulay 加群**, 略して, **I に関して special 加群**であるという.

また, Ulrich イデアルに対応するものとして, special イデアルの概念を次のように定義する.

Definition 16. $\mathfrak{m}$-準素イデアル I が good であり, さらに, I に関する special Cohen-Macaulay 加群が少なくとも 1 つ存在するならば, **special イデアル**であると言う.

基本的な問題は,

Ulrich 加群 (Special) Cohen-Macaulay 加群及び Ulrich (special) イデアルを分類せよ.

である. 以下, 2 次元有理二重点上の Ulrich 加群と Ulrich イデアルの分類定理を中心に紹介しよう.

2. 2次元有理二重点の Ulrich イデアルと Ulrich 加群

以下, この節では, A は2次元の有理二重点とし, $\pi\colon X \to \operatorname{Spec} A$ をその極小特異点解消とする. 最初に, 各有理二重点における Ulrich イデアルの分類方法を説明しよう.

Lemma 17. *I が Ulrich イデアルであるための必要十分条件は, $\mu(I) = 3$ であり, X 上のある反ネフ因子 Z が存在して, $I = H^0(X, \mathcal{O}_X(-Z))$ が good イデアルになることである.*

この補題を用いれば, 任意の Ulrich イデアルを決定することは難しくない. (A_3) 型の有理二重点 $A = k[[x, y, z]]/(x^2 + y^4 + z^2)$ の場合に説明しよう. まず, 基本因子は $Z_0 = E_1 + E_2 + E_3$ である. また, $Z = a_1E_1 + a_2E_2 + a_3E_3$ を反ネフ因子とすれば,

$$\begin{cases} ZE_1 &= -2a_1 + a_2 \leq 0, \\ ZE_2 &= a_1 - 2a_2 + a_3 \leq 0, \\ ZE_3 &= a_2 - 2a_3 \leq 0. \end{cases}$$

が成り立つことをすでにみた. 一方, $\mu(I) = 3$ という条件は, $ZZ_0 = -2$, すなわち, $(Z - Z_0)Z_0 = 0$ は

$$Z_0E_i \neq 0 \Longrightarrow a_i = n_i = 1$$

に翻訳される. この例では, $Z_0E_1 = Z_0E_3 = -1$, $Z_0E_2 = 0$ だから, $a_1 = a_3 = 1$ を得る. このとき, 上の連立不等式から, $a_2 = 1$ または $a_2 = 2$ を得る. 言い換えると, Ulrich イデアルに対応する反ネフ因子は,

$$Z_0 = E_1 + E_2 + E_3, \quad Z_1 = E_1 + 2E_2 + E_3$$

の2つである.

次に, Ulrich イデアルに関する Ulrich 加群を分類しよう. (その後, イデアル I に関する Ulrich 加群があれば, I は Ulrich イデアルであることを示すことで Ulrich 加群の完全な分類が完成する.)

基本的な道具は次の定理である.

Theorem 18 (**Kato's Riemann Roch** e.g. [13])**.** *A を2次元有理特異点とする. $I = H_0(X, \mathcal{O}_X(-Z))$ を $\pi\colon X \to \operatorname{Spec} A$ 上で表現される整閉イデアルとすると,*

$$\ell_A(A/I) = -\frac{Z^2 + KZ}{2}$$

が成り立つ. ここで, K は X 上の標準因子である. また, 極大 CM 加群 M に対して,

$$\ell_A(M/IM) = \operatorname{rank}_A M \cdot \ell_A(A/I) + c(\widetilde{M})Z.$$

が成り立つ. ここで, $\widetilde{M} = \pi^* M/torsion$ は M の引き戻しで定義される層であり, $c(\widetilde{M})$ はその第1 Chern 類を表す.

Remark 19. A を2次元有理特異点とする. $\{E_i\}_{i=1}^r$ を極小特異点解消に現れる例外因子全体とするとき, 標準因子 $K = \sum_{i=1}^r k_iE_i$ は, 方程式

$$0 = p_a(E_i) = \frac{E_i^2 + KE_i}{2} + 1 = 0 \quad (i = 1, \ldots, r)$$

で定まる.

Theorem 20 (McKay correspndence). X を極小な特異点解消とし, $\{E_i\}$ を例外因子全体とする. このとき, 各 E_i に対して, 直既約な極大 *Cohen-Macaulay* 加群 M_i が同型を除いて一意的に存在して, 次を満たす:

(1) $c(\widetilde{M_i})E_j = \delta_{ij}$. (クロネッカーのデルタ)
(2) $\mathrm{rank}_A M_i = n_i$, ここに, $Z_0 = \sum_i n_i E_i$ は基本因子を与える.

先に, Ulrich イデアルに対応する因子を決定した, (A_3) 型の有理二重点

$$A = k[[x,y,z]]/(x^2+y^4+z^2)$$

に対して, 各 Ulrich イデアルに関する Ulrich 加群をすべて決定しよう. I に関する Ulrich 加群の直和や直和因子も同じ性質を持つから, 直既約なものに限って調べればよい.

• Z_0 に対応する Ulrich 加群

$Z_0 = E_1 + E_2 + E_3$ に対応する good イデアルは極大イデアルである. M_1, M_2, M_3 が極大イデアル $\mathfrak{m}$ に関する Ulrich 加群であることは既に確認している.

• Z_1 に対応する Ulrich 加群

$Z_1 = E_1 + 2E_2 + E_3$ に対応する Ulrich イデアル $I = H^0(X, \mathcal{O}_X(-Z_1))$ に関する Ulrich 加群を決定すれば十分である. なお, 以下の計算において I の生成系を具体的に決定する必要性はないが, 具体的には, $I = (x, y^2, z)$ である. 実際, Kato の Riemann-Roch 定理から, $\ell_A(A/I) = 2$ であることが分かる. 一方, (x, y^2, z) がこの条件をみたす Ulrich イデアルであることが定義により確認できるので, $I = (x, y^2, z)$ であることが結論される.

まず, Kato の Riemann-Roch 定理から,

$$\ell_A(M_i/IM_i) = \ell_A(A/I) \cdot \mathrm{rank}_A M_i + c(\widetilde{M_i})Z = \ell_A(A/I) \cdot n_i + a_i$$

を得る. 他方, I が good イデアルであることに注意すると,

$$e_I^0(M_i) = e(I) \cdot \mathrm{rank}_A(M_i) = 2 \cdot \ell_A(A/I) \cdot n_i$$

を得る. ゆえに, もし M_i が I に関する Ulrich 加群ならば,

$$(*) \qquad a_i = \ell_A(A/I) \cdot n_i = \ell_A(A/I)$$

を得る。逆に, 条件 (*) が成り立てば, $\ell_A(M_i/IM_i) = e_I^0(M_i)$ が成立する. また, M_i は (極大イデアルに関する) Ulrich 加群だから,

$$\mu_A(M_i) = e_{\mathfrak{m}}^0(M_i) = e(\mathfrak{m}) \cdot \mathrm{rank}_A(M_i) = 2n_i$$

であり,

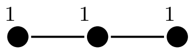
$$\mu_A(M_i)\ell_A(A/I) = 2n_i \cdot \ell_A(A/I) = \ell_A(M_i/IM_i)$$

が成り立つから, M_i/IM_i は自由 A/I-加群である. 従って, M_i が I に関して Ulrich 加群であるためには (*) が成り立てばよい. $\ell_A(A/I) = \frac{-Z_1^2}{2} = 2$ なので, この条件をみたす i は $i = 2$ に限る. 言い換えると, M_2 のみが I に関する Ulrich 加群である.

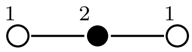
従って, $\mathfrak{m}$ に関する Ulrich 加群は, M_1, M_2, M_3 の有限直和であり, $I = (x, y^2, z)$ に関する Ulrich 加群は, M_2 の直和である.

以上の議論を次のように図示することができる.

(A_3) $f = x^2 + y^4 + z^2$

$Z_0 =$ 

$\mathfrak{m} = I_0 = (x, y, z)$

$Z_1 =$ 

$I_1 = (x, y^2, z)$

上記と同様の議論により, 与えられた有理二重点の Ulrich イデアルに関する Ulrich イデアルを完全に決定することができる.

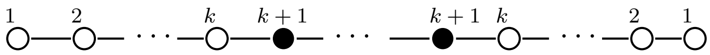
Ulrich 加群の分類を完成するには, Ulrich イデアル以外のイデアル I に関する Ulrich 加群を分類しなければならないが, 次の定理により, 2次元有理二重点の場合はその心配はない.

Theorem 21. *A を2次元の有理二重点とし, I を $\mathfrak{m}$-準素イデアルとする. このとき, 次の条件は同値である.*

(1) *I に関する Ulrich 加群が存在する.*
(2) *I は Ulrich イデアルである.*

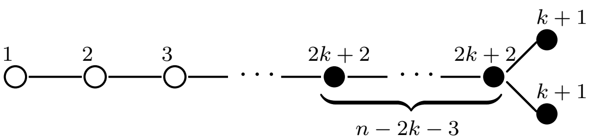
以上により, 2次元有理二重点上の Ulrich 加群を完全に分類できることが分かった. 以下では, 先に述べた (A_3) の場合を参考にして, 各有理二重点の場合の Ulrich イデアルとそれに関する Ulrich 加群の表 (一部省略) をあげておく.

Ulrich ideals in RDP of type (A_n) $f = x^2 + y^{n+1} + z^2$ $(n = 2m)$

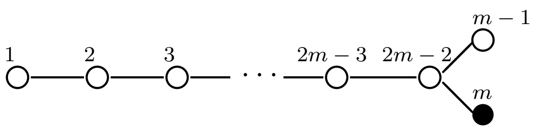
$Z_k =$ 

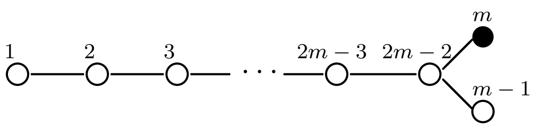
$(k = 0, 1, \cdots, m-1)$

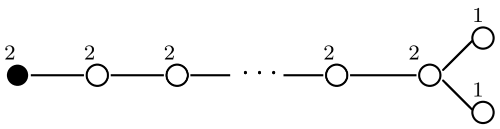
Ulrich ideals in RDP of type (D_n) $f = x^2y + y^{n-1} + z^2$ $(n = 2m \geq 4)$

$Z_k =$ 

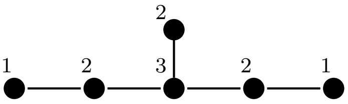
$(k = 0, 1, \ldots, m-2)$

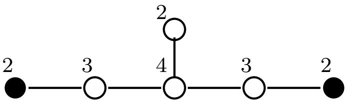
$Z_{m-1} =$ 

$Z_m =$ 

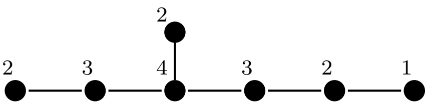
$Z_{m+1} =$ 

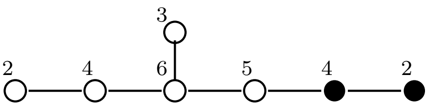
Ulrich ideals in RDP of type (E_6) $f = x^3 + y^4 + z^2$

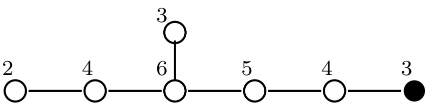
$Z_0 =$ 

$Z_1 =$ 

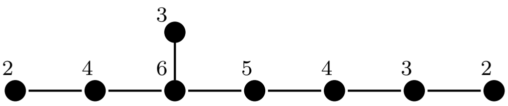
Ulrich ideals in RDP of type (E_7) $f = x^3 + xy^3 + z^2$

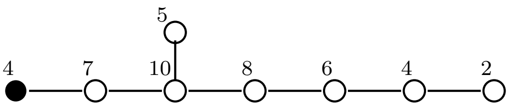
$Z_0 =$ 

$Z_1 =$ 

$Z_2 =$ 

Ulrich ideals in RDP of type (E_8) $x^3 + y^5 + z^2$

$Z_0 =$ 

$Z_1 =$ 

一方, イデアル論的な計算により具体的な Ulrich イデアルを見つけ, 個数を比較することにより, 次の定理のような Ulrich イデアルの完全リストを得る.

Theorem 22. 2次元有理二重点における *Ulrich* イデアルは次の表で与えられる：

(A_n) $\{(x,y,z),(x,y^2,z),\ldots,(x,y^m,z)\}$ *if* $n=2m$;
$\{(x,y,z),(x,y^2,z),\ldots,(x,y^{m+1},z)\}$ *if* $n=2m+1$.

(D_n) $\{(x,y,z),(x,y^2,z),\ldots,(x,y^{m-1},z),$
$(x+\sqrt{-1}y^{m-1},y^m,z),(x-\sqrt{-1}y^{m-1},y^m,z),(x^2,y,z)\}$
if $n=2m$;
$\{(x,y,z),(x,y^2,z),\ldots,(x,y^m,z),(x^2,y,z)\}$
if $n=2m+1$.

(E_6) $\{(x,y,z),(x,y^2,z)\}$.

(E_7) $\{(x,y,z),(x,y^2,z),(x,y^3,z)\}$.

(E_8) $\{(x,y,z),(x,y^2,z)\}$.

3. 有理二重点上の Special Cohen-Macaulay 加群

有理二重点において，「イデアル I に関する Ulrich Cohen-Macaulay 加群」は，実は，「M/IM が自由 A/I 加群である」ことに過ぎないことが分かる．

Theorem 23. A を2次元有理二重点とし，I をパラメーターイデアルでない，$\mathfrak{m}$ 準素イデアルとする．このとき，極大 *Cohen-Macaulay* 加群 M に関して，次の条件は同値である．

(1) M は I に関して，*Ulrich* 加群である．
(2) M は I に関して，*special* 加群である．
(3) M は自由加群を直和因子に持たず，M/IM は自由 A/I-加群である．

このとき，I は *Ulrich* イデアルで，$M^*=\mathrm{Hom}_A(M,A)$ も I に関する *Ulrich* 加群である．

4. non Gorenstein 有理特異点における Ulrich (special) イデアルの例

Example 24. G を $g=\begin{pmatrix}\varepsilon_7 & \\ & \varepsilon_7^3\end{pmatrix}$ で定義された巡回群として，$A=k[[s^7,s^4t,st^2,t^7]]=k[[x,y]]^G$ とおく．

このとき，極小特異点解消の双対グラフは，$E_1-E_2-E_3$ (ただし，$E_1^2=-3$, $E_2^2=E_3^2=-2$) の形をしている．

$M_a=(s^it^j\,|\,i+3j\equiv a \pmod 7)$ $(a=0,1,\ldots,6)$ とおくと，$\{M_a\}_{i=0}^6$ は A 上の直既約な極大 Cohen-Macaulay 加群の全体であり，M_a が special (resp. Ulrich) になるのは，$a=1,2,3$ (resp. $a=4,5,6$) の場合である．

一方，special cycles は $Z_0=E_1+E_2+E_3$ と $Z_1=E_1+2E_2+E_3$ である．$I_0=\mathfrak{m}$ に対する直既約な special 加群は M_1, M_2 と M_3 であるが，$I_1=H^0(X,\mathcal{O}_X(-Z_1))$ に対する直既約な special 加群は M_2 のみである．

一方，Ulrich イデアルは $\mathfrak{m}$ のみであることが分かるが，**「Ulrich 加群」は完全に決められていない.**

最後に, 未解決問題をあげておく.

(1) どんな局所環が Ulrich イデアルを持つか? (Ulrich イデアルを持つ Gorenstein 整域は完全交叉であるか?)
(2) 単純超曲面特異点上の Ulrich イデアルを分類せよ. (2 次元から来るもので決まるか?)
(3) 2 次元単純楕円特異点上の Ulrich イデアルを分類せよ.
(4) 2 次元 non Gorenstein 有理特異点上のあるイデアルに関する Ulrich 加群を完全に分類せよ.

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ON SEPARABLE POLYNOMIALS IN SKEW POLYNOMIAL RINGS

SATOSHI YAMANAKA

Abstract. Let B be a ring with identity element 1 of prime characteristic p, D a derivation of B, and $B[X; D]$ the skew polynomial ring in which the multiplication is given by $\alpha X = X\alpha + D(\alpha)$ for any $\alpha \in B$. We consider a condition for $X^p - Xa - b \in B[X; D]$ to be a Galois polynomial.

1. Introduction

This is based on a joint work with S. Ikehata [17].

In [12, 13, 14], T. Nagahara has studied separable and Galois polynomials of degree 2 in skew polynomial rings. He got several interesting results. The pourpose of this paper is to give a generalization of Nagahara's result for polynomials of degree 2 to general prime degree p.

Throughout this paper, B will mean a ring with identity element 1 and D a derivation of B, that is, D is an additive endomorphism of B such that $D(\alpha\beta) = D(\alpha)\beta + \alpha D(\beta)$ for any α, $\beta \in B$. We assume that B is of prime characteristic p. Let $B[X; D]$ be the skew polynomial ring in which the multiplication is given by $\alpha X = X\alpha + D(\alpha)$ ($\alpha \in B$).

A ring extension A/B is called separable if the A-A-homomorphism of $A \otimes_B A$ onto A defined by $a \otimes b \to ab$ splits, and A/B is called Hirata separable if $A \otimes_B A$ is A-A-isomorphic to a direct summand of a finite direct sum of copies of A. It is well known that a Hirata separable extension is a separable extension.

Let f be a monic polynomial in $B[X; D]$ such that $fB[X; D] = B[X; D]f$, then the residue ring $B[X; D]/fB[X; D]$ is a free ring extension of B. If $B[X; D]/fB[X; D]$ is a *separable* (resp. *Hirata separable*) extension of B, then f is called a *separable* (resp. *Hirata separable*) polynomial in $B[X; D]$. These provide typical and essential examples of separable and Hirata separable extensions. K. Kishimoto, T. Nagahara, Y. Miyashita, G. Szeto, L. Xue, and S. Ikehata studied extensively separable polynomials in skew polynomial rings. In [11], Y. Miyashita gave characterizations of separable and Hirata separable polynomials of general degree by the theory of (*)-positively filtered rings. He gave a method to study polynomials of general degree in skew polynomial rings. Then in [1, 2, 3, 4], Ikehata studied separable polynomials and Hirata separable plolynomials in skew polynomial rings by making use of Miyashita's method. Recently, the author and Ikehata gave an alternative proof of Miyashita's theorem in [16].

A ring extension A/B is called a G-Galois extension, provided that there exists a finite group of G of automorphisms of A such that $B = A^G$ (the fix ring of G in A) and $\sum_i x_i\sigma(y_i) = \delta_{1,\sigma}$ for some finite number of elements x_i, $y_i \in A$. We call $\{x_i, y_i\}$ a G-Galois coordinate system for A/B. It is well known that a G-Galois extension is a separable

The detailed version of this paper will be submitted for publication elsewhere.

extension. Let f be a monic polynomial in $B[X;D]$ such that $fB[X;D] = B[X;D]f$, then f is called a Galois polynomial in $B[X;D]$ if $B[X;D]/fB[X;D]$ is a G-Galois extension over B for some finite group G.

We shall use the following conventions:

$Z =$ the center of B.

$U(Z) =$ the set of all invertible elements in Z.

$B^D = \{\alpha \in B \mid D(\alpha) = 0\}$, $Z^D = \{\alpha \in Z \mid D(\alpha) = 0\}$.

$B[X;D]_{(0)} =$ the set of all monic polynomials g in $B[X;D]$ such that $gB[X;D] = B[X;D]g$.

2. Galois polynomials in $B[X;D]$

In [12, 13, 14], T. Nagahara has studied separable and Galois polynomials of degree 2 in skew polynomial rings. He proved the following

Proposition 1. ([12, Theorem 3.7]) *Assume* $2 = 0$, *and let* $f = X^2 - Xa - b$ *be in* $B[X;D]_{(0)}$. *Then* f *is a Galois polynomial in* $B[X;D]$ *if and only if there exists an element* s *in* $U(Z)$ *such that* $D(s) + as = 1$.

The purpose of this paper is to generalize the above result to the general prime degree.

We shall state some basic results which were already known. The following is easily verified by a direct computation.

Lemma 2. ([1, Corollary 1.7]) *Let* $f = X^p - Xa - b$ *be in* $B[X;D]$. *Then* f *is in* $B[X;D]_{(0)}$, *that is,* $fB[X;D] = B[X;D]f$, *if and only if*

(1) $a \in Z^D$, and $b \in B^D$.

(2) $D^p(\alpha) - D(\alpha)a = \alpha b - b\alpha \quad (\alpha \in B)$.

Concerning Galois polynomials, the following Kishimoto's result is fundamental.

Lemma 3. ([9, Theorem 1.1 and Corollary 1.7], [6, Lemma 2.3]) *Let* $f = X^p - X - b$ *be in* $B[X;D]_{(0)}$. *Then* f *is a Galois polynomial over* B.

Proof. For convenience, we outline the proof. Let $A = B[X;D]/fB[X;D]$ and $x = X + fB[X;D]$. The mapping $\sigma : A \to A$ defined by $\sigma(\sum_i x^i d_i) = \sum_i (x+1)^i d_i$ is a B-automorphism of A of order p. Let $G = <\sigma>$. It is easy to see that $A^G = B$. We put here

$$a_j = j^{-1}\sigma^j(x) \text{ and } b_j = (-j^{-1})x \quad (1 \le j \le p-1).$$

Then the expansions of

$$\Pi_{j=1}^{p-1}(a_j + b_j) = 1 \text{ and } \Pi_{j=1}^{p-1}(a_j + \sigma^k(b_j)) = 0 \quad (1 \le j \le p-1)$$

enable us to see the existence of a G-Galois coordinate system for A/B. Thus, A is a G-Galois extension over B.

In general, if we consider a polynomial $f = X^p - Xa - b \in B[X;D]_{(0)}$ and $a \neq 1$, it is not easy to check whether f is a Galois polynomial or not.

Now, we shall generalize Nagahara's theorem to general prime degree p case. In what follows we fix the following.

Let $f = X^p - Xa - b$ be in $B[X;D]_{(0)}$. We put here $A = B[X;D]/fB[X;D]$ and $x = X + fB[X;D]$.

First we shall state the following lemma.

Lemma 4.

$$D^{p-1}(s^{p-1}) = -s^{-1}(sD)^{p-1}(s) \textit{ for any element } s \textit{ in } U(Z).$$

Then we can prove the following theorem which is a generalization of Nagahara's theorem ([12, Theorem 3.7]).

Theorem 5. *Let $f = X^p - Xa - b$ be in $B[X;D]_{(0)}$. If f is a Galois polynomial in $B[X;D]$ with a Galois group $G =< \sigma_s >$ of order p, where $\sigma_s(x) = x + s^{-1}$ with an element $s \in U(Z)$, then $s^{-1}(sD)^{p-1}(s) + s^{p-1}a = 1$. Conversely, if there exists an element $s \in U(Z)$ such that $s^{-1}(sD)^{p-1}(s) + s^{p-1}a = 1$ then f is a Galois polynomial in $B[X;D]$ with a Galois group $G =< \sigma_s >$ of order p, where $\sigma_s(x) = x + s^{-1}$.*

The proofs of Lemma 4 and Theorem 5 are written in the detailed version of this paper which will be submitted for publication elsewhere. Theorem 5 is proved by making use of the following two formulas : For any $s \in Z$,

$$(X + s)^p = X^p + s^p + D^{p-1}(s), \quad \text{and}$$

$$(sD)^p = s^pD^p + (sD)^{p-1}(s)D \quad \text{(the Hochschild's formula)}.$$

Remark. In [12], T. Nagahara proved that if $f = X^2 - Xa - b$ is a Galois polynomial in $B[X;D]$, then necessarily the order of the Galois group is 2. However, in general case we do not prove yet that if $f = X^p - Xa - b$ is a Galois polynomial in $B[X;D]$, then the order of the Galois group is p.

In virtue of Lemma 4, we obtain the following corollary as a direct consequence of Theorem 5.

Corollary 6. *Let $f = X^p - Xa - b$ be in $B[X;D]_{(0)}$. If there exists an element $y \in Z$ such that $D^{p-1}(y) - ya = 1$ and $y = -s^{p-1}$ for some element $s \in U(Z)$, then f is a Galois polynomial in $B[X;D]$ with a Galois group $G =< \sigma_s >$ of oreder p, where $\sigma_s(x) = x + s^{-1}$. Conversely, if f is a Galois polynomial in $B[X;D]$ with a Galois group $G =< \sigma_s >$ of order p, where $\sigma_s(x) = x + s^{-1}$ with an element $s \in U(Z)$, then $D^{p-1}(y) - ya = 1$ and $y = -s^{p-1}$.*

Corollary 7. *Let $f = X^p - Xa - b$ be in $B[X;D]_{(0)}$. If there exists an invertible element $u \in Z^D$ such that $u^{p-1} = a$, then f is a Galois polynomial in $B[X;D]$ with a Galois group $G =< \sigma_{u^{-1}} >$, where $\sigma_{u^{-1}}(x) = x + u$.*

Finally we shall state the following theorem which is proved in [17].

Theorem 8. *Let $f = X^p - Xa - b$ be in $B[X;D]_{(0)}$. If there exists an element $z \in Z$ such that $D(z)$ is invertible in Z, then f is a Hirata separable polynomial in $B[X;D]$. In addition, if z is an invertible element in Z, then, f is a Galois polynomial in $B[X;D]$ with a Galois group $G =< \tau >$, where $\tau(x) = x + D(z)z^{-1}$.*

Lastly, as a direct consequence of Theorem 8, we obtain the following

Corollary 9. *If B is a simple ring and $D|Z \neq 0$, then $f = X^p - Xa - b$ in $B[X;D]_{(0)}$ is always a Hirata separable and Galois polynomial in $B[X;D]$.*

Corollary 10. *If B is a field and $D \neq 0$, then $f = X^p - Xa - b$ in $B[X;D]_{(0)}$ is always a Hirata separable and Galois polynomial in $B[X;D]$.*

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