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Organizing Committee of The Symposium on Ring Theory and Representation Theory

The Symposium on Ring Theory and Representation Theory has been held annually in Japan and the Proceedings have been published by the organizing committee. The first Symposium was organized in 1968 by H.Tominaga, H.Tachikawa, M.Harada and S.Endo. After their retirement, a new committee was organized in 1997 for managing the Symposium and committee members are listed in the web page

http://fuji.cec.yamanashi.ac.jp/˜ring/h-of-ringsymp.html.

The present members of the committee are H.Asashiba (Shizuoka Univ.), S.Ikehata (Okayama Univ.), S.Kawata (Osaka City Univ.), M.Sato (Yamanashi Univ.) and K.Yamagata (Tokyo Univ. of Agriculture and Technology).

The Proceedings of each Symposium is edited by program organizer. Anyone who wants these Proceedings should ask to the program organizer of each Symposium or one of the committee members.

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Concerning several information on ring theory and representation theory of group and algebras containing schedules of meetings and symposiums as well as ring mailing list service for registered members, you should refer to the following ring homepage, which is arranged by M.Sato (Yamanashi Univ.):

http://fuji.cec.yamanashi.ac.jp/˜ring/ (in Japanese)
(Mirror site: www.cec.yamanashi.ac.jp/˜ring/)

http://fuji.cec.yamanashi.ac.jp/˜ring/japan/ (in English)

Masahisa Sato
Yamanashi Japan
December, 2011

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Preface

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Masahisa Sato
Yamanashi
December, 2011

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10:50–11:30

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Room 24 Kazuho Ozeki (Meiji University)
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The Example by Stephens

TILTING MODULES ARISING FROM TWO-TERM TILTING COMPLEXES

HIROKI ABE

ABSTRACT. We see that every two-term tilting complex over an Artin algebra has a tilting module over a certain factor algebra as a homology group. Also, we determine the endomorphism algebra of such a homology group, which is given as a certain factor algebra of the endomorphism algebra of the two-term tilting complex. Thus, every derived equivalence between Artin algebras given by a two-term tilting complex induces a derived equivalence between the corresponding factor algebras.

Let A be an Artin algebra. We denote by mod-A the category of finitely generated right A-modules and by $\mathcal{P}_A$ the full subcategory of mod-A consisting of projective modules.

Definition 1. A pair $(\mathcal{T}, \mathcal{F})$ of full subcategories $\mathcal{T}$, $\mathcal{F}$ in mod-A is said to be *a torsion theory* for mod-A if the following conditions are satisfied:

(1) $\mathcal{T} \cap \mathcal{F} = \{0\}$;
(2) $\mathcal{T}$ is closed under factor modules;
(3) $\mathcal{F}$ is closed under submodules; and
(4) for any $X \in$ mod-A, there exists an exact sequence $0 \to X' \to X \to X'' \to 0$ with $X' \in \mathcal{T}$ and $X'' \in \mathcal{F}$.

If $\mathcal{T}$ is stable under the Nakayama functor ν, then $(\mathcal{T}, \mathcal{F})$ is said to be *a stable torsion theory* for mod-A.

Let $T^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ be a two-term complex:

$$T^\bullet : \cdots \to 0 \to T^{-1} \xrightarrow{\alpha} T^0 \to 0 \to \cdots ,$$

and set the following subcategories in mod-A:

$\mathcal{T}(T^\bullet) = \mathrm{Ker}\ \mathrm{Hom}_{\mathcal{K}(A)}(T^\bullet[-1], -) \cap \text{mod-}A,$
$\mathcal{F}(T^\bullet) = \mathrm{Ker}\ \mathrm{Hom}_{\mathcal{K}(A)}(T^\bullet, -) \cap \text{mod-}A.$

Proposition 2 ([1, Propositions 5.5 and 5.7])**.** *The following are equivalent.*

(1) $T^\bullet$ *is a tilting complex.*
(2) $(\mathcal{T}(T^\bullet), \mathcal{F}(T^\bullet))$ *is a stable torsion theory for* mod-A.

Furthermore, if these equivalent conditions hold, then the following hold.

(1) $\mathcal{T}(T^\bullet) = \mathrm{gen}(\mathrm{H}^0(T^\bullet))$*, the generated class by* $\mathrm{H}^0(T^\bullet)$*, and* $\mathrm{H}^0(T^\bullet)$ *is Ext-projective in* $\mathcal{T}(T^\bullet)$.
(2) $\mathcal{F}(T^\bullet) = \mathrm{cog}(\mathrm{H}^{-1}(\nu T^\bullet))$*, the cogenerated class by* $\mathrm{H}^{-1}(\nu T^\bullet)$ *and* $\mathrm{H}^{-1}(\nu T^\bullet)$ *is Ext-injective in* $\mathcal{F}(T^\bullet)$.

The detailed version of this note has been submitted for publication elsewhere.

Conversely, let $(\mathcal{T}, \mathcal{F})$ be a stable torsion theory for mod-A.

Proposition 3 ([1, Theorem 5.8]). *Assume that there exist $X \in \mathcal{T}$ and $Y \in \mathcal{F}$ satisfying the following conditions:*

(1) *$\mathcal{T} = \mathrm{gen}(X)$ and X is Ext-projective in $\mathcal{T}$; and*
(2) *$\mathcal{F} = \mathrm{cog}(Y)$ and Y is Ext-injective in $\mathcal{F}$.*

Let $P_X^\bullet$ be a minimal projective presentation of X and $I_Y^\bullet$ be a minimal injective presentation of Y, and set $T_{X,Y}^\bullet = P_X^\bullet \oplus \nu^{-1} I_Y^\bullet[1]$. Then $T_{X,Y}^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ is a tilting complex such that $\mathcal{T} = \mathcal{T}(T_{X,Y}^\bullet)$ and $\mathcal{F} = \mathcal{F}(T_{X,Y}^\bullet)$.

Let $T^\bullet$ be a two-term tilting complex. We set $\mathfrak{a} = \mathrm{ann}_A(\mathrm{H}^0(T^\bullet))$, the annihilator of $\mathrm{H}^0(T^\bullet)$. Note that $\mathrm{H}^0(T^\bullet)$ is faithful in mod-$A/\mathfrak{a}$ and the canonical full embedding mod-$A/\mathfrak{a} \hookrightarrow$ mod-A induces $\mathrm{gen}(\mathrm{H}^0(T^\bullet)_{A/\mathfrak{a}}) = \mathrm{gen}(\mathrm{H}^0(T^\bullet)_A)$ which is closed under extensions. Thus, the next lemma follows from Proposition 2.

Lemma 4. *The following hold.*

(1) *$\mathrm{proj\ dim\ H}^0(T^\bullet)_{A/\mathfrak{a}} \leq 1$.*
(2) *$\mathrm{Ext}^1_{A/\mathfrak{a}}(\mathrm{H}^0(T^\bullet), \mathrm{H}^0(T^\bullet)) = 0$.*
(3) *There exists an exact sequence $0 \to A/\mathfrak{a} \to X^0 \to X^1 \to 0$ in mod-$A/\mathfrak{a}$ such that $X^0 \in \mathrm{add}(\mathrm{H}^0(T^\bullet)_{A/\mathfrak{a}})$ and $X^1 \in \mathrm{gen}(\mathrm{H}^0(T^\bullet)_{A/\mathfrak{a}})$ which is Ext-projective in $\mathrm{gen}(\mathrm{H}^0(T^\bullet)_{A/\mathfrak{a}})$.*

We set $\mathfrak{a}' = \mathrm{ann}_A(\mathrm{H}^{-1}(\nu T^\bullet))$, the annihilator of $\mathrm{H}^{-1}(\nu T^\bullet)$. The next lemma follows by the dual arguments of Lemma 4

Lemma 5. *The following hold.*

(1) *$\mathrm{inj\ dim\ H}^{-1}(\nu T^\bullet)_{A/\mathfrak{a}'} \leq 1$.*
(2) *$\mathrm{Ext}^1_{A/\mathfrak{a}'}(\mathrm{H}^{-1}(\nu T^\bullet), \mathrm{H}^{-1}(\nu T^\bullet)) = 0$.*
(3) *There exists an exact sequence $0 \to Y^1 \to Y^0 \to A/\mathfrak{a}' \to 0$ in mod-$A/\mathfrak{a}'$ such that $Y^0 \in \mathrm{add}(\mathrm{H}^{-1}(\nu T^\bullet)_{A/\mathfrak{a}'})$ and $Y^1 \in \mathrm{cog}(\mathrm{H}^{-1}(\nu T^\bullet)_{A/\mathfrak{a}'})$ which is Ext-injective in $\mathrm{cog}(\mathrm{H}^{-1}(\nu T^\bullet)_{A/\mathfrak{a}'})$.*

Let X be the direct sum of all indecomposable non-projective Ext-projective modules in $\mathrm{gen}(\mathrm{H}^0(T^\bullet))$ which are not contained in $\mathrm{add}(\mathrm{H}^0(T^\bullet))$. Then $\mathrm{add}(\mathrm{H}^0(T^\bullet) \oplus X)$ coincides with the class of all Ext-projective modules in $\mathrm{gen}(\mathrm{H}^0(T^\bullet))$. Also, since $\mathrm{gen}(\mathrm{H}^0(T^\bullet)) = \mathrm{gen}(\mathrm{H}^0(T^\bullet) \oplus X)$, the pair $(\mathrm{gen}(\mathrm{H}^0(T^\bullet) \oplus X), \mathrm{cog}(\mathrm{H}^{-1}(\nu T^\bullet))$ is a stable torsion theory in mod-A. Let $P^\bullet$ be the minimal projective presentation of $\mathrm{H}^0(T^\bullet) \oplus X$ and $I^\bullet$ be the minimal injective presentation of $\mathrm{H}^{-1}(\nu T^\bullet)$, and set $U^\bullet = P^\bullet \oplus \nu^{-1} I^\bullet[1]$. Then $U^\bullet$ is a tilting complex such that $\mathcal{T}(U^\bullet) = \mathrm{gen}(\mathrm{H}^0(T^\bullet) \oplus X)$ and $\mathcal{F}(U^\bullet) = \mathrm{cog}(\mathrm{H}^{-1}(\nu T^\bullet))$ by Proposition 3. Note that the stable torsion theory induced by $U^\bullet$ coincides with that of $T^\bullet$. From this fact, we can prove that $\mathrm{add}(\mathrm{H}^0(U^\bullet)) = \mathrm{add}(\mathrm{H}^0(T^\bullet))$. Since there exist the inclusions $\mathrm{add}(\mathrm{H}^0(T^\bullet)) \subset \mathrm{add}(\mathrm{H}^0(T^\bullet) \oplus X) \subset \mathrm{add}(\mathrm{H}^0(U^\bullet))$, we conclude that $\mathrm{add}(\mathrm{H}^0(T^\bullet)) = \mathrm{add}(\mathrm{H}^0(T^\bullet) \oplus X)$. Thus, we have the next lemma.

Lemma 6. *For any $M, N \in$ mod-A, the following hold.*

(1) *$M \in \mathrm{add}(\mathrm{H}^0(T^\bullet))$ if and only if M is* Ext*-projective in* $\mathrm{gen}(\mathrm{H}^0(T^\bullet))$.
(2) *$N \in \mathrm{add}(\mathrm{H}^{-1}(\nu T^\bullet))$ if and only if N is* Ext*-injective in* $\mathrm{cog}(\mathrm{H}^{-1}(\nu T^\bullet))$.

The next theorem is a direct consequence of the previous three lemmas.

Theorem 7. *The following hold.*

(1) $\mathrm{H}^0(T^\bullet)$ *is a tilting module in* $\text{mod-}A/\mathfrak{a}$.
(2) $\mathrm{H}^{-1}(\nu T^\bullet)$ *is a cotilting module in* $\text{mod-}A/\mathfrak{a}'$*, i.e.,* $D(\mathrm{H}^{-1}(\nu T^\bullet))$ *is a tilting module in* $\text{mod-}(A/\mathfrak{a}')^{\mathrm{op}}$.

We determine the endomorphism algebras of $\mathrm{H}^0(T^\bullet)$. Set $B = \mathrm{End}_{\mathcal{K}(A)}(T^\bullet)$. Since there exists a surjective algebra homomorphism

$$\theta : B \to \mathrm{End}_{A/\mathfrak{a}}(\mathrm{H}^0(T^\bullet)),$$

which is induced by the functor $\mathrm{H}^0(-)$, we have an algebra isomorphism

$$\mathrm{End}_{A/\mathfrak{a}}(\mathrm{H}^0(T^\bullet)) \cong B/\mathrm{Ker}\ \theta.$$

Also, we can prove that $\mathrm{Ker}\ \theta = \mathrm{ann}_B(\mathrm{Hom}_{\mathcal{K}(A)}(A, T^\bullet)) = \mathrm{ann}_B(\mathrm{H}^0(T^\bullet))$. Thus, we have the next theorem.

Theorem 8. *We have the following algebra isomorphisms.*

(1) $\mathrm{End}_{A/\mathfrak{a}}(\mathrm{H}^0(T^\bullet)) \cong B/\mathfrak{b}$*, where* $\mathfrak{b} = \mathrm{ann}_B(\mathrm{H}^0(T^\bullet))$.
(2) $\mathrm{End}_{A/\mathfrak{a}'}(\mathrm{H}^{-1}(\nu T^\bullet)) \cong B/\mathfrak{b}'$*, where* $\mathfrak{b}' = \mathrm{ann}_B(\mathrm{H}^{-1}(\nu T^\bullet))$.

As the final of this note, we demonstrate our results through an example.

Example 9. Let A be the path algebra defined by the quiver

$$\begin{tikzcd}
 & 2 \arrow[dr, "\gamma"] & \\
1 \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & 4 \\
 & 3 \arrow[ur, "\delta"'] &
\end{tikzcd}$$

with relations $\alpha\gamma = \beta\delta = 0$. We denote by e_i the empty path corresponding to the vertex $i = 1, \cdots, 4$. The Auslander–Reiten quiver of A is given by the following:

$$\begin{tikzcd}[column sep=small]
 & \begin{smallmatrix}2\\4\end{smallmatrix} \arrow[dr] \arrow[rr, dotted, no head] & & 3 \arrow[dr] \arrow[rr, dotted, no head] & & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[dr] & \\
4 \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & \begin{smallmatrix}2&&3\\&4&\end{smallmatrix} \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & \begin{smallmatrix}&1&\\2&&3\end{smallmatrix} \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & 1 \\
 & \begin{smallmatrix}3\\4\end{smallmatrix} \arrow[ur] \arrow[rr, dotted, no head] & & 2 \arrow[ur] \arrow[rr, dotted, no head] & & \begin{smallmatrix}1\\3\end{smallmatrix} \arrow[ur] &
\end{tikzcd}$$

where each indecomposable module is represented by its composition factors. It is not difficult to see that the following pair gives a stable torsion theory for mod-A:

$$\mathcal{T} = \left\{ \begin{smallmatrix}&1&\\2&&3\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix}, \begin{smallmatrix}1\\3\end{smallmatrix}, 1 \right\} \text{ and } \mathcal{F} = \left\{ 4, \begin{smallmatrix}2\\4\end{smallmatrix}, \begin{smallmatrix}3\\4\end{smallmatrix}, \begin{smallmatrix}2&&3\\&4&\end{smallmatrix}, 3, 2 \right\},$$

where $\mathcal{T}$ is a torsion class and $\mathcal{F}$ is a torsion-free class. We set

$$X = \begin{smallmatrix}&1&\\2&&3\end{smallmatrix}, \quad Y = \begin{smallmatrix}2&&3\\&4&\end{smallmatrix} \oplus 3 \oplus 2.$$

Then $\mathcal{T} = \mathrm{gen}(X)$ and X is Ext-projective in $\mathcal{T}$, and $\mathcal{F} = \mathrm{cog}(Y)$ and Y is Ext-injective in $\mathcal{F}$. According to Proposition 3, we have a two-term tilting complex $T^\bullet = T_1^\bullet \oplus T_2^\bullet \oplus T_3^\bullet \oplus T_4^\bullet$, where

$$T_1^\bullet = 0 \to \begin{smallmatrix} & 1 & \\ 2 & & 3 \end{smallmatrix}, \quad T_2^\bullet = \begin{smallmatrix} 2 \\ 4 \end{smallmatrix} \to \begin{smallmatrix} & 1 & \\ 2 & & 3 \end{smallmatrix}, \quad T_3^\bullet = \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \to \begin{smallmatrix} & 1 & \\ 2 & & 3 \end{smallmatrix}, \quad T_4^\bullet = \begin{smallmatrix} 4 \end{smallmatrix} \to 0.$$

Thus, we have

$$\mathrm{H}^0(T^\bullet) = \begin{smallmatrix} & 1 & \\ 2 & & 3 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$$

as a right A-module. Since $\mathfrak{a} = \mathrm{ann}_A(\mathrm{H}^0(T^\bullet))$ is a two-sided ideal generated by e_4, γ, δ, the factor algebra $A/\mathfrak{a}$ is defined by the quiver

$$\begin{tikzcd} & 2 \\ 1 \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & \\ & 3 \end{tikzcd}$$

without relations. Next, it is not difficult to see that $B = \mathrm{End}_{\mathcal{K}(A)}(T^\bullet)$ is defined by the quiver

$$\begin{tikzcd} & 2 \arrow[dl, "\lambda"'] \arrow[dr, "\nu"] & \\ 1 & & 4 \\ & 3 \arrow[ul, "\mu"] \arrow[ur, "\xi"'] & \end{tikzcd}$$

without relations. Then we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{K}(A)}(A, T^\bullet) &= \bigoplus_{i=1}^{4} \mathrm{Hom}_{\mathcal{K}(A)}(e_i A, T^\bullet) \\ &= \begin{smallmatrix} & 1 & \\ 2 & & 3 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus 0 \end{aligned}$$

as a left B-module. Thus, $\mathfrak{b} = \mathrm{ann}_B(\mathrm{Hom}_{\mathcal{K}(A)}(A, T^\bullet))$ is a two-sided ideal generated by ν, ξ and the empty path corresponding to the vertex 4. Therefore, the factor algebra $B/\mathfrak{b}$ is defined by the quiver

$$\begin{tikzcd} & 2 \arrow[dl, "\lambda"'] \\ 1 & \\ & 3 \arrow[ul, "\mu"] \end{tikzcd}$$

without relations. It follows by Theorems 7 and 8 that $A/\mathfrak{a}$ and $B/\mathfrak{b}$ are derived equivalent to each other.

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DIMENSIONS OF DERIVED CATEGORIES

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Abstract. Several years ago, Bondal, Rouquier and Van den Bergh introduced the notion of the dimension of a triangulated category, and Rouquier proved that the bounded derived category of coherent sheaves on a separated scheme of finite type over a perfect field has finite dimension. In this paper, we study the dimension of the bounded derived category of finitely generated modules over a commutative Noetherian ring. The main result of this paper asserts that it is finite over a complete local ring containing a field with perfect residue field.

1. Introduction

The notion of the dimension of a triangulated category has been introduced by Bondal, Rouquier and Van den Bergh [4, 14]. Roughly speaking, it measures how quickly the category can be built from a single object. The dimensions of the bounded derived category of finitely generated modules over a Noetherian ring and that of coherent sheaves on a Noetherian scheme are called the *derived dimensions* of the ring and the scheme, while the dimension of the singularity category (in the sense of Orlov [12]; the same as the stable derived category in the sense of Buchweitz [5]) is called the *stable dimension.* These dimensions have been in the spotlight in the studies of the dimensions of triangulated categories.

The importance of the notion of derived dimension was first recognized by Bondal and Van den Bergh [4] in relation to representability of functors. They proved that smooth proper commutative/non-commutative varieties have finite derived dimension, which yields that every contravariant cohomological functor of finite type to vector spaces is representable.

As to upper bounds, the derived dimension of a ring is at most its Loewy length [14]. In particular, Artinian rings have finite derived dimension. Christensen, Krause and Kussin [6, 9] showed that the derived dimension is bounded above by the global dimension, whence rings of finite global dimension are of finite derived dimension. In relation to a conjecture of Orlov [13], a series of studies by Ballard, Favero and Katzarkov [1, 2, 3] gave in several cases upper bounds for derived and stable dimensions of schemes. For instance, they obtained an upper bound of the stable dimension of an isolated hypersurface singularity by using the Loewy length of the Tjurina algebra. On the other hand, there are a lot of triangulated categories having infinite dimension. The dimension of the derived category of perfect complexes over a ring (respectively, a quasi-projective scheme) is infinite unless

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the ring has finite global dimension (respectively, the scheme is regular) [14]. It has turned out by work of Oppermann and Šťovíček [11] that over a Noetherian algebra (respectively, a projective scheme) all proper thick subcategories of the bounded derived category of finitely generated modules (respectively, coherent sheaves) containing perfect complexes have infinite dimension. However, these do not apply for the finiteness of the derived dimension of a non-regular Noetherian ring of positive Krull dimension.

As a main result of the paper [14], Rouquier gave the following theorem.

Theorem 1 (Rouquier)**.** *Let X be a separated scheme of finite type over a perfect field. Then the bounded derived category of coherent sheaves on X has finite dimension.*

Applying this theorem to an affine scheme, one obtains:

Corollary 2. *Let R be a commutative ring which is essentially of finite type over a perfect field k. Then the bounded derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R)$ of finitely generated R-modules has finite dimension, and so does the singularity category $\mathsf{D}_{\mathsf{Sg}}(R)$ of R.*

The main purpose of this paper is to study the dimension and generators of the bounded derived category of finitely generated modules over a commutative Noetherian ring. We will give lower bounds of the dimensions over general rings under some mild assumptions, and over some special rings we will also give upper bounds and explicit generators. The main result of this paper is the following theorem. (See Definition 5 for the notation.)

Main Theorem. *Let R be either a complete local ring containing a field with perfect residue field or a ring that is essentially of finite type over a perfect field. Then there exist a finite number of prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ of R and an integer $m \geq 1$ such that*

$$\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R) = \langle R/\mathfrak{p}_1 \oplus \cdots \oplus R/\mathfrak{p}_n \rangle_m.$$

In particular, $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R)$ and $\mathsf{D}_{\mathsf{Sg}}(R)$ have finite dimension.

In Rouquier's result stated above, the essential role is played, in the affine case, by the Noetherian property of the enveloping algebra $R \otimes_k R$. The result does not apply to a complete local ring, since it is in general far from being (essentially) of finite type and therefore the enveloping algebra is non-Noetherian. Our methods not only show finiteness of dimensions over a complete local ring but also give a ring-theoretic proof of Corollary 2.

2. Preliminaries

This section is devoted to stating our convention, giving some basic notation and recalling the definition of the dimension of a triangulated category.

We assume the following throughout this paper.

Convention 3. (1) All subcategories are full and closed under isomorphisms.
(2) All rings are associative and with identities.
(3) A Noetherian ring, an Artinian ring and a module mean a right Noetherian ring, a right Artinian ring and a right module, respectively.
(4) All complexes are cochain complexes.

We use the following notation.

Notation 4. (1) Let $\mathcal{A}$ be an abelian category.

(a) For a subcategory $\mathcal{X}$ of $\mathcal{A}$, the smallest subcategory of $\mathcal{A}$ containing $\mathcal{X}$ which is closed under finite direct sums and direct summands is denoted by $\mathsf{add}_{\mathcal{A}}\,\mathcal{X}$.

(b) We denote by $\mathsf{C}(\mathcal{A})$ the category of complexes of objects of $\mathcal{A}$. The derived category of $\mathcal{A}$ is denoted by $\mathsf{D}(\mathcal{A})$. The left bounded, the right bounded and the bounded derived categories of $\mathcal{A}$ are denoted by $\mathsf{D}^+(\mathcal{A}), \mathsf{D}^-(\mathcal{A})$ and $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$, respectively. We set $\mathsf{D}^{\varnothing}(\mathcal{A}) = \mathsf{D}(\mathcal{A})$, and often write $\mathsf{D}^{\star}(\mathcal{A})$ with $\star \in \{\varnothing, +, -, \mathrm{b}\}$ to mean $\mathsf{D}^{\varnothing}(\mathcal{A})$, $\mathsf{D}^+(\mathcal{A})$, $\mathsf{D}^-(\mathcal{A})$ and $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$.

(2) Let R be a ring. We denote by $\mathsf{Mod}\,R$ and $\mathsf{mod}\,R$ the category of R-modules and the category of finitely generated R-modules, respectively. For a subcategory $\mathcal{X}$ of $\mathsf{mod}\,R$ (when R is Noetherian), we put $\mathsf{add}_R\,\mathcal{X} = \mathsf{add}_{\mathsf{mod}\,R}\,\mathcal{X}$.

The concept of the dimension of a triangulated category has been introduced by Rouquier [14]. Now we recall its definition.

Definition 5. Let $\mathcal{T}$ be a triangulated category.

(1) A triangulated subcategory of $\mathcal{T}$ is called *thick* if it is closed under direct summands.

(2) Let $\mathcal{X}, \mathcal{Y}$ be two subcategories of $\mathcal{T}$. We denote by $\mathcal{X} * \mathcal{Y}$ the subcategory of $\mathcal{T}$ consisting of all objects M that admit exact triangles

$$X \to M \to Y \to \Sigma X$$

with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. We denote by $\langle \mathcal{X} \rangle$ the smallest subcategory of $\mathcal{T}$ containing $\mathcal{X}$ which is closed under finite direct sums, direct summands and shifts. For a non-negative integer n, we define the subcategory $\langle \mathcal{X} \rangle_n$ of $\mathcal{T}$ by

$$\langle \mathcal{X} \rangle_n = \begin{cases} \{0\} & (n = 0), \\ \langle \mathcal{X} \rangle & (n = 1), \\ \langle \langle \mathcal{X} \rangle * \langle \mathcal{X} \rangle_{n-1} \rangle & (2 \leq n < \infty). \end{cases}$$

Put $\langle \mathcal{X} \rangle_\infty = \bigcup_{n \geq 0} \langle \mathcal{X} \rangle_n$. When the ground category $\mathcal{T}$ should be specified, we write $\langle \mathcal{X} \rangle_n^{\mathcal{T}}$ instead of $\langle \mathcal{X} \rangle_n$. For a ring R and a subcategory $\mathcal{X}$ of $\mathsf{D}(\mathsf{Mod}\,R)$, we put $\langle \mathcal{X} \rangle_n^R = \langle \mathcal{X} \rangle_n^{\mathsf{D}(\mathsf{Mod}\,R)}$.

(3) The *dimension* of $\mathcal{T}$, denoted by $\dim \mathcal{T}$, is the infimum of the integers d such that there exists an object $M \in \mathcal{T}$ with $\langle M \rangle_{d+1} = \mathcal{T}$.

3. Upper bounds

The aim of this section is to find explicit generators and upper bounds of dimensions for derived categories in several cases.

We observe that the dimensions of the bounded derived categories of finitely generated modules over quotient singularities are at most their (Krull) dimensions, particularly that they are finite.

Proposition 6. *Let S be either the polynomial ring $k[x_1, \dots, x_n]$ or the formal power series ring $k[[x_1, \dots, x_n]]$ over a field k. Let G be a finite subgroup of the general linear* *group* $\mathrm{GL}_n(k)$*, and assume that the characteristic of* k *does not divide the order of* G*. Let* $R = S^G$ *be the invariant subring. Then* $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R) = \langle S \rangle_{n+1}$ *holds, and hence one has*

$$\dim \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R) \leq n = \dim R < \infty.$$

For a commutative ring R, we denote the set of minimal prime ideals of R by $\operatorname{Min} R$. As is well-known, $\operatorname{Min} R$ is a finite set whenever R is Noetherian. Also, we denote by $\lambda(R)$ the infimum of the integers $n \geq 0$ such that there is a filtration

$$0 = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n = R$$

of ideals of R with $I_i/I_{i-1} \cong R/\mathfrak{p}_i$ for $1 \leq i \leq n$, where $\mathfrak{p}_i \in \operatorname{Spec} R$. If R is Noetherian, then such a filtration exists and $\lambda(R)$ is a non-negative integer.

Proposition 7. *Let* R *be a Noetherian commutative ring.*

(1) *Suppose that for every* $\mathfrak{p} \in \operatorname{Min} R$ *there exist an* $R/\mathfrak{p}$*-complex* $G(\mathfrak{p})$ *and an integer* $n(\mathfrak{p}) \geq 0$ *such that* $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R/\mathfrak{p}) = \langle G(\mathfrak{p}) \rangle_{n(\mathfrak{p})}$*. Then* $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R) = \langle G \rangle_n$ *holds, where* $G = \bigoplus_{\mathfrak{p} \in \operatorname{Min} R} G(\mathfrak{p})$ *and* $n = \lambda(R) \cdot \max\{\, n(\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Min} R \,\}$*.*

(2) *There is an inequality*

$$\dim \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R) \leq \lambda(R) \cdot \sup\{\, \dim \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R/\mathfrak{p}) + 1 \mid \mathfrak{p} \in \operatorname{Min} R \,\} - 1.$$

Let R be a commutative Noetherian ring. We set

$$\ell\ell(R) = \inf\{\, n \geq 0 \mid (\operatorname{rad} R)^n = 0 \,\},$$
$$r(R) = \min\{\, n \geq 0 \mid (\operatorname{nil} R)^n = 0 \,\},$$

where $\operatorname{rad} R$ and $\operatorname{nil} R$ denote the Jacobson radical and the nilradical of R, respectively. The first number is called the *Loewy length* of R and is finite if (and only if) R is Artinian, while the second one is always finite. Let $R_{\mathrm{red}} = R/\operatorname{nil} R$ be the associated reduced ring. When R is reduced, we denote by $\overline{R}$ the integral closure of R in the total quotient ring Q of R. Let C_R denote the *conductor* of R, i.e., C_R is the set of elements $x \in Q$ with $x\overline{R} \subseteq R$. We can give an explicit generator and an upper bound of the dimension of the bounded derived category of finitely generated modules over a one-dimensional complete local ring.

Proposition 8. *Let* R *be a Noetherian commutative complete local ring of Krull dimension one with residue field* k*. Then it holds that* $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R) = \langle \overline{R_{\mathrm{red}}} \oplus k \rangle_{r(R)\cdot(2\,\ell\ell(R_{\mathrm{red}}/C_{R_{\mathrm{red}}})+2)}$*. In particular,*

$$\dim \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R) \leq r(R) \cdot (2\,\ell\ell(R_{\mathrm{red}}/C_{R_{\mathrm{red}}}) + 2) - 1 < \infty.$$

Let R be a commutative Noetherian local ring of Krull dimension d with maximal ideal $\mathfrak{m}$. We denote by $e(R)$ the multiplicity of R, that is, $e(R) = \lim_{n\to\infty} \frac{d!}{n^d} \ell_R(R/\mathfrak{m}^{n+1})$. Recall that a *numerical semigroup* is defined as a subsemigroup H of the additive semigroup $\mathbb{N} = \{0, 1, 2, \dots\}$ containing 0 such that $\mathbb{N} \backslash H$ is a finite set. For a numerical semigroup H, let $c(H)$ denote the *conductor* of H, that is,

$$c(H) = \max\{\, i \in \mathbb{N} \mid i - 1 \notin H \,\}.$$

For a real number α, put $\lceil \alpha \rceil = \min\{\, n \in \mathbb{Z} \mid n \geq \alpha \,\}$. Making use of the above proposition, one can get an upper bound of the dimension of the bounded derived category of finitely generated modules over a *numerical semigroup ring* $k[[H]]$ over a field k, in terms of the conductor of the semigroup and the multiplicity of the ring.

Corollary 9. *Let k be a field and H be a numerical semigroup. Let R be the numerical semigroup ring $k[[H]]$, that is, the subring $k[[t^h | h \in H]]$ of $S = k[[t]]$. Then $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R) = \langle S \oplus k \rangle_{2\lceil \frac{c(H)}{e(R)} \rceil + 2}$ holds. Hence*

$$\dim \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R) \leq 2 \left\lceil \frac{c(H)}{e(R)} \right\rceil + 1.$$

4. Finiteness

In this section, we consider finiteness of the dimension of the bounded derived category of finitely generated modules over a complete local ring. Let R be a commutative algebra over a field k. Rouquier [14] proved the finiteness of the dimension of $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R)$ when R is an affine k-algebra, where the fact that the enveloping algebra $R \otimes_k R$ is Noetherian played a crucial role. The problem in the case where R is a complete local ring is that one cannot hope that $R \otimes_k R$ is Noetherian. Our methods instead use the completion of the enveloping algebra, that is, the complete tensor product $R \widehat{\otimes}_k R$, which is a Noetherian ring whenever R is a complete local ring with coefficient field k.

Let R and S be commutative Noetherian complete local rings with maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$, respectively. Suppose that they contain fields and have the same residue field k, i.e., $R/\mathfrak{m} \cong k \cong S/\mathfrak{n}$. Then Cohen's structure theorem yields isomorphisms

$$R \cong k[[x_1, \dots, x_m]]/(f_1, \dots, f_a),$$
$$S \cong k[[y_1, \dots, y_n]]/(g_1, \dots, g_b).$$

We denote by $R \widehat{\otimes}_k S$ the *complete tensor product* of R and S over k, namely,

$$R \widehat{\otimes}_k S = \varprojlim_{i,j} (R/\mathfrak{m}^i \otimes_k S/\mathfrak{n}^j).$$

For $r \in R$ and $s \in S$, we denote by $r \widehat{\otimes} s$ the image of $r \otimes s$ by the canonical ring homomorphism $R \otimes_k S \to R \widehat{\otimes}_k S$. Note that there is a natural isomorphism

$$R \widehat{\otimes}_k S \cong k[[x_1, \dots, x_m, y_1, \dots, y_n]]/(f_1, \dots, f_a, g_1, \dots, g_b).$$

Details of complete tensor products can be found in [15, Chapter V].

Recall that a ring extension $A \subseteq B$ is called *separable* if B is projective as a $B \otimes_A B$-module. This is equivalent to saying that the map $B \otimes_A B \to B$ given by $x \otimes y \mapsto xy$ is a split epimorphism of $B \otimes_A B$-modules.

Now, let us prove our main theorem.

Theorem 10. *Let R be a Noetherian complete local commutative ring containing a field with perfect residue field. Then there exist a finite number of prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n \in \operatorname{Spec} R$ and an integer $m \geq 1$ such that*

$$\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R) = \langle R/\mathfrak{p}_1 \oplus \cdots \oplus R/\mathfrak{p}_n \rangle_m.$$

Hence one has $\dim \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R) < \infty$.

Sketch of proof. We use induction on the Krull dimension $d := \dim R$. If $d = 0$, then R is an Artinian ring, and the assertion follows from [14, Proposition 7.37]. Assume $d \geq 1$. By [10, Theorem 6.4], we have a sequence

$$0 = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n = R$$

of ideals of R such that for each $1 \leq i \leq n$ one has $I_i/I_{i-1} \cong R/\mathfrak{p}_i$ with $\mathfrak{p}_i \in \operatorname{Spec} R$. Then every object X of $\mathsf{D}^{\mathrm{b}}(\operatorname{\mathsf{mod}} R)$ possesses a sequence

$$0 = XI_0 \subseteq XI_1 \subseteq \cdots \subseteq XI_n = X$$

of R-subcomplexes. Decompose this into exact triangles

$$XI_{i-1} \to XI_i \to XI_i/XI_{i-1} \to \Sigma XI_{i-1},$$

in $\mathsf{D}^{\mathrm{b}}(\operatorname{\mathsf{mod}} R)$, and note that each XI_i/XI_{i-1} belongs to $\mathsf{D}^{\mathrm{b}}(\operatorname{\mathsf{mod}} R/\mathfrak{p}_i)$. Hence one may assume that R is an integral domain. By [16, Definition-Proposition (1.20)], we can take a formal power series subring $A = k[[x_1, \ldots, x_d]]$ of R such that R is a finitely generated A-module and that the extension $Q(A) \subseteq Q(R)$ of the quotient fields is finite and separable.

Claim 1. *We have natural isomorphisms*

$$\begin{aligned}
R &\cong k[[x]][t]/(f(x,t)) = k[[x,t]]/(f(x,t)),\\
S := R \otimes_A R &\cong k[[x]][t,t']/(f(x,t), f(x,t')) = k[[x,t,t']]/(f(x,t), f(x,t')),\\
U := R \mathbin{\widehat{\otimes}_k} A &\cong k[[x,t,x']]/(f(x,t)),\\
T := R \mathbin{\widehat{\otimes}_k} R &\cong k[[x,t,x',t']]/(f(x,t), f(x',t')).
\end{aligned}$$

Here $x = x_1, \ldots, x_d$, $x' = x'_1, \ldots, x'_d$, $t = t_1, \ldots, t_n$, $t' = t'_1, \ldots, t'_n$ are indeterminates over k, and $f(x,t) = f_1(x,t), \ldots, f_m(x,t)$ are elements of $k[[x]][t] \subseteq k[[x,t]]$. In particular, the rings S, T, U are Noetherian commutative complete local rings.

There is a surjective ring homomorphism $\mu : S = R \otimes_A R \to R$ which sends $r \otimes r'$ to rr'. This makes R an S-module. Using Claim 1, we observe that μ corresponds to the map $k[[x,t,t']]/(f(x,t), f(x,t')) \to k[[x,t]]/(f(x,t))$ given by $t' \mapsto t$. Taking the kernel, we have an exact sequence

$$0 \to I \to S \xrightarrow{\mu} R \to 0$$

of finitely generated S-modules. Along the injective ring homomorphism $A \to S$ sending $a \in A$ to $a \otimes 1 = 1 \otimes a \in S$, we can regard A as a subring of S. Note that S is a finitely generated A-module. Put $W = A \setminus \{0\}$. This is a multiplicatively closed subset of A, R and S, and one can take localization $(-)_W$.

Claim 2. *The S_W-module R_W is projective.*

There are ring epimorphisms

$$\begin{aligned}
\alpha &: U \to R, \quad r \mathbin{\widehat{\otimes}} a \mapsto ra,\\
\beta &: T \to S, \quad r \mathbin{\widehat{\otimes}} r' \mapsto r \otimes r',\\
\gamma &: T \to R, \quad r \mathbin{\widehat{\otimes}} r' \mapsto rr'.
\end{aligned}$$

Identifying the rings R, S, T and U with the corresponding residue rings of formal power series rings made in Claim 1, we see that α, β are the maps given by $x' \mapsto x$, and γ

is the map given by $x' \mapsto x$ and $t' \mapsto t$. Note that $\gamma = \mu\beta$. The map α is naturally a homomorphism of (R, A)-bimodules, and β, γ are naturally homomorphisms of (R, R)-bimodules. The ring R has the structure of a finitely generated U-module through α. The Koszul complex on the U-regular sequence $x' - x$ gives a free resolution of the U-module R:

$$0 \to U \to U^{\oplus d} \to U^{\oplus \binom{d}{2}} \to \cdots \to U^{\oplus \binom{d}{2}} \to U^{\oplus d} \xrightarrow{x'-x} U \xrightarrow{\alpha} R \to 0. \tag{4.1}$$

This is an exact sequence of (R, A)-bimodules. Since the natural homomorphisms

$$\begin{aligned} &A \cong k[[x']] \to k[[x']][x,t]/(f(x,t)), \\ &k[[x']][x,t]/(f(x,t)) \to k[[x']][[x,t]]/(f(x,t)) \cong U \end{aligned}$$

are flat, so is the composition. Therefore U is flat as a right A-module. The exact sequence (4.1) gives rise to a chain map $\eta : F \to R$ of U-complexes, where

$$F = (0 \to U \to U^{\oplus d} \to U^{\oplus \binom{d}{2}} \to \cdots \to U^{\oplus \binom{d}{2}} \to U^{\oplus d} \xrightarrow{x'-x} U \to 0)$$

is a complex of finitely generated free U-modules. By Claim 1, we have isomorphisms

$$\begin{aligned} U \otimes_A R &\cong U \otimes_A A[t]/(f(x,t)) \cong U[t']/(f(x',t')) \cong U[[t']]/(f(x',t')) \\ &\cong (k[[x,t,x']]/(f(x,t)))[[t']]/(f(x',t')) \cong k[[x,t,x',t']]/(f(x,t), f(x',t')) \cong T. \end{aligned}$$

Note from [17, Exercise 10.6.2] that $R \otimes^{\mathbf{L}}_A R$ is an object of $\mathsf{D}^-(R\text{-}\mathsf{Mod}\text{-}R) = \mathsf{D}^-(\mathsf{Mod}\, R \otimes_k R)$. (Here, R-Mod-R denotes the category of (R, R)-bimodules, which can be identified with $\mathsf{Mod}\, R \otimes_k R$.) There are isomorphisms

$$\begin{aligned} R \otimes^{\mathbf{L}}_A R &\cong F \otimes_A R \\ &\cong (0 \to U \otimes_A R \to (U \otimes_A R)^{\oplus d} \to \cdots \to (U \otimes_A R)^{\oplus d} \xrightarrow{x'-x} U \otimes_A R \to 0) \\ &\cong (0 \to T \to T^{\oplus d} \to \cdots \to T^{\oplus d} \xrightarrow{x'-x} T \to 0) =: C \end{aligned}$$

in $\mathsf{D}^-(\mathsf{Mod}\, R \otimes_k R)$. Note that C can be regarded as an object of $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, T)$. Taking the tensor product $\eta \otimes_A R$, one gets a chain map $\lambda : C \to S$ of T-complexes. Thus, one has a commutative diagram

$$\begin{tikzcd} K \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "\lambda"'] & R \arrow[r, "\delta"] \arrow[d, equal] & \Sigma K \arrow[d, "\xi"'] \\ I \arrow[r] & S \arrow[r, "\mu"] & R \arrow[r, "\varepsilon"] & \Sigma I \end{tikzcd}$$

of exact triangles in $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, T)$.

Claim 3. *There exists an element $a \in W$ such that $\delta \cdot (1 \mathbin{\widehat{\otimes}} a) = 0$ in $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, T)}(R, \Sigma K)$. One can choose it as a non-unit element of A, if necessary.*

Let $a \in W$ be a non-unit element of A as in Claim 3. Since we regard R as a T-module through the homomorphism γ, we have an exact sequence $0 \to R \xrightarrow{1 \widehat{\otimes} a} R \to R/(a) \to 0$.

The octahedral axiom makes a diagram in $\mathsf{D}^{\mathrm{b}}(\operatorname{\mathsf{mod}} T)$

$$
\begin{tikzcd}
R \arrow[r, "1\widehat{\otimes} a"] \arrow[d, equal] & R \arrow[r] \arrow[d, "\delta"'] & R/(a) \arrow[r] \arrow[d] & \Sigma R \arrow[d, equal] \\
R \arrow[r, "0"] \arrow[d] & \Sigma K \arrow[r] \arrow[d, equal] & \Sigma K \oplus \Sigma R \arrow[r] \arrow[d] & \Sigma R \arrow[d] \\
R \arrow[r, "\delta"] \arrow[d] & \Sigma K \arrow[r] \arrow[d] & \Sigma C \arrow[r] \arrow[d, equal] & \Sigma R \arrow[d] \\
R/(a) \arrow[r] & \Sigma K \oplus \Sigma R \arrow[r] & \Sigma C \arrow[r] & \Sigma R/(a)
\end{tikzcd}
$$

with the bottom row being an exact triangle. Rotating it, we obtain an exact triangle

$$K \oplus R \to C \to R/(a) \to \Sigma(K \oplus R)$$

in $\mathsf{D}^{\mathrm{b}}(\operatorname{\mathsf{mod}} T)$. The exact functor $\mathsf{D}^{\mathrm{b}}(\operatorname{\mathsf{mod}} T) \to \mathsf{D}^{-}(\operatorname{\mathsf{Mod}} R \otimes_k R)$ induced by the canonical ring homomorphism $R \otimes_k R \to T$ sends this to an exact triangle

$$K \oplus R \to R \otimes^{\mathbf{L}}_A R \to R/(a) \to \Sigma(K \oplus R) \tag{4.2}$$

in $\mathsf{D}^{-}(\operatorname{\mathsf{Mod}} R \otimes_k R)$. As R is a local domain and a is a non-zero element of the maximal ideal of R, we have $\dim R/(a) = d-1 < d$. Hence one can apply the induction hypothesis to the ring $R/(a)$, and sees that

$$\mathsf{D}^{\mathrm{b}}(\operatorname{\mathsf{mod}} R/(a)) = \langle R/\mathfrak{p}_1 \oplus \cdots \oplus R/\mathfrak{p}_h \rangle_m^{R/(a)}$$

for some integer $m \geq 1$ and some prime ideals $\mathfrak{p}_1, \cdots, \mathfrak{p}_h$ of R that contain a. Now, let X be any object of $\mathsf{D}^{\mathrm{b}}(\operatorname{\mathsf{mod}} R)$. Applying the exact functor $X \otimes^{\mathbf{L}}_R -$ to (4.2) gives an exact triangle in $\mathsf{D}^{-}(\operatorname{\mathsf{Mod}} R)$

$$(X \otimes^{\mathbf{L}}_R K) \oplus X \to X \otimes^{\mathbf{L}}_A R \to X \otimes^{\mathbf{L}}_R R/(a) \to \Sigma((X \otimes^{\mathbf{L}}_R K) \oplus X). \tag{4.3}$$

Note that $X \otimes^{\mathbf{L}}_R R/(a)$ is an object of $\mathsf{D}^{\mathrm{b}}(\operatorname{\mathsf{mod}} R/(a)) = \langle R/\mathfrak{p}_1 \oplus \cdots \oplus R/\mathfrak{p}_h \rangle_m^{R/(a)}$. As an object of $\mathsf{D}^{\mathrm{b}}(\operatorname{\mathsf{mod}} R)$, the complex $X \otimes^{\mathbf{L}}_R R/(a)$ belongs to $\langle R/\mathfrak{p}_1 \oplus \cdots \oplus R/\mathfrak{p}_h \rangle_m^R$. We observe from (4.3) that X is in $\langle R \oplus R/\mathfrak{p}_1 \oplus \cdots \oplus R/\mathfrak{p}_h \rangle_{d+1+m}^R$. Thus we obtain $\mathsf{D}^{\mathrm{b}}(\operatorname{\mathsf{mod}} R) = \langle R \oplus R/\mathfrak{p}_1 \oplus \cdots \oplus R/\mathfrak{p}_h \rangle_{d+1+m}$. (As R is a domain, the zero ideal of R is a prime ideal.) $\square$

Now, we make sure that the proof of Theorem 10 also gives a ring-theoretic proof of the affine case of Rouquier's theorem. Actually, we obtain a more detailed result as follows. Recall that a commutative ring R is said to be *essentially of finite type* over a field k if R is a localization of a finitely generated k-algebra. Of course, every finitely generated k-algebra is essentially of finite type over k.

Theorem 11. (1) *Let R be a finitely generated algebra over a perfect field. Then there exist a finite number of prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \in \operatorname{Spec} R$ and an integer $m \geq 1$ such that*

$$\mathsf{D}^{\mathrm{b}}(\operatorname{\mathsf{mod}} R) = \langle R/\mathfrak{p}_1 \oplus \cdots \oplus R/\mathfrak{p}_n \rangle_m.$$

(2) *Let R be a commutative ring which is essentially of finite type over a perfect field. Then there exist a finite number of prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n \in \operatorname{Spec} R$ and an integer $m \geq 1$ such that*

$$\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R) = \langle R/\mathfrak{p}_1 \oplus \cdots \oplus R/\mathfrak{p}_n \rangle_m.$$

Now the following result due to Rouquier (cf. [14, Theorem 7.38]) is immediately recovered by Theorem 11(2).

Corollary 12 (Rouquier)**.** *Let R be a commutative ring essentially of finite type over a perfect field. Then the derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R)$ has finite dimension.*

Remark 13. In Corollary 12, the assumption that the base field is perfect can be removed; see [7, Proposition 5.1.2]. We do not know whether we can also remove the perfectness assumption of the residue field in Theorem 10. It seems that the techniques in the proof of [7, Proposition 5.1.2] do not directly apply to that case.

5. Lower bounds

In this section, we will mainly study lower bounds for the dimension of the bounded derived category of finitely generated modules. We shall refine a result of Rouquier over an affine algebra, and also give a similar lower bound over a general commutative Noetherian ring.

Throughout this section, let R be a commutative Noetherian ring. First, we consider refining a result of Rouquier.

Theorem 14. *Let R be a finitely generated algebra over a field. Suppose that there exists $\mathfrak{p} \in \operatorname{Assh} R$ such that $R_\mathfrak{p}$ is a field. Then one has the inequality $\dim \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R) \geq \dim R$.*

The following result of Rouquier [14, Proposition 7.16] is a direct consequence of Theorem 14.

Corollary 15 (Rouquier)**.** *Let R be a reduced finitely generated algebra over a field. Then $\dim \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R) \geq \dim R$.*

Next, we try to extend Theorem 14 to non-affine algebras. We do not know whether the inequality in Theorem 14 itself holds over non-affine algebras; we can prove that a similar but slightly weaker inequality holds over them.

Now we can show the following result.

Theorem 16. *Let R be a ring of finite Krull dimension such that $R_\mathfrak{p}$ is a field for all $\mathfrak{p} \in \operatorname{Assh} R$. Then we have $\dim \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R) \geq \dim R - 1$.*

Here is an obvious conclusion of the above theorem.

Corollary 17. *Let R be a reduced ring of finite Krull dimension. Then $\dim \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R) \geq \dim R - 1$.*

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QUIVER PRESENTATIONS OF GROTHENDIECK CONSTRUCTIONS

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Abstract. We give quiver presentations of the Grothendieck constructions of functors from a small category to the 2-category of $\mathbb{k}$-categories for a commutative ring $\mathbb{k}$.

Key Words: Grothendieck construction, functors, quivers.

1. Introduction

Throughout this report I is a small category, $\mathbb{k}$ is a commutative ring, and $\mathbb{k}$-**Cat** denotes the 2-category of all $\mathbb{k}$-categories, $\mathbb{k}$-functors between them and natural transformations between $\mathbb{k}$-functors.

The Grothendieck construction is a way to form a single category $\mathrm{Gr}(X)$ from a diagram X of small categories indexed by a small category I, which first appeared in [4, §8 of Exposé VI]. As is exposed by Tamaki [7] this construction has been used as a useful tool in homotopy theory (e.g., [8]) or topological combinatorics (e.g., [9]). This can be also regarded as a generalization of orbit category construction from a category with a group action.

In [2] we defined a notion of derived equivalences of (oplax) functors from I to $\mathbb{k}$-**Cat**, and in [3] we have shown that if (oplax) functors $X, X' \colon I \to \mathbb{k}\text{-}\mathbf{Cat}$ are derived equivalent, then so are their Grothendieck constructions $\mathrm{Gr}(X)$ and $\mathrm{Gr}(X')$. An easy example of a derived equivalent pair of functors is given by using diagonal functors: For a category $\mathcal{C}$ define the *diagonal* functor $\Delta(\mathcal{C}) \colon I \to \mathbb{k}\text{-}\mathbf{Cat}$ to be a functor sending all objects of I to $\mathcal{C}$ and all morphisms in I to the identity functor of $\mathcal{C}$. Then if categories $\mathcal{C}$ and $\mathcal{C}'$ are derived equivalent, then so are their diagonal functors $\Delta(\mathcal{C})$ and $\Delta(\mathcal{C}')$. Therefore, to compute examples of derived equivalent pairs using this result, it will be useful to present Grothendieck constructions of functors by quivers with relations. We already have computations in two special cases. First for a $\mathbb{k}$-algebra A, which we regard as a $\mathbb{k}$-category with a single object, we noted in [3] that if I is a semigroup G, a poset S, or the free category $\mathbb{P}Q$ of a quiver Q, then the Grothendieck construction $\mathrm{Gr}(\Delta(A))$ of the diagonal functor $\Delta(A)$ is isomorphic to the semigroup algebra AG, the incidence algebra AS, or the path-algebra AQ, respectively. Second in [1] we gave a quiver presentation of the orbit category $\mathcal{C}/G$ for each $\mathbb{k}$-category $\mathcal{C}$ with an action of a semigroup G in the case that $\mathbb{k}$ is a field, which can be seen as a computation of a quiver presentation of the Grothendieck construction $\mathrm{Gr}(X)$ of each functor $X \colon G \to \mathbb{k}\text{-}\mathbf{Cat}$.

In this report we generalize these two results as follows:

(1) We compute the Grothendieck construction $\mathrm{Gr}(\Delta(A))$ of the diagonal functor $\Delta(A)$ for each $\mathbb{k}$-algebra A and each small category I.

The final version of this paper has been submitted for publication elsewhere.

(2) We give a quiver presentation of the Grothendieck construction $\mathrm{Gr}(X)$ for each functor $X\colon I \to \Bbbk\text{-}\mathbf{Cat}$ and each small category I when $\Bbbk$ is a field.

2. Preliminaries

Throughout this report $Q = (Q_0, Q_1, t, h)$ is a quiver, where $t(\alpha) \in Q_0$ is the *tail* and $h(\alpha) \in Q_0$ is the *head* of each arrow α of Q. For each path μ of Q, the tail and the head of μ is denoted by $t(\mu)$ and $h(\mu)$, respectively. For each non-negative integer n the set of all paths of Q of length at least n is denoted by $Q_{\geq n}$. In particular $Q_{\geq 0}$ denotes the set of all paths of Q.

A category $\mathcal{C}$ is called a $\Bbbk$-*category* if for each $x, y \in \mathcal{C}$, $\mathcal{C}(x, y)$ is a $\Bbbk$-module and the compositions are $\Bbbk$-bilinear.

Definition 1. Let Q be a quiver.

(1) The *free* category $\mathbb{P}Q$ of Q is the category whose underlying quiver is $(Q_0, Q_{\geq 0}, t, h)$ with the usual composition of paths.
(2) The *path* $\Bbbk$-category of Q is the $\Bbbk$-linearization of $\mathbb{P}Q$ and is denoted by $\Bbbk Q$.

Definition 2. Let $\mathcal{C}$ be a category. We set

$$\mathrm{Rel}(\mathcal{C}) := \bigcup_{(i,j)\in\mathcal{C}_0\times\mathcal{C}_0} \mathcal{C}(i, j) \times \mathcal{C}(i, j),$$

elements of which are called *relations* of $\mathcal{C}$. Let $R \subseteq \mathrm{Rel}(\mathcal{C})$. For each $i, j \in \mathcal{C}_0$ we set

$$R(i, j) := R \cap (\mathcal{C}(i, j) \times \mathcal{C}(i, j)).$$

(1) The smallest congruence relation

$$R^c := \bigcup_{(i,j)\in\mathcal{C}_0\times\mathcal{C}_0} \{(dac, dbc) \mid c \in \mathcal{C}(-, i), d \in \mathcal{C}(j, -), (a, b) \in R(i, j)\}$$

containing R is called the *congruence relation* generated by R.

(2) For each $i, j \in \mathcal{C}_0$ we set

$$\begin{aligned}
R^{-1}(i, j) &:= \{(g, f) \in \mathcal{C}(i, j) \times \mathcal{C}(i, j) \mid (f, g) \in R(i, j)\}\\
1_{\mathcal{C}(i,j)} &:= \{(f, f) \mid f \in \mathcal{C}(i, j)\}\\
S(i, j) &:= R(i, j) \cup R^{-1}(i, j) \cup 1_{\mathcal{C}(i,j)}\\
S(i, j)^1 &:= S(i, j)\\
S(i, j)^n &:= \{(h, f) \mid \exists g \in \mathcal{C}(i, j), (g, f) \in S(i, j), (h, g) \in S(i, j)^{n-1}\} \quad \text{(for all } n \geq 2)\\
S(i, j)^\infty &:= \bigcup_{n\geq 1} S(i, j)^n, \text{ and set}
\end{aligned}$$

$$R^e := \bigcup_{(i,j)\in\mathcal{C}_0\times\mathcal{C}_0} S(i, j)^\infty.$$

R^e is called the *equivalence relation* generated by R.

(3) We set $R^\# := (R^c)^e$ (cf. [5]).

The following is well known (cf. [6]).

Proposition 3. *Let $\mathcal{C}$ be a category, and $R \subseteq \mathrm{Rel}(\mathcal{C})$. Then the category $\mathcal{C}/R^{\#}$ and the functor $F : \mathcal{C} \to \mathcal{C}/R^{\#}$ defined above satisfy the following conditions.*

(i) *For each $i, j \in \mathcal{C}_0$ and each $(f, f') \in R(i, j)$ we have $Ff = Ff'$.*
(ii) *If a functor $G : \mathcal{C} \to \mathcal{D}$ satisfies $Gf = Gf'$ for all $f, f' \in \mathcal{C}(i, j)$ and all $i, j \in \mathcal{C}_0$ with $(f, f') \in R(i, j)$, then there exists a unique functor $G' : \mathcal{C}/R^{\#} \to \mathcal{D}$ such that $G' \circ F = G$.*

Definition 4. Let Q be a quiver and $R \subseteq \mathrm{Rel}(\mathbb{P}Q)$. We set

$$\langle Q \mid R \rangle := \mathbb{P}Q/R^{\#}.$$

The following is straightforward.

Proposition 5. *Let $\mathcal{C}$ be a category, Q the underlying quiver of $\mathcal{C}$, and set*

$$R := \{(e_i, \mathbb{1}_i), (\mu, [\mu]) \mid i \in Q_0, \mu \in Q_{\geq 2}\} \subseteq \mathrm{Rel}(\mathbb{P}Q),$$

where e_i is the path of length 0 at each vertex $i \in Q_0$, and $[\mu] := \alpha_n \circ \cdots \circ \alpha_1$ (the composite in $\mathcal{C}$) for all paths $\mu = \alpha_n \ldots \alpha_1 \in Q_{\geq 2}$ with $\alpha_1, \ldots, \alpha_n \in Q_1$. Then

$$\mathcal{C} \cong \langle Q \mid R \rangle.$$

By this statement, an arbitrary category is presented by a quiver and relations. Throughout the rest of this report I is a small category with a presentation $I = \langle Q \mid R \rangle$.

3. Grothendieck constructions of Diagonal functors

Definition 6. Let $X : I \to \mathbb{k}\text{-}\mathbf{Cat}$ be a functor. Then a category $\mathrm{Gr}(X)$, called the *Grothendiek construction* of X, is defined as follows:

(i) $(\mathrm{Gr}(X))_0 := \bigcup\limits_{i \in I_0} \{(i, x) \mid x \in X(i)_0\}$
(ii) For $(i, x), (j, y) \in (\mathrm{Gr}(X))_0$

$$\mathrm{Gr}(X)((i, x), (j, y)) := \bigoplus_{a \in I(i,j)} X(j)(X(a)x, y)$$

(iii) For $f = (f_a)_{a \in I(i,j)} \in \mathrm{Gr}(X)((i, x), (j, y))$ and $g = (g_b)_{b \in I(j,k)} \in \mathrm{Gr}(X)((j, y), (k, z))$

$$g \circ f := (\sum_{\substack{c = ba \\ a \in I(i,j) \\ b \in I(j,k)}} g_b X(b) f_a)_{c \in I(i,k)}$$

Definition 7. Let $\mathcal{C} \in \mathbb{k}\text{-}\mathbf{Cat}_0$. Then the *diagonal functor* $\Delta(\mathcal{C})$ of $\mathcal{C}$ is a functor from I to $\mathbb{k}\text{-}\mathbf{Cat}$ sending each arrow $a\colon i \to j$ in I to $\mathbb{1}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ in $\mathbb{k}\text{-}\mathbf{Cat}$.

In this section, we fix a $\mathbb{k}$-algebra A which we regard as a $\mathbb{k}$-category with a single object $*$ and with $A(*, *) = A$. The *quiver algebra* AQ of Q over A is the A-linearization of $\mathbb{P}Q$, namely $AQ := A \otimes_{\mathbb{k}} \mathbb{k}Q$.

Theorem 8. *We have an isomorphism $\mathrm{Gr}(\Delta(A)) \cong AQ/\langle R \rangle_A$, where $\langle R \rangle_A$ is the ideal of AQ generated by the elements $g - h$ with $(g, h) \in R$.*

Remark 9. Theorem 8 can be written in the form

$$\mathrm{Gr}(\Delta(A)) \cong A \otimes_{\mathbb{k}} (\mathbb{k}Q/\langle R \rangle_{\mathbb{k}}).$$

4. The quiver presentation of Grothendieck constructions

In this section we give a quiver presentation of the Grothendieck construction of an arbitrary functor $I \to \Bbbk\text{-}\mathbf{Cat}$. Throughout this section we assume that $\Bbbk$ is a field.

Theorem 10. *Let $X : I \to \Bbbk\text{-}\mathbf{Cat}$ be a functor, and for each $i \in I$ set $X(i) = \Bbbk Q^{(i)}/\langle R^{(i)}\rangle$ with $\Phi^{(i)} \colon \Bbbk Q^{(i)} \to X(i)$ the canonical morphism, where $R^{(i)} \subseteq \Bbbk Q^{(i)}$, $\langle R^{(i)}\rangle \cap \{e_x \mid x \in Q(i)_0\} = \emptyset$. Then Grothendieck construction is presented by the quiver with relations (Q, R') defined as follows.*

Quiver: $Q' = (Q'_0, Q'_1, t', h')$, where

(i) $Q'_0 := \displaystyle\bigcup_{i\in I}\{{}_ix \mid x \in Q_0^{(i)}\}$.

(ii) $Q'_1 := \displaystyle\bigcup_{i\in I}\{\{{}_i\alpha \mid \alpha \in Q_1^{(i)}\} \cup \{(a, {}_ix) : {}_ix \to {}_j(ax) \mid a : i \to j \in Q_1, x \in Q_0^{(i)}, ax \neq 0\}\}$,
where we set $ax := X(\overline{a})(x)$.

(iii) *For $\alpha \in Q_1^{(i)}$, $t'({}_i\alpha) = t^{(i)}(\alpha)$ and $h'({}_i\alpha) = h^{(i)}(\alpha)$.*

(iv) *For $a : i \to j \in Q_1, x \in Q_0^{(i)}$, $t'(a, {}_ix) = {}_ix$ and $h'(a, {}_ix) = {}_j(ax)$.*

Relations: $R' := R'_1 \cup R'_2 \cup R'_3$, where

(i) $R'_1 := \{\sigma^{(i)}(\mu) \mid i \in Q_0, \mu \in R^{(i)}\}$,
where we set $\sigma^{(i)} : \Bbbk Q^{(i)} \hookrightarrow \Bbbk Q'$.

(ii) $R'_2 := \{\pi(g, {}_ix) - \pi(h, {}_ix) \mid i, j \in Q_0, (g, h) \in R(i, j), x \in Q_0^{(i)}\}$, *where for each path a in Q we set*

$$\pi(a, {}_ix) := (a_n, {}_{i_{n-1}}(a_{n-1}a_{n-2}\dots a_1x))\dots(a_2, {}_{i_1}(a_1x))(a_1, {}_ix)$$

if $a = a_n \dots a_2a_1$ for some $a_1, \dots, a_n$ arrows in Q, and

$$\pi(a, {}_ix) := e_{{}_ix}$$

if $a = e_i$ for some $i \in Q_0$.

(iii) $R'_3 := \{(a, {}_iy){}_i\alpha - {}_j(a\alpha)(a, {}_ix) \mid a : i \to j \in Q_1, \alpha : x \to y \in Q_1^{(i)}\}$, *where we take $a\alpha : ax \to ay$ so that $\Phi^{(j)}(a\alpha) \in X(\overline{a})\Phi^{(i)}(\alpha)$:*

$$\begin{tikzcd} \alpha \in \Bbbk Q^{(i)} \arrow[r, "\Phi^{(i)}"] & X(i) \arrow[d, "X(\overline{a})"] \\ a\alpha \in \Bbbk Q^{(j)} \arrow[r, "\Phi^{(j)}"] & X(j). \end{tikzcd}$$

Note that the ideal $\langle R'\rangle$ is independent of the choice of $a\alpha$ because $R'_1 \subseteq R'$.

5. Examples

In this section, we illustrate Theorems 8 and 10 by some examples.

Example 11. Let Q be the quiver

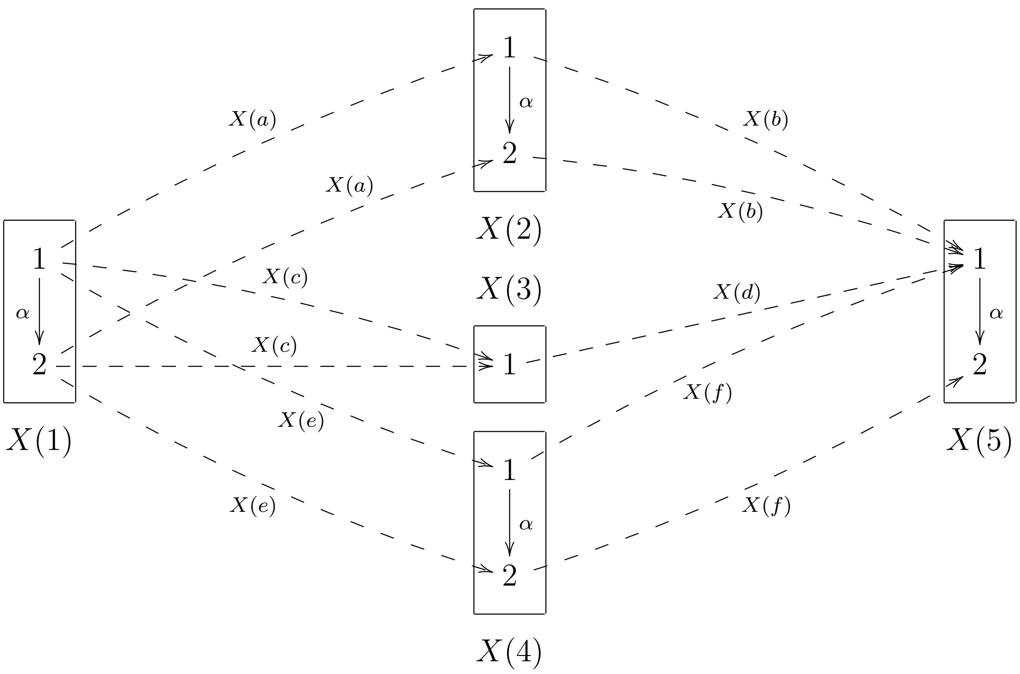
$$\begin{tikzcd}
 & 2 \arrow[dr, "b"] & \\
1 \arrow[ur, "a"] \arrow[r, "c"] \arrow[dr, "e"'] & 3 \arrow[r, "d"] & 5 \\
 & 4 \arrow[ur, "f"'] &
\end{tikzcd}$$

and let $R = \{(ba, dc)\}$. Then the category $I := \langle Q \mid R \rangle$ is not given as a semigroup, as a poset or as the free category of a quiver. For any algebra A consider the diagonal functor $\Delta(A)\colon I \to \Bbbk\text{-}\mathbf{Cat}$. Then by Theorem 8 the category $\mathrm{Gr}(\Delta(A))$ is given by

$$AQ/\langle ba - dc \rangle.$$

Remark 12. Let Q and Q' be quivers having neither double arrows nor loops, and let $f\colon Q_0 \to Q'_0$ be a map (a *vertex map* between Q and Q'). If $Q(x,y) \neq \emptyset$ $(x, y \in Q_0)$ implies $Q'(f(x), f(y)) \neq \emptyset$ or $f(x) = f(y)$, then f induces a $\Bbbk$-functor $\hat{f}\colon \Bbbk P \to \Bbbk P'$ defined by the following correspondence: For each $x \in Q_0$, $\hat{f}(e_x) := e_{f(x)}$, and for each arrow $a\colon x \to y$ in Q, $f(a)$ is the unique arrow $f(x) \to f(y)$ (resp. $e_{f(x)}$) if $f(x) \neq f(y)$ (resp. if $f(x) = f(y)$).

Example 13. Let $I = \langle Q \mid R \rangle$ be as in the previous example. Define a functor $X : I \to \Bbbk\text{-}\mathbf{Cat}$ by the $\Bbbk$-linearizations of the following quivers in frames and the $\Bbbk$-functors induced by the vertex maps expressed by broken arrows between them:



Then by Theorem 10 $\mathrm{Gr}(X)$ is presented by the quiver

$$
Q' = \left(\begin{tikzcd}
 & {}_2 1 \arrow[d, "{}_2\alpha"] \arrow[ddr, dashed, "{(b,{}_2 1)}"] & \\
 & {}_2 2 \arrow[dr, dashed, "{(b,{}_2 2)}"'] & \\
{}_1 1 \arrow[d, "{}_1\alpha"'] \arrow[uur, dashed, "{(a,{}_1 1)}"] \arrow[dr, dashed, "{(c,{}_1 1)}"] \arrow[ddr, dashed, "{(e,{}_1 1)}"'] & & {}_5 1 \arrow[d, "{}_5\alpha"] \\
{}_1 2 \arrow[uur, dashed, "{(a,{}_1 2)}"] \arrow[r, dashed, "{(c,{}_1 2)}"] \arrow[ddr, dashed, "{(e,{}_1 2)}"'] & {}_3 1 \arrow[ur, dashed, "{(d,{}_3 1)}"] & {}_5 2 \\
 & {}_4 1 \arrow[d, "{}_4\alpha"] \arrow[uur, dashed, "{(f,{}_4 1)}"'] & \\
 & {}_4 2 \arrow[uur, dashed, "{(f,{}_4 2)}"'] &
\end{tikzcd}\right)
$$

with relations

$$
\begin{aligned}
R' \;=\; & \{\pi(ba,{}_1 1) - \pi(dc,{}_1 1), \pi(ba,{}_1 2) - \pi(dc,{}_1 2)\} \\
& \cup\{(a,{}_i y)_i\alpha - {}_j(a\alpha)(a,{}_i x) \mid a : i \to j \in Q_1, \alpha : x \to y \in Q_1^{(i)}\},
\end{aligned}
$$

where the new arrows are presented by broken arrows.

Example 14. Let $Q = (\ 1 \xrightarrow{a} 2\ )$ and $I := \langle Q\rangle$. Define functors $X, X' \colon I \to \Bbbk\text{-}\mathbf{Cat}$ by the $\Bbbk$-linearizations of the following quivers in frames and the $\Bbbk$-functors induced by the vertex maps expressed by dotted arrows between them:

$$
X : \begin{tikzcd}
1 \arrow[dr, "\alpha"] \arrow[ddr, dotted, "X(a)"'] & & 2 \arrow[dl, "\beta"'] \arrow[ddl, dotted, "X(a)"] & X(1) \\
 & 3 \arrow[d, dotted, "X(a)"] & & \\
 & 1 & X(2), &
\end{tikzcd}
\qquad
X' : \begin{tikzcd}
 & 1 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] \arrow[dd, dotted, "X'(a)"] & & X'(1) \\
2 \arrow[dr, dotted, "X'(a)"'] & & 3 \arrow[dl, dotted, "X'(a)"] & \\
 & 1 & X'(2). &
\end{tikzcd}
$$

Then by Theorem 10 $\mathrm{Gr}(X)$ is given by the following quiver with no relations

$$
\left(\begin{tikzcd}
{}_1 1 \arrow[dr, "{}_1\alpha"] \arrow[ddr, dotted, "{(a,{}_1 1)}"'] & & {}_1 2 \arrow[dl, "{}_1\beta"'] \arrow[ddl, dotted, "{(a,{}_1 2)}"] \\
 & {}_1 3 \arrow[d, dotted, "{(a,{}_1 3)}"'] & \\
 & {}_2 1 &
\end{tikzcd}, (a,{}_1 3)_1\alpha - (a,{}_1 1), (a,{}_1 3)_1\beta - (a,{}_1 2)\right)
\cong
\left(\begin{tikzcd}
{}_1 1 \arrow[dr, "{}_1\alpha"] & & {}_1 2 \arrow[dl, "{}_1\beta"'] \\
 & {}_1 3 \arrow[d, "{(a,{}_1 3)}"'] & \\
 & {}_2 1 &
\end{tikzcd}\right),
$$

and $\mathrm{Gr}(X')$ is given by the following quiver with a commutativity relation

$$\left(\begin{tikzcd} & {}_1 1 \arrow[dl, "{}_1\alpha"'] \arrow[dr, "{}_1\beta"] \arrow[dd, dotted, "{(a,{}_1 1)}"'] & \\ {}_1 2 \arrow[dr, dotted, "{(a,{}_1 2)}"'] & & {}_1 3 \arrow[dl, dotted, "{(a,{}_1 3)}"] \\ & {}_2 1 & \end{tikzcd}, (a, {}_1 2){}_1\alpha - (a, {}_1 1), (a, {}_1 3){}_1\beta - (a, {}_1 1) \right) \cong \left(\begin{tikzcd} & {}_1 1 \arrow[dl, "{}_1\alpha"'] \arrow[dr, "{}_1\beta"] \arrow[dd, phantom, "\circlearrowright"] & \\ {}_1 2 \arrow[dr, dotted, "{(a,{}_1 2)}"'] & & {}_1 3 \arrow[dl, dotted, "{(a,{}_1 3)}"] \\ & {}_2 1 & \end{tikzcd} \right).$$

By using the main theorem in [3] derived equivalences between $X(1)$ and $X'(1)$ and between $X(2)$ and $X'(2)$ are glued together to have a derived equivalence between $\mathrm{Gr}(X)$ and $\mathrm{Gr}(X')$.

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THE LOEWY LENGTH OF TENSOR PRODUCTS FOR DIHEDRAL TWO-GROUPS

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Abstract. The indecomposable modules of a dihedral 2-group over a field of characteristic 2 were classified by Ringel over 30 years ago. However, relatively little is known about the tensor products of such modules, except in certain special cases. We describe here the main result of our recent work determining the Loewy length of a tensor product of modules for a dihedral 2-group. As a consequence of this result, we can determine precisely when a tensor product has a projective direct summand.

1. Introduction

Let k be a field of positive characteristic p and let G be a finite group. The group algebra kG is a Hopf algebra with coproduct and co-unit given by $\Delta(\sum_{g\in G} r_g g) = \sum_{g\in G} r_g g \otimes g$ and $\epsilon(\sum_{g\in G} r_g g) = \sum_{g\in G} r_g$ for $r_g \in k$. Thus, there is a tensor product operation on the category of kG-modules. If M and N are kG-modules, then the tensor product of M and N is the module with underlying vector space $M \otimes_k N$ and module structure given by $g(m \otimes n) = \Delta(g)(m \otimes n) = gm \otimes gn$ for $g \in G, m \in M, n \in N$. The tensor product is a frequently used tool in the representation theory of finite groups. However, the problem of determining the decomposition of a tensor product of two modules of a finite group G – the Clebsch-Gordan problem – can be extremely difficult.

One approach to understanding tensor products of kG-modules goes via the representation ring, or Green ring, of kG. The isomorphism classes of finite-dimensional kG-modules form a semiring, with addition given by the direct sum, and multiplication by the tensor product of kG-modules. The Green ring, $A(kG)$, is the Groethendieck ring of this semiring, i.e., the ring obtained by formally adjoining additive inverses to all elements in the semiring. Research in this direction was pioneered by J. A. Green [6], who proved the Green ring of a cyclic p-group is semisimple. The question of semisimplicity of the Green ring for other finite groups has been studied by several authors since. Benson and Carlson [2] gave a method to produce nilpotent elements in a Green ring, and determined a quotient of the Green ring which has no nilpotent elements.

This so-called Benson–Carlson quotient of the Green ring was studied by Archer [1] in the case of the dihedral 2-groups, who realised it as an integral group ring of an abelian, infinitely generated, torsion-free group. Archer gave a precise statement relating the multiplication of two elements in this infinite group to the Auslander–Reiten quiver of kD_{4q}. The Green ring of the Klein four group V_4 was completely determined by Conlon [4]; a summary of this result can be found in [1].

For the dihedral 2-groups D_{4q}, the indecomposable modules, over fields of characteristic 2, were classified by Ringel [7] over 30 years ago. However, very little progress has been made towards understanding the behaviour of the tensor product of the kD_{4q}-modules. In particular, the decomposition of a tensor product of two indecomposable kD_{4q}-modules is not known, other than in some very special cases. One example is the work of Bessenrodt [3], classifying the endotrivial kD_{4q}-modules, thus determining the kD_{4q}-modules M for which the tensor product of M with its dual M^* is the direct sum of a trivial and a projective module.

In recent work [5], we have continued the study of tensor products of kD_{4q}-modules, determining completely the Loewy length of the tensor product of any two indecomposable kD_{4q}-modules. This provides an additional piece of information towards the understanding of the Green rings of the dihedral 2-groups, and gives certain bounds on which modules can occur as direct summands of a tensor product. In particular, it determines precisely when a tensor product of two modules has a projective direct summand.

The *Loewy length* $\ell(M)$ of a module M is, by definition, the common length of the radical series and the socle series of M, that is, $\ell(M) = \min\{t \in \mathbb{N} \mid \mathrm{rad}^t(kD_{4q})M = 0\}$.

In the next section, we recall Ringel's classification of the indecomposable kD_{4q}-modules. Section 3 gives a summary of the results in [5], and in Section 4, we give examples illuminating our results and showing how they can be used to determine the direct sum decomposition of a tensor product in certain cases.

2. The indecomposable modules of dihedral 2-groups

Let q be a 2-power, and write $D_{4q} = \langle x, y \mid x^2 = y^2 = 1, (xy)^q = (yx)^q\rangle$ for the dihedral group of order $4q$. There is an isomorphism of algebras

$$kD_{4q} \xrightarrow{\sim} \Lambda_q := \frac{k\langle X, Y\rangle}{(X^2, Y^2, (XY)^q - (YX)^q)},$$

given by $x \mapsto 1 + X$ and $y \mapsto 1 + Y$. Setting $\Delta(X) = 1 \otimes X + X \otimes 1 + X \otimes X$ and $\Delta(Y) = 1 \otimes Y + Y \otimes 1 + Y \otimes Y$ defines a coproduct on Λ_q corresponding under this isomorphism to the Hopf algebra structure of kD_{4q}. Owing to the fact that Λ_q is a special biserial algebra, its non-projective modules split into two classes, known as string modules and band modules. We describe both classes of modules below.

Let $\bar{\mathcal{W}}$ be the set of words in letters a, b and inverse letters a^{-1}, b^{-1} such that a or a^{-1} are always followed by b or b^{-1} and b or b^{-1} are always followed by a or a^{-1}. A *directed subword* of a word $w \in \bar{\mathcal{W}}$ is a word w' in either the letters $\{a, b\}$ or $\{a^{-1}, b^{-1}\}$ such that $w = w_1 w' w_2$ for some words $w_1, w_2 \in \bar{\mathcal{W}}$. Let $\mathcal{W}$ be the subset of $\bar{\mathcal{W}}$ consisting of words in which all directed subwords are of length strictly less than $2q$. Define an equivalence relation $\sim_1$ on $\mathcal{W}$ by $w \sim_1 w'$ if, and only if, $w' = w$ or $w' = w^{-1}$. Given $w = l_1 \dots l_n \in \mathcal{W}$, the *string module determined by* w, denoted by $M(w)$, is the $n+1$-dimensional module with basis $e_0, \dots, e_n$ and Λ_q-action given by the following schema:

$$ke_0 \xleftarrow{l_1} ke_1 \xleftarrow{l_2} ke_2 \longleftarrow \dots \xleftarrow{l_{n-1}} ke_{n-1} \xleftarrow{l_n} ke_n \ .$$

If $l_i \in \{a^{-1}, b^{-1}\}$, the corresponding arrow should be interpreted as going in the opposite direction, from ke_{i-1} to ke_i, and having the label l_i^{-1}. Now X maps e_i to e_j ($j \in \{i-1, i+1\}$) if there is an arrow $ke_i \xrightarrow{a} ke_j$, and as zero if no arrow labelled with a starting in ke_i exists. Similarly, the action of Y is given by arrows labelled with b. Two modules $M(w)$ and $M(w')$, $w, w' \in \mathcal{W}$, are isomorphic if, and only if, $w \sim_1 w'$.

Next, let $\mathcal{W}'$ be the subset of $\mathcal{W}$ consisting of words w of even positive length containing letters from both $\{a, b\}$ and $\{a^{-1}, b^{-1}\}$, and such that w is not a power of a word of smaller length. Given $w = l_1 \ldots l_m \in \mathcal{W}'$, and φ an indecomposable linear automorphism of k^n, the *band module determined by* w *and* φ, $M(w, \varphi)$, is the Λ_q-module with underlying vector space $\bigoplus_{i=0}^{m-1} V_i$ where $V_i = k^n$, and Λ_q-action specified by the following schema:

$$\begin{tikzcd} V_0 \arrow[rrrrr, bend right=15, "l_m"'] & V_1 \arrow[l, "l_1=\varphi"'] & V_2 \arrow[l, "l_2"'] & \cdots \arrow[l] & V_{m-2} \arrow[l, "l_{m-2}"'] & V_{m-1} \arrow[l, "l_{m-1}"'] \end{tikzcd} .$$

The interpretation of the schema is similar to that for string modules. The elements $l_2, \ldots, l_m$ act as the identity map on k^n, l_1 acts as the linear automorphism φ (this means that if $l_1 \in \{a^{-1}, b^{-1}\}$ then V_0 is mapped onto V_1 by φ^{-1} under either X or Y).

Define $\sim_2$ to be the equivalence relation on $\mathcal{W}'$ defined by $w \sim_2 w'$ if, and only if, w of w^{-1} is a cyclic permutation of w'. Two band modules $M(w, \varphi)$ and $M(w', \psi)$ are isomorphic if, and only if, $w \sim_2 w'$ and $\varphi = \phi\psi\phi^{-1}$ for some linear automorphism ϕ. It may be noted that for every $w \in \mathcal{W}'$ there exists a $w' \in \mathcal{W}$ with an even number of maximal directed subwords such that $w \sim_2 w'$. While there are several different choices for w', its maximal directed subwords are uniquely determined, as elements in $\mathcal{W}/\sim_1$, by w.

Every indecomposable non-projective kD_{4q}-module is isomorphic either to a string or a band module, but not both. There is a single, indecomposable projective module, ${}_{kD_{4q}}kD_{4q}$. The Loewy length of any non-projective module is at most $2q$, while $\ell({}_{kD_{4q}}kD_{4q}) = 2q + 1$.

3. Loewy length of tensor products

Here, we give an overview of the results in [5]. We fix the the following conventions and notation. The least natural number is 0. For $l < 2q$, $A_l \in \mathcal{W}$ is the (unique) word of length l in the letters a, b ending in a, and similarly $B_l \in \mathcal{W}$ is the word of length l in the letters a, b ending in b.

If M is a module and $X \subset M$ any set of generators, then $\ell(M) = \max\{\ell(\langle x \rangle) \mid x \in X\}$, and if N is another module then, in a similar fashion, $\ell(M \otimes N) = \max\{\ell(\langle x \rangle \otimes N) \mid x \in X\}$. Any kD_{4q}-module that is generated by a single element is isomorphic to either $M(A_l B_m^{-1})$ or $M(A_l B_m^{-1}, \rho)$ for some $l, m < 2q$ and $\rho \in k \setminus \{0\}$, hence it is sufficient to determine Loewy lengths of tensor products of these types of modules, to solve the problem for arbitrary modules M and N. Refining these ideas a little, one can prove the following results.

Proposition 1. *Let M and N be kD_{4q}-modules. If M is a string module corresponding to a word $w \in \mathcal{W}$ with maximal directed subwords w_i, $i \in \{1, \ldots, m\}$, then*

$$\ell(M \otimes N) = \max\{\ell(M(w_i) \otimes N) \mid i \in \{1, \ldots, m\}\}.$$

Proposition 2. *Let $M = M(w, \varphi)$, where $w \in \mathcal{W}'$ and φ is an indecomposable automorphism of k^n, $n \geqslant 1$. Let $w' \in \mathcal{W}'$ be a word with an even number of maximal directed subwords w_i, $i \in \{1, \ldots, 2m\}$, such that $w \sim_2 w'$. If m and n are not both equal to 1, then*

$$\ell(M(w, \varphi) \otimes N) = \max\{\ell(M(w_i) \otimes N) \mid i \in \{1, \ldots, 2m\}\}.$$

Proposition 1 and 2 leave us with determining the Loewy lengths of tensor products of modules of the types $M(A_l)$, $M(B_m)$ and $M(A_lB_m^{-1}, \rho)$ for $l, m < 2q$, $\rho \in k \setminus \{0\}$. One can note that these are precisely the non-projective kD_{4q}-modules whose top and socle are simple modules.

Given $x \in \mathbb{N}$, denote by $[x]_i$ the ith term of its binary expansion, i.e., $[x]_i \in \{0,1\}$ such that $x = \sum_{i\in\mathbb{N}}[x]_i 2^i$. Let $l, m \in \mathbb{N}$, and take $a \in \mathbb{N}$ to be the smallest number such that $[l]_i + [m]_i \leqslant 1$ for all $i \geqslant a$. Set $\lambda = \sum_{i\geqslant a}[l]_i 2^i$ and $\mu = \sum_{j\geqslant a}[m]_j 2^j$. Now define a binary operation # on $\mathbb{N}$ by setting

$$l\#m = \lambda + \mu + 2^a - 1\,.$$

If the binary expansions of l and m are disjoint, that is, if $[l]_i + [m]_i \leqslant 1$ for all $i \in \mathbb{N}$, we write $l \perp m$. Now observe that if $l \perp m$ then $a = 0$ and $l\#m = l + m$, while $l\#m < l + m$ otherwise.

Example 3. We have $85\#38 = 119$. Namely, $85 = 2^0 + 2^2 + 2^4 + 2^6$ and $38 = 2^1 + 2^2 + 2^5$, hence $a = 3$ for these two numbers, and therefore $85\#38 = (2^4 + 2^6) + 2^5 + 2^3 - 1 = 119$. Clearly, $85\#38 = 119 < 123 = 85 + 38$, which was to be expected, since $85 \not\perp 38$.

The relevance of the operation # is that it neatly describes the Loewy length of a tensor product of uniserial modules, that is, modules of the type $M(A_l)$ and $M(B_l)$. If u is a generating element in the module $M(A_l)$, and v a generating element in $M(A_m)$, then $\ell(\langle u \otimes v\rangle) = l\#m + 1$ (observe that $u \otimes v$ does not generate $M(A_l) \otimes M(A_m)$, unless l or m equals zero). Showing this is the most important step in the proof of our principal theorem, which gives the Loewy lengths of tensor products of kD_{4q}-modules with simple top and simple socle.

Theorem 4. *Let $l, m \in \mathbb{N}$, $l_1, l_2, m_1, m_2 \in \mathbb{N} \setminus \{0\}$, $\rho, \sigma \in k \setminus \{0\}$.*

1. String with string:

$$\ell(M(A_l) \otimes M(B_m)) = \begin{cases} 1 + l\#m = 1 + l + m & \text{if} \quad l \perp m, \\ 2 + l\#m & \text{if} \quad l \not\perp m, \end{cases}$$

$$\ell(M(A_l) \otimes M(A_m)) = \begin{cases} 1 + l\#m & \text{if } [l]_t = [m]_t = 0 \text{ for all } 0 \leqslant t < a - 1, \\ 2 + l\#m & \text{otherwise.} \end{cases}$$

where $a = \min\{r \in \mathbb{N} \mid [l]_t + [m]_t \leqslant 1,\ \forall t \geqslant r\}$.

2. Band with string:

$$\ell\left(M(A_{l_1}B_{l_2}^{-1}, \rho) \otimes M(A_m)\right) = \begin{cases} 2 + (l_1 - 1)\#m & \text{if } \rho = 1,\ l_1 = l_2 \text{ and} \\ & \quad l_1 \perp m,\ l_1 \perp (m-1), \\ \ell\left(M\left(A_{l_1}B_{l_2}^{-1}\right) \otimes M(A_m)\right) & \text{otherwise.} \end{cases}$$

3. Band with band: *Let $M = M\left(A_{l_1}B_{l_2}^{-1}, \rho\right)$, $N = M\left(A_{m_1}B_{m_2}^{-1}, \sigma\right)$.*
   (a) *If $l_1 \neq l_2$, then*

$$\ell(M \otimes N) = \ell(M\left(A_{l_1}B_{l_2}^{-1}\right) \otimes N).$$

Assume $l_1 = l_2$, $m_1 = m_2$.

(b) *If $l_1 \not\perp m_1$, $l_1 \not\perp (m_1-1)$, $(l_1-1) \not\perp m_1$ then*

$$\ell(M \otimes N) = 2 + (l_1-1)\#(m_1-1).$$

(c) *If $l_1 \perp m_1$, $(l_1-1) \perp m_1$, then*

$$\ell(M \otimes N) = \begin{cases} 2 + (l_1-1)\#(m_1-1) & \text{if} \quad \sigma = 1, \\ l_1 + m_1 + 1 & \text{otherwise.} \end{cases}$$

(d) *If $l_1 \perp m_1$, $l_1 \perp (m_1-1)$, then*

$$\ell(M \otimes N) = \begin{cases} 2 + (l_1-1)\#(m_1-1) & \text{if} \quad \rho = 1, \\ l_1 + m_1 + 1 & \text{otherwise.} \end{cases}$$

(e) *If $(l_1-1) \perp m_1$, $l_1 \perp (m_1-1)$, then*

$$\ell(M \otimes N) = \begin{cases} 2 + (l_1-1)\#(m_1-1) & \text{if} \quad \rho = \sigma = 1, \\ l_1 + m_1 & \text{if} \quad \rho = \sigma \neq 1, \\ l_1 + m_1 + 1 & \text{otherwise.} \end{cases}$$

We remark that if any one of the statements $l \perp m$, $(l-1) \perp m$ and $l \perp (m-1)$ holds true, then so does precisely one of the remaining ones. Hence 3(b)–3(e) in the theorem give a complete list of cases. As a consequence of Theorem 4:1, it is not difficult to prove the following sequence of inequalities:

$$\ell(M(A_l) \otimes M(A_m)) \leqslant \ell(M(A_l) \otimes M(B_m)) \leqslant \ell(M(A_l) \otimes M(A_{m+1}))\,. \tag{3.1}$$

It is entirely possible that each of these inequalities are identities. This is the case for example if $l = 8$, $m = 9$: $\ell(M(A_8) \otimes M(A_9)) = 2 + 8\#9 = 2 + 15 = 2 + 8\#10 = \ell(M(A_8) \otimes M(A_{10}))$.

Corollary 5. *Let $l, m < 2q$, $0 < l_1, l_2, m_1, m_2 < 2q$, and $\rho, \sigma \in k \smallsetminus \{0\}$.*

1. *$M(A_l) \otimes M(B_m)$ has a projective direct summand if, and only if, $l + m \geqslant 2q$,*
2. *$M(A_l) \otimes M(A_m)$ has a projective direct summand if, and only if, $l + m \geqslant 2q + 1$.*
3. *$M(A_{l_1}B_{l_2}^{-1}, \rho) \otimes M(A_m)$ has a projective direct summand precisely when*

$$\max\{l_1 + m - 1, l_2 + m\} \geqslant 2q.$$

4. *If $l_1 \neq l_2$ or $m_1 \neq m_2$, then $M\left(A_{l_1}B_{l_2}^{-1}, \rho\right) \otimes M\left(A_{m_1}B_{m_2}^{-1}, \sigma\right)$ has a projective direct summand if, and only if,*

$$\max\{l_1 + m_1 - 1, l_1 + m_2, l_2 + m_1, l_2 + m_2 - 1\} \geqslant 2q\,.$$

5. *If $l_1 = l_2$, $m_1 = m_2$ then $M\left(A_{l_1}B_{l_2}^{-1}, \rho\right) \otimes M\left(A_{m_1}B_{m_2}^{-1}, \sigma\right)$ has projective direct summands if, and only if,*
   (a) *$l_1 \perp (m_1-1)$, $\rho \neq \sigma$ and $l_1 + m_1 = 2q$, or*
   (b) *$l_1 \not\perp (m_1-1)$, and $l_1 + m_1 \geqslant 2q$.*

We remark that, for $l, m < 2q$, the condition $l + m \geqslant 2q$ implies $l \not\perp m$. Thus, in particular, in 5(a) above, the condition $l_1 \perp (m_1-1)$ is equivalent to $(l_1-1) \perp m_1$, and similarly, in 5(b), $l_1 \not\perp (m_1-1)$ could be replaced by $(l_1-1) \not\perp m_1$.

4. Examples

Example 6. Let $M = M(A_5 B_7^{-1} B_4)$, $N = M(A_6 B_4^{-1})$. By Proposition 1,

$$\begin{aligned}\ell(M \otimes N) &= \max\left\{\ell(M(w) \otimes M(w')) \mid w \in \{A_5, B_7^{-1}, B_4\}, w' \in \{A_6, B_4^{-1}\}\right\} \\ &= \ell(M(B_7^{-1}) \otimes M(A_6)) = \ell(M(B_7) \otimes M(A_6))\,.\end{aligned}$$

Since $7 \not\preceq 6$, by Theorem 4:1 we have, $\ell(M(B_7) \otimes M(A_6)) = 2 + 7\#6 = 9$. Hence, the Loewy length of $M \otimes N$ is 9 and, seen as a kD_{16}-module, $M \otimes N$ has a projective direct summand.

Example 7. By Proposition 1 and the inequality (3.1), we have

$$\begin{aligned}\ell(M(A_l) \otimes M(A_{m+1} B_m^{-1})) &= \max\{\ell(M(A_l) \otimes M(A_{m+1})), \ell(M(A_l) \otimes M(B_m))\} \\ &= \ell(M(A_l) \otimes M(A_{m+1}))\,.\end{aligned}$$

for all $l, m \in \mathbb{N}$.

Example 8. While it is clear that $\ell(M(A_l B_l^{-1}, 1) \otimes N) \leqslant \ell(M(A_l B_l^{-1}) \otimes N)$, the difference between the lengths of the two tensor products may be zero, or arbitrarily large. For example, if $N = M(A_m)$ and $l \not\preceq m$, then $\ell(M(A_l B_l^{-1}, 1) \otimes N) = \ell(M(A_l B_l^{-1}) \otimes N)$ by Theorem 4:2. If, on the other hand, $l = 2^r$ and $m = 2^s$ with $r > s$ then

$$\ell(M(A_l B_l^{-1}, 1) \otimes M(A_m)) = 2 + (l-1)\#m = 2 + 2^r - 1 = 1 + 2^r,$$

while

$$\ell(M(A_l B_l^{-1}) \otimes M(A_m)) = \ell(M(B_l) \otimes M(A_m)) = 1 + l + m = 1 + 2^r + 2^s\,.$$

Example 9. Let $M = M(a) = M(A_1)$ and $N = M(b(ab)^l) = M(B_{2l+1})$ for some $l \in \mathbb{N}$. Now $1 \not\preceq (2l+1)$, and $1\#(2l+1) = 2l+1$, so by Theorem 4:1, $\ell(M \otimes N) = 2l+3$. In this case, the Loewy length actually provides the missing piece of information to compute the isomorphism type of $M \otimes N$.

Namely, since k is the unique simple module, we have

$$\begin{aligned}\dim \operatorname{soc}(M \otimes N) &= \dim \operatorname{Hom}_{kD_{4q}}(k, M \otimes N) = \dim \operatorname{Hom}_{kD_{4q}}(N^*, M) \\ &= \dim \operatorname{Hom}_{kD_{4q}}(N, M) = 1\end{aligned}$$

and similarly,

$$\dim \operatorname{top}(M \otimes N) = \dim \operatorname{Hom}_{kD_{4q}}(M \otimes N, k) = 1\,.$$

Hence $M \otimes N$ is a module with simple top and simple socle, of dimension $4(l+1)$, and Loewy length $2l+3$. A module satisfying these conditions is indecomposable, and must be isomorphic to $M(A_{2l+2} B_{2l+2}^{-1}, \rho)$ for some $\rho \in k \setminus \{0\}$. Now if k is the prime field, that is the Galois field with two elements, this means that $\rho = 1$. From this follows that $\rho = 1$ also in the general case, since extension of scalars commutes with taking tensor products. Hence, we have $M \otimes N \simeq M(A_{2l+2} B_{2l+2}^{-1}, 1)$.

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EXAMPLE OF CATEGORIFICATION OF A CLUSTER ALGEBRA

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Abstract. We present here two detailed examples of additive categorifications of the cluster algebra structure of a coordinate ring of a maximal unipotent subgroup of a simple Lie group. The first one is of simply-laced type (A_3) and relies on an article by Geiß, Leclerc and Schröer. The second is of non simply-laced type (C_2) and relies on an article by the author of this note. This is aimed to be accessible, specially for people who are not familiar with this subject.

1. Introduction: the total positivity problem

Let N be the subgroup of $\mathrm{SL}_4(\mathbb{C})$ consisting of upper triangular matrices with diagonal 1. We say that $X \in N$ is *totally positive* if its 12 non-trivial minors are positive real numbers (a minor is non-trivial if it is not constant on N and not product of other minors). As a consequence of various results of Fomin and Zelevinsky [3] (see also [1]), in a (very) special case, we get

Proposition 1 (Fomin-Zelevinsky). *$X \in N$ is totally positive if and only if the minors $\Delta_4^1(X)$, $\Delta_{34}^{12}(X)$, $\Delta_{234}^{123}(X)$, $\Delta_{24}^{12}(X)$, $\Delta_4^2(X)$, $\Delta_4^3(X)$ are positive.*

where $\Delta_{c_1 \dots c_k}^{\ell_1 \dots \ell_k}(X)$ is the minor of X with rows $\ell_1, \dots, \ell_k$ and columns $c_1, \dots, c_k$.

Remark that, as the algebraic variety N has dimension 6, we can not expect to find a criterion with less than 6 inequalities to check the total positivity of a matrix.

To prove this, just remark that we have the following equality:

$$\Delta_{24}^{12}\Delta_{34}^{23} = \Delta_{234}^{123}\Delta_4^2 + \Delta_4^3\Delta_{34}^{12}$$

which immediately implies that $\Delta_4^1(X)$, $\Delta_{34}^{12}(X)$, $\Delta_{234}^{123}(X)$, $\Delta_{24}^{12}(X)$, $\Delta_4^2(X)$, $\Delta_4^3(X)$ are positive if and only if $\Delta_4^1(X)$, $\Delta_{34}^{12}(X)$, $\Delta_{234}^{123}(X)$, $\Delta_{34}^{23}(X)$, $\Delta_4^2(X)$, $\Delta_4^3(X)$ are positive. Such an equality is called an *exchange identity*. In Figure 1, we wrote 14 sets of minors which are related by exchange identities whenever they are linked by an edge. As every minor appears in this graph, it induces the previous proposition.

These observations lead to the definition of a *cluster algebra* [4]. A cluster algebra is an algebra endowed with an additional combinatorial structure. Namely, a (generally infinite) set of distinguished elements called *cluster variables* grouped into subsets of the same cardinality n, called *clusters* and a finite set $\{x_{n+1}, x_{n+2}, \dots, x_m\}$ called the set of *coefficients*. For each cluster $\{x_1, x_2, \dots, x_n\}$, the *extended cluster* $\{x_1, \dots, x_n, x_{n+1}, \dots, x_m\}$ is a transcendence basis of the algebra. Moreover, each cluster $\{x_1, x_2, \dots, x_n\}$ has n

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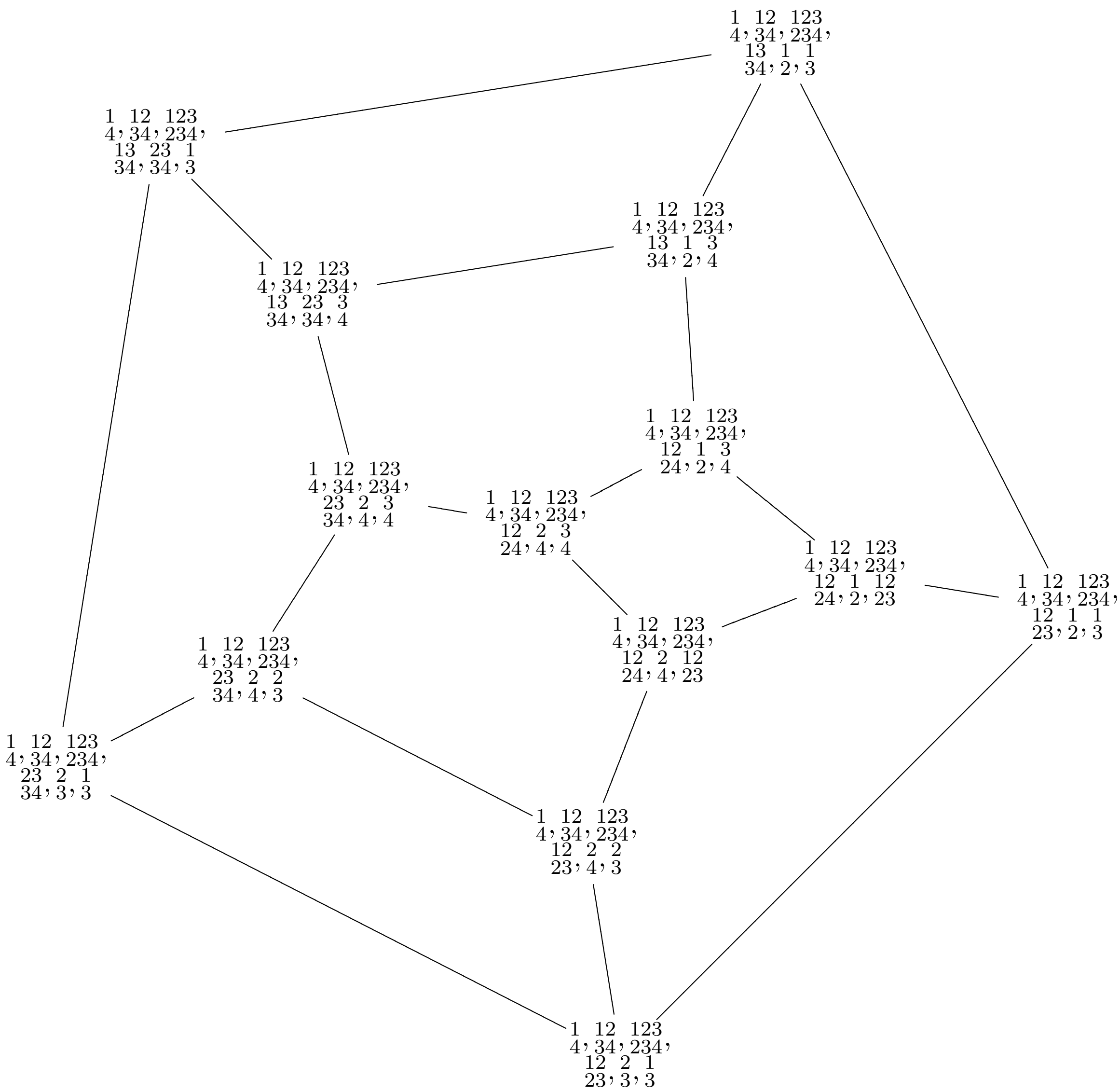

FIGURE 1. Exchange graph of minors

neighbours obtained by replacing one of its elements x_k by a new one x'_k related by a relation

$$x_k x'_k = M_1 + M_2$$

where M_1 and M_2 are mutually prime monomials in $\{x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_m\}$, given by precise combinatorial rules. These replacements, called *mutations* and denoted by μ_k are involutive. For precise definitions and details about these constructions, we refer to [4].

In the previous example, the coefficients are Δ_4^1, Δ_{34}^{12} and Δ_{234}^{123} and the cluster variables are all the other non-trivial minors. The extended clusters are the sets appearing at the vertices of Figure 1.

The aim of the following sections is to describe examples of *additive categorifications* of cluster algebras. It consists of enhancing the cluster algebra structure with an additive category, some objects of which reflect the combinatorial structure of the cluster algebra; moreover, there is an explicit formula, the *cluster character* associating to these particular objects elements of the algebra, in a way which is compatible with the combinatorial structure. The examples we develop here rely on (abelian) module categories. They are particular cases of categorifications by exact categories appearing in [6] (simply-laced case) and [2] (non simply-laced case). The study of cluster algebras and their categorifications has been particularly successful these last years. For a survey on categorification by triangulated categories and a much more complete bibliography, see [7].

2. The preprojective algebra and the cluster character

Let Q be the following quiver (oriented graph):

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, "\alpha^*"] \arrow[r, bend right, "\beta^*"'] & 3 \arrow[l, bend right, "\beta"'] \end{tikzcd}$$

As usual, denote by $\mathbb{C}Q$ the $\mathbb{C}$-algebra, a basis of which is formed by the paths (including 0-length paths supported by each of the three vertices) and the multiplication of which is defined by concatenation of paths when it is possible and vanishes when paths can not be composed (we write here the composition from left to right, on the contrary to the usual composition of maps). Thus, a (right) $\mathbb{C}Q$-module is naturally graded by idempotents (0-length paths) corresponding to vertices and the action of arrows seen as elements of the algebra can naturally be identified with linear maps between the corresponding homogeneous subspaces of the representation. We shall use the following right-hand side convenient notation:

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "\begin{pmatrix} 0 \\ -1 \end{pmatrix}"] & \mathbb{C}^2 \arrow[l, bend left, "\begin{pmatrix} 1 & 0 \end{pmatrix}"] \arrow[r, bend right, "\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}"'] & \mathbb{C}^2 \arrow[l, bend right, "\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}"'] \end{tikzcd} = \begin{tikzcd}[column sep=small, row sep=small] & 2 \arrow[dl] \arrow[dr] & \\ 1 \arrow[dr, "-1"'] & & 3 \arrow[dl] \\ & 2 \arrow[dr] & \\ & & 3 \end{tikzcd}$$

where each of the digits represents a basis vector of the representation and each arrow a non-zero scalar (1 when not specified) in the corresponding matrix entry.

Let us now introduce the preprojective algebra of Q:

Definition 2. The *preprojective algebra* of Q is defined by

$$\Pi_Q = \frac{\mathbb{C}Q}{(\alpha\alpha^*, \alpha^*\alpha + \beta^*\beta, \beta\beta^*)}$$

the representations of which are seen as particular representations of $\mathbb{C}Q$ (in other words, $\operatorname{mod}\Pi_Q$ is a full subcategory of $\operatorname{mod}\mathbb{C}Q$).

Example 3. Among the following representations of $\mathbb{C}Q$, the first one and the second one are representations of Π_Q:

$$
\begin{tikzcd}[column sep=small,row sep=small]
1 \arrow[dr] & & \\
& 2 \arrow[dr] & \\
& & 3
\end{tikzcd}
\; ; \;
\begin{tikzcd}[column sep=small,row sep=small]
& 2 \arrow[dl] \arrow[dr] & \\
1 \arrow[dr, "-1"'] & & 3 \arrow[dl] \\
& 2 &
\end{tikzcd}
\; ; \;
\begin{tikzcd}
1 \arrow[r, bend left] & 2 \arrow[l, bend left]
\end{tikzcd}
\; ; \;
\begin{tikzcd}[column sep=small,row sep=small]
& 2 \arrow[dl] \arrow[dr] & \\
1 \arrow[dr, "-1"'] & & 3 \arrow[dl] \\
& 2 \arrow[dr] & \\
& & 3
\end{tikzcd}
.
$$

One of the property, which is discussed in many places (for example in [6]), of the preprojective algebra of Q, fundamental for this categorification, is

Proposition 4. *The category* $\operatorname{mod}\Pi_Q$ *is* stably 2-Calabi-Yau. *In other words, for every* $X, Y \in \operatorname{mod}\Pi_Q$,

$$\operatorname{Ext}^1(X,Y) \simeq \operatorname{Ext}^1(Y,X)^*$$

functorially in X and Y, where $\operatorname{Ext}^1(Y,X)^*$ *is the $\mathbb{C}$-dual of* $\operatorname{Ext}^1(Y,X)$. *In particular, it is a Frobenius category (is has enough projective objects and enough injective objects and they coincide).*

Let us now define the three following one-parameter subgroups of N:

$$x_1(t) = \begin{pmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad x_2(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad x_3(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For $X \in \operatorname{mod}\Pi_Q$ and any sequence of vertices $a_1, a_2, \ldots, a_n$ of Q, we denote by

$$\Phi_{X,a_1a_2\ldots a_n} = \left\{ 0 = X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X \mid \forall i \in \{1,2,\ldots,n\}, \frac{X_i}{X_{i-1}} \simeq S_{a_i} \right\}$$

the *variety of composition series* of X of type $a_1a_2\ldots a_n$ (S_{a_i} is the simple module, of dimension 1, supported at vertex a_i). This is a closed algebraic subvariety of the product of Grassmannians

$$\operatorname{Gr}_1(X) \times \operatorname{Gr}_2(X) \times \cdots \times \operatorname{Gr}_n(X).$$

We denote by χ the Euler characteristic. Using results of Lusztig and Kashiwara-Saito, Geiß-Leclerc-Schroër proved the following result:

Theorem 5 ([6])**.** *Let $X \in \operatorname{mod}\Pi_Q$. There is a unique $\varphi_X \in \mathbb{C}[N]$ such that*

$$\varphi_X\left(x_{a_1}(t_1)x_{a_2}(t_2)\ldots x_{a_6}(t_6)\right) = \sum_{i_1,i_2,\ldots,i_6\in\mathbb{N}} \chi\left(\Phi_{X,a_1^{i_1}a_2^{i_2}\ldots a_6^{i_6}}\right) \frac{t_1^{i_1}t_2^{i_2}\ldots t_6^{i_6}}{i_1!i_2!\ldots i_6!}$$

for every word $a_1a_2a_3a_4a_5a_6$ representing the longest element of $\mathfrak{S}_4$ ($a_k^{i_k}$ is the repetition i_k times of a_k).

The map $\varphi : \operatorname{mod}\Pi_Q \to \mathbb{C}[N]$ is called a *cluster character.*

Remark 6. (1) The uniqueness in the previous theorem is easy because it is well known that

$$x_{a_1}(t_1)x_{a_2}(t_2)\ldots x_{a_6}(t_6)$$

runs over a dense subset of N ;

$X \in \operatorname{mod}\Pi_Q$	S_1	S_2	S_3	$\begin{smallmatrix}1&&\\&\searrow&\\&&2\end{smallmatrix}$	$\begin{smallmatrix}&&2\\&\swarrow&\\1&&\end{smallmatrix}$	$\begin{smallmatrix}2&&\\&\searrow&\\&&3\end{smallmatrix}$	$\begin{smallmatrix}&&3\\&\swarrow&\\2&&\end{smallmatrix}$
$\varphi_X \in \mathbb{C}[N]$	Δ_2^1	Δ_3^2	Δ_4^3	Δ_{23}^{12}	Δ_3^1	Δ_{34}^{23}	Δ_4^2

$X \in \operatorname{mod}\Pi_Q$	$\begin{smallmatrix}&&2&&\\&\swarrow&&\searrow&\\1&&&&3\end{smallmatrix}$	$\begin{smallmatrix}1&&&&3\\&\searrow&&\swarrow&\\&&2&&\end{smallmatrix}$	$1[dr]$ $\begin{smallmatrix}2&&\\&\searrow&\\&&3\end{smallmatrix}$	$\begin{smallmatrix}&&2&&\\&\swarrow&&\searrow&\\1&&&&3\\&\searrow&&\swarrow&\\&&2&&\end{smallmatrix}$	$\begin{smallmatrix}&&&&3\\&&&\swarrow&\\&&2&&\\&\swarrow&&&\\1&&&&\end{smallmatrix}$
$\varphi_X \in \mathbb{C}[N]$	Δ_{34}^{13}	Δ_{24}^{12}	Δ_{234}^{123}	Δ_{34}^{12}	Δ_4^1

FIGURE 2. Cluster character

(2) the existence is much harder and strongly relies on the construction of semi-canonical bases by Lusztig [8]. In particular, the fact that it does not depend on the choice of $a_1a_2a_3a_4a_5a_6$ is not clear *a priori* (see the following examples).

Example 7. We suppose that $a_1a_2a_3a_4a_5a_6 = 213213$. Then

$$x_{a_1}(t_1)x_{a_2}(t_2)x_{a_3}(t_3)x_{a_4}(t_4)x_{a_5}(t_5)x_{a_6}(t_6) = \begin{pmatrix} 1 & t_2+t_5 & t_2t_4 & t_2t_4t_6 \\ 0 & 1 & t_1+t_4 & t_1t_3+t_1t_6+t_4t_6 \\ 0 & 0 & 1 & t_3+t_6 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- The module S_1 has only one composition series, of type 1. Therefore $\Phi_1(S_1)$ is one point and $\Phi_{\mathbf{a}}(S_1) = \emptyset$ for any other $\mathbf{a}$. Identifying the two members in the formula of the previous theorem,

$$\varphi_{S_1}\left(x_{a_1}(t_1)x_{a_2}(t_2)x_{a_3}(t_3)x_{a_4}(t_4)x_{a_5}(t_5)x_{a_6}(t_6)\right) = t_2 + t_5 = \Delta_2^1.$$

- The module

$$P_2 = \begin{smallmatrix}&&2&&\\&\swarrow&&\searrow&\\1&&&&3\\&\searrow&&\swarrow&\\&&2&&\end{smallmatrix}$$

has two composition series, of type 2312 and 2132. Therefore,

$$\varphi_{P_2}\left(x_{a_1}(t_1)x_{a_2}(t_2)x_{a_3}(t_3)x_{a_4}(t_4)x_{a_5}(t_5)x_{a_6}(t_6)\right) = t_1t_2t_3t_4 = \Delta_{34}^{12}.$$

Remark that, in this case, the only composition series which is playing a role is 2132, even if the situation is symmetric. This justify the second part of the previous remark.

The other indecomposable representations of Π_Q and their cluster character values are collected in Figure 2.

Two important properties of this cluster character were proved by Geiß-Leclerc-Schroër (see for example [6]):

Proposition 8. *Let $X, Y \in \operatorname{mod}\Pi_Q$.*

(1) $\varphi_{X\oplus Y} = \varphi_X\varphi_Y$.

(2) *Suppose that* $\dim \mathrm{Ext}^1(X,Y) = 1$ *(and therefore* $\dim \mathrm{Ext}^1(Y,X) = 1$*) and let*

$$0 \to X \to T_a \to Y \to 0 \quad \textit{and} \quad 0 \to Y \to T_b \to X \to 0$$

be two (unique up to isomorphism) non-split short exact sequences. Then

$$\varphi_X \varphi_Y = \varphi_{T_a} + \varphi_{T_b}.$$

3. Minimal approximations

This section recall the definition and elementary properties of approximations. It is there for the sake of ease. In what follows, $\operatorname{mod} \Pi_Q$ can be replaced by any additive Hom-finite category over a field.

Definition 9. Let X and T be two objects of $\operatorname{mod} \Pi_Q$. A *left* $\mathrm{add}(T)$*-approximation* of X is a morphism $f : X \to T'$ such that

- $T' \in \mathrm{add}(T)$ (which means that every indecomposable summand of T' is an indecomposable summand of T) ;
- every morphism $g : X \to T$ factors through f.

If, moreover, there is no strict direct summand T'' of T' and left *add*(T)-approximation $f' : X \to T''$, then f is said to be a *minimal left* $\mathrm{add}(T)$*-approximation.*

In the same way, we can define

Definition 10. Let X and T be two objects in $\operatorname{mod} \Pi_Q$. A *right* $\mathrm{add}(T)$*-approximation* of X is a morphism $f : T' \to X$ such that

- $T' \in \mathrm{add}(T)$;
- every morphism $g : T \to X$ factors through f.

If, moreover, there is no strict direct summand T'' of T' and right *add*(T)-approximation $f' : T'' \to X$, then f is said to be a *minimal right* $\mathrm{add}(T)$*-approximation.*

Now, a classical proposition which permits to explicitly compute approximations:

Proposition 11. *Let* X *and* $T \simeq T_1^{i_1} \oplus T_2^{i_2} \oplus \cdots \oplus T_n^{i_n}$ *be two objects in* $\operatorname{mod} \Pi_Q$ *(the* T_i*'s are non-isomorphic indecomposable). For* $i, j \in \{1, \ldots, n\}$*, we denote by* I_{ij} *the subvector space of* $\mathrm{Hom}(T_i, T_j)$ *consisting of the non-invertible morphisms (*$I_{ij} = \mathrm{Hom}(T_i, T_j)$ *if* $i \neq j$*). Thus, for* $j \in \{1, \ldots, n\}$*, we obtain a linear map*

$$\bigoplus_{i \in \{1,\ldots,n\}} I_{ij} \otimes \mathrm{Hom}(X, T_i) \xrightarrow{\varphi_j} \mathrm{Hom}(X, T_j)$$

$$(g, f) \mapsto g \circ f.$$

Let $\mathcal{B}_j$ *be a basis of* $\operatorname{coker} \varphi_j$ *lifted to* $\mathrm{Hom}(X, T_j)$*. Then the morphism*

$$X \xrightarrow{(f)_{j \in \{1,\ldots,n\}, f \in \mathcal{B}_j}} \bigoplus_{j \in \{1,\ldots,n\}} T_j^{\# \mathcal{B}_j}$$

is a minimal left $\mathrm{add}(T)$*-approximation of* X*. Moreover, any minimal left* $\mathrm{add}(T)$*-approximation of* X *is isomorphic to it.*

The previous proposition has a dual version which permits to compute minimal right approximations. In practice, this computation relies on searching morphisms up to factorization through other objects. There is an explicit example of computation in Example 19.

4. Maximal rigid objects and their mutations

Let us introduce the objects the combinatorics of which will play the role of the cluster algebra structure.

Definition 12. Let $X \in \operatorname{mod} \Pi_Q$.

- The module X is said to be *rigid* if it has no self-extension, (*i.e.*, $\operatorname{Ext}^1(X, X) = 0$).
- The module X is said to be *basic maximal rigid* if it is basic (*i.e.*, it does not have two isomorphic indecomposable summands), rigid, and maximal for these two properties.

Remark 13. A basic maximal rigid Π_Q-module contains Π_Q as a direct summand (because Π_Q is both projective and injective and therefore has no extension with any module).

Example 14. The object

$$\begin{tikzcd}[row sep=small, column sep=small] 1 \arrow[dr] & & 3 \arrow[dl] \\ & 2 & \end{tikzcd} \oplus \begin{tikzcd}[row sep=small, column sep=small] & 3 \arrow[dl] \\ 2 & \end{tikzcd} \oplus \begin{tikzcd}[row sep=small, column sep=small] 1 \arrow[dr] & \\ & 2 \end{tikzcd} \oplus \begin{tikzcd}[row sep=small, column sep=small] & 1[dr] \\ 2 \arrow[dr] & \\ & 3 \end{tikzcd} \oplus \begin{tikzcd}[row sep=small, column sep=small] & 2 \arrow[dl] \arrow[dr] & \\ 1 \arrow[dr] & & 3 \arrow[dl] \\ & 2 & \end{tikzcd} \oplus \begin{tikzcd}[row sep=small, column sep=small] & & 3 \arrow[dl] \\ & 2 \arrow[dl] & \\ 1 & & \end{tikzcd},$$

the last three summands of which are the indecomposable projective-injective Π_Q-modules, is basic maximal rigid. It is easy to check that it is basic and rigid, but more difficult to prove that it is maximal for these properties (see [6] for more details).

Remark 15. We can prove that all basic maximal rigid objects have the same number of indecomposable summands (six in the example we are talking about).

The following result permits to define a mutation on basic maximal rigid objects. Considered as an operation on isomorphism classes of basic maximal rigid objects, the induced combinatorial structure will correspond to the one of a cluster algebra.

Theorem 16 ([6]). *Let $T \simeq T_1 \oplus T_2 \oplus T_3 \oplus P_1 \oplus P_2 \oplus P_3 \in \operatorname{mod} \Pi_Q$ be basic maximal rigid such that P_1, P_2 and P_3 are the indecomposable projective Π_Q-modules and T_1, T_2 and T_3 are indecomposable non-projective Π_Q-modules. Then, for $i \in \{1, 2, 3\}$, there are two (unique) short exact sequences*

$$0 \to T_i \xrightarrow{f} T_a \xrightarrow{f'} T_i^* \to 0 \quad \textit{and} \quad 0 \to T_i^* \xrightarrow{g} T_b \xrightarrow{g'} T_i \to 0$$

such that

1. *f and g are minimal left $\operatorname{add}(T/T_i)$-approximations ;*
2. *f' and g' are minimal right $\operatorname{add}(T/T_i)$-approximations ;*
3. *T_i^* is indecomposable and non-projective ;*
4. $\dim \operatorname{Ext}^1(T_i, T_i^*) = \dim \operatorname{Ext}^1(T_i^*, T_i) = 1$ *and the two short exact sequences do not split ;*
5. *$\mu_i(T) = T/T_i \oplus T_i^*$ is basic maximal rigid ;*

(6) *T_a and T_b do not have common summands.*

Remark 17. In the previous theorem, the existence and uniqueness, regarding the first two conditions, are automatic, except the fact that the extremities of the two short exact sequences coincide up to order. This fact strongly relies on the stably 2-Calabi-Yau property. It implies that μ_i is involutive.

Definition 18. In the previous theorem, μ_i is called the *mutation in direction i*. The short exact sequences appearing are called *exchange sequences.*

Example 19. Let

$$T = \begin{smallmatrix}1&&3\\&\searrow\swarrow&\\&2&\end{smallmatrix} \oplus \begin{smallmatrix}&&3\\&\swarrow&\\2&&\end{smallmatrix} \oplus \begin{smallmatrix}1&&\\&\searrow&\\&&2\end{smallmatrix} \oplus \begin{smallmatrix}&1[dr]&\\2&&\\&\searrow&\\&&3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\ \swarrow&&\searrow\\1&&3\\ \searrow&&\swarrow\\&2&\end{smallmatrix} \oplus \begin{smallmatrix}&&&&3\\&&&\swarrow&\\&&2&&\\&\swarrow&&&\\1&&&&\end{smallmatrix}.$$

Using Proposition 11, we get a left add $\left(T/\begin{smallmatrix}&&3\\&\swarrow&\\2&&\end{smallmatrix}\right)$-approximation of $\begin{smallmatrix}&&3\\&\swarrow&\\2&&\end{smallmatrix}$:

$$\begin{smallmatrix}&&3\\&\swarrow&\\2&&\end{smallmatrix} \to \begin{smallmatrix}1&&3\\&\searrow\swarrow&\\&2&\end{smallmatrix}$$

and computing the cokernel, we get the exchange sequence:

$$0 \to \begin{smallmatrix}&&3\\&\swarrow&\\2&&\end{smallmatrix} \to \begin{smallmatrix}1&&3\\&\searrow\swarrow&\\&2&\end{smallmatrix} \to S_1 \to 0$$

so that

$$\mu_2(T) = \begin{smallmatrix}1&&3\\&\searrow\swarrow&\\&2&\end{smallmatrix} \oplus S_1 \oplus \begin{smallmatrix}1&&\\&\searrow&\\&&2\end{smallmatrix} \oplus \begin{smallmatrix}&1[dr]&\\2&&\\&\searrow&\\&&3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\ \swarrow&&\searrow\\1&&3\\ \searrow&&\swarrow\\&2&\end{smallmatrix} \oplus \begin{smallmatrix}&&&&3\\&&&\swarrow&\\&&2&&\\&\swarrow&&&\\1&&&&\end{smallmatrix}.$$

Doing mutation in the reverse direction:

$$0 \to S_1 \to \begin{smallmatrix}&&&&3\\&&&\swarrow&\\&&2&&\\&\swarrow&&&\\1&&&&\end{smallmatrix} \to \begin{smallmatrix}&&3\\&\swarrow&\\2&&\end{smallmatrix} \to 0.$$

Let us now compute $\mu_1\mu_2(T)$ with its two exchange sequences:

$$0 \to \begin{smallmatrix}1&&3\\&\searrow\swarrow&\\&2&\end{smallmatrix} \to S_1 \oplus \begin{smallmatrix}&2&\\ \swarrow&&\searrow\\1&&3\\ \searrow&&\swarrow\\&2&\end{smallmatrix} \to \begin{smallmatrix}&&2\\&\swarrow&\\1&&\end{smallmatrix} \to 0$$

$$0 \to \begin{smallmatrix}&&2\\&\swarrow&\\1&&\end{smallmatrix} \to \begin{smallmatrix}1&&\\&\searrow&\\&&2\end{smallmatrix} \oplus \begin{smallmatrix}&&&&3\\&&&\swarrow&\\&&2&&\\&\swarrow&&&\\1&&&&\end{smallmatrix} \to \begin{smallmatrix}1&&3\\&\searrow\swarrow&\\&2&\end{smallmatrix} \to 0$$

$$\mu_1\mu_2(T) = \begin{smallmatrix}&&2\\&\swarrow&\\1&&\end{smallmatrix} \oplus S_1 \oplus \begin{smallmatrix}1&&\\&\searrow&\\&&2\end{smallmatrix} \oplus \begin{smallmatrix}&1[dr]&\\2&&\\&\searrow&\\&&3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\ \swarrow&&\searrow\\1&&3\\ \searrow&&\swarrow\\&2&\end{smallmatrix} \oplus \begin{smallmatrix}&&&&3\\&&&\swarrow&\\&&2&&\\&\swarrow&&&\\1&&&&\end{smallmatrix}.$$

Computing inductively all the mutations, we obtain the *exchange graph of maximal rigid objects of* Π_Q (Figure 3).

Then, using Proposition 8 and Theorem 16 together with other technical results, we get the following proposition:

Proposition 20 ([6])**.** *If we project the mutation of maximal rigid objects to* $\mathbb{C}[N]$ *through the cluster character* φ*, we get a cluster algebra structure on* $\mathbb{C}[N]$ *(in the sense of* [4]*). Moreover, this structure is the one proposed combinatorially in* [1]*. Under this projection, we get the correspondence:*

$$\begin{aligned} \{\textit{non projective indecomposable objects}\} &\leftrightarrow \{\textit{cluster variables}\} \\ \{\textit{projective indecomposable objects}\} &\leftrightarrow \{\textit{coefficients}\} \\ \{\textit{basic maximal rigid objects}\} &\leftrightarrow \{\textit{extended clusters}\} \end{aligned}$$

Example 21. Taking the notation of Example 19 and looking at Figure 2, we get:

$$\Delta_2^1\Delta_4^2 = \varphi_{S_1}\varphi \begin{smallmatrix} & & 3 \\ & \swarrow & \\ 2 & & \end{smallmatrix} = \varphi \begin{smallmatrix} 1 & & & & 3 \\ & \searrow & & \swarrow & \\ & & 2 & & \end{smallmatrix} + \varphi \begin{smallmatrix} & & & & 3 \\ & & & \swarrow & \\ & & 2 & & \\ & \swarrow & & & \\ 1 & & & & \end{smallmatrix} = \Delta_{24}^{12} + \Delta_4^1$$

and

$$\begin{aligned} \Delta_{24}^{12}\Delta_3^1 &= \varphi \begin{smallmatrix} 1 & & & & 3 \\ & \searrow & & \swarrow & \\ & & 2 & & \end{smallmatrix} \varphi \begin{smallmatrix} & & 2 \\ & \swarrow & \\ 1 & & \end{smallmatrix} = \varphi \left(S_1 \oplus \begin{smallmatrix} & & 2 & & \\ & \swarrow & & \searrow & \\ 1 & & & & 3 \\ & \searrow & & \swarrow & \\ & & 2 & & \end{smallmatrix} \right) + \varphi \left(\begin{smallmatrix} 1 & & \\ & \searrow & \\ & & 2 \end{smallmatrix} \oplus \begin{smallmatrix} & & & & 3 \\ & & & \swarrow & \\ & & 2 & & \\ & \swarrow & & & \\ 1 & & & & \end{smallmatrix} \right) \\ &= \varphi_{S_1}\varphi \begin{smallmatrix} & & 2 & & \\ & \swarrow & & \searrow & \\ 1 & & & & 3 \\ & \searrow & & \swarrow & \\ & & 2 & & \end{smallmatrix} + \varphi \begin{smallmatrix} 1 & & \\ & \searrow & \\ & & 2 \end{smallmatrix} \varphi \begin{smallmatrix} & & & & 3 \\ & & & \swarrow & \\ & & 2 & & \\ & \swarrow & & & \\ 1 & & & & \end{smallmatrix} = \Delta_2^1\Delta_{34}^{12} + \Delta_{23}^{12}\Delta_4^1. \end{aligned}$$

which can be easily checked by hand. These are part of the equalities which appear in the proof of Proposition 1.

5. From simply-laced case to general one

Define the following symplectic form:

$$\Psi = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

and the subgroup

$$N' = \{M \in N | {}^tM\Psi M = \Psi\} \quad \text{or, equivalently} \quad N' = N^{\mathbb{Z}/2\mathbb{Z}}$$

where $\mathbb{Z}/2\mathbb{Z} = \langle g \rangle$ acts on N by $M \mapsto \Psi^{-1}\,({}^tM^{-1})\,\Psi$. The group N' is a maximal unipotent subgroup of a symplectic group of type C_2.

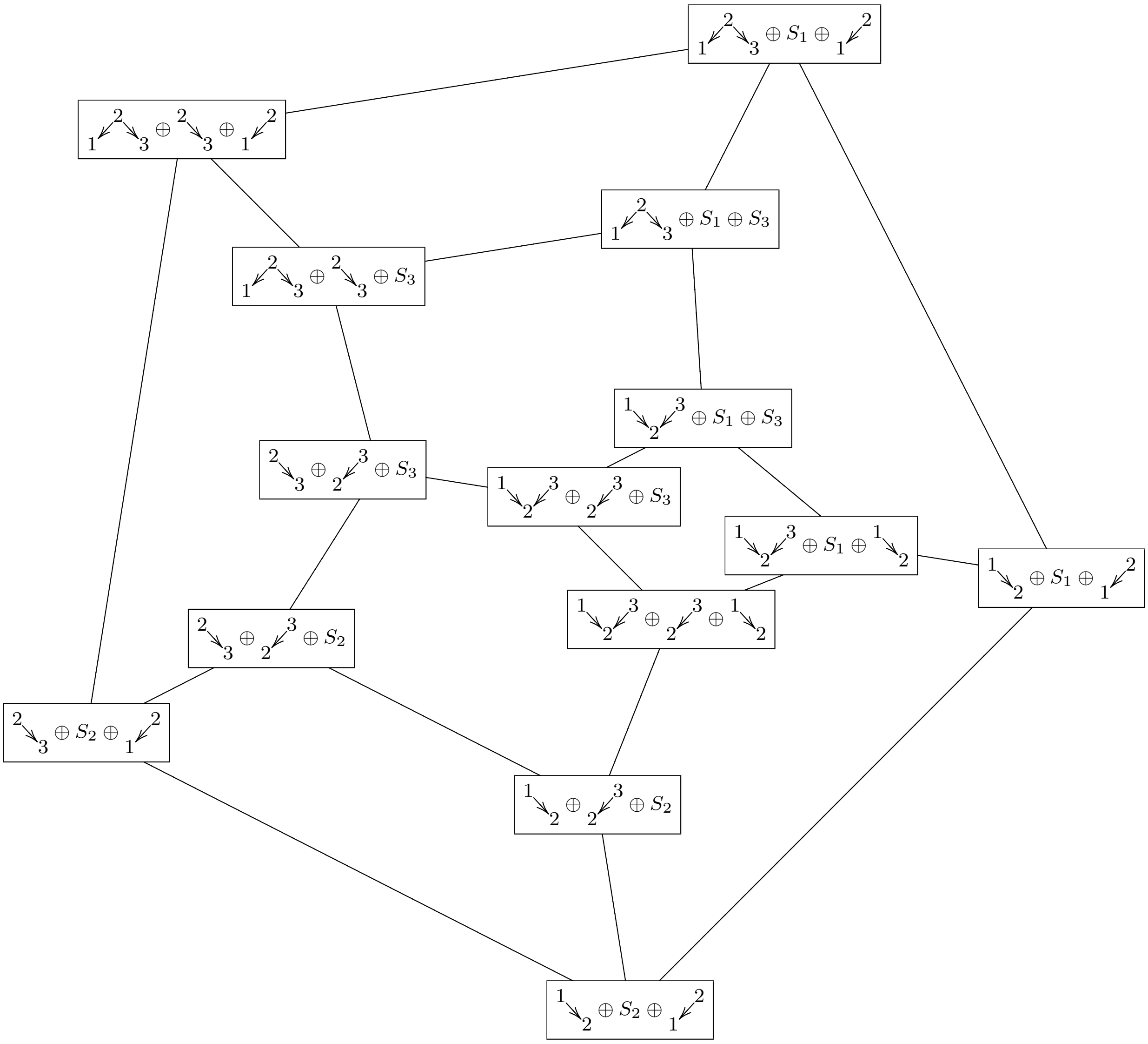

FIGURE 3. Exchange graph of maximal rigid objects (up to projective summands)

The only non-trivial action of $\mathbb{Z}/2\mathbb{Z}$ on Q induces an action on Π_Q and therefore on $\operatorname{mod}\Pi_Q$. Denote by $\pi : \mathbb{C}[N] \to \mathbb{C}[N']$ the canonical projection. We can now formulate the following result:

Theorem 22 ([2]). (1) *If T is a $\mathbb{Z}/2\mathbb{Z}$-stable basic maximal rigid Π_Q-module, then $\mu_1\mu_3(T) = \mu_3\mu_1(T)$. Moreover, $\mu_1\mu_3(T)$ and $\mu_2(T)$ are also $\mathbb{Z}/2\mathbb{Z}$-stable.*

(2) *If $X \in \operatorname{mod}\Pi_Q$, then $\pi(\varphi_X) = \pi(\varphi_{gX})$.*

(3) *If we denote $\bar{\mu}_2 = \mu_2$ and $\bar{\mu}_1 = \mu_1\mu_3 = \mu_3\mu_1$, acting on the set of $\mathbb{Z}/2\mathbb{Z}$-stable maximal rigid Π_Q-modules, $\bar{\mu}$ induces through $\pi \circ \varphi$ the structure of a cluster algebra on $\mathbb{C}[N']$, the clusters of which are projections of the $\mathbb{Z}/2\mathbb{Z}$-stable ones of $\mathbb{C}[N]$.*

Example 23. We have

$$\Delta^{12}_{23}\begin{pmatrix}1 & a_{12} & a_{13} & a_{14}\\ 0 & 1 & a_{23} & a_{24}\\ 0 & 0 & 1 & a_{34}\\ 0 & 0 & 0 & 1\end{pmatrix} = a_{12}a_{23} - a_{13} \quad \text{and} \quad \Delta^{2}_{4}\begin{pmatrix}1 & a_{12} & a_{13} & a_{14}\\ 0 & 1 & a_{23} & a_{24}\\ 0 & 0 & 1 & a_{34}\\ 0 & 0 & 0 & 1\end{pmatrix} = a_{24}.$$

Moreover,

$$\Psi^{-1}\ {}^{t}\!\begin{pmatrix}1 & a_{12} & a_{13} & a_{14}\\ 0 & 1 & a_{23} & a_{24}\\ 0 & 0 & 1 & a_{34}\\ 0 & 0 & 0 & 1\end{pmatrix}^{-1}\Psi$$

$$= \begin{pmatrix}1 & a_{34} & a_{23}a_{34} - a_{24} & a_{12}a_{23}a_{34} - a_{12}a_{24} - a_{13}a_{34} + a_{14}\\ 0 & 1 & a_{23} & a_{12}a_{23} - a_{13}\\ 0 & 0 & 1 & a_{12}\\ 0 & 0 & 0 & 1\end{pmatrix}$$

which implies that, as expected,

$$\pi\left(\Delta^{12}_{23}\right) = \pi\varphi_{\substack{1\searrow \\ \quad 2}} = \pi\varphi_{\substack{\quad 3 \\ 2\swarrow}} = \pi\left(\Delta^{2}_{4}\right).$$

The exchange graph of the $\mathbb{Z}/2\mathbb{Z}$-stable basic maximal rigid objects of $\operatorname{mod}\Pi_Q$ is presented on Figure 4, in relation to the exchange graph of the basic maximal rigid objects. It permits, in view of Figure 1 to describe the clusters of $\mathbb{C}[N']$:

$$\begin{tikzcd}[column sep=tiny]
 & \substack{\Delta^{1}_{4} = \Delta^{123}_{234},\, \Delta^{12}_{34},\\ \Delta^{12}_{24},\, \Delta^{1}_{2} = \Delta^{3}_{4}} \arrow[dl, dash] \arrow[dr, dash] & \\
\substack{\Delta^{1}_{4} = \Delta^{123}_{234},\, \Delta^{12}_{34},\\ \Delta^{12}_{24},\, \Delta^{2}_{4} = \Delta^{12}_{23}} \arrow[d, dash] & & \substack{\Delta^{1}_{4} = \Delta^{123}_{234},\, \Delta^{12}_{34},\\ \Delta^{13}_{34},\, \Delta^{1}_{2} = \Delta^{3}_{4}} \arrow[d, dash] \\
\substack{\Delta^{1}_{4} = \Delta^{123}_{234},\, \Delta^{12}_{34},\\ \Delta^{2}_{3},\, \Delta^{2}_{4} = \Delta^{12}_{23}} \arrow[dr, dash] & & \substack{\Delta^{1}_{4} = \Delta^{123}_{234},\, \Delta^{12}_{34},\\ \Delta^{13}_{34},\, \Delta^{23}_{34} = \Delta^{1}_{3}} \arrow[dl, dash] \\
 & \substack{\Delta^{1}_{4} = \Delta^{123}_{234},\, \Delta^{12}_{34},\\ \Delta^{2}_{3},\, \Delta^{23}_{34} = \Delta^{1}_{3}} &
\end{tikzcd}.$$

6. Scope of these results and consequences

The example presented here can be generalized to the coordinate rings of:

- The groups of the form

$$N(w) = N \cap \left(w^{-1}N_{-}w\right) \quad \text{and} \quad N^{w} = N \cap \left(B_{-}wB_{-}\right)$$

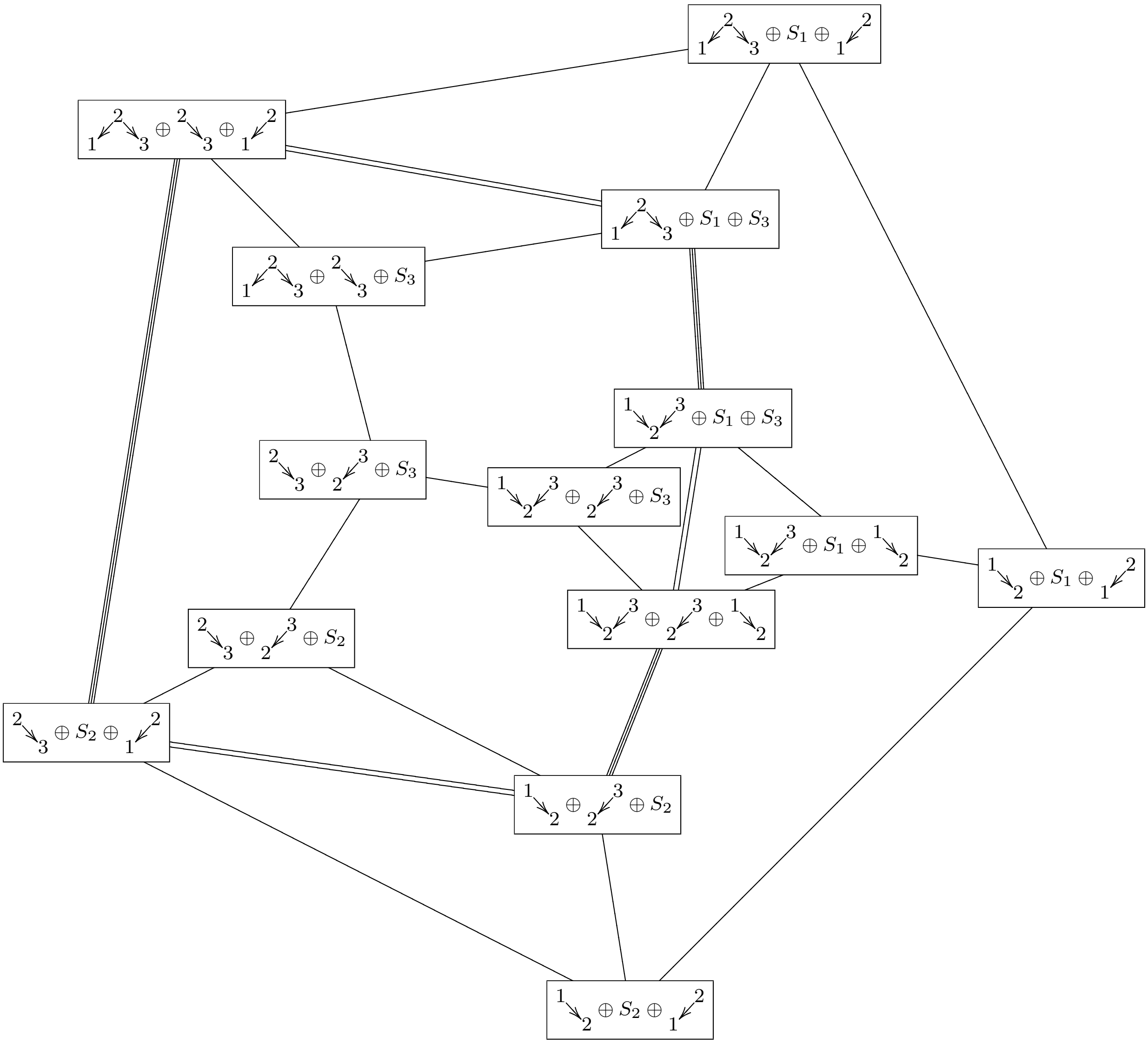

FIGURE 4. Exchange graph of $\mathbb{Z}/2\mathbb{Z}$-stable maximal rigid objects

where N is a maximal unipotent subgroup of a Kac-Moody group, N_- its opposite unipotent group, B_- the corresponding Borel subgroup, and w is an element of the corresponding Weyl group. In particular, if N is of Lie type and w is the longest element, then $N(w) = N$.

- Partial flag varieties corresponding to classical Lie groups.

These results were obtained in [5] and [6] for the simply-laced cases and in [2] for the non simply-laced cases.

It permits for example to prove in these cases that all the cluster monomials (products of elements of a same extended cluster) are linearly independent (result which is now generalized but was new at that time) and other more specific results (for example the classification of partial flag varieties the coordinate rings of which have finite cluster type, that is a finite number of clusters).

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HOCHSCHILD COHOMOLOGY OF CLUSTER-TILTED ALGEBRAS OF TYPES $\mathbb{A}_n$ AND $\mathbb{D}_n$

TAKAHIKO FURUYA AND TAKAO HAYAMI

Abstract. In this note, we study the Hochschild cohomology for cluster-tilted algebras of Dynkin types $\mathbb{A}_n$ and $\mathbb{D}_n$. We first show that all cluster-tilted algebras of type $\mathbb{A}_n$ are (D, A)-stacked monomial algebras (with $D = 2$ and $A = 1$), and then investigate their Hochschild cohomology rings modulo nilpotence. Also we describe the Hochschild cohomology rings modulo nilpotence for some cluster-tilted algebras of type $\mathbb{D}_n$ which are derived equivalent to a (D, A)-stacked monomial algebra. Finally we determine the structures of the Hochschild cohomology rings modulo nilpotence for algebras in a class of some special biserial algebras which contains a cluster-tilted algebra of type $\mathbb{D}_4$.

1. Introduction

The purpose in this note is to study the Hochschild cohomology for cluster-tilted algebras of Dynkin types $\mathbb{A}_n$ and $\mathbb{D}_n$.

Throughout this note, let K denote an algebraically closed field. Let A be a finite-dimensional K-algebra, and let A^e be the enveloping algebra $A^{op} \otimes_K A$ of A (hence right A^e-modules correspond to A-A-bimodules). Then the *Hochschild cohomology ring* $\mathrm{HH}^*(A)$ of A is defined by the graded ring

$$\mathrm{HH}^*(A) := \mathrm{Ext}^*_{A^e}(A, A) = \bigoplus_{i \geq 0} \mathrm{Ext}^i_{A^e}(A, A),$$

where the product is given by the Yoneda product. It is well-known that $\mathrm{HH}^*(A)$ is a graded commutative K-algebra.

Let $\mathcal{N}_A$ be the ideal in $\mathrm{HH}^*(A)$ generated by all homogeneous nilpotent elements. The following question is important in the study of the Hochschild cohomology rings for finite-dimensional algebras:

Question ([23]). When is the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(A)/\mathcal{N}_A$ finitely generated as an algebra?

It is shown that the Hochschild cohomology rings modulo nilpotence are finitely generated in the following cases: blocks of a group ring of a finite group [12, 25], monomial algebras [16], self-injective algebras of finite representation type [17], finite-dimensional hereditary algebras ([19]). On the other hand, Xu [26] gave an algebra whose Hochschild cohomology ring modulo nilpotence is infinitely generated (see also [23]).

In [7], Buan, Marsh and Reiten introduced cluster-tilted algebras, and since then they have been the subjects of many investigations (see for example [1, 3, 6, 7, 8, 9, 10, 11, 21]). We briefly recall their definition. Let $H = KQ$ be the path algebra of a finite acyclic

The detailed version of this paper will be submitted for publication elsewhere.

quiver Q over K, and let $D^b(H)$ the bounded derived category of H. Then the *cluster category* $\mathcal{C}_H$ *associated with* H is defined to be the orbit category $D^b(H)/\tau^{-1}[1]$, where τ denotes the Auslander-Reiten translation in $D^b(H)$, and $[1]$ is the shift functor in $D^b(H)$ ([5, 10]). Note that, by [5], $\mathcal{C}_H$ is a Krull-Schmidt category, and by Keller [20] it is also a triangulated category. A basic object T in $\mathcal{C}_H$ is called a *cluster tilting object*, if it satisfies the following conditions ([5]):

(1) $\mathrm{Ext}^1_{\mathcal{C}_H}(T,T) = 0$; and
(2) the number of the indecomposable summands of T equals the number of vertices of Q.

Let Δ be the underlying graph of Q. Then the endomorphism ring $\mathrm{End}_{\mathcal{C}_H}(T)$ of a cluster tilting object T in $\mathcal{C}_H$ is called a cluster-tilted algebra of type Δ ([7]). In this note, we deal with cluster-tilted algebras of Dynkin types $\mathbb{A}_n$ and $\mathbb{D}_n$. Note that by [7] these algebras are of finite representation type.

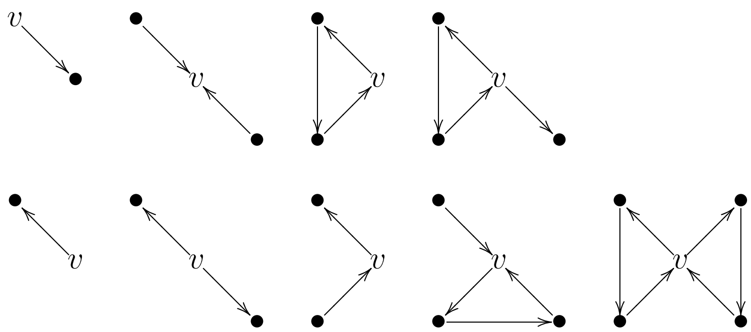
In Section 2, we show that cluster-tilted algebras of type $\mathbb{A}_n$ are (D, A)-stacked monomial algebras (with $D = 2$ and $A = 1$) of [18] (Lemma 3), and then describe the structures of their Hochschild cohomology rings modulo nilpotence by using [18] (Theorem 4). In Section 3, we determine the Hochschild cohomology rings modulo nilpotence for some cluster-tilted algebras of type $\mathbb{D}_n$ which are derived equivalent to a (D, A)-stacked monomial algebra (Proposition 7). We also describe the Hochschild cohomology rings modulo nilpotence for algebras in a class of some special biserial algebras which contains a cluster-tilted algebra of type $\mathbb{D}_4$ (Theorem 9).

2. Cluster-tilted algebras of type $\mathbb{A}_n$ and the Hochschild cohomology rings modulo nilpotence

In this section we describe the structure of the Hochschild cohomology rings modulo nilpotence for cluster-tilted algebras of type $\mathbb{A}_n$ ($n \geq 1$).

First we recall the presentation by the quiver and relations of cluster-tilted algebras of type $\mathbb{A}_n$ given in [3, 9]. For a vertex x in a quiver Γ, the *neighborhood* of x is the full subquiver of Γ consisting of x and the vertices which are end-points of arrows starting at x or start-points of arrows ending with x. Let $n \geq 2$ be an integer, and let Q_n be the class of quivers Q satisfying the following:

(1) Q has n vertices.
(2) The neighborhood of each vertex v of Q is one of the following forms:



(3) There is no cycles in the underlying graph of Q apart from those induced by oriented cycles contained in neighborhoods of vertices of Q.

Let $Q_1 = \{Q'\}$, where Q' is the quiver which has a single vertex and no arrows. It is shown in [9, Proposition 2.4] that a quiver Γ is mutation equivalent $\mathbb{A}_n$ if and only if $\Gamma \in Q_n$.

In [9], Buan and Vatne proved the following (see also [3]):

Proposition 1 ([9, Proposition 3.1])**.** *The cluster-tilted algebras of type $\mathbb{A}_n$ are exactly the algebras KQ/I, where $Q \in Q_n$, and*

$$I = \langle p \mid p \text{ is a path of length } 2, \text{ and on an oriented } 3\text{-cycle in } Q \rangle \tag{2.1}$$

As a consequence we see that cluster-tilted algebras of type $\mathbb{A}_n$ are gentle algebras of [2]:

Corollary 2 ([9, Corollary 3.2])**.** *The cluster-tilted algebras of type $\mathbb{A}_n$ are gentle algebras.*

Green and Snashall [18] introduced (D, A)-*stacked monomial algebras* by using the notion of overlaps of paths, where D and A are positive integers with $D \geq 2$ and $A \geq 1$, and gave generators and relations of the Hochschild cohomology rings modulo nilpotence for (D, A)-stacked monomial algebras completely. (In this note, we do not state the definition of (D, A)-stacked algebras and the result of [18]; see for their details [13, Section 1], [18, Section 3], or [23, Section 3].)

It is known that $(2, 1)$-stacked monomial algebras are precisely Koszul monomial algebras (equivalently, quadratic monomial algebras), and also $(D, 1)$-stacked monomial algebras are exactly D-Koszul monomial algebras (see [4]). By the definition, we directly see that all gentle algebras are $(2, 1)$-stacked monomial algebras (see [13]). Hence, by Corollary 2, we have the following:

Lemma 3. *All cluster-tilted algebras of type $\mathbb{A}_n$ are $(2, 1)$-stacked monomial algebras, and so are Koszul monomial algebras.*

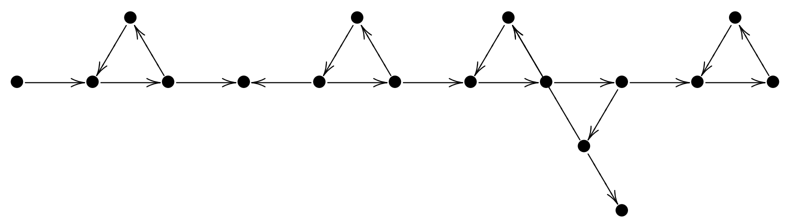
By Lemma 3, we can apply the result of [18] to describe the Hochshild cohomology rings of cluster-tilted algebras of type $\mathbb{A}_n$. Applying [18, Theorem 3.4] with Proposition 1, we have the following theorem:

Theorem 4. *Let n be a positive integer, and let $\Lambda = KQ/I$ be a cluster-tilted algebra of type $\mathbb{A}_n$, where $Q \in Q_n$ and I is the ideal given by (2.1). Suppose that $\operatorname{char} K \neq 2$. Moreover, let r be the number of oriented 3-cycles in Q. Then*

$$\mathrm{HH}^*(\Lambda)/\mathcal{N}_\Lambda \simeq \begin{cases} K[x_1, \dots, x_r]/\langle x_i x_j \mid i \neq j \rangle & \text{if } r > 0 \\ K & \text{if } r = 0, \end{cases}$$

where $\deg x_i = 6$ for $i = 1, \dots, r$.

Example 5. Let Q be the following quiver with 17 vertices and five oriented 3-cycles:



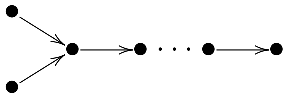
Then $Q \in Q_{17}$. Suppose $\operatorname{char} K \neq 2$, and let $\Lambda := KQ/I$, where I is the ideal generated by all possible paths of length 2 on oriented 3-cycles. Then Λ is a cluster-tilted algebra of type $\mathbb{A}_{17}$, and by Theorem 4 we have $\mathrm{HH}^*(\Lambda)/\mathcal{N}_\Lambda \simeq K[x_1, \dots, x_5]/\langle x_i x_j \mid i \neq j \rangle$, where $\deg x_i = 6$ $(1 \leq i \leq 5)$.

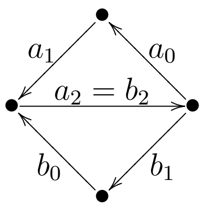
3. Cluster-tilted algebras of type $\mathbb{D}_n$ and the Hochschild cohomology rings modulo nilpotence

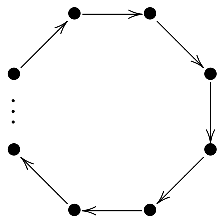
The purpose in this section is to describe the Hochschild cohomology rings modulo nilpotence for some cluster-tilted algebras of type $\mathbb{D}_n$ $(n \geq 4)$ which are derived equivalent to a (D, A)-stacked monomial algebra.

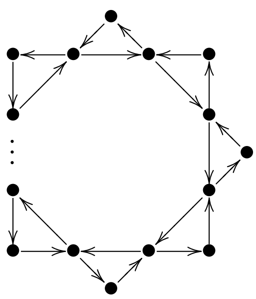
In [3, Theorem 2.3], Bastian, Holm and Ladkani introduced specific quivers, called "standard forms" for derived equivalences, and proved that any cluster-tilted algebra of type $\mathbb{D}_n$ is derived equivalent to one of cluster-tilted algebras of type $\mathbb{D}_n$ whose quiver is a standard form.

It is known that Hochschild cohomology ring is invariant under derived equivalence, so that it suffices to deal with cluster-tilted algebras of type $\mathbb{D}_n$ whose quivers are standard forms. In this note, we consider the following quivers Γ_i $(1 \leq i \leq 4)$. Clearly these quivers are standard forms of [3, Theorem 2.3].

Γ_1 :  with $m\,(\geq 4)$ vertices,

Γ_2 : 

Γ_3 :  with m vertices, where $m\,(\geq 5)$ is odd, or $m = 4$,

Γ_4 :  with $2m$ vertices, where $m \geq 3$.

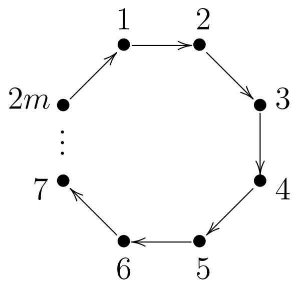
Remark 6. For $i = 1, \dots, 4$, let $\Lambda_i = K\Gamma_i/I_i$ be the cluster-tilted algebra of type $\mathbb{D}_n$ corresponding to Γ_i. Then we see from [3, 24] that

(1) Λ_1 is the path algebra of a Dynkin quiver of type $\mathbb{D}_m$.

(2) Λ_2 is of type $\mathbb{D}_4$, and $I_2 = \langle a_1a_2, b_1b_2, a_2a_0, b_2b_0, a_0a_1 - b_0b_1\rangle$. We immediately see that Λ_2 is a special biserial algebra of [22], but not a self-injective algebra.

(3) Λ_3 is of type $\mathbb{D}_m$, and $I_3 = \langle p \mid p \text{ is a path of length } m-1\rangle$. Hence Λ_3 is a $(m-1,1)$-stacked monomial algebra, and is also a self-injective Nakayama algebra.

(4) Λ_4 is of type $\mathbb{D}_{2m}$, and it follows by [3, Lemma 4.5] that Λ_4 is derived equivalent to the $(2m-1,1)$-stacked monomial algebra $\Lambda' = KQ'/I'$, where Q' is the cyclic quiver with $2m$ vertices



and I' is generated by all paths of length $2m-1$. Note that Λ' is a self-injective Nakayama algebra, and moreover is a cluster-tilted algebra of type $\mathbb{D}_{2m}$ ([21, 24]).

In [19], Happel described the Hochschild cohomology for path algebras. Using this result and [18, Theorem 3.4], we have the following proposition:

Proposition 7. *For the algebras Λ_1, Λ_3 and Λ_4 above, we have*

$$\mathrm{HH}^*(\Lambda_1) \simeq \mathrm{HH}^*(\Lambda_1)/\mathcal{N}_{\Lambda_1} \simeq K$$
$$\mathrm{HH}^*(\Lambda_3)/\mathcal{N}_{\Lambda_3} \simeq \mathrm{HH}^*(\Lambda_4)/\mathcal{N}_{\Lambda_4} \simeq K[x].$$

Finally we describe the Hochschild cohomology ring modulo nilpotence of the algebra $A_k := \Gamma_2/J_k$, where $k \geq 0$ and J_k is the ideal generated by the following elements:

$$(a_1a_2a_0)^k a_1a_2, \quad b_1b_2, \quad (a_2a_0a_1)^k a_2a_0, \quad b_2b_0, \quad (a_0a_1a_2)^k a_0a_1 - b_0b_1.$$

If $k = 0$, then $J_0 = I_2$, and so $A_0 = \Gamma_2/J_0$ coincides with the algebra Λ_2. Note that, for all $k \geq 0$, A_k is a special biserial algebra and not a self-injective algebra.

Now the dimensions of the Hochschild cohomology groups of A_k are given as follows:

Theorem 8 ([14])**.** *For $k \geq 0$ and $i \geq 0$ we have*

$$\dim_K \mathrm{HH}^i(A_k) = \begin{cases} k+1 & \textit{if } i \equiv 0 \pmod 6 \\ k+1 & \textit{if } i \equiv 1 \pmod 6 \\ k & \textit{if } i \equiv 2 \pmod 6 \\ k+1 & \textit{if } i \equiv 3 \pmod 6 \ \textit{and} \ \mathrm{char}\, K \mid 3k+2 \\ k & \textit{if } i \equiv 3 \pmod 6 \ \textit{and} \ \mathrm{char}\, K \nmid 3k+2 \\ k+1 & \textit{if } i \equiv 4 \pmod 6 \ \textit{and} \ \mathrm{char}\, K \mid 3k+2 \\ k & \textit{if } i \equiv 4 \pmod 6 \ \textit{and} \ \mathrm{char}\, K \nmid 3k+2 \\ k & \textit{if } i \equiv 5 \pmod 6. \end{cases}$$

Moreover the Hochschild cohomology ring modulo nilpotence of A_k is given as follows:

Theorem 9 ([15]). *For $k \geq 0$, we have*

$$\mathrm{HH}^*(A_k)/\mathcal{N}_{A_k} \simeq K[x], \quad \textit{where} \quad \deg x = \begin{cases} 3 & \textit{if } k = 0 \textit{ and } \mathrm{char}\, K = 2 \\ 6 & \textit{otherwise.} \end{cases}$$

Hence $\mathrm{HH}^*(A_k)/\mathcal{N}_{A_k}$ $(k \geq 0)$ *is finitely generated as an algebra.*

Remark 10. It seems that most of computations of the Hochschild cohomology rings modulo nilpotence for cluster-tilted algebras of type $\mathbb{D}_n$ except those in the derived equivalence classes of Λ_i $(1 \leq i \leq 4)$ are open questions.

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DERIVED AUTOEQUIVALENCES AND BRAID RELATIONS

JOSEPH GRANT

Abstract. We will consider braid relations between autoequivalences of derived categories of symmetric algebras. We first recall the construction of spherical twists for symmetric algebras and the braid relations that they satisfy, as illustrated by Brauer tree algebras. Then we explain the construction of periodic twists, which generalise spherical twists for symmetric algebras. Finally, we explain a lifting theorem for periodic twists, and show how this gives a new interpretation of the action on the derived Picard group of lifts of longest elements of the symmetric group to the braid group.

1. Preliminaries

Let k be an algebraically closed field. All algebras we consider will be finite-dimensional k-algebras, and for simplicity we will also assume that A is basic. We will denote the category of finite-dimensional left A-modules by A-mod, and of finite-dimensional right A-modules by mod-A.

Given an algebra A and a left (or right) A-module M, we have a right (or left, respectively) A-module $M^* = \mathrm{Hom}_k(M, k)$ with A-action $fa(m) = f(am)$ for $m \in M$, $f \in M^*$, and $a \in A$. This gives a duality

$$(-)^* : A\text{-mod} \xrightarrow{\sim} \text{mod-}A.$$

Similarly, if M is an A-B-bimodule for algebras A and B, then M^* is a B-A-bimodule.

There is another way to construct a right module from $M \in A$-mod: we set $M^\vee = \mathrm{Hom}_A(M, A)$, where the action is given here by $fa(m) = f(m)a$ for $m \in M$, $f \in M^\vee$, and $a \in A$. This defines a functor

$$(-)^\vee : A\text{-mod} \to \text{mod-}A.$$

but in general this is not an equivalence. However, in the cases we consider below this will be an equivalence.

Any algebra A has a natural structure of an A-A-bimodule given by the multiplication. We say that A is a *symmetric* algebra if there exists an isomorphism of A-A-bimodules $A \xrightarrow{\sim} A^*$. Symmetric algebras have various equivalent definitions: one is that $(-)^*$ and $(-)^\vee$ are natually isomorphic functors, and another is a Calabi-Yau type condition on the derived category. For more information on this, we refer the reader to [Ric2, Section 3].

We will be interested in bounded derived categories of module categories over algebras A, which we will denote $\mathrm{D^b}(A)$. We refer the reader to [Wei, Chapter 10] for their definition and basic properties. In partuicular, we will study autoequivalences of $\mathrm{D^b}(A)$. Clearly the autoequivalences form a group, but in fact we can restrict ourselves to a particular subset. One way to define an endofunctor of $\mathrm{D^b}(A)$ is to take the derived tensor product with

The detailed version of this paper will be submitted for publication elsewhere.

a cochain complex X of A-A-bimodules. If this gives us an equivalence of triangulated categories, we call X a *two-sided tilting complex* [Ric1]. Rickard showed that tensoring with two-sided tilting complexes does give a subgroup of the group of autoequivalences [Ric1]. We call this subgroup the *derived Picard group of* A, and denote it $\mathrm{DPic}(A)$. Here we can work with ordinary tensor products, and will not need to consider derived tensor products, as all our two-sided tilting complexes will be presented as cochain complexes of A-A-bimodules which are projective on both sides.

2. Spherical Twists and Braid Relations

Let A be a symmetric algebra and let P be a projective A-module. Following [ST], we say that P is *spherical* if $\mathrm{End}_A(P) \cong k[x]/\langle x^2\rangle$. In this case, consider the cochain complex of A-A-bimodules

$$P \otimes_k P^\vee \to A$$

concentrated in degrees 1 and 0, where the nonzero map is given by evaluation. We will denote this complex by X_P. It defines an object in the bounded derived category $\mathrm{D^b}(A)$, which we will also denote by X_P. Then tensoring with X_P defines an endofunctor

$$X_P \otimes_A - : \mathrm{D^b}(A) \to \mathrm{D^b}(A)$$

which we denote by F_P.

Theorem 1 ([RZ] for Brauer tree algebras, [ST] in general)**.** *If the projective A-module P is spherical then F_P is an autoequivalence.*

Now let $P_1, \dots, P_n$ be a collection of n spherical projective A-modules. Following [ST], we say that $\{P_1, \dots, P_n\}$ is an A_n-collection if

$$\dim_k \mathrm{Hom}_A(P_i, P_j) = \begin{cases} 0 & \text{if } |i-j| > 1; \\ 1 & \text{if } |i-j| = 1 \end{cases}$$

for all $1 \le i, j \le n$.

Theorem 2 ([RZ] for Brauer tree algebras, [ST] in general)**.** *If $\{P_1, \dots, P_n\}$ is an A_n-collection then the spherical twists $F_i = F_{P_i}$ satisfy the braid relations*

- $F_iF_j \cong F_jF_i$ *if* $|i-j| > 1$*;*
- $F_iF_jF_i \cong F_jF_iF_j$ *if* $|i-j| = 1$

for all $1 \le i, j \le n$.

Another way to say this is as follows: let B_{n+1} be the braid group on the letters $\{1, \dots, n, n+1\}$. This is generated by elements $s_1, \dots, s_n$ and has relations

- $s_is_j = s_js_i$ if $|i-j| > 1$;
- $s_is_js_i = s_js_is_j$ if $|i-j| = 1$.

If A has an A_n-collection then we have a group homomorphism

$$B_{n+1} \to \mathrm{DPic}(A)$$

which sends s_i to the spherical twist F_i.

Let S_{n+1} be the symmetric group on the letters $\{1, \dots, n, n+1\}$. We also denote the generators of S_i by $s_1, \dots, s_n$, and there is an obvious group epimorphism $B_{n+1} \twoheadrightarrow S_{n+1}$.

3. Periodic Twists

We now describe a generalization of the spherical twists described above.

An algebra E is called *twisted periodic* if there is an algebra automorphism $\sigma : E \xrightarrow{\sim} E$ and an exact sequence of E-E-bimodules

$$0 \to E_\sigma \to Y_{n-1} \to Y_{n-2} \to \cdots \to Y_1 \to Y_0 \to E \to 0$$

where each Y_i is a projective E-E-bimodule. This just says that the E-E-bimodule E has a periodic resolution which is projective up to some automorphism (twist). We say that E has a period n.

Let A be a symmetric algebra and P a projective A-module. Let $E = \mathrm{End}_A(P)^{\mathrm{op}}$, so P is an A-E-bimodule, and suppose that E is a periodic algebra. We denote the cochain complex

$$Y_{n-1} \to Y_{n-2} \to \cdots \to Y_1 \to Y_0$$

concentrated in degrees $n-1$ to 0 by Y. Then we have a natural map $f : Y \to E$ of cochain complexes of E-E-bimodules. We use this to construct a map $g : P \otimes_E Y \otimes_E P^\vee \to A$ of cochain complexes of A-A-bimodules defined as the following composition

$$P \otimes_E Y \otimes_E P^\vee \to P \otimes_E E \otimes_E P^\vee \xrightarrow{\sim} P \otimes_E P^\vee \to A$$

where the first map is given by $P \otimes_E f \otimes_E P^\vee$ and the last is given by an evaluation map.

We take the cone of the map g to obtain a cochain complex

$$P \otimes_E Y_{n-1} \otimes_E P^\vee \to P \otimes_E Y_{n-2} \otimes_E P^\vee \to \cdots \to P \otimes_E Y_0 \otimes_E P^\vee \to A$$

concentrated in degrees n to 0, which we denote X. By tensoring over A we obtain an endofunctor

$$X \otimes_A - : \mathrm{D^b}(A) \to \mathrm{D^b}(A)$$

which we denote by Ψ_P.

Theorem 3 ([Gra])**.** *If the algebra E is twisted periodic then Ψ_P is an autoequivalence.*

Note that the functor Ψ_P depends on the resolution Y that we choose.

If $E \cong k[x]/\langle x^2 \rangle$ then we recover the spherical twists described above by using the exact sequence

$$0 \to E_\sigma \to E \otimes_k E \to E \to 0$$

where σ is the algebra automorphism which sends x to $-x$.

4. Brauer Tree Algebras of Lines

We define a collection of algebras Γ_n, $n \geq 1$, which are isomorphic to the Brauer tree algebras of lines without multiplicity. Let $\Gamma_1 = k[x]/\langle x^2 \rangle$ and let $\Gamma_2 = kQ_2/I_2$, where Q_2 is the quiver

$$Q_2 \quad = \quad 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$$

and I_2 is the ideal generated by $\alpha\beta\alpha$ and $\beta\alpha\beta$. For $n \geq 3$, let $\Gamma_n = kQ_n/I_n$ where Q_n is the quiver

$$Q_n \quad = \quad \begin{tikzcd} 1 \arrow[r, bend left, "\alpha_1"] & 2 \arrow[l, bend left, "\beta_2"] \arrow[r, bend left, "\alpha_2"] & \cdots \arrow[l, bend left, "\beta_3"] \arrow[r, bend left, "\alpha_{n-1}"] & n \arrow[l, bend left, "\beta_n"] \end{tikzcd}$$

and I_n is the ideal generated by $\alpha_{i-1}\alpha_i$, $\beta_{i+1}\beta_i$, and $\alpha_i\beta_{i+1} - \beta_i\alpha_{i-1}$ for $2 \leq i \leq n-1$. Note that if we take the indecomposable projective Γ_n-modules $P_i, P_{i+1}, \ldots, P_j$ for $1 \leq i < j \leq n$, we have $\mathrm{End}_A(P_i \oplus P_{i+1} \oplus \ldots \oplus P_j)^{\mathrm{op}} \cong \Gamma_{j-i}$.

One can check that for all $n \geq 1$, each indecomposable projective Γ_n-module is spherical. Spherical twists for these algebras were studied in detail in [RZ].

We have the following observation:

Lemma 4. *Let A be a symmetric algebra. A collection $\{P_1, \ldots, P_n\}$ of projective A-modules is an A_n-collection if and only if*

$$\mathrm{End}_A(\bigoplus_{i=1}^{n} P_i)^{\mathrm{op}} \cong \Gamma_n.$$

The algebras Γ_n are of finite representation type, and hence are twisted periodic, but in fact we can say more.

Theorem 5 ([BBK]). *The algebra n is twisted periodic with period n and automorphism σ_n induced by the quiver automorphism which sends the vertex i to $n - i + 1$.*

A natural question is: what do the associated periodic twists look like? It was noted in [Gra] that periodic twists associated to Γ_2 are isomorphic to the composition $F_1F_2F_1$ of spherical twists. We will show that this pattern continues.

5. A Lifting Theorem

Let A be a symmetric algebra and let $P_1, \ldots, P_n$ be a collection of indecomposable projective A-modules. We will use the following notation:

- $P = \bigoplus_{i=1}^{n} P_i$;
- $E = \mathrm{End}_A(P)^{\mathrm{op}}$;
- $E_i = \mathrm{End}_A(P_i)^{\mathrm{op}}$;
- $Q_i = \mathrm{Hom}_A(P, P_i)$,
- $Q = \bigoplus_{i=1}^{n} Q_i$;

so $\{Q_i | 1 \leq i \leq n\}$ is a complete set of representatives of the isoclasses of indecomposable projective E-modules. Note that $\mathrm{End}_E(Q_i)^{\mathrm{op}} \cong E_i$. We will explain a connection between compositions of periodic twists for E and compositions of corresponding periodic twists for A.

Theorem 6 (Lifting Theorem). *Suppose that E and each E_i are twisted periodic with fixed truncated resolutions Y and Y_i. Let $\Psi_i = \Psi_{P_i} : \mathrm{D^b}(A) \xrightarrow{\sim} \mathrm{D^b}(A)$ and $\Psi'_i = \Psi_{Q_i} : \mathrm{D^b}(E) \xrightarrow{\sim} \mathrm{D^b}(E)$. If*

$$\Psi_Q \cong \Psi'_{i_\ell} \ldots \Psi'_{i_2}\Psi'_{i_1}$$

for some $1 \leq i_1, \ldots, i_\ell \leq n$ then

$$\Psi_P \cong \Psi_{i_\ell} \ldots \Psi_{i_2}\Psi_{i_1}.$$

We now specialise to the case where $\{P_1, \ldots, P_n\}$ is an A_n-collection, so $E \cong \Gamma_n$, and $F_i = \Psi_i$ and $F'_i = \Psi'_i$ are spherical twists.

Recall that the symmetric group S_{n+1} has a unique longest element, often denoted w_0. We choose a particular presentation

$$w_0 = s_1(s_2s_1)\ldots(s_n \ldots s_2s_1) \in S_{n+1}$$

and define an element w_0 of the braid group by the same presentation. Rouquier and Zimmermann showed how this element acts on the derived Picard group of an algebra Γ_n:

Theorem 7 ([RZ, Theorem 4.5]). *The image of the element w_0 under the group morphism $B_{n+1} \to \mathrm{DPic}(\Gamma_n)$ is the functor $-_{\sigma_n}[n]$ which twists on the right by the automorphism σ_n and shifts cochain complexes n places to the left.*

By Theorem 5 we see that $\Psi_{\Gamma_n} : \mathrm{D^b}(\Gamma_n) \to \mathrm{D^b}(\Gamma_n)$ is the same functor, and hence by applying the lifting theorem we obtain the following:

Corollary 8. *Suppose the symmetric algebra A has an A_n-collection $\{P_1, \ldots, P_n\}$. Then the image of w_0 in the group morphism $B_{n+1} \to \mathrm{DPic}(A)$ is Ψ_P, where $P = \bigoplus_{i=1}^n P_i$.*

We also obtain a new proof of the braid relations by using Theorem 7 in the case $n = 2$, or alternatively by performing a straightforward calculation with Γ_2, and then applying the lifting theorem.

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n-REPRESENTATION INFINITE ALGEBRAS

MARTIN HERSCHEND

Abstract. We introduce the class of n-representation infinite algebras and discuss some of their homological properties. We also present the family of n-representation infinite algebras of type $\widetilde{A}$.

1. Introduction

This brief survey contains the results from my presentation at the *44th Symposium on Ring Theory and Representation Theory* in Okayama. It is based on joint work Osamu Iyama and Steffen Oppermann. A detailed final version will be published elsewhere.

The class of hereditary finite dimensional algebras is one of the best understood in terms of representation theory, especially in the context of Auslander-Reiten theory. This applies in particular to representation finite hereditary algebras. In higher dimensional Auslander-Reiten theory an analogue of these algebras is given by the class of n-representation finite algebras [1, 2]. Recall that a finite dimensional algebra is called n-representation finite if it has global dimension at most n and admits an n-cluster tilting module. Since a 1-cluster tilting module is the same as an additive generator of the module category, 1-representation finite means precisely hereditary and representation finite.

The aim of this report is to define the class of n-representation infinite algebras, that will in a similar way be a higher dimensional analogue of representation infinite hereditary algebras. To do this we begin by recalling some properties of n-representation finite algebras.

Let K be a field and Λ a finite dimensional K-algebra with $\operatorname{gl.dim}\Lambda \leq n$. We always assume that Λ is ring indecomposable. Denote by $\operatorname{mod}\Lambda$ the category of finite dimensional left Λ-modules and by $\mathcal{D}^{\mathrm{b}}(\Lambda)$ the bounded derived category of $\operatorname{mod}\Lambda$. Combining the K-dual $D := \operatorname{Hom}_K(-, K)$ with the Λ-dual we obtain the Nakayama functor

$$\nu := D\mathbf{R}\mathrm{Hom}(-, \Lambda) : \mathcal{D}^{\mathrm{b}}(\Lambda) \to \mathcal{D}^{\mathrm{b}}(\Lambda).$$

It is a Serre functor in the sense that there is a functorial ismorphism

$$\operatorname{Hom}_{\mathcal{D}^{\mathrm{b}}(\Lambda)}(X, Y) \simeq D \operatorname{Hom}_{\mathcal{D}^{\mathrm{b}}(\Lambda)}(Y, \nu(X)).$$

We combine ν with the shift functor on $\mathcal{D}^{\mathrm{b}}(\Lambda)$ to obtain the autoequivalence

$$\nu_n := \nu \circ [-n] : \mathcal{D}^{\mathrm{b}}(\Lambda) \to \mathcal{D}^{\mathrm{b}}(\Lambda).$$

It plays the role of the higher Auslander-Reiten translation in $\mathcal{D}^{\mathrm{b}}(\Lambda)$. More precisely, define

$$\tau_n := D \operatorname{Ext}^n_\Lambda(-, \Lambda) : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda$$

The detailed version of this paper will be submitted for publication elsewhere.

and

$$\tau_n^- := \operatorname{Ext}^n_\Lambda(D\Lambda, -) : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda.$$

Then $\tau_n = H^0(\nu_n -)$ and $\tau_n^- = H^0(\nu_n^{-1} -)$. Using these functors we can capture the notion of n-representation finiteness in the following way.

Proposition 1. [3] *Let Λ be a finite dimensional K-algebra with* $\operatorname{gl.dim}\Lambda \leq n$. *Then the following conditions are equivalent.*

(a) Λ *is n-representation finite.*
(b) *For every indecomposable projective Λ-module P, there is a non-negative integer ℓ_P such that $\nu_n^{-\ell_P} P$ is an indecomposable injective Λ-module.*

We remark that if condition (b) is satisfied then $\nu_n^{-i} P \simeq \tau_n^{-i} P$ for all $0 \leq i \leq \ell_P$ and

$$\bigoplus_P \bigoplus_{i=0}^{\ell_P} \tau_n^{-i} P = \bigoplus_P \bigoplus_{i=0}^{\ell_P} \nu_n^{-i} P$$

is an n-cluster tilting Λ-module [1]. Furthermore, since ν^{-1} sends injectives to projectives we have

$$\nu_n^{-(\ell_P+1)} P = \nu^{-1}(\nu_n^{-\ell_P} P)[n] = P'[n] \in \operatorname{mod}\Lambda[n]$$

for some indecomposable projective P'. We conclude that knowing the τ_n^--orbits of the indecomposable projectives in $\operatorname{mod}\Lambda$ is enough to determine their ν_n^{-1}-orbits. Comparing this to the classical case $n = 1$ gives us a hint how to define n-representation infinite algebras.

2. n-representation infinite algebras

Recall that if $n = 1$ and Λ is representation infinite, then $\tau^{-i} P$ is never injective for an indecomposable projective Λ-module P. In fact $\nu_1^{-i} P = \tau^{-i} P \in \operatorname{mod}\Lambda$ for all $i \geq 0$. Inspired by this we make the following definition.

Definition 2. Let Λ be a finite dimensional K-algebra with $\operatorname{gl.dim}\Lambda \leq n$. We say that Λ is n-representation infinite if

$$\nu_n^{-i}\Lambda \in \operatorname{mod}\Lambda$$

for all $i \geq 0$.

We remark that this condition is equivalent to $\nu_n^i(D\Lambda) \in \operatorname{mod}\Lambda$ for all $i \geq 0$. In the classical setting of $n = 1$ every indecomposable module is either preprojective, preinjective or regular. We define higher analogues of these classes of modules as follows.

Definition 3. Let Λ be an n-representation infinite algebra. The full subcategories of n-preprojective, n-preinjective and n-regular modules are defined as

$$\begin{aligned}
\mathcal{P} &:= \operatorname{add}\{\nu_n^{-i}\Lambda \mid i \geq 0\}, \\
\mathcal{I} &:= \operatorname{add}\{\nu_n^{i}(D\Lambda) \mid i \geq 0\}, \\
\mathcal{R} &:= \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(\mathcal{P}, X) = 0 = \operatorname{Ext}^i_\Lambda(X, \mathcal{I}) \text{ for all } i \geq 0\},
\end{aligned}$$

respectively.

Note that $\mathcal{P}$ and $\mathcal{I}$ are well-defined as subcategories of $\operatorname{mod}\Lambda$ since Λ is n-representation infinite. Many properties of representation infinite hereditary algebras generalize to n-representation infinite algebras. For instance n-regular modules can be characterized by $\mathcal{R} = \{X \in \operatorname{mod}\Lambda \mid \nu_n^i(X) \in \operatorname{mod}\Lambda \text{ for all } i \in \mathbb{Z}\}$. Moreover, one has the following result about vanishing of homomorphisms and extensions.

Theorem 4. *Let Λ be an n-representation infinite algebra. Then the following holds:*

$$\begin{array}{llllll} \operatorname{Hom}_\Lambda(\mathcal{R},\mathcal{P}) & = 0, & \operatorname{Hom}_\Lambda(\mathcal{I},\mathcal{P}) & = 0, & \operatorname{Hom}_\Lambda(\mathcal{I},\mathcal{R}) & = 0, \\ \operatorname{Ext}^n_\Lambda(\mathcal{P},\mathcal{R}) & = 0, & \operatorname{Ext}^n_\Lambda(\mathcal{P},\mathcal{I}) & = 0, & \operatorname{Ext}^n_\Lambda(\mathcal{R},\mathcal{I}) & = 0. \end{array}$$

3. n-representation infinite algebras of type $\widetilde{A}$

In this section we assume that K is an algebraically closed field of characteristic zero. We shall present a family of n-representation infinite algebras by generalizing one of the simplest classes of representation infinite hereditary algebras, namely path algebras of extended Dynkin quivers of type $\widetilde{A}$.

On can construct extended Dynkin quivers of type $\widetilde{A}$ by taking the following steps. Start with the double quiver of A_∞^∞:

$$\begin{tikzcd} \cdots & -2 \arrow[r, shift left] & -1 \arrow[l, shift left] \arrow[r, shift left] & 0 \arrow[l, shift left] \arrow[r, shift left] & 1 \arrow[l, shift left] \arrow[r, shift left] & 2 \arrow[l, shift left] & \cdots \end{tikzcd}$$

Identify vertices and arrows modulo m for some $m \geq 1$ and remove one arrow from each 2-cycle. For instance, choosing $m = 2$ and removing the arrows starting in the odd vertex gives the Kronecker quiver:

$$\begin{tikzcd} 0 \arrow[r, shift left] \arrow[r, shift right] & 1. \end{tikzcd}$$

We shall construct the n-representation infinite algebras of type $\widetilde{A}$ similarly. First we define the covering quiver Q. As vertices in Q we take the lattice

$$Q_0 = G := \left\{ v \in \mathbb{Z}^{n+1} \;\middle|\; \sum_{i=1}^{n+1} v_i = 0 \right\}.$$

It is freely generated as an abelian group by the elements $f_i := e_{i+1} - e_i$ for $1 \leq i \leq n$. We also define $f_{n+1} := e_1 - e_{n+1}$, so that $\sum_{i=1}^{n+1} f_i = 0$. As arrows in Q we take

$$Q_1 := \{a_i : v \to v + f_i \mid v \in G,\ 1 \leq i \leq n+1\}.$$

Then Q is the double of A_∞^∞ for $n = 1$. For $n \geq 2$ we need to introduce certain relations. Let $v \in Q_0$ and $i, j \in \{1, \dots, n+1\}$. We consider the relation $r_{ij}^v := a_i a_j - a_j a_i$ from v to $v + f_i + f_j$ and let I be the two-sided ideal in KQ generated by

$$\{r_{ij}^v \mid v \in Q_0,\ 1 \leq i, j \leq n+1\}.$$

Since G is an abelian group it acts on itself by translations. This extends to a unique G-action on the quiver Q. We say that a subgroup $B \leq G$ is cofinite if G/B is finite. In that case we define $\Gamma(B)$ as the orbit algebra of KQ/I. More explicitly we define $Q/B := (Q_0/B, Q_1/B)$ and set

$$\Gamma(B) := K(Q/B)/\langle \overline{r}_{ij}^{\overline{v}} \mid \overline{v} \in Q_0/B,\ 1 \leq i, j \leq n+1 \rangle$$

where $\overline{r}_{ij}^{\overline{v}} := \overline{a}_i\overline{a}_j - \overline{a}_j\overline{a}_i$ and $\overline{a}$ denotes the B-orbit of a. As motivation for this construction we remark that $\Gamma(B)$ is isomorphic to a skew group algebra $K[x_1, \ldots, x_{n+1}] * H$ for some finite abelian subgroup $H < SL_{n+1}(K)$.

Next we consider the analogue of 2-cycles. For every $v \in Q_0$ and permutation σ of $1, \ldots, n+1$, there is a cyclic path $a_{\sigma(1)} \cdots a_{\sigma(n+1)}$ from v to v. We call such cyclic paths small cycles. A subset $C \subset Q_1$ is called a cut if it contains precisely one arrow from every small cycle. The symmetry group of C is defined as

$$S_C := \{g \in G \mid gC = C\} \leq G.$$

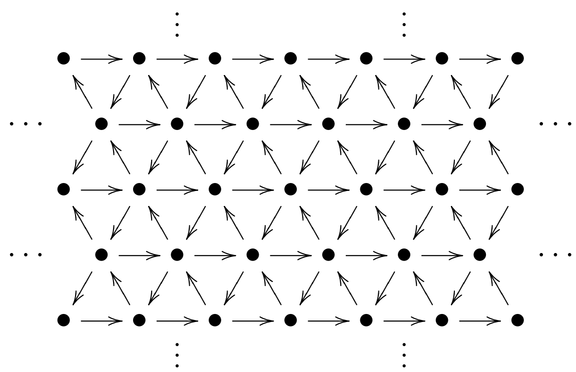
We say that a cut C is acyclic if all paths in $Q_C := (Q_0, Q_1 \setminus C)$ have length bounded by some $N \geq 0$, and periodic if S_C is cofinite in G. If both these conditions are satisfied and $B \leq S_C$ is cofinite we say that

$$\Gamma(B)_C := \Gamma(B)/\langle \overline{a} \mid \overline{a} \in C/B\rangle$$

is n-representation finite of type $\widetilde{A}$. The name is justified by the following Theorem.

Theorem 5. *If C is an acyclic periodic cut and $B \leq S_C$ is cofinite, then $\Gamma(B)_C$ is n-representation finite.*

We remark that if $n = 1$, then $\Gamma(B)_C$ is a path algebra of an acyclic quiver of type $\widetilde{A}$ constructed exactly as explained above. For $n = 2$, Q_0 is a triangular lattice in the plane and Q is



where the small cycles are formed by the small triangles.

Finally we shall generalize the alternating orientation of A_∞^∞. To do this define $\omega : G \to \mathbb{Z}/(n+1)\mathbb{Z}$ by $\omega(f_i) = 1$ and set

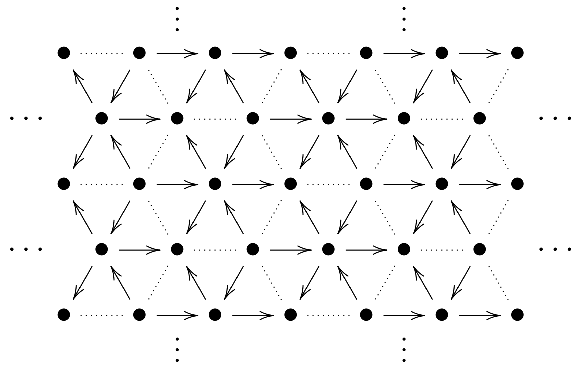
$$C := \{a_i : v \to v + f_i \mid \omega(v) = 0,\ 1 \leq i \leq n+1\}.$$

Then every path in Q of length $n+1$ intersects C and so C is acyclic. Moreover, $S_C = \ker\omega$ and so C is periodic.

For $n = 1$, Q_C is

$$\cdots\ -2 \leftarrow -1 \rightarrow 0 \leftarrow 1 \rightarrow 2\ \cdots$$

For $n = 2$, Q_C is



where the dotted lines indicate commutativity relations in $\Gamma/\langle C\rangle$.

Now let's consider $\Gamma(B)_C$ for $B = S_C$. Then we can identify Q_0/B with $\mathbb{Z}/(n+1)\mathbb{Z}$ via ω and C/B consists of all arrows from $n+1$ to 1. Hence $\Gamma(B)_C$ is the Beilinson algebra:

$$\begin{tikzcd} 1 \arrow[r, bend left, "a_1"] \arrow[r, bend right, "a_{n+1}"'] \arrow[r, phantom, "\vdots"] & 2 \arrow[r, bend left, "a_1"] \arrow[r, bend right, "a_{n+1}"'] \arrow[r, phantom, "\vdots"] & 3 \cdots n \arrow[r, bend left, "a_1"] \arrow[r, bend right, "a_{n+1}"'] \arrow[r, phantom, "\vdots"] & n+1 \end{tikzcd}, \qquad a_i a_j = a_j a_i.$$

and for $n = 1$ we obtain the Kronecker algebra:

$$1 \rightrightarrows 2.$$

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ON A DEGENERATION PROBLEM FOR COHEN-MACAULAY MODULES

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1. Introduction

The aim of this article is to give an outline of the paper [3], which is a joint work with Yuji Yoshino.

In this note, we would like to give several examples of degenerations of maximal Cohen-Macaulay modules and to show how we can describe them (Theorem 12). This result depends heavily on the recent work by Yoshino about the stable analogue of degenerations for Cohen-Macaulay modules over a Gorenstein local algebra [9]. In Section 3 we also investigate the relation among the extended versions of the degeneration order, the extension order and the AR order (Theorem 22).

2. Examples of degenerations

In this section, we recall the definition of degeneration and state several known results on degenerations.

Definition 1. Let R be a noetherian algebra over a field k, and let M and N be finitely generated left R-modules. We say that M degenerates to N, or N is a degeneration of M, if there is a discrete valuation ring (V, tV, k) that is a k-algebra (where t is a prime element) and a finitely generated left $R \otimes_k V$-module Q which satisfies the following conditions:

(1) Q is flat as a V-module.
(2) $Q/tQ \cong N$ as a left R-module.
(3) $Q[1/t] \cong M \otimes_k V[1/t]$ as a left $R \otimes_k V[1/t]$-module.

The following characterization of degenerations has been proved by Yoshino [7].

Theorem 2. [7, Theorem 2.2] *The following conditions are equivalent for finitely generated left R-modules M and N.*

(1) *M degenerates to N.*
(2) *There is a short exact sequence of finitely generated left R-modules*

$$0 \longrightarrow Z \xrightarrow{\binom{\varphi}{\psi}} M \oplus Z \longrightarrow N \longrightarrow 0,$$

such that the endomorphism ψ of Z is nilpotent, i.e. $\psi^n = 0$ for $n \gg 1$.

Remark 3. Let R be a noetherian k-algebra.

(1) Suppose that a finitely generated R-module M degenerates to a finitely generated module N. Then as a discrete valuation ring V in Definition 1 we can always take the ring $k[t]_{(t)}$. See [7, Corollary 2.4.]. Thus we always take $k[t]_{(t)}$ as V.

(2) Assume that there is an exact sequence of finitely generated left R-modules

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0.$$

Then M degenerates to $L \oplus N$. See [7, Remark 2.5] for the detail.

(3) Let M and N be finitely generated R-modules and suppose that M degenerates to N. Then the modules M and N give the same class in the Grothendieck group, *i.e.* $[M] = [N]$ as an element of $K_0(\mathrm{mod}(R))$, where $\mathrm{mod}(R)$ denotes the category of finitely generated R-modules and R-homomorphisms.

We are mainly interested in degenerations of modules over commutative rings. Henceforth, in the rest of the paper, all the rings are assumed to be commutative.

Definition 4. Let M and N be finitely generated modules over a commutative noetherian k-algebra R.

(1) We denote by $M \leq_{deg} N$ if N is obtained from M by iterative degenerations, i.e. there is a sequence of finitely generated R-modules $L_0, L_1, \dots, L_r$ such that $M \cong L_0$, $N \cong L_r$ and each L_i degenerates to L_{i+1} for $0 \leq i < r$.

(2) We say that M degenerates by an extension to N if there is a short exact sequence $0 \to U \to M \to V \to 0$ of finitely generated R-modules such that $N \cong U \oplus N$.

We denote by $M \leq_{ext} N$ if N is obtained from M by iterative degenerations by extensions, i.e. there is a sequence of finitely generated R-modules $L_0, L_1, \dots, L_r$ such that $M \cong L_0$, $N \cong L_r$ and each L_i degenerates by an extension to L_{i+1} for $0 \leq i < r$.

If R is a local ring, then $\leq_{deg}$ and $\leq_{ext}$ are known to be partial orders on the set of isomorphism classes of finitely generated R-modules, which are called the degeneration order and the extension order respectively. See [6] for the detail.

Remark 5. By virtue of Remark 3, if $M \leq_{ext} N$ then $M \leq_{deg} N$. However the converse is not necessarily true.

For example, consider a ring $R = k[[x, y]]/(x^2)$. A pair of matrices over $k[[x, y]]$;

$$(\varphi, \psi) = \left(\begin{pmatrix} x & y^2 \\ 0 & x \end{pmatrix}, \begin{pmatrix} x & -y^2 \\ 0 & x \end{pmatrix} \right)$$

is a matrix factorization of the equation x^2, hence it gives a maximal Cohen-Macaulay R-module N that is isomorphic to the ideal $(x, y^2)R$. It is known that N is indecomposable. Then we can show that R degenerates to $(x, y^2)R$ in this case, and hence $R \leq_{deg} (x, y^2)R$. See [3, Remark 2.5.].

In general if $M \leq_{ext} N$ and if $M \ncong N$, then N is a non-trivial direct sum of modules. Since $N \cong (x, y^2)R$ is indecomposable, we see that $R \leq_{ext} (x, y^2)R$ can never happen.

Remark 6. We remark that if finitely generated R-modules M and N satisfy the relation $M \leq_{ext} N$, then M degenerates to N.

Now we note that the following lemma holds.

Lemma 7. *Let I be a two-sided ideal of a noetherian k-algebra R, and let M and N be finitely generated left R/I-modules. Then $M \leq_{deg} N$ (resp. $M \leq_{ext} N$) as R-modules if and only if so does as R/I-modules.* □

We make several other remarks on degenerations for the later use.

Remark 8. Let R be a noetherian k-algebra, and let M and N be finitely generated R-modules. Suppose that M degenerates to N. The ith Fitting ideal of M contains that of N for all $i \geq 0$. Namely, denoting the ith Fitting ideal of an R-module M by $\mathcal{F}_i^R(M)$, we have $\mathcal{F}_i^R(M) \supseteq \mathcal{F}_i^R(N)$ for all $i \geqq 0$. (See [9, Theorem 2.5]).

Let $R = k[[x]]$ be a formal power series ring over a field k with one variable x and let M be an R-module of length n. It is easy to see that there is an isomorphism

$$M \cong R/(x^{p_1}) \oplus \cdots \oplus R/(x^{p_n}), \tag{2.1}$$

where

$$p_1 \geq p_2 \geq \cdots \geq p_n \geq 0 \quad \text{and} \quad \sum_{i=1}^{n} p_i = n. \tag{2.2}$$

In this case the finite presentation of M is given as follows:

$$0 \longrightarrow R^n \xrightarrow{\begin{pmatrix} x^{p_1} & & \\ & \ddots & \\ & & x^{p_n} \end{pmatrix}} R^n \longrightarrow M \longrightarrow 0.$$

Note that we can easily compute the ith Fitting ideal of M from this presentation;

$$\mathcal{F}_i^R(M) = (x^{p_{i+1}+\cdots+p_n}) \quad (i \geq 0).$$

We denote by p_M the sequence $(p_1, p_2, \cdots, p_n)$ of non-negative integers. Recall that such a sequence satisfying (2.2) is called a partition of n.

Conversely, given a partition $p = (p_1, p_2, \cdots, p_n)$ of n, we can associate an R-module of length n by (2.1), which we denote by $M(p)$. In such a way we see that there is a one-one correspondence between the set of partitions of n and the set of isomorphism classes of R-modules of length n.

Definition 9. Let n be a positive integer and let $p = (p_1, p_2, \cdots, p_n)$ and $q = (q_1, q_2, \cdots, q_n)$ be partitions of n. Then we denote $p \succeq q$ if it satisfies $\sum_{i=1}^{j} p_i \geq \sum_{i=1}^{j} q_i$ for all $1 \leq j \leq n$.

We note that $\succeq$ is known to be a partial order on the set of partitions of n and called the dominance order (see for example [4, page 7]).

In the following proposition we show the degeneration order for R-modules of length n coincides with the opposite of the dominance order of corresponding partitions.

Proposition 10. *Let $R = k[[x]]$ as above, and let M, N be R-modules of length n. Then the following conditions are equivalent:*

(1) $M \leq_{deg} N$,
(2) $M \leq_{ext} N$,
(3) $p_M \succeq p_N$.

Proof. First of all, we assume M degenerates to N, and let $p_M = (p_1, p_2, \cdots, p_n)$ and $p_N = (q_1, q_2, \cdots, q_n)$. Then, by definition, we have the equalities of the Fitting ideals; $\mathcal{F}_i^R(M) = (x^{p_{i+1}+\cdots+p_n})$ and $\mathcal{F}_i^R(M) = (x^{q_{i+1}+\cdots+q_n})$ for all $i \geq 0$. Since M degenerates to N, it follows from Remark 8 that $\mathcal{F}_i^R(M) \supseteq \mathcal{F}_i^R(N)$ for all i. Thus $p_{i+1} + \cdots + p_n \leq q_{i+1} + \cdots + q_n$. Since $\sum_{i=1}^n p_i = n = \sum_{i=1}^n q_i$, it follows that $p_1 + \cdots + p_i \geq q_1 + \cdots + q_i$ for all $i \geq 0$. Therefore $p_M \succeq p_N$, so that we have proved the implication (1) $\Rightarrow$ (3).

Finally we shall prove (3) $\Rightarrow$ (2). To this end let $p = (p_1, p_2, \cdots, p_n)$ and $q = (q_1, q_2, \cdots, q_n)$ be partitions of n. Note that it is enough to prove that the corresponding R-module $M(p)$ degenerates by an extension to $M(q)$ whenever q is a predecessor of p under the dominance order. (Recall that q is called a predecessor of p if $p \succeq q$ and there are no partitions r with $p \succeq r \succeq q$ other than p and q.)

Assume that q is a predecessor of p under the dominance order. Then it is easy to see that there are numbers $1 \leq i < j \leq n$ with $p_i - p_j \geq 2$, $p_i > p_{i+1}$, $p_{j-1} > p_j$ such that the equality $q = (p_1, \cdots, p_i - 1, p_{i+1}, \cdots, p_j + 1, \cdots, p_n)$ holds. In this case, setting $L = M((p_1, \cdots, p_{i-1}, p_{i+1}, \cdots, p_{j-1}, p_j, \cdots, p_n))$, we have $M(p) = L \oplus M((p_i, p_j))$ and $M(q) = L \oplus M((p_i - 1, p_j + 1))$. Note that, in general, if M degenerates by an extension to N, then $M \oplus L$ degenerates by an extension to $N \oplus L$, for any R-modules L. Hence it is enough to show that $M((a, b))$ degenerates by an extension to $M((a-1, b+1))$ if $a \geq b + 2$. However there is a short exact sequence of the form:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R/(x^{a-1}) & \longrightarrow & R/(x^a) \oplus R/(x^b) & \longrightarrow & R/(x^{b+1}) & \longrightarrow & 0 \\ & & 1 & \longrightarrow & (x, 1) & & & & \end{array}$$

Thus $M((a, b)) = R/(x^a) \oplus R/(x^b)$ degenerates by an extension to $M((a-1, b+1)) = R/(x^{a-1}) \oplus R/(x^{b+1})$. $\square$

Combining Proposition 10 with Lemma 7, we have the following corollary which will be used latter.

Corollary 11. *Let $R = k[[x]]/(x^m)$, where k is a field and m is a positive integer, and let M, N be finitely generated R-modules. Then $M \leq_{deg} N$ holds if and only if $M \leq_{ext} N$ holds.* $\square$

Next we describe another example.

Let k be a field of characteristic 0 and $R = k[[x_0, x_1, x_2, \cdots, x_d]]/(f)$, where f is a polynomial of the form

$$f = x_0^{n+1} + x_1^2 + x_2^2 + \cdots + x_d^2 \qquad (n \geq 1).$$

Recall that such a ring R is call the ring of simple singularity of type (A_n). Note that R is a Gorenstein complete local ring and has finite Cohen-Macaulay representation type. (Recall that a Cohen-Macaulay k-algebra R is said to be of finite Cohen-Macaulay representation type if there are only a finite number of isomorphism classes of objects in $\mathrm{CM}(R)$. See [5].) We shall show the following whose proof will be given in the last part of this section.

Theorem 12. *Let k be an algebraically closed field of characteristic 0 and let $R = k[[x_0, x_1, x_2, \cdots, x_d]]/(x_0^{n+1} + x_1^2 + x_2^2 + \cdots + x_d^2)$ as above, where we assume that d is even. For maximal Cohen-Macaulay R-modules M and N, if $M \leq_{deg} N$, then $M \leq_{ext} N$.*

To prove the theorem, we need several results concerning the stable degeneration which was introduced by Yoshino in [9].

Let A be a commutative Gorenstein ring. We denote by $\mathrm{CM}(A)$ the category of all maximal Cohen-Macaulay A-module with all A-homomorphisms. And we also denote by $\underline{\mathrm{CM}}(A)$ the stable category of $\mathrm{CM}(A)$. For a maximal Cohen-Macaulay module M we denote it by $\underline{M}$ to indicate that it is an object of $\underline{\mathrm{CM}}(A)$. Since A is Gorenstein, it is known that $\underline{\mathrm{CM}}(A)$ has a structure of triangulated category.

The following theorem proved by Yoshino [9] shows the relation between stable degenerations and ordinary degenerations.

Theorem 13. [9, Theorem 5.1, 6.1, 7.1] *Let $(R, \mathfrak{m}, k)$ be a Gorenstein complete local k-algebra, where k is an infinite field. Consider the following four conditions for maximal Cohen-Macaulay R-modules M and N:*

(1) *$R^m \oplus M$ degenerates to $R^n \oplus N$ for some $m, n \in \mathbb{N}$.*
(2) *There is a triangle*

$$\underline{Z} \xrightarrow{\binom{\underline{\varphi}}{\underline{\psi}}} \underline{M} \oplus \underline{Z} \longrightarrow \underline{N} \longrightarrow \underline{Z}[1]$$

in $\underline{\mathrm{CM}}(R)$, where $\underline{\psi}$ is a nilpotent element of $\underline{\mathrm{End}}_R(Z)$.
(3) *$\underline{M}$ stably degenerates to $\underline{N}$.*
(4) *There exists an $X \in \mathrm{CM}(R)$ such that $M \oplus R^m \oplus X$ degenerates to $N \oplus R^n \oplus X$ for some $m, n \in \mathbb{N}$.*

Then, in general, the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) hold. If R is an isolated singularity, then (2) and (3) are equivalent. Furthermore, if R is an artinian ring, then the conditions (1), (2) and (3) are equivalent.

Corollary 14. [9, Corollary 6.6] *Let $(R_1, \mathfrak{m}_1, k)$ and $(R_2, \mathfrak{m}_2, k)$ be Gorenstein complete local k-algebras. Assume that the both R_1 and R_2 are isolated singularities, and that k is an infinite field. Suppose there is a k-linear equivalence $F : \underline{\mathrm{CM}}(R_1) \to \underline{\mathrm{CM}}(R_2)$ of triangulated categories. Then, for $\underline{M}$, $\underline{N} \in \underline{\mathrm{CM}}(R_1)$, $\underline{M}$ stably degenerates to $\underline{N}$ if and only if $F(\underline{M})$ stably degenerates to $F(\underline{N})$.* □

We now consider the stable analogue of the degeneration by an extension.

Definition 15.

(1) We denote by $\underline{M} \leq_{st} \underline{N}$ if $\underline{N}$ is obtained from $\underline{M}$ by iterative stable degenerations, i.e. there is a sequence of objects $\underline{L}_0, \underline{L}_1, \ldots, \underline{L}_r$ in $\underline{\mathrm{CM}}(R)$ such that $\underline{M} \cong \underline{L}_0$, $\underline{N} \cong \underline{L}_r$ and each $\underline{L}_i$ stably degenerates to $\underline{L}_{i+1}$ for $0 \leq i < r$.
(2) We say that $\underline{M}$ stably degenerates by a triangle to $\underline{N}$, if there is a triangle of the form $\underline{U} \to \underline{M} \to \underline{V} \to \underline{U}[1]$ in $\underline{\mathrm{CM}}(R)$ such that $\underline{U} \oplus \underline{V} \cong \underline{N}$. We denote by $\underline{M} \leq_{tri} \underline{N}$ if there is a finite sequence of modules $\underline{L_0}, \underline{L_1}, \cdots, \underline{L_r}$ in $\underline{\mathrm{CM}}(R)$ such that $\underline{M} \cong \underline{L_0}$, $\underline{N} \cong \underline{L_r}$ and each $\underline{L_i}$ stably degenerates by a triangle to $\underline{L_{i+1}}$ for $0 \leq i < r$.

Remark 16. Let R be a Gorenstein local ring that is a k-algebra.

(1) Let $M, N \in \mathrm{CM}(R)$. If M degenerates to N, then $\underline{M}$ stably degenerates to $\underline{N}$. Therefore that $M \leq_{deg} N$ forces that $\underline{M} \leq_{st} \underline{N}$. (See [9, Lemma 4.2].)

(2) Suppose that there is a triangle

$$\underline{L} \longrightarrow \underline{M} \longrightarrow \underline{N} \longrightarrow \underline{L}[1],$$

in $\underline{\mathrm{CM}}(R)$. Then $\underline{M}$ stably degenerates to $\underline{L} \oplus \underline{N}$, thus $\underline{M} \le_{st} \underline{L} \oplus \underline{N}$. (See [9, Proposition 4.3].)

We need the following proposition to prove Theorem 12.

Proposition 17. *Let $(R, \mathfrak{m}, k)$ be a Gorenstein complete local ring and let $M, N \in \mathrm{CM}(R)$. Assume $[M] = [N]$ in $K_0(\mathrm{mod}(R))$. Then $\underline{M} \le_{tri} \underline{N}$ if and only if $M \le_{ext} N$.* $\square$

Now we proceed to the proof of Theorem 12.

Let k be an algebraically closed field of characteristic 0 and let

$$R = k[[x_0, x_1, x_2, \cdots, x_d]]/(x_0^{n+1} + x_1^2 + x_2^2 + \cdots + x_d^2)$$

as in the theorem, where we assume that d is even. Suppose that $M \le_{deg} N$ for maximal Cohen-Macaulay R-modules M and N. We want to show $M \le_{ext} N$.

Since $M \le_{deg} N$, we have $\underline{M} \le_{st} \underline{N}$ in $\underline{\mathrm{CM}}(R)$ and $[M] = [N]$ in $K_0(\mathrm{mod}(R))$, by Remarks 16(1) and 3(3). Now let us denote $R' = k[[x_0]]/(x_0^{n+1})$, and we note that $\underline{\mathrm{CM}}(R)$ and $\underline{\mathrm{CM}}(R')$ are equivalent to each other as triangulated categories. In fact this equivalence is given by using the lemma of the Knörrer's periodicity (cf. [5]), since d is even. Let $\Omega : \underline{\mathrm{CM}}(R) \to \underline{\mathrm{CM}}(R')$ be a triangle functor which gives the equivalence. Then, by virtue of Corollary 14, we have $\Omega(\underline{M}) \le_{st} \Omega(\underline{N})$ in $\underline{\mathrm{CM}}(R')$. Since R' is an artinian algebra, the equivalence (1) $\Leftrightarrow$ (3) holds in Theorem 13, and thus we have $\tilde{M} \oplus R'^m \le_{deg} \tilde{N} \oplus R'^n$, where $\tilde{M}$ (resp. $\tilde{N}$) is a module in $\mathrm{CM}(R')$ with $\underline{\tilde{M}} \cong \Omega(\underline{M})$ (resp. $\underline{\tilde{N}} \cong \Omega(\underline{N})$) and m, n are non-negative integers. It then follows from Corollary 11 that $\tilde{M} \oplus R'^m \le_{ext} \tilde{N} \oplus R'^n$. Hence, by Proposition 17, we have that $\Omega(\underline{M}) \le_{tri} \Omega(\underline{N})$ in $\underline{\mathrm{CM}}(R')$. Noting that the partial order $\le_{tri}$ is preserved under a triangle functor, we see that $\underline{M} \le_{tri} \underline{N}$ in $\underline{\mathrm{CM}}(R)$. Since $[M] = [N]$ in $K_0(\mathrm{mod}(R))$, applying Proposition 17, we finally obtain that $M \le_{ext} N$. $\square$

Example 18. Let $R = k[[x_0, x_1, x_2]]/(x_0^3 + x_1^2 + x_2^2)$, where k is an algebraically closed field of characteristic 0. Let $\mathfrak{p}$ and $\mathfrak{q}$ be the ideals generated by $(x_0, x_1 - \sqrt{-1}\ x_2)$ and $(x_0^2, x_1 + \sqrt{-1}\ x_2)$ respectively. It is known that the set $\{R, \mathfrak{p}, \mathfrak{q}\}$ is a complete list of the isomorphism classes of indecomposable maximal Cohen-Macaulay modules over R. The Hasse diagram of degenerations of maximal Cohen-Macaulay R-modules of rank 3 is a disjoint union of the following diagrams:

$$\begin{tikzcd}[column sep=small]
\mathfrak{p} \oplus \mathfrak{p} \oplus \mathfrak{p} \arrow[dr, dash] & & \mathfrak{q} \oplus \mathfrak{q} \oplus \mathfrak{q} \arrow[dl, dash] \\
& R \oplus \mathfrak{p} \oplus \mathfrak{q} \arrow[d, dash] & \\
& R^3, &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{p} \oplus \mathfrak{p} \oplus \mathfrak{q} \arrow[d, dash] \\
R \oplus \mathfrak{q} \oplus \mathfrak{q} \arrow[d, dash] \\
R^2 \oplus \mathfrak{p},
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{p} \oplus \mathfrak{q} \oplus \mathfrak{q} \arrow[d, dash] \\
R \oplus \mathfrak{p} \oplus \mathfrak{p} \arrow[d, dash] \\
R^2 \oplus \mathfrak{q}.
\end{tikzcd}$$

3. Extended orders

In the rest of this paper R denotes a (commutative) Cohen-Macaulay complete local k-algebra, where k is any field.

We shall show that any extended degenerations of maximal Cohen-Macaulay R-modules are generated by extended degenerations of Auslander-Reiten (abbr.AR) sequences if R is of finite Cohen-Macaulay representation type. For the theory of AR sequences of maximal Cohen-Macaulay modules, we refer to [5]. First of all we recall the definitions of the extended orders generated respectively by degenerations, extensions and AR sequences.

Definition 19. [6, Definition 4.11, 4.13] The relation $\leq_{DEG}$ on $\mathrm{CM}(R)$, which is called the extended degeneration order, is a partial order generated by the following rules:

(1) If $M \leq_{deg} N$ then $M \leq_{DEG} N$.
(2) If $M \leq_{DEG} N$ and if $M' \leq_{DEG} N'$ then $M \oplus M' \leq_{DEG} N \oplus N'$
(3) If $M \oplus L \leq_{DEG} N \oplus L$ for some $L \in \mathrm{CM}(R)$ then $M \leq_{DEG} N$.
(4) If $M^n \leq_{DEG} N^n$ for some natural number n then $M \leq_{DEG} N$.

Definition 20. [6, Definition 3.6] The relation $\leq_{EXT}$ on $\mathrm{CM}(R)$, which is called the extended extension order, is a partial order generated by the following rules:

(1) If $M \leq_{ext} N$ then $M \leq_{EXT} N$.
(2) If $M \leq_{EXT} N$ and if $M' \leq_{EXT} N'$ then $M \oplus M' \leq_{EXT} N \oplus N'$
(3) If $M \oplus L \leq_{EXT} N \oplus L$ for some $L \in \mathrm{CM}(R)$ then $M \leq_{EXT} N$.
(4) If $M^n \leq_{EXT} N^n$ for some natural number n then $M \leq_{EXT} N$.

Definition 21. [6, Definition 5.1] The relation $\leq_{AR}$ on $\mathrm{CM}(R)$, which is called the extended AR order, is a partial order generated by the following rules:

(1) If $0 \to X \to E \to Y \to 0$ is an AR sequence in $\mathrm{CM}(R)$, then $E \leq_{AR} X \oplus Y$.
(2) If $M \leq_{AR} N$ and if $M' \leq_{AR} N'$ then $M \oplus M' \leq_{AR} N \oplus N'$
(3) If $M \oplus L \leq_{AR} N \oplus L$ for some $L \in \mathrm{CM}(R)$ then $M \leq_{AR} N$.
(4) If $M^n \leq_{AR} N^n$ for some natural number n then $M \leq_{AR} N$.

The following is the main theorem of this section.

Theorem 22. *Let R be a Cohen-Macaulay complete local k-algebra as above. Adding to this, we assume that R is of finite Cohen-Macaulay representation type.Then the following conditions are equivalent for $M, N \in \mathrm{CM}(R)$:*

(1) $M \leq_{DEG} N$,
(2) $M \leq_{EXT} N$,
(3) $M \leq_{AR} N$.

Proof. The implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are clear from the definitions.

To prove (1) $\Rightarrow$ (2), it suffices to show that $M \leq_{EXT} N$ whenever M degenerates to N. If M degenerates to N, then, by virtue of Theorem 2, we have a short exact sequence $0 \to Z \to M \oplus Z \to N \to 0$ with $Z \in \mathrm{CM}(R)$. Thus $M \oplus Z \leq_{ext} N \oplus Z$, hence $M \leq_{EXT} N$.

It remains to prove that (2) $\Rightarrow$ (3), for which we need the following lemma which is essentially due to Auslander and Reiten [1].

Lemma 23. *Under the same assumptions on R as in Theorem 22, let $0 \to L \to M \to N \to 0$ be a short exact sequence in* $\mathrm{CM}(R)$*. Then there are a finite number of AR sequences in* $\mathrm{CM}(R)$*;*

$$0 \to X_i \to E_i \to Y_i \to 0 \quad (1 \leq i \leq n),$$

such that there is an equality in $\mathrm{G}(\mathrm{CM}(R))$*;*

$$L - M + N = \sum_{i=1}^{n} (X_i - E_i + Y_i).$$

Here, $\mathrm{G}(\mathrm{CM}(R)) = \bigoplus \mathbb{Z} \cdot X$ *where X runs through all isomorphism classes of indecomposable objects in* $\mathrm{CM}(R)$*.*

To prove this lemma, we consider the functor category $\mathrm{Mod}(\mathrm{CM}(R))$ and the Auslander category $\mathrm{mod}(\mathrm{CM}(R))$ of $\mathrm{CM}(R)$. □

Remark 24. In the paper [6], Yoshino introduced the order relation $\leq_{hom}$ as well. Adding to the assumption that R is of finite Cohen-Macaulay representation type, if we assume further conditions on R, such as R is an integral domain of dimension 1 or R is of dimension 2, then he showed that $\leq_{hom}$ is also equal to any of $\leq_{AR}$, $\leq_{EXT}$ and $\leq_{DEG}$.

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WEAK GORENSTEIN DIMENSION FOR MODULES AND GORENSTEIN ALGEBRAS

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Abstract. We will generalize the notion of Gorenstein dimension and introduce that of weak Gorenstein dimension. Using this notion, we will characterize Gorenstein algebras.

1. Introduction

1.1. **Notation and definitions.** For a ring A we denote by $\mathrm{rad}(A)$ the Jacobson radical of A. Also, we denote by Mod-A the category of right A-modules, by mod-A the full subcategory of Mod-A consisting of finitely presented modules and by $\mathcal{P}_A$ the full subcategory of mod-A consisting of projective modules. For each $X \in$ Mod-A we denote by $E_A(X)$ its injective envelope. Left A-modules are considered as right A^{op}-modules, where A^{op} denotes the opposite ring of A. In particular, we denote by inj dim A (resp., inj dim A^{op}) the injective dimension of A as a right (resp., left) A-module and by $\mathrm{Hom}_A(-,-)$ (resp., $\mathrm{Hom}_{A^{\mathrm{op}}}(-,-)$) the set of homomorphisms in Mod-A (resp., Mod-A^{op}). Sometimes, we use the notation X_A (resp., ${}_AX$) to stress that the module considered is a right (resp., left) A-module.

In this note, complexes are cochain complexes and modules are considered as complexes concentrated in degree zero. For a complex $X^\bullet$ and an integer $n \in \mathbb{Z}$, we denote by $\mathrm{H}^n(X^\bullet)$ the nth cohomology. We denote by $\mathcal{K}(\text{Mod-}A)$ the homotopy category of complexes over Mod-A, by $\mathcal{K}^-(\mathcal{P}_A)$ (resp., $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$) the full triangulated subcategory of $\mathcal{K}(\text{Mod-}A)$ consisting of bounded above (resp., bounded) complexes over $\mathcal{P}_A$ and by $\mathcal{K}^{-,\mathrm{b}}(\mathcal{P}_A)$ the full triangulated subcategory of $\mathcal{K}^-(\mathcal{P}_A)$ consisting of complexes with bounded cohomology. We denote by $\mathcal{D}(\text{Mod-}A)$ the derived category of complexes over Mod-A. Also, we denote by $\mathrm{Hom}^\bullet_A(-,-)$ (resp., $- \otimes^\bullet -$) the associated single complex of the double hom (resp., tensor) complex and by $\mathbf{R}\mathrm{Hom}^\bullet_A(-,A)$ the right derived functor of $\mathrm{Hom}^\bullet_A(-,A)$. We refer to [4], [9] and [15] for basic results in the theory of derived categories.

Definition 1 ([5])**.** A module $X \in$ Mod-A is said to be coherent if it is finitely generated and every finitely generated submodule of it is finitely presented. A ring A is said to be left (resp., right) coherent if it is coherent as a left (resp., right) A-module.

Throughout the first three sections, A is a left and right coherent ring. Note that mod-A consists of the coherent modules and is a thick abelian subcategory of Mod-A in the sense of [9].

We denote by $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)$ the full triangulated subcategory of $\mathcal{D}(\text{Mod-}A)$ consisting of complexes over mod-A with bounded cohomology.

The detailed version of this paper will be submitted for publication elsewhere.

Definition 2 ([9])**.** A complex $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A)$ is said to have finite projective dimension if $\mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(X^\bullet[i], -)$ vanishes on mod-A for $i \ll 0$. We denote by $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$ the épaisse subcategory of $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)$ consisting of complexes of finite projective dimension.

Note that the canonical functor $\mathcal{K}(\text{Mod-}A) \to \mathcal{D}(\text{Mod-}A)$ gives rise to equivalences of triangulated categories

$$\mathcal{K}^{-,\mathrm{b}}(\mathcal{P}_A) \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(\text{mod-}A) \quad \text{and} \quad \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A) \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}.$$

We denote by $D(-)$ both $\mathbf{R}\mathrm{Hom}^\bullet_A(-, A)$ and $\mathbf{R}\mathrm{Hom}^\bullet_{A^{\mathrm{op}}}(-, A)$. There exists a bifunctorial isomorphism

$$\theta_{M^\bullet, X^\bullet} : \mathrm{Hom}_{\mathcal{D}(\text{Mod-}A^{\mathrm{op}})}(M^\bullet, DX^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(X^\bullet, DM^\bullet)$$

for $X^\bullet \in \mathcal{D}(\text{Mod-}A)$ and $M^\bullet \in \mathcal{D}(\text{Mod-}A^{\mathrm{op}})$. For each $X^\bullet \in \mathcal{D}(\text{Mod-}A)$ we set

$$\eta_{X^\bullet} = \theta_{DX^\bullet, X^\bullet}(\mathrm{id}_{DX^\bullet}) : X^\bullet \to D^2X^\bullet = D(DX^\bullet).$$

Definition 3. A complex $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A)$ is said to have bounded dual cohomology if $DX^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A^{\mathrm{op}})$. We denote by $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}$ the full triangulated subcategory of $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)$ consisting of complexes with bounded dual cohomology.

Definition 4 ([2] and [12])**.** A complex $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A^{\mathrm{op}})_{\mathrm{bdh}}$ is said to have finite Gorenstein dimension if $\eta_{X^\bullet}$ is an isomorphism. We denote by $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fGd}}$ the full triangulated subcategory of $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)$ consisting of complexes of finite Gorenstein dimension.

For a module $X \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fGd}}$, we set

$$\text{G-dim}\ X = \sup\{\ i \geq 0 \mid \mathrm{Ext}^i_A(X, A) \neq 0\}$$

if $X \neq 0$, and G-dim $X = 0$ if $X = 0$. Also, we set G-dim $X = \infty$ for a module $X \in \text{mod-}A$ with $X \notin \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fGd}}$. Then G-dim X is called the Gorenstein dimension of $X \in \text{mod-}A$. We denote by $\mathcal{G}_A$ the full additive subcategory of mod-A consisting of modules of Gorenstein dimension zero.

Remark 5. A module $X \in \text{mod-}A$ has Gorenstein dimension zero if and only if X is reflexive, i.e., the canonical homomorphism

$$X \to \mathrm{Hom}_{A^{\mathrm{op}}}(\mathrm{Hom}_A(X, A), A), x \mapsto (f \mapsto f(x))$$

is an isomorphism and $\mathrm{Ext}^i_A(X, A) = \mathrm{Ext}^i_{A^{\mathrm{op}}}(\mathrm{Hom}_A(X, A), A) = 0$ for $i \neq 0$.

Remark 6. The following hold.

(1) $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}} \subseteq \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fGd}} \subseteq \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}$ and $\mathcal{P}_A \subseteq \mathcal{G}_A$.

(2) The pair of functors $\mathbf{R}\mathrm{Hom}^\bullet_A(-, A)$ and $\mathbf{R}\mathrm{Hom}^\bullet_{A^{\mathrm{op}}}(-, A)$ defines a duality between $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fGd}}$ and $\mathcal{D}^{\mathrm{b}}(\text{mod-}A^{\mathrm{op}})_{\mathrm{fGd}}$ and a duality between $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$ and $\mathcal{D}^{\mathrm{b}}(\text{mod-}A^{\mathrm{op}})_{\mathrm{fpd}}$.

(3) The pair of functors $\mathrm{Hom}_A(-, A)$ and $\mathrm{Hom}_{A^{\mathrm{op}}}(-, A)$ defines a duality between $\mathcal{G}_A$ and $\mathcal{G}_{A^{\mathrm{op}}}$ and a duality between $\mathcal{P}_A$ and $\mathcal{P}_{A^{\mathrm{op}}}$.

1.2. **Introduction.** The notion of Gorenstein dimension has played an important role in the study of Gorenstein algebras (see e.g. [2], [10], [11] and so on). In this note, generalizing this, we will introduce the notion of weak Gorenstein dimension and characterize Gorenstein algebras in terms of weak Gorenstein dimension.

A complex $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}$ with $\sup\{\, i \mid \mathrm{H}^i(X^\bullet) \neq 0\} = d < \infty$ is said to have finite weak Gorenstein dimension if $\mathrm{H}^i(\eta_{X^\bullet})$ is an isomorphism for all $i < d$ and $\mathrm{H}^d(\eta_{X^\bullet})$ is a monomorphism. Obviously, every $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fGd}}$ has finite weak Gorenstein dimension, the converse of which does not hold true in general (see Example 9 and Proposition 15). Extending the fact announced by Avramov [3], we will characterize complexes of finite weak Gorenstein dimension. Denote by $\mathcal{G}_A/\mathcal{P}_A$ the residue category of $\mathcal{G}_A$ over $\mathcal{P}_A$. Also, denote by $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fGd}}/\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$ the quotient category of $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fGd}}$ over the épaisse subcategory $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$. Avramov [3] announced that the embedding $\mathcal{G}_A \to \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fGd}}$ gives rise to an equivalence

$$\mathcal{G}_A/\mathcal{P}_A \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fGd}}/\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}.$$

We will extend this fact. Denote by $\hat{\mathcal{G}}_A$ the full additive subcategory of mod-A consisting of modules $X \in \text{mod-}A$ with $\mathrm{Ext}^i_A(X, A) = 0$ for $i \neq 0$, by $\hat{\mathcal{G}}_A/\mathcal{P}_A$ the residue category of $\hat{\mathcal{G}}_A$ over $\mathcal{P}_A$ and by $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}/\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$ the quotient category of $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}$ over the épaisse subcategory $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$. We will show that the embedding $\hat{\mathcal{G}}_A \to \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}$ gives rise to a full embedding

$$F : \hat{\mathcal{G}}_A/\mathcal{P}_A \to \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}/\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$$

(see Proposition 8), that a complex $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}$ has finite weak Gorenstein dimension if and only if there exists a homomorphism $Z[m] \to X^\bullet$ in $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}$ inducing an isomorphism in $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}/\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$ for some $Z \in \hat{\mathcal{G}}_A$ and $m \in \mathbb{Z}$ (see Lemma 12) and that F is an equivalence if and only if $\hat{\mathcal{G}}_A = \mathcal{G}_A$ (see Proposition 15).

Using the notion of weak Gorenstein dimension, we will characterize Gorenstein algebras. Let R be a commutative noetherian local ring and A a noetherian R-algebra, i.e., A is a ring endowed with a ring homomorphism $R \to A$ whose image is contained in the center of A and A is finitely generated as an R-module. We say that A satisfies the condition (G) if the following equivalent conditions are satisfied: (1) Every simple $X \in \text{mod-}A$ has finite weak Gorenstein dimension; and (2) $A/\mathrm{rad}(A)$ has finite weak Gorenstein dimension (see Definition 18). Our main theorem states that the following are equivalent: (1) $\mathrm{inj\ dim}\, A = \mathrm{inj\ dim}\, A^{\mathrm{op}} < \infty$; and (2) $A_{\mathfrak{p}}$ satisfies the condition (G) for all $\mathfrak{p} \in \mathrm{Supp}_R(A)$ (see Theorem 19). Furthermore, in case A is a local ring, we will show that for any $d \geq 0$ the following are equivalent: (1) $\mathrm{inj\ dim}\, A = \mathrm{inj\ dim}\, A^{\mathrm{op}} = d$; (2) $\mathrm{inj\ dim}\, A = \mathrm{depth}\, A = d$; and (3) $A/\mathrm{rad}(A)$ has weak Gorenstein dimension d (see Theorem 20). Note that if $\mathrm{inj\ dim}\, A = \mathrm{depth}\, A < \infty$ then A is a Gorenstein R-algebra in the sense of Goto and Nishida [8].

This note is organized as follows. In Section 2, we will extend the fact announced by Avramov [3] quoted above. Also, we will include an example of A with $\hat{\mathcal{G}}_A \neq \mathcal{G}_A$ which is due to J.-I. Miyachi. In Section 3, we will introduce the notion of weak Gorenstein dimension and study finitely presented modules of finite weak Gorenstein dimension. In Section 4, we will study noetherian algebras of finite selfinjective dimension and prove the main theorem. In Section 5, we will characterize local noetherian algebras of finite selfinjective dimension. Also, we will provide several examples showing what rich properties local noetherian algebras of finite selfinjective dimension enjoy.

2. Full embedding

Let $\mathcal{G}_A/\mathcal{P}_A$ be the residue category of $\mathcal{G}_A$ over the full additive subcategory $\mathcal{P}_A$ and $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fGd}}/\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$ the quotient category of $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fGd}}$ over the épaisse subcategory $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$. Then, as Avramov [3] announced, the embedding $\mathcal{G}_A \to \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fGd}}$ gives rise to an equivalence

$$\mathcal{G}_A/\mathcal{P}_A \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fGd}}/\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}.$$

In this section, we will extend this fact.

Definition 7. We denote by $\hat{\mathcal{G}}_A$ the full additive subcategory of mod-A consisting of modules $X \in \text{mod-}A$ with $\mathrm{Ext}^i_A(X, A) = 0$ for $i \neq 0$.

We denote by $\hat{\mathcal{G}}_A/\mathcal{P}_A$ the residue category of $\hat{\mathcal{G}}_A$ over the full additive subcategory $\mathcal{P}_A$ and by $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)/\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$ the quotient category of $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)$ over the épaisse subcategory $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$. Also, we denote by $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}/\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$ the quotient category of $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}$ over the épaisse subcategory $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$.

Proposition 8. *The embedding $\hat{\mathcal{G}}_A \to \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}$ gives rise to a full embedding*

$$F : \hat{\mathcal{G}}_A/\mathcal{P}_A \to \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}/\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}.$$

In the next section, we will characterize a complex $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}$ which admits a homomorphism $Z[m] \to X^\bullet$ in $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}$ inducing an isomorphism $Z[m] \xrightarrow{\sim} X^\bullet$ in $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}/\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$ for some $Z \in \hat{\mathcal{G}}_A$ and $m \in \mathbb{Z}$. Such a complex does not necessarily belong to $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fGd}}$. Namely, $\hat{\mathcal{G}}_A \neq \mathcal{G}_A$ in general (see Proposition 15 below), which has been pointed out by J.-I. Miyachi in oral communication.

Example 9 (Miyachi)**.** Let k be a field and fix a nonzero element $c \in k$ which is not a root of unity. Let $S = k < x, y >$ be a non-commutative polynomial ring and $I = (x^2, y^2, cxy + yx)$ a two-sided ideal generated by x^2, y^2 and $cxy + yx$. Set $R = S/I$, $z_n = x + c^n y + I \in R$ for $n \in \mathbb{Z}$ and $w = xy + I \in R$. Then R is a selfinjective algebra and for each $n \in \mathbb{Z}$ there exist exact sequences $R \xrightarrow{z_{n+1}} R \xrightarrow{z_n} R$ in mod-R and $R \xrightarrow{z_n} R \xrightarrow{z_{n+1}} R$ in mod-R^{op}. Since c is not a root of unity, $z_n R \ncong z_m R$ and $\mathrm{Hom}_R(z_n R, z_m R) \cong k$ unless $n = m$. Thus, since we have a projective resolution $\cdots \to R \xrightarrow{z_3} R \xrightarrow{z_2} R \xrightarrow{z_1} z_1 R \to 0$ in mod-R, applying $\mathrm{Hom}_R(-, z_0 R)$ we have $\mathrm{Ext}^i_R(z_1 R, z_0 R) = 0$ for all $i \geq 1$ (see [14]).

Now, we set

$$A = \begin{pmatrix} k & z_0 R \\ 0 & R \end{pmatrix} \text{ and } e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in A.$$

Then a module $X \in \text{mod-}A$ is given by a triple $(X_1, X_2; \phi)$ of $X_1 \in \text{mod-}k$, $X_2 \in \text{mod-}R$ and $\phi \in \mathrm{Hom}_R(X_1 \otimes_k z_0 R, X_2)$, and a module $M \in \text{mod-}A^{\mathrm{op}}$ is given by a triple $(M_1, M_2; \psi)$ of $M_1 \in \text{mod-}k$, $M_2 \in \text{mod-}R^{\mathrm{op}}$ and $\psi \in \mathrm{Hom}_k(z_0 R \otimes_R M_2, M_1)$ (see [7]). Set $X = (0, z_1 R; 0) \in \text{mod-}A$. Since we have a projective resolution $\cdots \xrightarrow{z_3} e_2 A \xrightarrow{z_2}$

$e_2A \xrightarrow{z_1} X \to 0$ in mod-A, it follows that $\mathrm{Ext}^i_A(X, e_1A) \cong \mathrm{Ext}^i_R(z_1R, z_0R) = 0$ and $\mathrm{Ext}^i_A(X, e_2A) \cong \mathrm{Ext}^i_R(z_1R, R) = 0$ for $i > 0$. Thus $X \in \hat{\mathcal{G}}_A$. On the other hand, we have

$$\begin{aligned} \mathrm{Hom}_A(X, A) &\cong \mathrm{Ker}(Ae_2 \xrightarrow{z_2} Ae_2) \\ &\cong (wR, Rz_1; 0) \\ &\cong (wR, 0; 0) \oplus (0, Rz_1; 0) \end{aligned}$$

in mod-A^{op} and hence $\mathrm{Hom}_{A^{\mathrm{op}}}(\mathrm{Hom}_A(X, A), A)$ is decomposable, so that we have $X \ncong \mathrm{Hom}_{A^{\mathrm{op}}}(\mathrm{Hom}_A(X, A), A)$ and $X \notin \mathcal{G}_A$.

3. Weak Gorenstein dimension

In this section, we will introduce the notion of weak Gorenstein dimension for finitely presented modules and study finitely presented modules of finite weak Gorenstein dimension.

Definition 10. A complex $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A)$ with $\sup\{\, i \mid \mathrm{H}^i(X^\bullet) \neq 0\} = d < \infty$ is said to have finite weak Gorenstein dimension if $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}$, $\mathrm{H}^i(\eta_{X^\bullet})$ is an isomorphism for $i < d$ and $\mathrm{H}^d(\eta_{X^\bullet})$ is a monomorphism.

For a module $X \in$ mod-A of finite weak Gorenstein dimension we set

$$\hat{\mathrm{G}}\text{-dim } X = \sup\{\, i \mid \mathrm{Ext}^i_A(X, A) \neq 0\}$$

if $X \neq 0$ and $\hat{\mathrm{G}}\text{-dim } X = 0$ if $X = 0$. Also, we set $\hat{\mathrm{G}}\text{-dim } X = \infty$ if $X \in$ mod-A does not have finite weak Gorenstein dimension. Then $\hat{\mathrm{G}}\text{-dim } X$ is called the weak Gorenstein dimension of $X \in$ mod-A.

Remark 11. For any $X \in$ mod-A the following hold.

(1) $\hat{\mathrm{G}}\text{-dim } X = 0$ if and only if X is embedded in some $P \in \mathcal{P}_A$, i.e., the canonical homomorphism

$$X \to \mathrm{Hom}_{A^{\mathrm{op}}}(\mathrm{Hom}_A(X, A), A), x \mapsto (f \mapsto f(x))$$

is a monomorphism and $X \in \hat{\mathcal{G}}_A$.

(2) If $\mathrm{G}\text{-dim } X = d < \infty$ then $\hat{\mathrm{G}}\text{-dim } X = d$.

(3) If $\hat{\mathrm{G}}\text{-dim } X = d < \infty$ then $\hat{\mathrm{G}}\text{-dim } X' \le d$ for all $X' \in \mathrm{add}(X)$, the full additive subcategory of mod-A consisting of direct summands of finite direct sums of copies of X.

Lemma 12. *A complex $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A)$ with $\sup\{\, i \mid \mathrm{H}^i(X^\bullet) \neq 0\} = d < \infty$ has finite weak Gorenstein dimension if and only if there exists a distinguished triangle in $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)$*

$$X^\bullet \to Y^\bullet \to Z[-d] \to$$

with $Y^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$, $Y^i = 0$ for $i > d$, and $Z \in \hat{\mathcal{G}}_A$.

Corollary 13 (cf. [6, Lemma 2.17])**.** *For any $X \in$ mod-A with $\hat{\mathrm{G}}\text{-dim } X < \infty$ there exists an exact sequence $0 \to X \to Y \to Z \to 0$ in mod-A with $\hat{\mathrm{G}}\text{-dim } X = \mathrm{proj\ dim}\ Y$ and $Z \in \hat{\mathcal{G}}_A$.*

Lemma 14. *For any exact sequence $0 \to X \to Y \to Z \to 0$ in mod-A the following hold.*

(1) *If* $\hat{\mathrm{G}}$-dim $Z < \infty$, *then* $\hat{\mathrm{G}}$-dim $X < \infty$ *if and only if* $\hat{\mathrm{G}}$-dim $Y < \infty$.
(2) *If* $\hat{\mathrm{G}}$-dim $Y < \infty$, *then* $\hat{\mathrm{G}}$-dim $X < \infty$ *if and only if* $Z \in \mathcal{D}^{\mathrm{b}}(\mathrm{mod}\text{-}A)_{\mathrm{bdh}}$ *and* $\mathrm{H}^i(D^2 Z) = 0$ *for* $i < -1$.
(3) *If* $\hat{\mathrm{G}}$-dim $X < \infty$ *and* $\mathrm{H}^0(\eta_X)$ *is an isomorphism, then* $\hat{\mathrm{G}}$-dim $Y < \infty$ *if and only if* $\hat{\mathrm{G}}$-dim $Z < \infty$.

Proposition 15. *The following are equivalent.*

(1) $\mathcal{G}_A = \hat{\mathcal{G}}_A$.
(2) $\hat{\mathrm{G}}$-dim $X = 0$ *for all* $X \in \hat{\mathcal{G}}_A$.
(3) $\mathcal{D}^{\mathrm{b}}(\mathrm{mod}\text{-}A)_{\mathrm{fGd}} = \mathcal{D}^{\mathrm{b}}(\mathrm{mod}\text{-}A)_{\mathrm{bdh}}$.
(4) *The embedding* $\hat{\mathcal{G}}_A/\mathcal{P}_A \to \mathcal{D}^{\mathrm{b}}(\mathrm{mod}\text{-}A)_{\mathrm{bdh}}/\mathcal{D}^{\mathrm{b}}(\mathrm{mod}\text{-}A)_{\mathrm{fpd}}$ *is dense.*

4. Finiteness of selfinjective dimension

Throughout the rest of this note, A is a left and right noetherian ring.

In this section, using the notion of weak Gorenstein dimension, we will characterize noetherian algebras of finite selfinjective dimension.

Lemma 16. *For any injective* $I \in \mathrm{Mod}\text{-}A$ *the following hold.*

(1) flat dim $I \leq$ inj dim A^{op} *and the equality holds if* I *is an injective cogenerator.*
(2) *Let* $d \geq 0$ *and assume that there exists a direct system* $(\{X_\lambda\}, \{f^\lambda_\mu\})$ *in* mod-A *over a directed set* Λ *such that* $\varinjlim X_\lambda \cong I$ *and* $\hat{\mathrm{G}}$-dim $X_\lambda \leq d$ *for all* $\lambda \in \Lambda$. *Then* flat dim $I \leq d$.

Corollary 17. *For any* $d \geq 0$ *the following are equivalent.*

(1) inj dim $A =$ inj dim $A^{\mathrm{op}} \leq d$.
(2) $\hat{\mathrm{G}}$-dim $X \leq d$ *for all* $X \in$ mod-A.

Throughout the rest of this note, R is a commutative noetherian local ring with the maximal ideal $\mathfrak{m}$ and A is a noetherian R-algebra, i.e., A is a ring endowed with a ring homomorphism $R \to A$ whose image is contained in the center of A and A is finitely generated as an R-module. It should be noted that $A/\mathfrak{m}A$ is a finite dimensional algebra over a field $R/\mathfrak{m}$.

We denote by $\mathrm{Spec}(R)$ the set of prime ideals of R. For each $\mathfrak{p} \in \mathrm{Spec}(R)$ we denote by $(-)_\mathfrak{p}$ the localization at $\mathfrak{p}$ and for each $X \in \mathrm{Mod}\text{-}R$ we denote by $\mathrm{Supp}_R(X)$ the set of $\mathfrak{p} \in \mathrm{Spec}(R)$ with $X_\mathfrak{p} \neq 0$. Also, we denote by dim X the Krull dimension of $X \in \mathrm{mod}\text{-}R$. We refer to [13] for basic commutative ring theory.

Definition 18. We say that A satisfies the condition (G) if the following equivalent conditions are satisfied:

(1) $\hat{\mathrm{G}}$-dim $X < \infty$ for all simple $X \in$ mod-A.
(2) $\hat{\mathrm{G}}$-dim $A/\mathrm{rad}(A) < \infty$.

Theorem 19. *The following are equivalent.*

(1) inj dim $A =$ inj dim $A^{\mathrm{op}} < \infty$.
(2) $A_\mathfrak{p}$ *satisfies the condition (G) for all* $\mathfrak{p} \in \mathrm{Supp}_R(A)$.

5. Gorenstein algebras

In this section, we will deal with the case where inj dim $A =$ inj dim $A^{\mathrm{op}} =$ depth A. In that case, A is a Gorenstein R-algebra in the sense of Goto and Nishida (see [8]). Also, we will characterize local Gorenstein algebras in terms of weak Gorenstein dimension.

We set $S = A/\mathrm{rad}(A)$ and denote by depth X the depth of $X \in$ mod-R. Throughout the rest of this note, we assume that A is a local ring, i.e., S is a division ring. Note that $S \in$ mod-A is a unique simple module up to isomorphism and that every $X \in$ mod-A admits a minimal projective resolution.

Theorem 20. *For any $d \geq 0$ the following are equivalent.*

(1) inj dim $A =$ inj dim $A^{\mathrm{op}} = d$.
(2) inj dim $A =$ depth $A = d$.
(3) $\hat{\mathrm{G}}$-dim $S_A = d$.

Corollary 21. *Assume that* inj dim $A =$ inj dim $A^{\mathrm{op}} = d < \infty$. *Then A is Cohen-Macaulay as an R-module and $I^d \cong \mathrm{Hom}_R(A, E_R(R/\mathfrak{m}))$ for a minimal injective resolution $A \to I^\bullet$ in* Mod-A.

Example 22. Even if inj dim $A =$ inj dim $A^{\mathrm{op}} < \infty$, it may happen that A is not Cohen-Macaulay as an R-module. For instance, let R be a Gorenstein local ring with dim $R \geq 1$ and set

$$A = \begin{pmatrix} R & R/xR \\ 0 & R/xR \end{pmatrix}$$

with $x \in \mathfrak{m}$ a regular element. Then A is not Cohen-Macaulay as an R-module but inj dim $A =$ inj dim $A^{\mathrm{op}} < \infty$ (see [1, Example 4.7]).

Example 23. Even if A is a Cohen-Macaulay R-module and inj dim $A =$ inj dim $A^{\mathrm{op}} < \infty$, it may happen that inj dim $A \neq$ depth A. For instance, let R be a Gorenstein local ring with dim $R = d$ and set

$$A = \begin{pmatrix} R & R \\ 0 & R \end{pmatrix}.$$

Then A is a Cohen-Macaulay R-module with depth $A = d$ but inj dim $A =$ inj dim $A^{\mathrm{op}} = d + 1$.

Example 24. Even if A is a Cohen-Macaulay R-module and inj dim $A =$ inj dim $A^{\mathrm{op}} =$ depth $A = d < \infty$, it may happen that $I^d \ncong \mathrm{Hom}_R(A, E_R(R/\mathfrak{m}))$ for a minimal injective resolution $A \to I^\bullet$ in Mod-A. For instance, let R be a Gorenstein local ring with dim $R = d$ and A a free R-module with a basis $\{e_{ij}\}_{1 \leq i,j \leq 3}$. Define a multiplication on A subject to the following axioms: (A1) $e_{ij}e_{kl} = 0$ unless $j = k$; (A2) $e_{ii}e_{ij} = e_{ij} = e_{ij}e_{jj}$ for all i, j; (A3) $e_{12}e_{21} = e_{11}$ and $e_{21}e_{12} = e_{22}$; and (A4) $e_{i3}e_{3j} = e_{3j}e_{i3} = 0$ for all $i, j \neq 3$. Set $e_i = e_{ii}$ for all i. Then A is an R-algebra with $1 = e_1 + e_2 + e_3$ and Cohen-Macaulay as an R-module. Also, setting $\Omega = \mathrm{Hom}_R(A, R)$, we have $e_1 A \cong e_2 A \cong e_3 \Omega$ and $e_1 \Omega \cong e_2 \Omega \cong e_3 A$. It follows that inj dim $A =$ inj dim $A^{\mathrm{op}} = d$ but $I^d \cong \mathrm{Hom}_R(\Omega, E_R(R/\mathfrak{m})) \ncong \mathrm{Hom}_R(A, E_R(R/\mathfrak{m}))$ in Mod-A.

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ON Ω-PERFECT MODULES AND SEQUENCES OF BETTI NUMBERS

OTTO KERNER AND DAN ZACHARIA

Abstract. Let R be a selfinjective algebra. In this paper we consider Ω-perfect modules and show how to use them to get information about the shapes of the Auslander-Reiten components containing modules of finite complexity. We also look at the growth of the sequence of Betti numbers for modules belonging to certain types of Auslander-Reiten components.

1. Introduction, background and motivation

The notion of complexity of a module has been around for more than thirty years. In depth studies have started in parallel at around the same time for group representations (see [1, 2, 7, 8, 21] for instance) and also in commutative algebra (see [4, 5, 16, 23] and [24]). In both cases the interest in complexity arose from the desire to understand the growth of minimal projective resolutions.

We will recall now the definition of complexity. For this definition we don't need to restrict ourselves to finite dimensional algebras, so R can be either a finite dimensional algebra over a field $\Bbbk$, or $R = (R, \mathbf{m}, \Bbbk)$ can be a local noetherian ring with maximal ideal $\mathbf{m}$ and residue field $\Bbbk$. Let M be a finitely generated R-module and let

$$P^\bullet : \quad \cdots \to P^2 \xrightarrow{\delta^2} P^1 \xrightarrow{\delta^1} P^0 \xrightarrow{\delta^0} M \to 0$$

be a minimal projective (free in the local case) resolution of M. The *i-th Betti number* of M, denoted $\beta_i(M)$, is the number of indecomposable summands of P^i. Then, the *complexity of M* is defined as

$$\operatorname{cx} M = \inf\{n \in \mathbb{N} | \beta_i(M) \leq ci^{n-1} \text{ for some positive } c \in \mathbb{Q} \text{ and all } i \geq 0\}$$

For instance $\operatorname{cx} M = 0$ is equivalent to M having finite projective dimension, and $\operatorname{cx} M = 1$ means that the Betti numbers of M are all bounded. If no such n exists, then we say that the complexity of M is infinite (at some point in time people also used to say that the complexity does not exist in this case). Let Ω denote the syzygy operator. Then it is clear from the definition that if M is a finitely generated R-module, then $\operatorname{cx} M = \operatorname{cx} \Omega M$, and

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an immediate application of the horseshoe lemma also shows that if $0 \to A \to B \to C \to 0$ is a short exact sequence of R-modules, then $\operatorname{cx} B \leq \max\{\operatorname{cx} A, \operatorname{cx} C\}$.

Note also that every Ω-periodic module (that is a module M with the property that $\Omega^k M \cong M$ for some positive integer k) has complexity 1. In fact, Eisenbud has proved that if $R = \Bbbk G$ is the group algebra of a finite group, or if R is a complete intersection, then every module of complexity 1 is Ω-periodic [10]. The converse need not hold in general, not even in the symmetric local case; we have the following example due to Liu and Schulz, [22]: Consider $R = \Bbbk\langle x, y\rangle/\langle x^2, y^2, xy + qyx\rangle$ where $0 \neq q \in \Bbbk$ is not a root of 1, and let T be the trivial extension of R by $\operatorname{Hom}_{\Bbbk}(R, \Bbbk)$. Then T is a local symmetric algebra. Let M be the T-module $(x+y)T$. For all $i \in \mathbb{Z}$ the modules $\Omega^i M$ have dimension 4, and are pairwise non-isomorphic. Since T is symmetric, $\tau M = \Omega^2 M$ holds, where τ is the Auslander-Reiten translation. Hence the module M has complexity 1 and is neither Ω- nor τ-symmetric. The module M therefore is contained in a $\mathbb{Z}A_\infty$ component. There are also counterexamples in the commutative case (see Gasharov and Peeva [15]).

Throughout this paper, R will denote a finite dimensional selfinjective algebra over an algebraically closed field $\Bbbk$ with Jacobson radical $\mathbf{r}$. Then, by an induction on the Loewy length, it follows readily from the definition and the above remarks, that for every finitely generated R-module M, we have $\operatorname{cx} M \leq \operatorname{cx} R/\mathbf{r}$. D will denote the usual duality$D = \operatorname{Hom}_{\Bbbk}(-, \Bbbk)$, and ν will denote the Nakayama equivalence

$$\nu = D\operatorname{Hom}_R(-, R)$$

Also, since R is selfinjective, then $\nu\Omega = \Omega\nu$. Moreover in this case, the Auslander-Reiten translate τ is given by $\tau = \nu\Omega^2$. Since ν is a dimension preserving equivalence that takes projective modules into projective modules, we have that $\operatorname{cx} M = \operatorname{cx} \nu M$, hence $\operatorname{cx} M = \operatorname{cx} \tau M$ for every finitely generated R-module M.

The paper is organized as follows. In the second section we talk about the shape of the Auslander-Reiten components containing modules of finite complexity as obtained in [20] and about the methods used in approaching this problem. In particular, we talk about a very special class of modules called Ω-perfect. In section three, we study the existence of Ω-perfect modules. Finally, in the last section we look at some special cases where we analyze the growth of the Betti numbers.

2. Auslander-Reiten components containing modules of finite complexity

We start this section with the following easy observation:

Lemma 1. *Let R be a selfinjective algebra and let $\mathcal{C}_s$ be a stable component of its Auslander-Reiten quiver. The complexity is constant on $\mathcal{C}_s$.*

Proof. Let $B \to C \in \mathcal{C}_s$ be an irreducible morphism. Then there exists an Auslander-Reiten sequence $0 \to \tau C \to B \oplus E \to C \to 0$ for some module E. Hence we have $\operatorname{cx} B \leq \operatorname{cx}(B \oplus C) \leq \max\{\operatorname{cx} C, \operatorname{cx} \tau C\} = \operatorname{cx} C$. Since there is an irreducible morphism from τC to B we use the same reasoning to get the reverse inequality. There is also an "extreme" case to prove: the case when the only irreducible morphisms to modules $C \in \mathcal{C}_s$ are from projective modules. But it is not hard to prove that this corresponds to the case when R is a Nakayama algebra of Loewy length two. In that case, the only non projective modules are the simple modules and they are all periodic, hence their complexity is 1. $\square$

In order to describe the shapes of the stable Auslander-Reiten components containing modules of finite complexity we recall first the notion of Ω-perfect modules introduced in [17, 18]. We observe first that if $g\colon B \to C$ is an irreducible epimorphism between two nonprojective modules, then we have an induced irreducible map $\Omega g\colon \Omega B \to \Omega C$, see [3] for instance These modules have a particularly nice behaviour under the syzygy operator. However, there is no reason why Ωg should be again an epimorphism. Being irreducible, we know though that it must be either an epimorphism or a monomorphism. And one could ask the same question about an irreducible monomorphism f: when can we guarantee that its syzygy Ωf is gain an irreducible monomorphism? We have the following definition:

Definition. *An irreducible map $g\colon B \to C$ is called Ω-perfect if for all $n \geq 0$ the induced maps $\Omega^n g\colon \Omega^n B \to \Omega^n C$ are all monomorphisms or are all epimorphisms. An irreducible map g is eventually Ω-perfect if, for some $i > 0$, the induced map $\Omega^i g\colon \Omega^i B \to \Omega^i C$ is Ω-perfect. An indecomposable non projective R-module C is called Ω-perfect, if each irreducible map into C is Ω-perfect. We say that C is it eventually Ω-perfect if some syzygy of C is an Ω-perfect module.*

It was proved in [17] that if $g\colon B \to C$ is an irreducible epimorphism, then Ωg is again an epimorphism if and only if its kernel is not a simple module. Thus, an irreducible map $g\colon B \to C$ is eventually Ω-perfect, if and only if there exists a positive integer n such that for each $i \geq n$, the induced map $\Omega^i g\colon \Omega^i B \to \Omega^i C$ has a non simple kernel. We have the following consequence, see [18]:

Proposition 2. *Let R be a selfinjective algebra having no periodic simple modules. Then every nonprojective R-module is eventually Ω-perfect.* $\square$

We can specialize to the local finite dimensional case to obtain the following:

Corollary 3. *Let $R = (R, \mathbf{m}, \Bbbk)$ be a local selfinjective algebra, and assume that there are modules of complexity two or higher. Then every indecomposable non projective R-module is eventually Ω-perfect.* $\square$

One very nice feature of Ω-perfect maps is that they behave very nice under the syzygy operator. We have the following (see [17]):

Proposition 4. *Let R be a selfinjective algebra, and let $0 \to A \to B \xrightarrow{g} C \to 0$ be a short exact sequence of R-modules where g is an irreducible Ω-perfect map. Then, for each $i \geq 0$ we have induced exact sequences $0 \to \Omega^i A \to \Omega^i B \xrightarrow{\Omega^i g} \Omega^i C \longrightarrow 0$, and thus $\beta_i(B) = \beta_i(A) + \beta_i(C)$ for each $i \geq 0$.* $\square$

It turns out that every indecomposable not τ-periodic module of complexity one is eventually Ω-perfect ([17]). The proof of this result is somehow involved and it would be interesting to have a more direct and possibly elementary proof. Note also that a recent result of Dugas ([9]), proves that if a *simple* module over a selfinjective algebra has complexity 1, then it must be periodic. As mentioned above in the introduction, this need not hold for all modules with bounded Betti numbers so the assumption that the module is simple, is essential.

We would also like to mention the following facts. Let C be an Ω-perfect module. Then it is easy to show that τC is also Ω-perfect. Let now B be an indecomposable module, and assume that there is an irreducible monomorphism $B \to C$. Then it was shown in [17] that B must also be Ω-perfect. We would like to know the answer to the following question:

Question 5. Let B and C be two indecomposable R-modules, let $B \to C$ be an irreducible epimorphism and assume C is Ω-perfect. Is B also Ω-perfect?

We will first look at Auslander-Reiten components containing modules that are not eventually Ω-perfect since this is the much easier case. We will show that these components must have a very predictable shape. First, we recall the following definition and theorem, see [19].

Definition. Let R be an artin algebra and let $\mathcal{C}_s$ be a stable component of its Auslander-Reiten quiver. A function $d\colon \mathcal{C}_s \to \mathbb{Q}$ is *additive* if it satisfies the following properties:
(a) $d(C) > 0$ for each $C \in \mathcal{C}_s$.
(b) $2d(C) = \sum_i d(E_i)$ for each indecomposable non projective module C, where the sequence $0 \to \tau C \to \bigoplus_i E_i \bigoplus P \to C \to 0$ is an Auslander-Reiten sequence and P is a (possibly 0) projective R-module.
(c) $d(C) = d(\tau C)$ for each $C \in \mathcal{C}_s$.

The following theorem was proved by Happel-Preiser-Ringel in [19]:

Theorem 6. *Let R be an artin algebra over an algebraically closed field and let $\mathcal{C}_s$ be a stable component of its Auslander-Reiten quiver. Assume that there exists an additive function on $\mathcal{C}_s$. Then the tree class of $\mathcal{C}_s$ is either an extended Dynkin diagram of type $\widetilde{A}_n, \widetilde{D}_n, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$, or an infinite Dynkin tree of type A_∞, D_∞ or A_∞^∞.*

Assume that a non-periodic stable component $\mathcal{C}_s$ contains a module C that is not eventually Ω-perfect. This means that some syzygy of C is a simple periodic module. Let us denote that module by S, and let n be the Ω-period of S. It is clear that S is also ν-periodic since the Nakayama functor preserves lengths, so let m denote the ν-period of S. Let $T = S \oplus \Omega S \oplus \ldots \oplus \Omega^{n-1} S$, and let $W = T \oplus \nu T \oplus \ldots \oplus \nu^{m-1} T$. It is now immediate that $\tau W = W$. Also, it is not hard to show that the function $d\colon \mathcal{C}_s \to \mathbb{Q}$ given by $d(M) = \dim \underline{\mathrm{Hom}}_R(W, M)$ is an additive function, see [13, 20]. Using the Happel-Preiser-Ringel theorem and the above observations we have the following surprising application (see [20]):

Theorem 7. *Let R be a selfinjective algebra and let $\mathcal{C}_s$ be a stable component of the Auslander-Reiten quiver of R containing a module that is not eventually Ω-perfect. Assume in addition that the component is not τ-periodic. Then $\mathcal{C}_s$ is of the form $\mathbb{Z}\Delta$ where Δ is of type $\widetilde{A}_n, \widetilde{D}_n, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$, or an infinite Dynkin tree of type D_∞ or A_∞^∞.* □

We should make a few remarks here. First, note the excluded case when the component is τ-periodic is also well understood. They are either infinite tubes or they are periodic components whose tree class is a Dynkin diagram (see [19, 26]). Note also that the theorem says that components of type $\mathbb{Z}A_\infty$ cannot occur. In fact, we shall see in the next section that we cannot have components of type $\mathbb{Z}\Delta$ for $\Delta = \widetilde{A}_1, \widetilde{E}_6, \widetilde{E}_7$, or $\widetilde{E}_8$ either. At this point we would like to state a second question that has actually been around in the area for some time.

Question 8. Let R be a selfinjective algebra and assume that its Auslander-Reiten quiver contains a component of type $\mathbb{Z}\Delta$ where $\Delta = \widetilde{A}_n, \widetilde{D}_n, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8, D_\infty$ or A_∞^∞. Does this imply that R is a tame algebra?

The answer to the above question is affirmative in the group algebra case, see [12]. Therefore it seems that given a selfinjective algebra, almost all the indecomposable modules are eventually Ω-perfect. We will discuss more about this phenomenon in the next section.

Considering Theorem 2.7, it turns out that a similar result holds for components containing modules of finite complexity. The following result was proved in [20]. It is a generalization of Webb's theorem who had proved it first for group algebras [28].

Theorem 9. *Let R be a selfinjective algebra and let $\mathcal{C}_s$ be a stable component of the Auslander-Reiten quiver of R containing a module of finite complexity. Assume in addition that the component is not τ-periodic. Then $\mathcal{C}_s$ is of the form $\mathbb{Z}\Delta$ where Δ is of type $\widetilde{A}_n, \widetilde{D}_n, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$, or an infinite Dynkin tree of type A_∞, D_∞ or A_∞^∞.* □

3. Ω-perfect modules

In this section we continue the study of Ω-perfect modules over a selfinjective algebra and show that every component of type $\mathbb{Z}\widetilde{E}_i$ for $i = 6, 7, 8$ or $\mathbb{Z}\widetilde{A}_1$ consists of eventually Ω-perfect modules. We also give an example of a component containing only modules that are not Ω-perfect, and discuss possible values for complexities. We also pose some new questions. We will need the notion of τ-perfect irreducible map. It is obviously very similar to the one of Ω-perfect map: we say that an irreducible map $g\colon B \to C$ is called *τ-perfect* if for all $n \geq 0$ the induced maps $\tau^n g\colon \tau^n B \to \tau^n C$ are all monomorphisms or are all epimorphisms.

If $\mathcal{C}$ is a component of the Auslander-Reiten quiver of R, we will denote by $\mathcal{C}_s$ its stable part, and by $\Omega\mathcal{C}$ the component containing all the modules of the form ΩX for $X \in \mathcal{C}$ non projective. We have the following:

Proposition 10. *Let R be a selfinjective artin algebra and let $\mathcal{C}$ be an Auslander-Reiten component. If the module $X \in \mathcal{C}$ does not have any projective or simple predecessors in $\mathcal{C}$, then ΩX does not have either any simple or projective predecessors in $\Omega\mathcal{C}$.*

Proof. Assume that ΩX has a simple predecessor S in the component $\Omega\mathcal{C}$. By applying the inverse syzygy operator we obtain in $\mathcal{C}$ a chain of irreducible maps $\Omega^{-1}S \to \cdots \to X$. Denote by P the indecomposable projective-injective with socle S. We have an Auslander-Reiten sequence $0 \to \mathbf{r}P \to P \oplus \mathbf{r}P/S \to P/S \to 0$, and since $P/S \cong \Omega^{-1}S$, we see that P is a predecessor of X in $\mathcal{C}$. Assume now that ΩX has a projective predecessor in its component, so there exists a chain of irreducible maps $P \to P/S \to \cdots \to \Omega X$ where S is the socle of P. As before, we have that $P/S \cong \Omega^{-1}S$, so there is a chain of irreducible maps in $\Omega^2\mathcal{C}$ from S to $\Omega^2 X$. Applying the Nakayama functor, we obtain that τX, and hence X have a simple predecessor since the Nakayama functor preserves lengths. □

One can prove in a similar fashion that for a selfinjective algebra, a component $\mathcal{C}$ contains a simple (projective) module, if and only if the component $\Omega\mathcal{C}$ contains a projective (respectively simple) module.

Remark 11. Let $\mathcal{C}$ be an Auslander-Reiten component having a boundary, that is, a component containing indecomposable modules whose Auslander-Reiten sequences have indecomposable middle terms. Assume that $\mathcal{C}$ is not a tube, and let C be an indecomposable module lying on the boundary of $\mathcal{C}$. Without loss of generality we may assume that neither C nor ΩC has a simple module in the positive direction of their τ-orbit. This means that if $0 \to \tau C \to B \to C \to 0$ is the Auslander-Reiten sequence ending at C, then both maps $\tau C \to B$ and $B \to C$ are Ω-perfect and so C is an Ω-perfect module. So we see that nonperiodic components with boundaries, always contain Ω-perfect maps and Ω-perfect modules. As we will see soon, this need not happen in components of the type $\mathbb{Z}A_\infty^\infty$.

Lemma 12. *Let $g\colon B \to C$ be an irreducible map that is not eventually Ω-perfect, where neither B nor C has a nonzero projective summand. Then, there exists a positive integer α such that for each $i \geq 0$ we have $|\ell(\Omega^i B) - \ell(\Omega^i C)| \leq \alpha$.*

Proof. By taking enough powers of the Auslander-Reiten translate τ, we may assume without loss of generality that g is onto, and that its kernel S is a simple periodic module. Note that by applying Ω we obtain an induced exact sequence $0 \to \Omega B \xrightarrow{\Omega g} \Omega C \to S \to 0$. If the induced map $\Omega^2 g$ is again a monomorphism, then we get the commutative exact diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega^2 B \arrow[r, "\Omega^2 g"] \arrow[d] & \Omega^2 C \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_{\Omega B} \arrow[r] \arrow[d] & P_{\Omega C} \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega B \arrow[r, "\Omega g"] \arrow[d] & \Omega C \arrow[r] \arrow[d] & S \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

hence the two modules L and ΩS are isomorphic, and we have a short exact sequence $0 \to \Omega^2 B \to \Omega^2 C \to \Omega S \to 0$. If on the other hand, the map $\Omega^2 g$ is an epimorphism, then we obtain a commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & L \arrow[r] \arrow[d, equal] & \Omega^2 B \arrow[r, "\Omega^2 g"] \arrow[d] & \Omega^2 C \arrow[r] \arrow[d] & 0 & \\
0 \arrow[r] & L \arrow[r] & P_{\Omega B} \arrow[r] \arrow[d] & P_{\Omega C} \arrow[r] \arrow[d] & S \arrow[r] \arrow[d, equal] & 0 \\
 & 0 \arrow[r] & \Omega B \arrow[r, "\Omega g"] \arrow[d] & \Omega C \arrow[r] \arrow[d] & S \arrow[r] & 0 \\
 & & 0 & 0 & &
\end{tikzcd}$$

and therefore we obtain a short exact sequence $0 \to \Omega^2 S \to \Omega^2 B \to \Omega^2 C \to 0$. Continuing in this fashion we see that for each integer $i \geq 0$ we get either short exact sequences $0 \to \Omega^i S \to \Omega^i B \to \Omega^i C \to 0$, or of the form $0 \to \Omega^{i+1} B \to \Omega^{i+1} C \to \Omega^i S \to 0$. By letting $\alpha = \max_{i \in \mathbb{N}}\{\ell(\Omega^i S)\}$, our result follows, since the simple module S is periodic. $\square$

The following (most probably) well-known lemma will be used to characterize Ω-perfect maps in terms of the "τ-perfect" property. As usual, if M is an indecomposable non-projective module, $\alpha(M)$ denotes the number of non-projective indecomposable direct summands of the middle term of the Auslander-Reiten sequence ending in M.

Lemma 13. *Let Λ be a selfinjective artin algebra and let M be an indecomposable non projective and non simple Λ-module with $\alpha(M) = 2$ with $n = \ell(M) = \ell(\tau M)$. Assume also that there exists an irreducible map $E \to M$ where E is indecomposable and that $\ell(E) = \ell(M) - 1$.*

(a) *The middle term of the Auslander-Reiten sequence ending at M has no nonzero projective summand.*

(b) *If E, M and τM are uniserial, then the remaining summand F of the Auslander-Reiten sequence ending at M is uniserial too and its length is $\ell(F) = \ell(M) + 1$.*

Proof. Let $0 \to \tau M \to E \oplus F \oplus P \to M \to 0$ be the Auslander-Reiten sequence ending at M, where F is indecomposable non projective, and P is a nonzero projective module. Note first that P must be indecomposable since the algebra is selfinjective. Using now the fact that $\tau M = \mathbf{r}P$, a length argument shows that

$$\ell(F) = 2n - \ell(E) - \ell(P) = 2n - (n-1) - (n+1) = 0$$

contradicting our assumption. This proves the first part of the lemma.

For part (b), note first that the Auslander-Reiten sequence ending at M has the form $0 \to \tau M \to E \oplus F \to M \to 0$ where E and F are both indecomposable and $\ell(F) = n+1$. Hence τM is a maximal submodule of F. To prove the uniseriality of F, it suffices to show that $\tau M = \mathbf{r}F$. It is folklore (see also [17], Proposition 2.5.) that, since τM is not simple, we have an induced exact sequence

$$0 \to \mathbf{r}\tau M \to \mathbf{r}E \oplus \mathbf{r}F \to \mathbf{r}M \to 0.$$

Counting lengths, we get $\ell(\mathbf{r}F) = 2(n-1) - (n-2) = n$. Since the image of τM in F contains the radical of F, it follows that $\tau M \cong \mathbf{r}F$, and F is also an uniserial module. $\square$

We are now ready to prove the promised characterization of Ω-perfect maps.

Proposition 14. *Let R be a selfinjective algebra of infinite representation type, and let C be an indecomposable module and let $g\colon B \to C$ be an irreducible map. Then g is eventually Ω-perfect if and only if both g and Ωg are eventually τ-perfect.*

Proof. Obviously, if g is eventually Ω-perfect then both maps g and Ωg are eventually τ-perfect. For the reverse direction, assume that both maps g and Ωg are eventually τ-perfect, but that g is not eventually Ω-perfect. By applying enough powers of Ω, we may assume that $g, \Omega g$ are both τ-perfect, and that for each $i \geq 0$, the maps $\tau^i g$ are onto and $\Omega^i \tau g$ are one-to-one. Thus, for each $i \geq 0$, there exist simple modules S_i and exact sequences $0 \to S_i \to \Omega^{2i} B \xrightarrow{\Omega^{2i} g} \Omega^{2i} C \to 0$ and $0 \to \Omega^{2i+1} B \xrightarrow{\Omega^{2i+1} g} \Omega^{2i+1} C \to S_i \to 0$. But $\Omega^{2i+2} g$ is again surjective so we infer from the proof of lemma 13 that for each $i \geq 0$, $\Omega^2 S_i \cong S_{i+1}$. Since there are only finitely many nonisomorphic simple modules, the sequence $\{S_1, \nu S_2 = \tau S_1, \nu^2 S_3 = \tau^2 S_1, \cdots\}$ is eventually periodic. Therefore without loss of generality we may assume that there is a periodic simple module S, say of period n, whose τ-powers are all simple.

We claim first that the simple modules $S, \tau S, \cdots, \tau^{n-1} S$ lie on the boundary of a regular tube $\mathcal{C}$. To see this, observe first that we can deduce that the middle term E of the Auslander-Reiten sequence $0 \to \tau S \to E \to S \to 0$ is indecomposable. Moreover, E cannot be projective, since otherwise the middle term of each Auslander-Reiten sequence ending at a $\tau^i S$ would be an indecomposable projective-injective module of length two. This would imply that our algebra is selfinjective Nakayama of Loewy length two, contradicting our assumption on the representation type of R. By construction, all the modules in the same τ-orbit of $\mathcal{C}$ have the same length, and these lengths increase by one from a τ-orbit to the next one. We may apply now the previous lemma and infer that the component is a regular component. By the second part of the lemma we get that all the modules in $\mathcal{C}$ are uniserial, a contradiction since we cannot have uniserial module of arbitrary large length. $\square$

Let Δ be a quiver. A vertex x of Δ is called a *tip*, if only one arrow of Δ starts or ends at x. If $\mathcal{C}$ is a component whose stable part is of type $\mathbb{Z}\Delta$, then a module M corresponds to a tip of Δ if and only if the Auslander-Reiten sequence ending at M is of the form:

$$0 \to \tau M \to Y \oplus P \to M \to 0$$

for some projective (possibly zero) module P and indecomposable non projective module Y. Assume that $\mathcal{C}$ is a connected component of the Auslander-Reiten quiver, and that we have $\mathcal{C}_s \cong \mathbb{Z}\Delta$ for some quiver Δ. Since an Auslander-Reiten component contains at most finitely many indecomposable projective or simple modules, for each indecomposable module $M \in \mathcal{C}_s$ there exists a positive integer r such that $\tau^r M$ has no projective or simple predecessors in $\mathcal{C}$. We have the following immediate consequence:

Corollary 15. *Let $\mathcal{C}_s$ be a stable component of type $\mathbb{Z}\Delta$ and let M be an indecomposable module in $\mathcal{C}_s$. Assume that M corresponds to a tip of Δ. Then M is eventually Ω-perfect.* $\square$

We have the following:

Proposition 16. *Let $\mathcal{C}_s$ be a stable component of type $Z\Delta$ where Δ is one of $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$, $\tilde{A}_1$, A_∞. Then every module in $\mathcal{C}_s$ is eventually Ω-perfect.*

Proof. The case where $\Delta = A_\infty$ was treated in [20], (Theorem 2.11. and Lemma 2.6.). Consider now the only case when the connected component $\mathbb{Z}\Delta$ has no tip, that is the case when $\Delta = \tilde{A}_1$, that is, the Kronecker quiver. Let $M \in \mathcal{C}_s$ with no projective or simple predecessors. The Auslander-Reiten sequence ending at M is of the form

$$0 \to \tau M \overset{[f_1,f_2]^t}{\longrightarrow} E \oplus E \overset{[g_1,g_2]}{\longrightarrow} M \to 0$$

and it is obvious that all of the irreducible maps f_1, f_2, g_1, g_2 are epimorphisms, or all are monomorphisms. We claim that they are all epimorphisms. If they are monomorphisms, then in the Auslander-Reiten sequence

$$0 \to \tau E \overset{[\tau g_1,\tau g_2]^t}{\longrightarrow} \tau M \oplus \tau M \overset{[f_1,f_2]}{\longrightarrow} E \to 0$$

the maps $\tau g_1, \tau g_2$ are also monomorphisms. Continuing in the positive τ direction we obtain an arbitrary long chain or irreducible monomorphisms

$$\cdots \tau^i E \hookrightarrow \tau^i M \hookrightarrow \tau^{i-1} E \hookrightarrow \cdots \hookrightarrow E \hookrightarrow M$$

which is absurd. We can make the same argument for ΩM, and it follows that M is Ω-perfect.
Assume now that Δ is one of the remaining finite quiver $\tilde{E}_i$, take M in $\mathcal{C}_s$ with $\alpha(M) = 3$, such that M has no projective or simple predecessor in $\mathcal{C}$ and let $\mathcal{C}_M$ be the full subquiver of $\mathcal{C}$, defined by the vertices which are predecessors of M. If $X \in \mathcal{C}_M$ with $\alpha(X) = 1$ (hence X corresponds to one of the 3 tips of Δ, then X is Ω-perfect by Remark 3.2, the irreducible map $Y \to \tau X$ is an Ω-perfect epimorphism, and $\tau X \to Y$ is injective and Ω-perfect. Consequently all irreducible maps between indecomposable modules in $\mathcal{C}_M$ are Ω-perfect, hence all indecomposable modules $N \in \mathcal{C}_M$ with $\alpha(N) \leq 2$ are Ω-perfect. Take finally $V \in \mathcal{C}_M$ with $\alpha(V) = 3$ and let

$$0 \to \tau V \overset{[f_1,f_2,f_3]^t}{\longrightarrow} \oplus_{i=1}^3 X_i \overset{[g_1,g_2,g_3]}{\longrightarrow} V \to 0$$

the Auslander-Reiten sequence ending in V. The irreducible maps f_i all are surjective, while the g_i are injective. Choose $j \leq 3$, let $\oplus_i X_i = Y \oplus X_j$ and let $[g, g_j] : Y \oplus X_j \to V$ be the sink map. Since f_j is an Ω-perfect epimorphism, the same holds for the "parallel" morphism g, hence V is Ω-perfect, too. □

As we will see very soon, it turns out that if $\mathcal{C}$ is a component in which no irreducible map is Ω-perfect, then every non projective module in $\mathcal{C}$ has complexity at most 2. In fact, we have a slightly more general result. We start with the following:

Proposition 17. *Let C be an indecomposable non projective, and non τ-periodic module and assume that there exist irreducible morphisms $B \to C$ and $\tau C \to B$ that are not eventually Ω-perfect. Then, there exists a positive integer α such that for each $n \geq 0$, $\ell(\Omega^{2n} C) \leq \ell(C) + n\alpha$. In particular, C and every nonprojective module in the same Auslander-Reiten component has complexity 2.*

Proof. Observe first that for each indecomposable non projective R-module M, we have $\ell(\Omega^2 M) = \ell(\tau M)$. Now, applying the previous Lemma 3.3, we obtain for each $i \geq 0$ that $\ell(\Omega^i B) \leq \ell(\Omega^i C) + \alpha/2$, and $\ell(\Omega^{i+2} C) \leq \ell(\Omega^i B) + \alpha/2$ for some positive number α. Hence, for each $i \geq 0$ we have $\ell(\Omega^{i+2} C) - \ell(\Omega^i C) \leq \alpha$. In particular, for each $n \geq 0$ we have $\ell(\Omega^{2n} C) - \ell(C) \leq n\alpha$. This means that the complexity of C is bounded by 2. If $\operatorname{cx} C = 1$, then, since it is not τ periodic, the module C must lie in a $\mathbb{Z}A_\infty$-component by [17], but for these components every irreducible map is eventually Ω-perfect. Hence $\operatorname{cx} C = 2$. $\square$

We obtain the following immediate consequence: assume that we have a component $\mathcal{C}$, whose stable part $\mathcal{C}_s$ is of the form $\mathbb{Z}A_\infty^\infty$, and assume also that there exists an Auslander-Reiten sequence $0 \to \tau C \to E \oplus F \to C \to 0$ where E and F are indecomposable, and neither $E \to C$ nor $F \to C$ is eventually Ω-perfect. Observe also that in this case, no irreducible map in $\mathcal{C}_s$ between indecomposable modules is eventually Ω-perfect. It follows immediately from the previous proposition that every non projective module in $\mathcal{C}$ has complexity 2. This situation can actually occur. The following example is due to Ringel.

Example 18. Let R be the finite dimensional selfinjective string algebra given by the quiver

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, "\gamma"] \arrow[r, bend left, "\beta"] & 3 \arrow[l, bend left, "\delta"] \end{tikzcd}$$

modulo the relations $\alpha\beta = 0$, $\delta\gamma = 0$, $\gamma\alpha\gamma\alpha = \beta\delta\beta\delta$ and $\alpha\gamma\alpha\gamma\alpha = \delta\beta\delta\beta\delta = 0$. There exists a $\mathbb{Z}A_\infty^\infty$ component where none of the irreducible maps between the indecomposable modules is eventually Ω-perfect, (or even τ-perfect). For instance, consider the string module $M = \mathbf{r}^3 P_2$. It is easy to see that M is not eventually Ω-perfect, that $\alpha(M) = 2$, and that no irreducible map from an indecomposable module to M is eventually Ω-perfect. Moreover, by [6], M lies in a component consisting entirely of string modules. But the only string modules lying on the boundary of an Auslander-Reiten component can lie on tubes (see [12], II.6.4), so this module belongs to a $\mathbb{Z}A_\infty^\infty$ component. Note also that the simple modules S_1 and S_3 are Ω-periodic of period 6, and that they both lie on tubes of rank 3.

Example 19. Following Erdmann [12], for each positive integer m, we denote by Λ_m the local symmetric string algebra over a field K,

$$\Lambda_m = K\langle x, y\rangle / \langle x^2, (xy)^{m+1} - (yx)^{m+1}, x^2 - (yx)^m y, x^3\rangle$$

If the characteristic of K is 2, and $m+1 = 2^n \geq 4$, then the algebra Λ_m modulo its socle is isomorphic to the group algebra of the semidihedral group of order 2^{n+2} modulo its socle. Motivated by this fact, Erdmann calls this algebra *semidihedral.* She proves that Λ_m has infinitely many stable components of type $\mathbb{Z}A_\infty^\infty$ and $\mathbb{Z}D_\infty$ ([12], Propositions II,10.1 and II,10.2), and that the other stable components are tubes of rank 1 and 2. Moreover, she shows that the unique simple module lies in a component of type $\mathbb{Z}D_\infty$ so it is not periodic. Therefore, every indecomposable non projective Λ_m-module is eventually Ω-perfect by [18]. Note that in the same book, Erdmann generalizes the notion of semidihedral algebra to that of algebras of *semidihedral type* and one also obtains interesting examples for the non local case ([12], Lemma VIII. 2.1.).

3.1. **$\mathbb{Z}D_\infty$-components.** We assume for the remainder of this section that $\mathcal{C}$ is a connected Auslander-Reiten component whose stable part is of the form $\mathbb{Z}D_\infty$. Let C be an indecomposable module lying on the boundary of $\mathcal{C}$. Then, without loss of generality we may assume that C is Ω-perfect, by Remark 11. In this context we have the following:

Lemma 20. *Let A and B be two indecomposable modules lying on the boundary of $\mathcal{C}$ with Auslander-Reiten sequences $0 \to \tau A \xrightarrow{f_1} M \xrightarrow{g_1} A \to 0$ and $0 \to \tau B \xrightarrow{f_2} M \xrightarrow{g_2} B \to 0$. Then the irreducible map $[g_1, g_2]^t \colon M \to A \oplus B$ is an epimorphism if and only if the map $[f_1, f_2] \colon \tau A \oplus \tau B \to M$ is also an epimorphism.*

Proof. Counting lengths, we have $\ell(\tau A) + \ell(A) + \ell(\tau B) + \ell(B) = 2\ell(M)$. This means that $\ell(A) + \ell(B) < \ell(M)$ if and only if $\ell(\tau A) + \ell(\tau B) > \ell(M)$. The result follows, since an irreducible map is either a monomorphism, or an epimorphism. $\square$

Keeping the notation from the lemma, we may clearly assume that the modules A and B lying on the boundary of the component are Ω-perfect, and that the Auslander-Reiten sequence ending at M is $0 \to \tau M \to \tau A \oplus \tau B \oplus \tau X \to M \to 0$ for some indecomposable module X. Since the irreducible epimorphisms $M \to A$ and $M \to B$ are Ω-perfect, then the irreducible epimorphisms $\tau A \oplus \tau X \to M$ and $\tau B \oplus \tau X \to M$ are also Ω-perfect being "parallel" to Ω-perfect epimorphisms. Similarly, the irreducible monomorphisms $\tau M \to \tau X \oplus \tau A$ and $\tau M \to \tau X \oplus \tau B$ are also Ω-perfect. Putting together our remarks, we have:

Proposition 21. *Let $\mathcal{C}$ be an Auslander-Reiten component whose stable part is of type $\mathbb{Z}D_\infty$. Assume that there is an irreducible map between indecomposable non projective modules $X \to Y$ that is not eventually Ω-perfect. Then each non projective module in $\mathcal{C}$ has complexity 2.*

Proof. From the shape of our component, it follows by looking at "parallel" maps one at a time, that we may assume that there exists an irreducible map of the form $\tau M \to \tau A \oplus \tau B$ or $\tau A \oplus \tau B \to M$ that is not eventually Ω-perfect, where A and B are indecomposable modules lying on the boundary of $\mathcal{C}$, and M is an indecomposable module. Observe that, neither $\tau M \to \tau A \oplus \tau B$ nor $\tau A \oplus \tau B \to M$ can be eventually Ω-perfect by Lemma 20. Being of type $\mathbb{Z}D_\infty$ means also that $\mathcal{C}$ cannot contain modules of complexity 1 by [17]. We apply now 17. and the result follows. $\square$

We would like to propose the following questions summarizing the discussion in the first three sections. The first one has been around for some time and is due to Rickard [25].

Questions 22. Let R be a selfinjective algebra.

1. Assume that there exists an indecomposable R-module of complexity greater than 2. Is R is of wild representation type?
2. Assume that R has stable components of type $\mathbb{Z}D_\infty$ or $\mathbb{Z}A_\infty^\infty$. Is R of tame representation type? Must these components have complexity 2?
3. Assume that R has a stable component of type $\mathbb{Z}A_\infty$. Is R necessarily of wild representation type?

The answer to the first question is known to be yes if R admits a theory of support varieties, for instance in the group algebras case. See also [14]. The answer to the second and third question is also known to be affirmative in the group algebra case [12] but almost nothing is known outside this case.

4. Growth of Betti numbers. The local case

Let us return to the situation where $R = (R, \mathbf{m}, \Bbbk)$ is a local noetherian $\Bbbk$-algebra. The following questions were among questions posed in the late 1970s and the early 1980s. They are still open even in the commutative artinian case, and even if we also add the selfinjective assumption.

Questions 23. Let $R = (R, \mathbf{m}, \Bbbk)$ be a local noetherian $\Bbbk$-algebra. Let M be an indecomposable finitely generated R-module of infinite projective dimension.

1. Assume that M has complexity 1. Is the sequence of Betti numbers $\{\beta_i(M)\}_i$ eventually constant?
2. Is the sequence of Betti numbers $\{\beta_i(M)\}_i$ eventually nondecreasing?

The first question has an affirmative answer if R is a complete intersection ([10]). In the radical square zero case the answer is also affirmative. We sketch the proof below (see also [15])

Proposition 24. *Let $R = (R, \mathbf{m}, \Bbbk)$ is a local artinian ring with $\mathbf{m}^2 = 0$ and let M be a finitely generated R-module with $\operatorname{cx} M = 1$. Then the Betti numbers of M are eventually constant.*

Proof. Let F be a finitely generated free R-module. We observe first that since $\mathbf{m}^2 = 0$, every submodule of $\mathbf{m}F$ is semisimple [3], so all the syzygies of M must be semisimple. Let k denote the largest possible value of a Betti number of M and assume that it corresponds to the i-th Betti number, that is $\beta_i(M) = k$. This means that the i-th syzygy of M is a direct sum of k simple modules, hence $\beta_{i+1}(M) \geq k$. Our choice of k implies now that $\beta_{i+j}(M) = k$ for all $j \geq 0$ and the result follows. $\square$

Question 1 also has an affirmative answer in the case where $R = (R, \mathbf{m}, \Bbbk)$ is a commutative Gorenstein artinian ring with $\mathbf{m}^3 = 0$, see [15]. Question 2 is also pretty much unresolved. In the local commutative artinian case, Gasharov and Peeva have shown ([15]) that for a finitely generated module M, we have the following:

$$\beta_{i+1}(M) \geq (2e - \ell(R) + h - 1)\beta_i(M)$$

for large enough i. Here $e = \dim_{\Bbbk} \mathbf{m}/\mathbf{m}^2$, h is the Loewy length of R, and $\ell(R)$ is the length of R. They have also shown that if the constant $2e - \ell(R) + h - 1 \geq 2$, then the sequence of Betti numbers has exponential growth. However it is not hard to produce examples of local commutative artinian rings where the constant $2e - \ell(R) + h - 1$ is a negative number. We also want to mention the following two results due to Ramras [23, 24]:

Theorem 25. *Let $(R, \mathbf{m}, \Bbbk)$ be a regular local ring of dimension at least two, and let $S = R/\mathbf{m}^k$ for some $k \geq 2$. Let M be a finitely generated non free S-module. Then, for each $i \geq 1$ we have $\beta^S_{i+2}(M) > \beta^S_i(M)$.* $\square$

and

Theorem 26. *Let R be a local artinian ring, and let M be a finitely generated non free R-module. Then, for each $i \geq 1$ we have*

$$\ell(R)\beta_i(M) > \beta_{i+1}(M) > \frac{\ell(socR)}{\ell(R)}\beta_i(M)$$

Observe that if we assume in the last theorem that R is also selfinjective, then its socle has length equal to 1, so we don't get any extremely useful information about the growth of the sequence of Betti numbers.

It turns out that in certain cases we can prove a similar theorem to Ramras' first theorem. For this type of result we might restrict ourselves only to the local selfinjective case $R = (R, \mathbf{m}, \Bbbk)$ but this is not necessary. Recall that since R is selfinjective, then for each integer $n \geq 0$ we have that $\beta_i(\tau M) = \beta_{i+2}(M)$ if M is an indecomposable non projective R-module. We will assume that $\operatorname{cx} M > 1$. Next we want to make sure that the stable component of M consists of modules that are eventually Ω-perfect. As mentioned in the introduction, this can be easily achieved if we assume that every simple R-module is non periodic ($\operatorname{cx} \Bbbk > 1$ for the local case) by [18].

We have the following:

Lemma 27. *Let R be a selfinjective algebra and let M be a finitely generated non projective indecomposable R-module. Assume that the stable component of the Auslander-Reiten quiver containing M is of the form $\mathbb{Z}A_\infty^\infty$ and that it consists entirely of eventually Ω-perfect modules. Then the sequences $\{\beta_{2n}(M)\}_n$ and $\{\beta_{2n+1}(M)\}_n$ are eventually strictly increasing.*

Proof. Let M be a module in this component. We may assume that M is Ω-perfect by taking enough powers of theAuslander-Reiten translate. The Auslander-Reiten sequence ending at M must have the following form [18, 20]

$$\begin{tikzcd} & X \arrow[dr, two heads] & \\ \tau M \arrow[ur, two heads] \arrow[dr, two heads] & & M \\ & Y \arrow[ur, two heads] & \end{tikzcd}$$

so we have an epimorphism $\tau M \to M$ that is the composition of two Ω-perfect epimorphisms. But we can infer from 4 that whenever we have an Ω-perfect epimorphism $f\colon B \to C$, then for each i we have $\beta_i(B) > \beta_i(C)$ since $\beta_i(\mathrm{Ker} f) > 0$. This implies that $\beta_{i+2}(M) = \beta_i(\tau M) > \beta_i(X) > \beta_i(M)$ for all $i \geq 0$ and the result follows. ☐

We now treat the D_∞ case.

Lemma 28. *Let R be a selfinjective algebra. Let $\mathcal{C}_s$ be a stable component of the Auslander-Reiten quiver of the form $\mathbb{Z}D_\infty$ consisting entirely of eventually Ω-perfect modules.*

(1) *Let M be a module in $\mathcal{C}_s$ not lying on the border of the component. Then the sequences $\{\beta_{2n}(M)\}_n$ and $\{\beta_{2n+1}(M)\}_n$ are eventually strictly increasing.*
(2) *Let Y and Z be two indecomposable modules in $\mathcal{C}_s$ lying in the two different τ-orbits that form the border of the component. Then the sequences $\{\beta_{2n}(Y \oplus Z)\}_n$ and $\{\beta_{2n+1}(Y \oplus Z)\}_n$ are eventually strictly increasing.*

Proof. Let M be an indecomposable module in this component. We may assume that M is Ω-perfect by taking enough powers of the Auslander-Reiten translate. If the Auslander-Reiten sequence ending at M has three indecomposable terms in the middle, the Auslander-Reiten sequence ending at M must have the following form [18, 20]

$$\begin{tikzcd} & X \arrow[dr, two heads] & \\ \tau M \arrow[ur, two heads] \arrow[r, two heads] \arrow[dr, two heads] & Y \arrow[r, hook] & M \\ & Z \arrow[ur, hook] & \end{tikzcd}$$

so as in the previous lemma we have an epimorphism $\tau M \to M$ that is the composition of two Ω-perfect epimorphisms. and $\beta_{i+2}(M) = \beta_i(\tau M) > \beta_i(M)$ for all $i \geq 0$. So the sequences of odd ,and of even Betti numbers for M are strictly increasing. Next we look at the module X. It is clear that we may assume that X is also Ω-perfect. The Auslander-Reiten sequence ending at X is of the form

$$0 \to \tau X \to \tau M \oplus X_1 \to X \to 0$$

where the irreducible map $X_1 \to X$ is an epimorphism. We proceed as in the proof of the previous lemma and obtain that for large enough n, the sequences $\{\beta_{2n}(X)\}_n$ and $\{\beta_{2n+1}(X)\}_n$ are strictly increasing. We proceed by induction along the sectional path of irreducible epimorphisms

$$\cdots X_n \to \cdots \to X_2 \to X_1 \to X$$

and we conclude that for each module X_j the two sequences $\{\beta_{2n}(X_j)\}_n$ and $\{\beta_{2n+1}(X_j)\}_n$ are eventually strictly increasing. This implies that the result holds for every module in the component, whose Auslander-Reiten sequence has the middle term decomposing into two indecomposable summands. This proves the first part of the lemma. By 20 we see that we have a composition of two irreducible epimorphisms fro $\tau Y \oplus \tau Z \to Y \oplus Z$ and we may also assume that both Y and Z are Ω-perfect. This shows that $\{\beta_{2n}(Y \oplus Z)\}_n$ and $\{\beta_{2n+1}(Y \oplus Z)\}_n$ are eventually strictly increasing. □

For the case when the stable component is of type $\mathbb{Z}\widetilde{A}_n$ or $\mathbb{Z}\widetilde{D}_n$ we proceed as above. We have the following similar proposition:

Proposition 29. *Let R be a selfinjective algebra and let M be a finitely generated non projective indecomposable R-module. Assume that the stable component of the Auslander-Reiten quiver containing M is of the form $\mathbb{Z}\widetilde{A}_n$ or $\mathbb{Z}\widetilde{D}_n$ and consists entirely of eventually Ω-perfect modules. Assume that the Auslander-Reiten sequence ending at M has a decomposable middle term. Then the sequences $\{\beta_{2n}(M)\}_n$ and $\{\beta_{2n+1}(M)\}_n$ are eventually increasing.*

Proof. Note first that M has complexity 2, by [20]. We use now the fact that a component of type $\mathbb{Z}\widetilde{A}_n$ is of tree type A_∞^∞ and use the same argument as above. For the case when the component is of type $\mathbb{Z}\widetilde{D}_n$ with $n > 4$, we can use the same proof as in the $\mathbb{Z}\widetilde{D}_\infty$ case, so it remains to look at the case when $n = 4$. In that case, if M is an Ω-perfect module, by [18] the Auslander-Reiten sequence ending at M has the form

$$0 \to \tau M \xrightarrow{[f_1,f_2,f_3,f_4]^T} E_1 \oplus E_2 \oplus E_3 \oplus E_4 \xrightarrow{[g_1,g_2,g_3,g_4]} M \to 0$$

where each f_i is an irreducible epimorphism and each g_i is an irreducible monomorphism. We show first that at least one of the two induced irreducible maps $E_1 \oplus E_2 \to M$ or $E_3 \oplus E_4 \to M$ is an irreducible epimorphism. Assume they are both monomorphisms. Then both $\ell(E_1 \oplus E_2) < \ell(M)$ and $\ell(E_3 \oplus E_4) < \ell(M)$ hence

$$\ell(M) + \ell(\tau M) = \ell(E_1) + \ell(E_2) + \ell(E_3) + \ell(E_4) < 2\ell(M)$$

implying $\ell(\tau M) < \ell(M)$. Since M is Ω-perfect we can repeat this argument and we obtain that the sequence $\{\ell(\tau^n M)\}$ is strictly decreasing; clearly a contradiction. Therefore we may assume that $E_3 \oplus E_4 \to M$ is an irreducible epimorphism. This means that we can look at our sequence as being

$$0 \to \tau M \xrightarrow{[f_1, f_2, f_3']^T} E_1 \oplus E_2 \oplus E_3' \xrightarrow{[g_1, g_2, g_3']} M \to 0$$

where $E_3' = E_3 \oplus E_4$. Now we obtain again from [18] that the induced map f_3' is an irreducible epimorphism and since M is Ω-perfect, $\beta_i(\tau M) > \beta_i(M)$ for all $i \geq 0$. The result follows now immediately. $\square$

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QUANTUM UNIPOTENT SUBGROUP AND DUAL CANONICAL BASIS

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Abstract. In a series of works [13, 16, 14, 15, 18, 19], Geiß-Leclerc-Schröer defined the cluster algebra structure on the coordinate ring $\mathbb{C}[N(w)]$ of the unipotent subgroup, associated with a Weyl group element w. And they proved cluster monomials are contained in Lusztig's *dual semicanonical basis* $\mathcal{S}^*$. We give a set up for the quantization of their results and propose a conjecture which relates the quantum cluster algebras in [3] to the *dual canonical basis* $\mathbf{B}^{\mathrm{up}}$. In particular, we prove that the quantum analogue $\mathcal{O}_q[N(w)]$ of $\mathbb{C}[N(w)]$ has the induced basis from $\mathbf{B}^{\mathrm{up}}$, which contains quantum flag minors and satisfies a factorization property with respect to the 'q-center' of $\mathcal{O}_q[N(w)]$. This generalizes Caldero's results [4, 5, 6] from finite type to an arbitrary symmetrizable Kac-Moody Lie algebra.

1. Introduction

1.1. **The canonical basis B and the dual canonical basis $\mathbf{B}^{\mathrm{up}}$.** Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra, $\mathbf{U}_q(\mathfrak{g})$ its associated quantized enveloping algebra, and $\mathbf{U}_q^-(\mathfrak{g})$ its negative part. In [24], Lusztig constructed the canonical basis $\mathbf{B}$ of $\mathbf{U}_q^-(\mathfrak{g})$ by a geometric method when $\mathfrak{g}$ is symmetric. In [21], Kashiwara constructed the (lower) global basis $G^{\mathrm{low}}(\mathcal{B}(\infty))$ by a purely algebraic method. Grojnowski-Lusztig [20] showed that the two bases coincide when $\mathfrak{g}$ is symmetric. We call the basis the *canonical basis*. There are two remarkable properties of the canonical basis, one is the positivity of structure constants of multiplication and comultiplication, and another is Kashiwara's crystal structure $\mathcal{B}(\infty)$, which is a combinatorial machinery useful for applications to representation theory, such as tensor product decomposition.

Since $\mathbf{U}_q^-(\mathfrak{g})$ has a natural pairing which makes it into a (twisted) self-dual bialgebra, we consider the dual basis $\mathbf{B}^{\mathrm{up}}$ of the canonical basis in $\mathbf{U}_q^-(\mathfrak{g})$. We call it the *dual canonical basis.*

1.2. **Cluster algebras.** Cluster algebras were introduced by Fomin and Zelevinsky [10] and intensively studied also with Berenstein [11, 1, 12] with an aim of providing a concrete and combinatorial setting for the study of Lusztig's (dual) canonical basis and total positivity. Quantum cluster algebras were also introduced by Berenstein and Zelevinsky [3], Fock and Goncharov [8, 9, 7] independently. The definition of (quantum) cluster algebra was motivated by Berenstein and Zelevinsky's earlier work [2] where combinatorial and multiplicative structures of the dual canonical basis were studied for $\mathfrak{g} = \mathfrak{sl}_n$ $(2 \leq n \leq 4)$. In [1], it was shown that the coordinate ring of the double Bruhat cell contains a cluster algebra as a subalgebra, which is conjecturally equal to the whole algebra.

The detailed version of this paper [22] will be published from Kyoto Journal of Mathematics.

A cluster algebra $\mathcal{A}$ is a subalgebra of rational function field $\mathbb{Q}(x_1, x_2, \cdots, x_r)$ of r indeterminates which is equipped with a distinguished set of generators (*cluster variables*) which is grouped into overlapping subsets (*clusters*) consisting of precisely r elements. Each subset is defined inductively by a sequence of certain combinatorial operation (*seed mutations*) from the initial seed. The monomials in the variables of a given single cluster are called *cluster monomials*. However, it is not known whether a cluster algebra have a basis, related to the dual canonical basis, which includes all cluster monomials in general.

1.3. **Cluster algebra and the semicanonical basis.** In a series of works [13, 16, 14, 15, 18, 19], Geiß, Leclerc and Schröer introduced a cluster algebra structure on the coordinate ring $\mathbb{C}[N(w)]$ of the unipotent subgroup associated with a Weyl group element w. Furthermore they show that the *dual semicanonical basis* $\mathcal{S}^*$ is compatible with the inclusion $\mathbb{C}[N(w)] \subset U(\mathfrak{n})^*_{\mathrm{gr}}$ and contains all cluster monomials. Here the dual semicanonical basis is the dual basis of the semicanonical basis of $U(\mathfrak{n})$, introduced by Lusztig [25, 28], and "compatible" means that $\mathcal{S}^* \cap \mathbb{C}[N(w)]$ forms a $\mathbb{C}$-basis of $\mathbb{C}[N(w)]$. It is known that canonical and semicanonical bases share similar combinatorial properties (crystal structure), but they are different. Geiß, Leclerc and Schröer conjecture that certain dual semicanonical basis elements are specialization of the corresponding dual canonical basis elements. This is called the *open orbit conjecture*.

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2. Quantum unipotent subgroup and the dual canonical basis

2.1. **Notations.** Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra and $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_- = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ be its triangular decomposition and its root decomposition. Let W be a Weyl group which is associated with $\mathfrak{g}$. Let $\Delta_\pm$ be the set of positive (resp. negative) roots. For a Weyl group element $w \in W$, we set $\Delta(w) := \Delta_+ \cap w\Delta_- = \{\alpha \in \Delta_+ \mid w^{-1}\alpha < 0\} \subset \Delta_+$. For a Weyl group element w, let $\overrightarrow{w} = (i_1, i_2, \ldots, i_\ell)$ be a reduced expression of w. We set $\beta_k := s_{i_1} \ldots s_{i_{k-1}}(\alpha_{i_k})$ for each $1 \le k \le \ell$. Then it is known that $\Delta(w) = \{\beta_k \mid 1 \le k \le \ell\}$. Let $\mathfrak{n}(w)$ be the nilpotent Lie subalgebra which is associated with $\Delta(w)$, that is

$$\mathfrak{n}(w) = \bigoplus_{1 \le k \le \ell} \mathfrak{g}_{\beta_k}.$$

For $i \in I$, we have Lusztig's braid symmetry T_i on $\mathbf{U}_q(\mathfrak{g})$, see [26, Chapter 32] for more details. It is known that $\{T_i\}_{i \in I}$ satisfies braid relations. Hence the composite $T_w := T_{i_1} \cdots T_{i_\ell}$does not depend on a choice of reduced word $\overrightarrow{w} = (i_1, i_2, \ldots, i_\ell)$ of w. In this article, we set $T_i = T'_{i,-1}$.

2.2. **Poincaré-Birkhoff-Witt basis.** Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra and $\mathbf{U}_q(\mathfrak{g})$ be the corresponding quantized enveloping algebra. We have a standard generators $\{E_i\}_{i \in I} \cup \{q^h\} \cup \{F_i\}_{i \in I}$ Let $\mathbf{U}_q^-(\mathfrak{g})$ be the $\mathbb{Q}(q)$-subalgebra which is generated by $\{F_i\}_{i \in I}$. It is known that $\mathbf{U}_q^-(\mathfrak{g})$ is isomophic to the $\mathbb{Q}(q)$-algebra which is defined by

$\{F_i\}_{i\in I}$ and q-Serre relations

$$\sum_{k=0}^{1-a_{ij}} (-1)^k F_i^{(k)} F_j F_i^{(1-a_{ij}-k)},$$

where $\{a_{ij}\}$ is the generalized Cartamn matrix which defines $\mathfrak{g}$ and $F_i^{(k)}$ is the divided power which is defined by $F_i^{(k)} := F_i^k/[k]_i!$. Let $\mathbf{U}_q^-(\mathfrak{g})_{\mathbb{Q}}$ be the $\mathbb{Q}[q^{\pm 1}]$-subalgebra which is generated by $\{F_i^{(n)}\}_{i\in I, n\in\mathbb{Z}_{\geq 0}}$. This $\mathbb{Q}[q^{\pm 1}]$-algebra is called Lusztig's $\mathbb{Q}[q^{\pm 1}]$-form.

We define root vectors associated with a reduced word$\overrightarrow{w} = (i_1, i_2, \ldots, i_\ell)$ for a Weyl group element $w \in W$. See [26, Proposition 40.1.3, Proposition 41.1.4] for more detail. For a Weyl group element$w \in W$ and a reduced word $\overrightarrow{w} = (i_1, i_2, \ldots, i_\ell)$, we define β_k as above. We define the *root vectors* $F(\beta_k)$ associated with $\beta_k \in \Delta(w)$

$$F(\beta_k) = T_{i_1} \ldots T_{i_{k-1}}(F_{i_k}).$$

It is known that $F(\beta_k) \in \mathbf{U}_q^-(\mathfrak{g})$ for all $1 \leq k \leq \ell$. We also define its divided power by $F(c\beta_k) = T_{i_1} \ldots T_{i_{k-1}}(F_{i_k}^{(c)})$. For an ℓ tuple of non-negative integers $\mathbf{c} = (c_1, c_2, \ldots, c_\ell)$, we set

$$F(\mathbf{c}, \overrightarrow{w}) := F(c_\ell \beta_\ell) \cdots F(c_1\beta_1).$$

It is known that $F(\mathbf{c}, \overrightarrow{w}) \in \mathbf{U}_q^-(\mathfrak{g})_{\mathbb{Q}}$.

Theorem 1 ([26, Proposition 40.2.1, Proposition 41.1.3])**.**

Theorem 2. (1) *Then* $\{F(\mathbf{c}, \overrightarrow{w})\}_{\mathbf{c}\in\mathbb{Z}_{\geq 0}^\ell}$ *forms a* $\mathbb{Q}(q)$*-basis of a subspace defined to be* $\mathbf{U}_q^-(w)$ *of* $\mathbf{U}_q^-(\mathfrak{g})$ *which does not depend on* $\overrightarrow{w}$*.*

(2) *We have* $F(\mathbf{c}, \overrightarrow{w}) \in \mathbf{U}_q^-(\mathfrak{g})_{\mathbb{Q}}$ *for all* $\mathbf{c} \in \mathbb{Z}_{\geq 0}^\ell$*.*

We consider the total order on $\Delta(w)$ as follows:

$$\beta_1 < \beta_2 < \cdots < \beta_\ell.$$

We have the following convex properties on $\{F(\beta_k)\}_{1\leq k\leq \ell}$.

Theorem 3 ([29, Proposition 3.6], [23, 5.5.2 Proposition])**.** *For $j < k$, let us write*

$$F(c_j\beta_j)F(c_k\beta_k) - q^{-(c_j\beta_j, c_k\beta_k)} F(c_k\beta_k)F(c_j\beta_j) = \sum_{\mathbf{c}'\in\mathbb{Z}_{\geq 0}^\ell} f_{\mathbf{c}'} F(\mathbf{c}', \overrightarrow{w})$$

$f_{\mathbf{c}'} \in \mathbb{Q}(q)$*. If* $f_{\mathbf{c}'} \neq 0$*, then* $c'_j < c_j$ *and* $c'_k < c_k$ *with* $\sum_{j\leq m\leq k} c'_m\beta_m = c_j\beta_j + c_k\beta_k$*.*

By the above formula, it is shown that $\mathbf{U}_q^-(w)$ is a $\mathbb{Q}(q)$-algebra which is generated by $\{F(\beta_k)\}_{1\leq k\leq\ell}$.

2.3. **PBW basis and crystal basis.** Let $\mathcal{L}(\infty)$ be the crystal lattice of $\mathbf{U}_q^-(\mathfrak{g})$ and $\mathcal{B}(\infty)$ be the crystal basis and $\mathbf{B}$ the canoncial basis.

The following result is due to Saito and Lusztig.

Theorem 4 ([30, Theorem 4.1.2], [27, Proposition 8.2]). (1) *We have* $F(\mathbf{c}, \overrightarrow{w}) \in \mathcal{L}(\infty)$ *and*

$$b(\mathbf{c}, \overrightarrow{w}) := F(\mathbf{c}, \overrightarrow{w}) \mod q\mathcal{L}(\infty) \in \mathcal{B}(\infty).$$

(2) *The map* $\mathbb{Z}_{\geq 0}^{\ell} \to \mathcal{B}(\infty)$ *which is defined by* $\mathbf{c} \mapsto b(\mathbf{c}, \overrightarrow{w})$ *is injective and the image* $\mathcal{B}(w)$ *does not depend on the choice of* $\overrightarrow{w}$.

2.4. **Dual canonical basis.** Let $(\ , \ )_K$ be the inner product on $\mathbf{U}_q^-(\mathfrak{g})$ defined by Kashiwara and $\mathbf{U}_q^-(\mathfrak{g})_{\mathbb{Q}}^{\mathrm{up}}$ be the dual $\mathbb{Q}[q^{\pm 1}]$-lattice of $\mathbf{U}_q^-(\mathfrak{g})_{\mathbb{Q}}$. Let $\mathbf{B}^{\mathrm{up}}$be the dual basis of $\mathbf{B}$ with respect to $(\ , \ )_K$ and this is called *dual canonical basis.* We set

$$F^{\mathrm{up}}(\mathbf{c}, \overrightarrow{w}) := \frac{1}{(F(\mathbf{c}, \overrightarrow{w}), F(\mathbf{c}, \overrightarrow{w}))_K} F(\mathbf{c}, \overrightarrow{w}).$$

Proposition 5. (1) *We have* $F^{\mathrm{up}}(\beta_k) \in \mathbf{B}^{\mathrm{up}}$.

(2) *Let* $\mathbf{U}_q^-(w)_{\mathbb{Q}}^{\mathrm{up}}$ *be the* $\mathbb{Q}[q^{\pm 1}]$*-span of* $\{F^{\mathrm{up}}(\mathbf{c}, \overrightarrow{w})\}_{\mathbf{c} \in \mathbb{Z}_{\geq 0}^{\ell}}$. *Then* $\mathbf{U}_q^-(w)_{\mathbb{Q}}^{\mathrm{up}}$ *is the* $\mathbb{Q}[q^{\pm 1}]$*-algebra generated by* $\{F^{\mathrm{up}}(\beta_k)\}_{1 \leq k \leq \ell}$.

Using the above proposition we obtain the following compabitility. This is a quantum analogue of the Geiß-Leclerc-Schroër's result.

Theorem 6. *Let* $\mathbf{B}^{\mathrm{up}}(w) := \mathbf{B}^{\mathrm{up}} \cap \mathbf{U}_q^-(w)_{\mathbb{Q}}^{\mathrm{up}}$. *Then* $\mathbf{B}^{\mathrm{up}}(w)$ *is a* $\mathbb{Q}[q^{\pm 1}]$*-basis of* $\mathbf{B}^{\mathrm{up}}(w)$.

2.5. **Specialization at** $q = 1$**.** For the Lusztig form, we have the specilization isomorphism $\mathbb{C} \otimes_{\mathbb{Q}[q^{\pm 1}]} \mathbf{U}_q^-(\mathfrak{g})_{\mathbb{Q}} \simeq U(\mathfrak{n})$. Dually, we have the $\mathbb{C}$-algebra isomorphism $\Phi^{\mathrm{up}} \colon \mathbb{C} \otimes_{\mathbb{Q}[q^{\pm 1}]} \mathbf{U}_q^-(\mathfrak{g})_{\mathbb{Q}}^{\mathrm{up}} \simeq \mathbb{C}[N]$.

Under the isomorphism $\mathbb{C} \otimes_{\mathbb{Q}[q^{\pm 1}]} \mathbf{U}_q^-(\mathfrak{g})_{\mathbb{Q}}^{\mathrm{up}} \simeq \mathbb{C}[N]$, as a corollay of the above theorem, we obtain the following result for $\mathbf{U}_q^-(w)$ which concerns the specialization at $q = 1$.

Corollary 7. *Under the* $\mathbb{C}$*-algebra isomorphism* Φ^{up}, *we have*

$$\mathbb{C} \otimes_{\mathbb{Q}[q^{\pm 1}]} \mathbf{U}_q^-(w)_{\mathbb{Q}}^{\mathrm{up}} \simeq \mathbb{C}[N(w)],$$

where $N(w)$ *is the unipotent subgroup associated with the nilpotent Lie algebra* $\mathfrak{n}(w)$.

3. Quantum closed unipotent subgroup and Dual canonical basis

For a Weyl group element $w \in W$ and a reduced word $\overrightarrow{w} = (i_1, \ldots, i_\ell)$, we set

$$\mathbf{U}_w^- := \sum_{\mathbf{a} = (a_1, \ldots, a_\ell) \in \mathbb{Z}_{\geq 0}^{\ell}} \mathbb{Q}(q) F_{i_1}^{(a_1)} \ldots F_{i_\ell}^{(a_\ell)}.$$

This is called Demazure-Schubert filtration. It is known that $\mathbf{U}_w^-$ is compatible with the canonical basis $\mathbf{B}$, that is $\mathbf{B} \cap \mathbf{U}_w^-$ is a $\mathbb{Q}[q^{\pm 1}]$-basis of $\mathbf{U}_w^-$. We denote the correponding subset by $\mathcal{B}(w, \infty)$. Hence we set

$$\mathcal{O}_q[\overline{N_w}] := \mathbf{U}_q^-(\mathfrak{g}) / (\mathbf{U}_w^-)^{\perp},$$

where $(\mathbf{U}_w^-)^{\perp}$ is the annhilator of $\mathbf{U}_w^-$ with respect to Kashiwara's bilinear form $(\ , \ )_K$. Since $(\mathbf{U}_w^-)^{\perp}$ is compatible with $\mathbf{B}^{\mathrm{up}}$, the canonical projection induces the dual canonical basis on $\mathcal{O}_q[\overline{N_w}]$.

Theorem 8. (1)*Let* $\mathbf{U}_q^-(w) \to \mathbf{U}_q^-(\mathfrak{g}) \to \mathcal{O}_q[\overline{N_w}]$ *be the inclusion and the canonical projectinon. Then the composite is monomorphism of algebra.*

(2) *We have* $\mathcal{B}(w) \subset \mathcal{B}(w,\infty)$.

4. Quantum flag minor and Its multiplicative proprerties

For a dominant integral weight $\lambda \in P_+$, let $V(\lambda)$ be the corresponding integrable highest weight module with highest weight vector u_λ. We have symmetric bilinear form $(\ ,\ )_\lambda$on $V(\lambda)$. Let $\pi_\lambda\colon \mathbf{U}_q^-(\mathfrak{g}) \twoheadrightarrow V(\lambda)$ be the projection defined by $x \mapsto xu_\lambda$. Let j_λ be dual of π_λ, that is $j_\lambda\colon V(\lambda) \hookrightarrow \mathbf{U}_q^-(\mathfrak{g})$. For a Weyl group element $w \in W$, we have the extremal vector $u_{w\lambda}$ of weight λ. It is known that $u_{w\lambda}$ is contained in the canoical basis and the dual canoncial basis. We set quantum unipotent minro $D_{w\lambda,\lambda}$by

$$D_{w\lambda,\lambda} := j_\lambda(u_{w\lambda}).$$

It is known that $D_{w\lambda,\lambda} \in \mathbf{B}^{\mathrm{up}}$. The following is main result in our study.

Theorem 9. (1) *For* $w \in W$ *and* $\lambda \in P_+$, *we have* $D_{w\lambda,\lambda} \in \mathbf{U}_q^-(w)$.

(2) *For arbitrary* $b \in \mathcal{B}(w)$, *there exists* $N \in \mathbb{Z}$ *such that* $q^N G^{\mathrm{up}}(b) D_{w\lambda,\lambda} \in \mathbf{B}^{\mathrm{up}}(w)$, *there* $G^{\mathrm{up}}(b)$ *is the dual canonical basis element which is associated with* $b \in \mathcal{B}(w)$.

Using the above theorem, we obtain the following quantum seed.

For a Weyl group element w, a reduced word $\overrightarrow{w} = (i_1, i_2, \ldots, i_\ell)$ and $\mathbf{c} = (c_1, \ldots, c_\ell) \in \mathbb{Z}_{\geq 0}^\ell$, we set

$$D^{\overrightarrow{w}}(\mathbf{c}) := \prod_{1 \leq k \leq \ell} D_{s_{i_1} \cdots s_{i_k} c_k \varpi_{i_k}, c_k \varpi_{i_k}}.$$

Then $\{D^{\overrightarrow{w}}(\mathbf{c})\}_{\mathbf{c} \in \mathbb{Z}_{\geq 0}^\ell}$ forms a mutually commmuting familty and $\{D^{\overrightarrow{w}}(\mathbf{c})\}_{\mathbf{c} \in \mathbb{Z}_{\geq 0}^\ell}$ is linear independent over $\mathbb{Z}[q^{\pm 1}]$. $\{D^{\overrightarrow{w}}(\mathbf{c})\}_{\mathbf{c} \in \mathbb{Z}_{\geq 0}^\ell}$ can be considered as a quantum analogue of the initial seed in [18] and we can form the corresponding quantum cluster algebra by it. Our conjecture is an $\mathbb{Q}[q^{\pm 1}]$-algebra isomorphism between the quantum cluster algebra and the quantum unipotent subgroup $\mathcal{O}_q[N(w)]$ and the set of quantum cluster monomials is contained by the dual canonical basis $\mathbf{B}^{\mathrm{up}}(w)$. This is just a quantum analogue of [18] and this is compatible with their open orbit conjecture for symmetric $\mathfrak{g}$. Recently the $\mathbb{Q}(q)$-algebra isomorphism is obtained by [17].

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WEAKLY CLOSED GRAPH

KAZUNORI MATSUDA

ABSTRACT. We introduce the notion of weak closedness for connected simple graphs. This notion is a generalization of closedness introduced by Herzog-Hibi-Hreindóttir-Kahle-Rauh. We give a characterization of weakly closed graphs and prove that the binomial edge ideal J_G is F-pure for weakly closed graph G.

Key Words: binomial edge ideal, F-purity, weakly closed graph.

2000 *Mathematics Subject Classification*: 05C25, 05E40, 13A35, 13C05.

1. Introduction

This article is based on [6].

Throughout this article, let k be an F-finite field of positive characteristic. Let G be a graph on the vertex set $V(G) = [n]$ with edge set $E(G)$. We assume that a graph G is always connected and simple, that is, G is connected and has no loops and multiple edges. And the term "labeling" means numbering of $V(G)$ from 1 to n.

For each graph G, we call $J_G := ([i, j] = X_iY_j - X_jY_i \mid \{i, j\} \in E(G))$ the *binomial edge ideal* of G (see [4], [8]). J_G is an ideal of $S := k[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$.

2. Weakly closed graph

In this section, we give the definition of weakly closed graphs and the first main theorem of this chapter, which is a characterization of weakly closed graphs.

Until we define the notion of weak closedness, we fix a graph G and a labeling of $V(G)$.

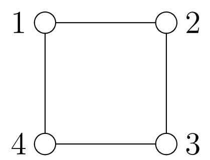
Let $(a_1, \ldots, a_n)$ be a sequence such that $1 \le a_i \le n$ and $a_i \ne a_j$ if $i \ne j$.

Definition 1. We say that a_i is *interchangeable with* a_{i+1} if $\{a_i, a_{i+1}\} \in E(G)$. And we call the following operation $\{a_i, a_{i+1}\}$-*interchanging* :

$$(a_1, \ldots, a_{i-1}, \underline{a_i}, \underline{a_{i+1}}, a_{i+2}, \ldots, a_n) \to (a_1, \ldots, a_{i-1}, \underline{a_{i+1}}, \underline{a_i}, a_{i+2}, \ldots, a_n)$$

Definition 2. Let $\{i, j\} \in E(G)$. We say that i is *adjacentable with* j if the following assertion holds: for a sequence $(1, 2, \ldots, n)$, by repeating interchanging, one can find a sequence $(a_1, \ldots, a_n)$ such that $a_k = i$ and $a_{k+1} = j$ for some k.

Example 3. About the following graph G, 1 is adjacentable with 4:



The detailed version of this paper will be submitted for publication elsewhere.

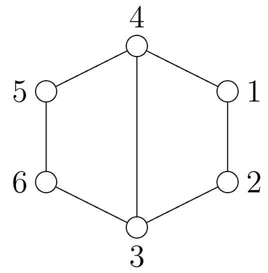
Indeed,

$$(\underline{1}, 2, 3, \underline{4}) \xrightarrow{\{1,2\}} (2, \underline{1}, 3, \underline{4}) \xrightarrow{\{3,4\}} (2, \underline{1}, \underline{4}, 3).$$

Now, we can define the notion of weakly closed graph.

Definition 4. Let G be a graph. G is said to be *weakly closed* if there exists a labeling which satisfies the following condition: for all i, j such that $\{i, j\} \in E(G)$, i is adjacentable with j.

Example 5. The following graph G is weakly closed:



Indeed,

$$(\underline{1}, 2, 3, \underline{4}, 5, 6) \xrightarrow{\{1,2\}} (2, \underline{1}, 3, \underline{4}, 5, 6) \xrightarrow{\{3,4\}} (2, \underline{1}, \underline{4}, 3, 5, 6),$$
$$(1, 2, \underline{3}, 4, 5, \underline{6}) \xrightarrow{\{3,4\}} (1, 2, 4, \underline{3}, 5, \underline{6}) \xrightarrow{\{5,6\}} (1, 2, 4, \underline{3}, \underline{6}, 5).$$

Hence 1 is adjacentable with 4 and 3 is adjacentable with 6.

Before stating the first main theorem of this chapter, which is a characterization of weakly closed graphs, we recall that the definition of closed graphs.

Definition 6 (See [4])**.** G is *closed with respect to the given labeling* if the following condition is satisfied: for all $\{i, j\}, \{k, l\} \in E(G)$ with $i < j$ and $k < l$ one has $\{j, l\} \in E(G)$ if $i = k$ but $j \neq l$, and $\{i, k\} \in E(G)$ if $j = l$ but $i \neq k$.

In particular, G is *closed* if there exists a labeling for which it is closed.

Remark 7.
(1) [4, Theorem 1.1] G is closed if and only if J_G has a quadratic Gröbner basis. Hence if G is closed then S/J_G is Koszul algebra.
(2) [2, Theorem 2.2] Let G be a graph. Then the following conditions are equivalent:
   (a) G is closed.
   (b) There exists a labeling of $V(G)$ such that all facets of $\Delta(G)$ are intervals $[a, b] \subset [n]$, where $\Delta(G)$ is the clique complex of G.

The following characterization of closed graphs is a reinterpretation of Crupi and Rinaldo's one. This is relevant to the first main theorem of this chapter deeply.

Proposition 8 (See [1, Proposition 2.6])**.** *Let G be a graph. Then the following conditions are equivalent:*

(1) *G is closed.*
(2) *There exists a labeling which satisfies the following condition: for all i, j such that $\{i, j\} \in E(G)$ and $j > i + 1$, the following assertion holds: for all $i < k < j$, $\{i, k\} \in E(G)$* and *$\{k, j\} \in E(G)$.*

Proof. (1) $\Rightarrow$ (2): Let $\{i, j\} \in E(G)$. Since G is closed, there exists a labeling satisfying $\{i, i+1\}, \{i+1, i+2\}, \ldots, \{j-1, j\} \in E(G)$ by [HeHiHrKR, Proposition 1.4]. Then we have that $\{i, i+2\}, \ldots, \{i, j-2\}, \{i, j-1\} \in E(G)$ by the definition of closedness. Similarly, we also have that $\{k, j\} \in E(G)$ for all $i < k < j$.

(2) $\Rightarrow$ (1): Assume that $i < k < j$. If $\{i, k\}, \{i, j\} \in E(G)$, then $\{k, j\} \in E(G)$ by assumption. Similarly, if $\{i, j\}, \{k, j\} \in E(G)$, then $\{i, k\} \in E(G)$. Therefore G is closed. □

The following theorem characterizes weakly closed graph.

Theorem 9. *Let G be a graph. Then the following conditions are equivalent:*

(1) *G is weakly closed.*
(2) *There exists a labeling which satisfies the following condition: for all i, j such that $\{i, j\} \in E(G)$ and $j > i + 1$, the following assertion holds: for all $i < k < j$, $\{i, k\} \in E(G)$* or *$\{k, j\} \in E(G)$.*

Proof. (1) $\Rightarrow$ (2): Assume that $\{i, j\} \in E(G)$, $\{i, k\} \notin E(G)$ and $\{k, j\} \notin E(G)$ for some $i < k < j$. Then i is not adjacentable with j, which is in contradiction with weak closedness of G.

(2) $\Rightarrow$ (1): Let $\{i, j\} E(G)$. By repeating interchanging along the following algorithm, we can see that i is adjacentable with j:

(a): Let $A := \{k \mid \{k, j\} \in E(G), i < k < j\}$ and $C := \emptyset$.

(b): If $A = \emptyset$ then go to (g), otherwise let $s := \max\{A\}$.

(c): Let $B := \{t \mid \{s, t\} \in E(G), s < t \leq j\} \setminus C = \{t_1, \ldots, t_m = j\}$, where $t_1 < \ldots < t_m = j$.

(d): Take $\{s, t_1\}$-interchanging, $\{s, t_2\}$-interchanging, …, $\{s, t_m = j\}$-interchanging in turn.

(e): Let $A := A \setminus \{s\}$ and $C := C \cup \{s\}$.

(f): Go to (b).

(g): Let $U := \{u \mid i < u < j, \{i, u\} \in E(G)$ and $\{u, j\} \notin E(G)\}$ and $W := \emptyset$.

(h): If $U = \emptyset$ then go to (m), otherwise let $u := \min\{U\}$.

(i): Let $V := \{v \mid \{v, u\} \in E(G), i \leq v < u\} \setminus W = \{v_1 = i, \ldots, v_l\}$, where $v_1 = i < \ldots < v_l$.

(j): Take $\{v_1 = i, u\}$-interchanging, $\{v_2, u\}$-interchanging, …, $\{v_l, u\}$-interchanging in turn.

(k): Let $U := U \setminus \{u\}$ and $W := W \cup \{u\}$.

(l): Go to (h).

(m): Finished. □

By comparing this theorem and Proposition 8, we get the following corollary. A graph G is said to be *complete r-partite* if there exists a partition $V(G) = \coprod_{i=1}^{r} V_i$ such that $\{i, j\} \in E(G)$ if and only of $a \neq b$ for all $i \in V_a$ and $j \in V_b$.

Corollary 10. *Closed graphs and complete r-partite graphs are weakly closed.*

Proof. Assume that G is complete r-partite and $V(G) = \coprod_{i=1}^{r} V_i$. Let $\{i, j\} \in E(G)$ with $i \in V_a$ and $j \in V_b$. Then $a \neq b$. Hence for all $i < k < j$, $k \notin V_a$ or $k \notin V_b$. This implies that $\{i, k\} \in E(G)$ or $\{k, j\} \in E(G)$. □

3. F-purity of binomial edge ideals

In this section, we study about F-purity of binomial edge ideals. Firstly, we recall that the definition of F-purity of a ring R.

Definition 11 (See [5]). Let R be an F-finite reduced Noetherian ring of characteristic $p > 0$. R is said to be *F-pure* if the Frobenius map $R \to R$, $x \mapsto x^p$ is pure, equivalently, the natural inclusion $\tau : R \hookrightarrow R^{1/p}$, $(x \mapsto (x^p)^{1/p})$ is pure, that is, $M \to M \otimes_R R^{1/p}$, $m \mapsto m \otimes 1$ is injective for every R-module M.

The following proposition, which is called the Fedder's criterion, is useful to determine the F-purity of a ring R.

Proposition 12 (See [3]). *Let $(S, \mathfrak{m})$ be a regular local ring of characteristic $p > 0$. Let I be an ideal of S. Put $R = S/I$. Then R is F-pure if and only if $I^{[p]} : I \not\subseteq \mathfrak{m}^{[p]}$, where $J^{[p]} = (x^p \mid x \in J)$ for an ideal J of S.*

In this section, we consider the following question:

Question. When is S/J_G F-pure ?

In [8], Ohtani proved that if G is complete r-partite graph then S/J_G is F-pure. Moreover, it is easy to show that if G is closed then S/J_G is F-pure. However, there are many examples of G such that G is neither complete r-partite nor closed but S/J_G is F-pure. Namely, there is room for improvement about the above studies.

The second main theorem of this chapter is as follows:

Theorem 13. *If G is weakly closed, then S/J_G is F-pure.*

Proof. For a sequence $v_1, v_2, \ldots, v_s$, we put

$$Y_{v_1}(v_1, v_2, \ldots, v_s)X_{v_s} := (Y_{v_1}[v_1, v_2][v_2, v_3] \cdots [v_{s-1}, v_s]X_{v_s})^{p-1}.$$

Let $\mathfrak{m} = (X_1, \ldots, X_n, Y_1, \ldots, Y_n)S$. By taking completion and using Proposition 2.2, it is enough to show that $Y_1(1, 2, \ldots, n)X_n \in (J_G^{[p]} : J_G) \setminus \mathfrak{m}^{[p]}$. It is easy to show that $Y_1(1, 2, \ldots, n)X_n \notin \mathfrak{m}^{[p]}$ by considering its initial monomial.

Next, we use the following lemmas (see [8]):

Lemma 14 ([8, Formula 1])**.** *If* $\{a,b\} \in E(G)$*, then*

$$Y_{v_1}(v_1,\dots,c,\underline{a},\underline{b},d,\dots,v_n)X_{v_n} \equiv Y_{v_1}(v_1,\dots,c,\underline{b},\underline{a},d,\dots,v_n)X_{v_n}$$

modulo $J_G^{[p]}$*.*

Lemma 15 ([8, Formula 2])**.** *If* $\{a,b\} \in E(G)$*, then*

$$Y_a(\underline{a},\underline{b},c,\dots,v_n)X_{v_n} \equiv Y_b(\underline{b},\underline{a},c,\dots,v_n)X_{v_n},$$
$$Y_{v_1}(v_1,\dots,c,\underline{a},\underline{b})X_b \equiv Y_{v_1}(v_1,\dots,c,\underline{b},\underline{a})X_a$$

modulo $J_G^{[p]}$*.*

Let $\{i,j\} \in E(G)$. Since G is weakly closed, i is adjacentable with j. Hence there exists a polynomial $g \in S$ such that

$$Y_1(1,2,\dots,n)X_n \equiv g \cdot [i,j]^{p-1}$$

modulo $J_G^{[p]}$ from the above lemmas. This implies $Y_1(1,2,\dots,n)X_n \in (J_G^{[p]} : J_G)$. □

4. Difference between closedness and weak closedness and some examples

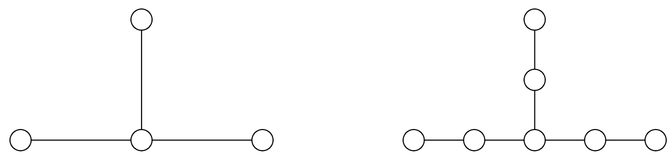
In this section, we state the difference between closedness and weak closedness and give some examples.

Proposition 16. *Let* G *be a graph.*

1. *[4, Proposition 1.2] If* G *is closed, then* G *is chordal, that is, every cycle of* G *with length* $t > 3$ *has a chord.*
2. *If* G *is weakly closed, then every cycle of* G *with length* $t > 4$ *has a chord.*

Proof. (2) It is enough to show that the pentagon graph G with edges $\{a,b\}$, $\{b,c\}$, $\{c,d\}$, $\{d,e\}$ and $\{a,e\}$ is not weakly closed. Suppose that G is weakly closed. We may assume that $a = \min\{a,b,c,d,e\}$ without loss of generality. Then $b \neq \max\{a,b,c,d,e\}$. Indeed, if $b = \max\{a,b,c,d,e\}$, then c,d,e are connected with a or b by the definition of weak closedness, but this is a contradiction. Similarly, $e \neq \max\{a,b,c,d,e\}$. Hence we may assume that $c = \max\{a,b,c,d,e\}$ by symmetry. If $b = \min\{b,c,d\}$, then d,e are connected with b or c, a contradiction. Therefore, $b \neq \min\{b,c,d\}$. Similarly, $b \neq \max\{b,c,d\}$. Hence we may assume that $d = \min\{b,c,d\}$ and $e = \max\{b,c,d\}$ by symmetry. Then $\{a,b\}$ and $a < d < b$, but $\{a,d\},\{d,b\} \notin E(G)$. This is a contradiction. □

Next, we give a characterization of closed (resp. weakly closed) tree graphs in terms of claw (resp. bigclaw). A graph G is said to be *tree* if G has no cycles. We consider the following graphs (a) and (b). We call the graph (a) a *claw* and the graph (b) a *bigclaw*.



(a) (b)

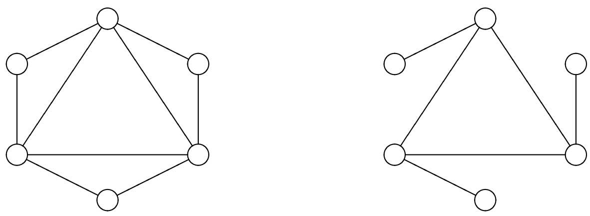
Proposition 17. *Let G be a tree.*

(1) *[4, Corollary 1.3] The following conditions are equivalent:*
   (a) *G is closed.*
   (b) *G is a path.*
   (c) *G is a claw-free graph.*

(2) *The following conditions are equivalent:*
   (a) *G is weakly closed.*
   (b) *G is a caterpillar, that is, a tree for which removing the leaves and incident edges produces a path graph.*
   (c) *G is a bigclaw-free graph.*

Proof. (2) One can see that a bigclaw graph is not weakly closed. □

Remark 18. From Proposition 17(2), we have that chordal graphs are not always weakly closed. As other examples, the following graphs are chordal, but not weakly closed:



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POLYCYCLIC CODES AND SEQUENTIAL CODES

MANABU MATSUOKA

Abstract. In this paper we generalize the notion of cyclicity of codes, that is, polycyclic codes and sequential codes. We study the relation between polycyclic codes and sequential codes over finite commutative QF rings. Furthermore, we characterized the family of some constacyclic codes.

Key Words: finite rings, (θ, δ)-codes, skew polynomial rings.

2010 *Mathematics Subject Classification*: Primary 94B60; Secondary 94B15.

1. Introduction

Let R be a finite commutative ring. A linear code C of length n over R is a submodule of the R-module $R^n = \{(a_0, \cdots, a_{n-1}) | a_i \in R\}$. If C is a free R-module, C is said to be a free code. A linear code $C \subseteq R^n$ is called cyclic if $(a_0, a_1, \cdots, a_{n-1}) \in C$ implies $(a_{n-1}, a_0, a_1, \cdots, a_{n-2}) \in C$. The notion of cyclicity has been extended in various directions.

In [6], S. R. López-Permouth, B. R. Parra-Avila and S. Szabo studied the duality between polycyclic codes and sequential codes. By the way, J. A. Wood establish the extension theorem and MacWilliams identities over finite frobenius rings in [9]. M. Greferath and M. E. O'Sullivan study bounds for block codes on finite frobenius rings in [2]. In this paper, we generalize the result of [6] to codes with finite commutative QF rings.

In section 2 we define polycyclic codes over finite commutative rings. And we study the properties of polycyclic codes. In section 3 we define sequential codes and consider the properties of sequential codes. In section 4 we study the relation between polycyclic codes and sequential codes over finite commutative QF rings. And we characterized the family of some constacyclic codes.

Throughout this paper, R denotes a finite commutative ring with $1 \neq 0$, n denotes a natural number with $n \geq 2$, unless otherwise stated.

2. Polycyclic codes

A linear $[n, k]$-code over a finite commutative ring R is a submodule $C \subseteq R^n$ of rank k. We define polycyclic codes over a finite commutative ring.

Definition 1. Let C be a linear code of length n over R. C is a polycyclic code induced by c if there exists a vector $c = (c_0, c_1, \cdots, c_{n-1}) \in R^n$ such that for every $(a_0, a_1, \cdots, a_{n-1}) \in C$, $(0, a_0, a_1, \cdots, a_{n-2}) + a_{n-1}(c_0, c_1, \cdots, c_{n-1}) \in C$. In this case we call c an associated vector of C.

The detailed version of this paper will be submitted for publication elsewhere.

As cyclic codes, polycyclic codes may be understood in terms of ideals in quotient rings of polynomial rings. Given $c = (c_0, c_1, \cdots, c_{n-1}) \in R^n$, if we let $f(X) = X^n - c(X)$, where $c(X) = c_{n-1}X^{n-1} + \cdots + c_1 X + c_0$ then the R-module homomorphism $\rho : R^n \to R[X]/(f(X))$ sending the vector $a = (a_0, a_1, \cdots, a_{n-1})$ to the equivalence class of polynomial $\overline{a_{n-1}X^{n-1} + \cdots + a_1 X + a_0}$, allows us to identify the polycyclic codes induced by c with the ideal of $R[X]/(f(X))$.

Definition 2. Let C be a polycyclic code in $R[X]/(f(X))$. If there exist monic polynomials g and h such that $\rho(C) = (g)/(f)$ and $f = hg$, then C is called a principal polycyclic code.

Proposition 3. *A code $C \subseteq R^n$ is a principal polycyclic code induced by some $c \in C$ if and only if C is a free R-module and has a $k \times n$ generator matrix of the form*

$$G = \begin{pmatrix} g_0 & g_1 & \cdots & g_{n-k} & 0 & \cdots & 0 \\ 0 & g_0 & g_1 & \cdots & g_{n-k} & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & & & \vdots \\ 0 & \cdots & 0 & g_0 & g_1 & \cdots & g_{n-k} \end{pmatrix}$$

with an invertible g_{n-k}. In this case

$$\rho(C) = \left(\overline{g_{n-k}X^{n-k} + \cdots + g_1 X + g_0}\right)$$

is the ideal of $R[X]/(f(X))$.

Definition 4. Let $C = (g)/(f) \subseteq R[X]/(f(X))$ be a principal polycyclic code. If the constant term of g is invertible, then C is called a principal polycyclic code with an invertible constant term.

For a $c = (c_0, c_1, \cdots, c_{n-1}) \in R^n$, let D_c be the following square matrix;

$$D_c = \begin{pmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ 0 & & & 1 \\ c_0 & c_1 & \cdots & c_{n-1} \end{pmatrix}.$$

It follows that a code $C \subseteq R^n$ is polycyclic with an associated vector $c \in R^n$ if and only if it is invariant under right multiplication by D_c.

3. Sequential codes

Definition 5. Let C be a linear code of length n over R. C is a sequential code induced by c if there exists a vector $c = (c_0, c_1, \cdots, c_{n-1}) \in R^n$ such that for every $(a_0, a_1, \cdots, a_{n-1}) \in C$, $(a_1, a_2, \cdots, a_{n-1}, a_0c_0 + a_1c_1 + \cdots + a_{n-1}c_{n-1}) \in C$. In this case we call c an associated vector of C.

Let C be a sequential code with an associated vector $c = (c_0, c_1, \cdots, c_{n-1})$. Then C is invariant under right multiplication by the matrix

$$
{}^tD_c = \begin{pmatrix} 0 & & 0 & c_0 \\ 1 & & & c_1 \\ & \ddots & & \vdots \\ 0 & & 1 & c_{n-1} \end{pmatrix}
$$

On R^n define the standard inner product by

$$
< x, y >= \sum_{i=0}^{n-1} x_i y_i
$$

for $x = (x_0, x_1, \cdots, x_{n-1})$, $y = (y_0, y_1, \cdots, y_{n-1}) \in R^n$.

The dual code $C^\perp$ of a linear code C is defined by

$$
C^\perp = \{a \in R^n | < c, a >= 0 \text{ for any } c \in C\}.
$$

Clearly, $C^\perp$ is a linear code over R.

Theorem 6. *For a code $C \subseteq R^n$, we have the following assertions:*
(1) *If C is polycyclic, then $C^\perp$ is sequential.*
(2) *If C is sequential, then $C^\perp$ is polycyclic.*

4. Codes over finite commutative QF rings

Let R be a (not necessarily commutative) ring. A left R-module P is projective if for every R-epimorphism $g : M \to N$ and every R-homomorphism $f : P \to N$, there exists a R-homomorphism $h : P \to M$ with $f = g \circ h$.

A left R-module Q is injective if for every R-monomorphism $g : N \to M$ and every R-homomorphism $f : N \to Q$, there exists a R-homomorphism $h : M \to Q$ with $f = h \circ g$.

The ring R is said to be left (resp. right) self-injective if R itself is injective as left (resp. right) R-module. If both conditions hold, R is said to be a self-injective ring.

A left R-module M is Artinian if M is satisfies the descending chain condition on submodules. A ring R is left (resp. right) Artinian if R itself is Artinian as left (resp. right) R-module. If both conditions hold, R is said to be an Artinian ring.

It is clear that a finite ring is an Artinian ring.

Definition 7. For a (not necessarily commutative) ring R, R is called a QF (quasi-Frobenius) ring if R is left Artinian and left self-injective.

It is well-known that the definition of a QF ring is left-right symmetric.

For any R-submodule $C \subseteq R^n$, C° is defined by

$$
C^\circ = \{\lambda \in Hom_R(R^n, R) | \lambda(C) = 0\}.
$$

Theorem 8. *For a (not necessarily commutative) ring R, the following conditions are equivalent:*
(1) *R is a QF ring.*
(2) *For submodules $M \subseteq R^n$, $M^{\circ\circ} = M$.*

Theorem 9. *For a (not necessarily commutative) ring R, the following are equivalent:*
(1) *R is a QF ring.*
(2) *A left module is projective if and only if it is injective.*

We define an R-module homomorphism $\delta_x : R^n \to R$ as $\delta_x(y) =< y, x >$ for any $x \in R^n$.

Proposition 10. *The homomorphism $\delta : C^{\perp} \to C^{\circ}$ sending x to δ_x is an isomorphism of R-modules.*

Theorem 11. *Let R be a finite commutative QF ring. For a submodule $C \subseteq R^n$, $(C^{\perp})^{\perp} = C$.*

By Theorem 1 and Theorem 4, we can get the following corollary.

Corollary 12. *Let R be a finite commutative QF ring. Then C is a polycyclic code if and only if $C^{\perp}$ is a sequential code.*

Theorem 13. *Let R be a finite commutative QF ring. If $C \subseteq R^n$ is a free R-module of finite rank, then $C^{\perp}$ is a free R-module of* $\mathrm{rank} C^{\perp} = n - \mathrm{rank} C$.

We determine the parity check matrix of a constacyclic code.

Proposition 14. *Let R be a finite commutative QF ring and $f = X^n - \alpha \in R[X]$. Suppose $f = hg \in R[X]$ where g and h are polynomials of degree $n-k$ and k, respectively. Let C be the linear $[n,k]$-code corresponding to the ideal generated by g in $R[X]/(X^n - \alpha)$ and $h(X) = h_k X^k + h_{k-1}X^{k-1} + \cdots + h_1 X + h_0$. Then C has the $(n-k) \times n$ parity check matrix H given by*

$$H = \begin{pmatrix} h_k & \cdots & h_1 & h_0 & 0 & \cdots & 0 \\ 0 & h_k & \cdots & h_1 & h_0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & & & \vdots \\ 0 & \cdots & 0 & h_k & \cdots & h_1 & h_0 \end{pmatrix}.$$

Definition 15. Let R be a finite commutative QF ring. For a sequential code $C \subseteq R^n$, C is called a principal sequential code if $C^{\perp}$ is a principal polycyclic code. And C is called a principal sequential code with an invertible constant term if $C^{\perp}$ is a principal polycyclic code with an invertible constant term.

Now we can get the main theorem.

Theorem 16. *Let R be a finite commutative QF ring. Suppose C is a free codes of R^n. Then the following conditions are equivalent:*
(1) *Both C and $C^{\perp}$ are principal polycyclic codes with invertible constant terms.*
(2) *Both C and $C^{\perp}$ are principal sequential codes with invertible constant terms.*
(3) *C is a principal polycyclic and sequential code with an invertible constant term.*
(4) *$C^{\perp}$ is a principal polycyclic and sequential code with an invertible constant term.*
(5) *$C = (g)/(X^n - \alpha)$ is a constacyclic code with an invertible α.*
(6) *$C^{\perp} = (q)/(X^n - \beta)$ is a constacyclic code with an invertible β.*

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A NOTE ON DIMENSION OF TRIANGULATED CATEGORIES.

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Abstract. In this note we study the behavior of the dimension of the perfect derived category $\mathrm{Perf}(A)$ of a dg-algebra A over a field k under a base field extension K/k. In particular we show that the dimension of a perfect derived category is invariant under a separable algebraic extension K/k. As an application we prove the following statement: Let A be a self-injective algebra over a perfect field k. If the dimension of the stable category $\underline{\mathrm{mod}}A$ is 0, then A is of finite representation type. This theorem is proved by M. Yoshiwaki in the case when k is an algebraically closed field. Our proof depends on his result.

1. Introduction

In [3] R. Rouquier introduced the dimension of triangulated categories and showed that it gives an upper bound or a lower bound of other dimensions in algebraic geometry or in representation theory(see also [4]). The dimension of triangulated categories is studied by many researchers.

In this note we study the behavior of the dimension of the perfect derived category $\mathrm{Perf}(A)$ of a dg-algebra A over a field k under a base field extension K/k. For a field extension K/k, we denote $A \otimes_k K$ by A_K.

Theorem 1. (1) *For an algebraic extension K/k, we have*

$$\mathrm{tridim}\ \mathrm{Perf}(A) \leq \mathrm{tridim}\ \mathrm{Perf}(A_K).$$

(2) *If moreover K/k is separable, then equality holds.*

As an application we prove the following theorem, which gives evidence that dimension of triangulated categories captures some representation theoretic properties.

The stable category $\underline{\mathrm{mod}}A$ plays an important role in the study of self-injective algebra A (cf. [2, 4]). If a self-injective algebra A is of finite representation type then the dimension of the stable category $\underline{\mathrm{mod}}A$ is zero. Then a natural question arises as to whether the converse should also hold.

Theorem 2. *Let A be a self-injective finite dimensional algebra over a perfect field k. If* $\mathrm{tridim}\ \underline{\mathrm{mod}}A = 0$*, then A is of finite representation type.*

In the case when k is an algebraically closed field, this theorem is proved by M. Yoshiwaki in [5]. Our proof depends on his result.

The final version of this paper has been submitted for publication elsewhere.

2. Dimension of triangulated categories.

We review the definition of dimension of triangulated categories due to R. Rouquier We need to prepare a bit of notations.

Let $\mathcal{T}$ be a triangulated category. For a full subcategory $\mathcal{I}$ of $\mathcal{T}$ we denote by $\langle \mathcal{I} \rangle$ the smallest full subcategory of $\mathcal{T}$ containing $\mathcal{I}$ which is closed under taking shifts, finite direct sums, direct summands and isomorphisms. For full subcategories $\mathcal{I}$ and $\mathcal{J}$ of $\mathcal{T}$ we denote by $\mathcal{I} * \mathcal{J}$ the full subcategory of $\mathcal{T}$ consisting of those object $M \in \mathcal{T}$ such that there exists an exact triangle $I \to M \to J \xrightarrow{[1]}$ with $I \in \mathcal{I}$ and $J \in \mathcal{J}$. Set $\mathcal{I} \diamond \mathcal{J} := \langle \mathcal{I} * \mathcal{J} \rangle$. For $n \geq 1$ we define inductively

$$\langle \mathcal{I} \rangle_n := \begin{cases} \langle \mathcal{I} \rangle & \text{for } n = 1; \\ \langle \mathcal{I} \rangle \diamond \langle \mathcal{I} \rangle_{n-1} & \text{for } n \geq 2. \end{cases}$$

Now we define the dimension of a triangulated category $\mathcal{T}$ to be

$$\operatorname{tridim} \mathcal{T} := \min\{n \mid \langle E \rangle_{n+1} = \mathcal{T} \text{ for some } E \in \mathcal{T}\}.$$

3. Sketch of proof of Theorem 1 and 2

First we consider the case when K/k is a finite extension. Let $\overline{E}$ be an object of $\operatorname{Perf}(A_K)$ such that $\langle \overline{E} \rangle_n = \operatorname{Perf}(A_K)$ for some $n \in \mathbb{N}$. Then we see that $\langle U\overline{E} \rangle_n = \operatorname{Perf}(A)$ where $U : \operatorname{Perf}(A_K) \to \operatorname{Perf}(A)$ is the forgetful functor.

In the case K/k is an infinite algebraic extension, the key of the proof is the following lemma.

Lemma 3. *Let K/k be an algebraic extension and E an object of $\mathcal{D}(A)$.*

If an object $\overline{G}$ of $\mathcal{D}(A_K)$ belongs to $\langle E \otimes_k K \rangle_n$, then there exists an intermediate field $k \subset K_0 \subset K$ which is finite dimensional over k such that there exists an object G' of $\langle E \otimes_k K_0 \rangle_n$, such that $G' \otimes_{K_0} K \cong \overline{G}$ in $\mathcal{D}(A_K)$.

Let $\overline{E}$ be an object of $\operatorname{Perf}(A_K)$ such that $\langle \overline{E} \rangle_n = \operatorname{Perf}(A_K)$ for some $n \in \mathbb{N}$. Since $\operatorname{Perf}(A_K) = \cup_{i \in \mathbb{N}} \langle A_K \rangle_i$, by the above lemma there exists an intermediate field $k \subset K_0 \subset K$ which is finite dimensional over k such that there exists an object E' of $\operatorname{Perf}(A_{K_0})$ such that $E' \otimes_{K_0} K \simeq \overline{E}$. Then we see that $\langle U_0(E') \rangle_n = \operatorname{Perf}(A)$ where $U_0 : \operatorname{Perf}(A_{K_0}) \to \operatorname{Perf}(A)$ is the forgetful functor.

To prove the second statement, we use the fact that when K/k is a finite separable field extension, the canonical morphism $K \otimes_k K \to K$ splits as $K - K$ bimodules. In the case when K/k is an infinite separable field extension, we reduce to the finite separable extension case by the above lemma.

Theorem 2 is reduced to the case when the base field k is an algebraically closed field by Theorem 1 and the following lemma.

Lemma 4. *Let A be a finite dimensional k-algebra. If $A_{\overline{k}}$ is of finite representation type, then A is of finite representation type.*

4. Examples which show that we need to impose conditions on Theorem 1

To conclude this note we give examples which show that we need to impose conditions on Theorem 1.

Example 5. If an algebraic extension K/k is not separable, then the dimension tridim $\mathrm{Perf}(A_K)$ is possibly larger than the dimension tridim $\mathrm{Perf}(A)$.

Here is an example. Let F be a field of characteristic $p > 0$. Let $K := F(t)$ be a rational function field in one variable and define $k := F(t^p) \subset K = F(t)$. Set $A := K$. Then it is easy to see $A_K \cong K[x]/(x^p)$. Since $\operatorname{gldim} A_K = \infty$, we see that tridim $\mathrm{Perf}(A_K) = \infty$ by [3, Proposition 7.26]. However since $A = K$ is a field, we have tridim $\mathrm{Perf}(A) = 0$.

Example 6. In the case when the extension K/k is not algebraic, the dimension tridim $\mathrm{Perf}(A_K)$ is possibly larger than tridim $\mathrm{Perf}(A)$ even if an extension K/k is separable.

Here is an example. Assume that for simplicity k is algebraically closed. Let $K = k(y)$ and $A = k(x)$ be rational function fields in one variable over k. Then we can easily see that tridim $\mathrm{Perf}(A_K) = 1$ by the method of the proof of [3, Theorem 7.17]. However since $A = k(x)$ is a field, we see that tridim $\mathrm{Perf}(A) = 0$.

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APR TILTING MODULES AND QUIVER MUTATIONS

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Abstract. We study the quiver with relations of the endomorphism algebra of an APR tilting module. We give an explicit description of the quiver with relations by graded quivers with potential (QPs) and mutations. Consequently, mutations of QPs provide a rich source of derived equivalence classes of algebras.

1. Introduction

Derived categories have been one of the important tools in the study of many areas of mathematics. In the representation theory of algebras, tilting modules play an essential role to give an equivalence of derived categories. More precisely, the endomorphism algebra of a tilting module is derived equivalent to the original algebra. Therefore the relationship of quivers with relations of these algebras has been investigated for a long time.

The first well-known result of these studies appears in the work of [5]. It is the origin of tilting theory and formulated in terms of an APR tilting module now [4]. Let us recall an important property of APR tilting modules.

Theorem 1. [4] *Let KQ be a path algebra of a finite acyclic quiver Q and T_k be the APR tilting KQ-module associated with a source $k \in Q$. Then we have an algebra isomorphism*

$$\mathrm{End}_{KQ}(T_k) \cong K(\mu_k Q),$$

where μ_k is a mutation at k.

Thus the quiver of the endomorphism algebra is completely determined by combinatorial methods and the *mutation* can be considered as a generalization of BGP reflection. The notion of mutation was introduced by Fomin-Zelevinsky [11], which is an important ingredient of cluster algebras, and many links with other subjects have been discovered and widely investigated. In particular, Derksen-Weyman-Zelevinsky applied mutations to quivers with potential (QPs). It has been found that mutations of QPs have close connections with tilting theory, for example [9, 17].

The main purposes of this paper is to generalize the above result for a more general class of algebras by using mutations of QPs. Since we have gl.dim$KQ \leq 1$, it is natural to consider algebras Λ with gl.dim$\Lambda \leq 2$. In this case, we can describe the quiver and relations by the following steps.

1. Define the associated graded QP $(Q_\Lambda, W_\Lambda, C_\Lambda)$.
2. Apply left mutation μ_k^L to $(Q_\Lambda, W_\Lambda, C_\Lambda)$.
3. Take the truncated Jacobian algebras $\mathcal{P}(\mu_k^L(Q_\Lambda, W_\Lambda, C_\Lambda))$.

The detailed version of this paper will be submitted for publication elsewhere.

Then we have the following result.

Theorem 2. (*Theorem 7*) *Let* Λ *be a finite dimensional algebra with* $\text{gl.dim}\Lambda \leq 2$ *and* T_k *be the APR tilting* Λ*-module associated with a source* k*. Then we have an algebra isomorphism*

$$\mathrm{End}_\Lambda(T_k) \cong \mathcal{P}(\mu_k^L(Q_\Lambda, W_\Lambda, C_\Lambda)).$$

We give three remarks about the theorem. First, we can show that $\mathcal{P}(\mu_k^L(Q_\Lambda, W_\Lambda, C_\Lambda))$ coincides with $K(\mu_k Q)$ if $\text{gl.dim}\Lambda = 1$, so that Theorem 2 gives a generalization of Theorem 1. Second, the condition $\text{gl.dim}\Lambda \leq 2$ is actually not necessary, and it is enough to assume that the associated projective module has the injective dimension at most 2. Finally, this isomorphic provides a bridge of the two notions which have entirely different origins, and it implies that the contemporary concepts have a profound connection with the classical ones.

Conventions and notations. We always suppose that K is an algebraically closed field for simplicity. All modules are left modules and the composition fg of morphisms means first f, then g. We denote the set of vertices by Q_0 and the set of arrows by Q_1 of a quiver Q. We denote by $a : s(a) \to e(a)$ the start and end vertices of an arrow or path a.

2. Preliminaries

In this section, we give a brief summary of the definitions and results we will use in the next sections. See references for more detailed arguments and precise definitions.

2.1. **Quivers with potentials.** We review the notions initiated in [10].

• Let Q be a finite connected quiver. We denote by KQ_i the K-vector space with basis consisting of paths of length i in Q, and by $KQ_{i,cyc}$ the subspace of KQ_i spanned by all cycles. We denote *complete path algebra* by

$$\widehat{KQ} = \prod_{i \geq 0} KQ_i.$$

A *quiver with potential* (QP) is a pair (Q, W) consisting of a quiver Q and an element $W \in \prod_{i\geq 2} KQ_{i,\mathrm{cyc}}$, called a *potential*. For each arrow a in Q, the *cyclic derivative* $\partial_a : \widehat{KQ}_{cyc} \to \widehat{KQ}$ is defined by the continuous linear map which sends $\partial_a(a_1 \cdots a_d) = \sum_{a_i = a} a_{i+1} \cdots a_d a_1 \cdots a_{i-1}$. For a QP (Q, W), we define the *Jacobian algebra* by

$$\mathcal{P}(Q, W) = \widehat{KQ}/\mathcal{J}(W),$$

where $\mathcal{J}(W) = \overline{\langle \partial_a W \mid a \in Q_1 \rangle}$ is the closure of the ideal generated by $\partial_a W$ with respect to the $J_{\widehat{KQ}}$-adic topology.

• A QP (Q, W) is called *reduced* if $W \in \prod_{i\geq 3} KQ_{i,\mathrm{cyc}}$.

• For two QPs (Q', W') and (Q'', W''), we define a new QP (Q, W) as a direct sum $(Q', W') \oplus (Q'', W'')$, where $Q_0 = Q'_0 (= Q''_0)$, $Q_1 = Q'_1 \coprod Q''_1$ and $W = W' + W''$.

Definition 3. For each vertex k in Q not lying on a loop nor 2-cycle, we define a *mutation* $\mu_k(Q, W)$ as a *reduced part* of $\widetilde{\mu}_k(Q, W) = (Q', W')$, where (Q', W') is given as follows.

(1) Q' is a quiver obtained from Q by the following changes.
• Replace each arrow $a : k \to v$ in Q by a new arrow $a^* : v \to k$.
• Replace each arrow $b : u \to k$ in Q by a new arrow $b^* : k \to u$.
• For each pair of arrows $u \xrightarrow{b} k \xrightarrow{a} v$, add a new arrow $[ba] : u \to v$

(2) $W' = [W] + \Delta$ is defined as follows.
• $[W]$ is obtained from the potential W by replacing all compositions ba by the new arrows $[ba]$ for each pair of arrows $u \xrightarrow{b} k \xrightarrow{a} v$.
• $\Delta = \sum_{\substack{a,b \in Q_1 \\ e(b)=k=s(a)}} [ba]a^*b^*$.

2.2. **Truncated Jacobian algebras.** We introduce the notion of cuts and the truncated Jacobian algebras.

Definition 4. [14] Let (Q, W) be a QP. A subset $C \subset Q_1$ is called a *cut* if each cycle appearing W contains exactly one arrow of C. Then we define the *truncated Jacobian algebra* by

$$\mathcal{P}(Q, W, C) := \mathcal{P}(Q, W)/\overline{\langle C \rangle} = \widehat{KQ_C}/\overline{\langle \partial_c W \mid c \in C \rangle},$$

where Q_C is the subquiver of Q with vertex set Q_0 and arrow set $Q_1 \setminus C$.

Then, we can naturally define a QP with a cut from a given algebra as follows.

Definition 5. [16] Let Q be a finite connected quiver and $\Lambda = \widehat{KQ}/\overline{\langle R \rangle}$ be a finite dimensional algebra with a minimal set of relations.

Then we define a QP (Q_Λ, W_Λ) as follows:

(1) $(Q_\Lambda)_0 = Q_0$
(2) $(Q_\Lambda)_1 = Q_1 \coprod C_\Lambda$, where $C_\Lambda := \{\rho_r : e(r) \to s(r) \mid r \in R\}$.
(3) $W_\Lambda = \sum_{r \in R} \rho_r r$.

Then the set C_Λ gives a cut of (Q_Λ, W_Λ).

2.3. **APR tilting modules.** We call a Λ-module T *tilting module* if $\mathrm{proj.dim}_\Lambda T \leq 1$, $\mathrm{Ext}^1_\Lambda(T, T) = 0$, and there exists a short exact sequence $0 \to \Lambda \to T_0 \to T_1 \to 0$ with T_0, T_1 in $\mathrm{add}T$.

Definition 6. Let Λ be a basic finite dimensional algebra and P_k be a simple projective non-injective Λ-module associated with a source k of the quiver Λ. Then Λ-module $T := \tau^- P_k \oplus \Lambda/P_k$ is called an *APR tilting module*, where τ^- denotes the inverse of the Auslander-Reiten translation.

3. Main theorem

3.1. **Main result.** Let Q be a finite connected quiver and $\Lambda = \widehat{KQ}/\overline{\langle R \rangle}$ be a finite dimensional algebra with a minimal set of relations. Assume that P_k is the simple projective non-injective Λ-module associated with a source $k \in Q$. Our aim is to determine the quiver and the set of relations giving $\mathrm{End}_\Lambda(T_k)$.

Consider the associated QP $(Q_\Lambda, W_\Lambda, C_\Lambda)$ of Λ and we put $\widetilde{\mu}_k(Q_\Lambda, W_\Lambda) = (Q', W')$. Then W' is given by

$$W' = [\sum_{r\in R}\rho_r r] + \sum_{\substack{a\in Q_1, r\in R\\ s(a)=k=s(r)}} [\rho_r a]a^*\rho_r^*,$$

and it is easy to check that subset

$$C' = \{\ \rho_r \mid r \in R, s(r) \neq k\} \coprod \{\ [\rho_r a] \mid a \in Q_1, r \in R, s(a) = k = s(r)\}$$

of Q' is a cut of (Q', W').

Then we have the following.

Theorem 7. *Let $\Lambda = \widehat{KQ}/\overline{\langle R\rangle}$ be a finite dimensional algebra with a minimal set of relations. Let $T_k := \tau^- P_k \oplus \Lambda/P_k$ be the APR tilting module. Then if* inj.dim$P_k \leq 2$*, we have an algebra isomorphism*

$$\mathrm{End}_\Lambda(T_k) \cong \mathcal{P}(\widetilde{\mu}_k(Q_\Lambda, W_\Lambda), C').$$

Notice that the assumption inj.dim$P_k \leq 2$ is automatic if gl.dim$\Lambda = 2$. Thus our theorem give a generalization from gl.dim$\Lambda = 1$ to gl.dim$\Lambda = 2$.

Here we will explain the choice of C'. In fact C' is naturally obtained by using graded mutations. For this purpose, we recall graded QPs, as introduced by [1].

Graded quivers with potentials. Let (Q, W) be a QP and we define a map $d : Q_1 \to \mathbb{Z}$. We call a QP (Q, W, d) *$\mathbb{Z}$-graded QP* if each arrow $a \in Q_1$ has a degree $d(a) \in \mathbb{Z}$, and *homogeneous of degree l* if each term in W is a degree l.

Definition 8. Let QP (Q, W, d) be a $\mathbb{Z}$-graded QP of degree l. For each vertex k in Q not lying on a loop nor 2-cycle, we define a *left mutation* $\mu_k^L(Q, W, d)$ as a reduced part of $\widetilde{\mu}_k^L(Q, W, d) = (Q', W', d')$, where (Q', W', d') is given as follows.

(1) $(Q', W') = \widetilde{\mu}_k(Q, W)$
(2) The new degree d' is defined as follows:
- $d'(a) = d(a)$ for each arrow $a \in Q \cap Q'$.
- $d'(a^*) = -d(a)$ for each arrow $a : k \to v$ in Q.
- $d'(b^*) = -d(b) + l$ for each arrow $b : u \to k$ in Q.
- $d'([ba]) = d(a) + d(b)$ for each pair of arrows $u \xrightarrow{b} k \xrightarrow{a} v$ in Q.

In particular, $\widetilde{\mu}_k^L(Q, W, d)$ also has a potential of degree l. Similarly, we can define $\widetilde{\mu}_k^R$ at k. In this case, we define $d'(b^*) = -d(b)$ for each arrow $b : u \to k$ in Q and $d'(a^*) = -d(a) + l$ for each arrow $a : k \to v$ in Q.

If (Q, W) has a cut C, we can identify the QP with a $\mathbb{Z}$-graded QP of degree 1 associating a grading on Q by

$$d_C(a) = \begin{cases} 1 & a \in C \\ 0 & a \notin C. \end{cases}$$

We denote by (Q, W, C) the graded QP of degree 1 with this grading. If any arrow of $\widetilde{\mu}_k^L(Q, W, C)$ has degree 0 or 1, degree 1 arrows give a cut of $\widetilde{\mu}_k(Q, W)$ since $\widetilde{\mu}_k^L(Q, W, C)$ is homogeneous of degree 1. Therefore a cut of $\widetilde{\mu}_k(Q_\Lambda, W_\Lambda)$ is naturally induced as degree

1 arrows of $\widetilde{\mu}_k^L(Q_\Lambda, W_\Lambda, C_\Lambda)$ and the above C' is obtained in this way. Thus we identify degree 1 arrows as a cut.

Because we have $\mathcal{P}(\widetilde{\mu}_k^L(Q_\Lambda, W_\Lambda, C_\Lambda)) \cong \mathcal{P}(\mu_k^L(Q_\Lambda, W_\Lambda, C_\Lambda))$, we can rewrite Theorem 7 that we have an algebra isomorphism

$$\mathrm{End}_\Lambda(T_k) \cong \mathcal{P}(\mu_k^L(Q_\Lambda, W_\Lambda, C_\Lambda)).$$

3.2. **Examples.** We explain the theorem with some examples.

Example 9. We keep the assumption of Theorem 7. If gl.dim$\Lambda = 1$, then we have $\Lambda = KQ$ and

$$\mathcal{P}(\mu_k^L(Q_\Lambda, W_\Lambda, C_\Lambda)) = \mathcal{P}(\mu_k^L(Q, 0, 0)) = K(\mu_k Q),$$

so that the mutation procedure is just reversing arrows having k. Thus the above theorem coincides with the classical result (Theorem 1).

Example 10. Let $\Lambda = \widehat{KQ}/\overline{\langle R\rangle}$ be a finite dimensional algebra given by the following quiver with a relation.

$$\begin{tikzcd} & 2 \arrow[dr, "b"] & \\ 1 \arrow[ur, "a"] \arrow[dr, "c"'] \arrow[rr, dotted, no head] & & 4 \\ & 3 \arrow[ur, "d"'] & \end{tikzcd} \qquad \langle R\rangle = \langle ab\rangle.$$

Then we consider the APR tilting module $T_1 := \tau^- P_1 \oplus \Lambda/P_1$ and calculate Q' and R' satisfying $\widehat{KQ'}/\overline{\langle R'\rangle} \cong \mathrm{End}_\Lambda(T_1)$ by the following steps.

$$\begin{tikzcd} & 2 \arrow[dr, "b"] & \\ 1 \arrow[ur, "a"] \arrow[dr, "c"'] \arrow[rr, dotted, no head] & & 4 \\ & 3 \arrow[ur, "d"'] & \end{tikzcd} \quad \overset{(Q_\Lambda, W_\Lambda, C_\Lambda)}{\Longrightarrow} \quad \begin{tikzcd} & 2 \arrow[dr, "b"] & \\ 1 \arrow[ur, "a"] \arrow[dr, "c"'] & & 4 \arrow[ll, "\rho"'] \\ & 3 \arrow[ur, "d"'] & \end{tikzcd} \quad \overset{\widetilde{\mu}_1^L}{\Longrightarrow} \quad \begin{tikzcd} & 2 \arrow[dl, "a^*"'] \arrow[dr, "b"] & \\ 1 \arrow[rr, "\rho^*"] & & 4 \arrow[ul, bend right, "{[\rho a]}"'] \arrow[dl, bend left, "{[\rho c]}"] \\ & 3 \arrow[ul, "c^*"] \arrow[ur, "d"'] & \end{tikzcd}$$

$$\langle R\rangle = \langle ab\rangle. \qquad W_\Lambda = \rho ab. \qquad W' = [\rho a]b + [\rho a]a^*\rho^* + [\rho c]c^*\rho^*.$$

$$\overset{\mu_1^L}{\Longrightarrow} \quad \begin{tikzcd} & 2 \arrow[dl, "a^*"'] & \\ 1 \arrow[rr, "\rho^*"] & & 4 \arrow[dl, bend left, "{[\rho c]}"] \\ & 3 \arrow[ul, "c^*"] \arrow[ur, "d"'] & \end{tikzcd} \quad \overset{\mathcal{P}(\mu_1^L(Q_\Lambda, W_\Lambda, C_\Lambda))}{\Longrightarrow} \quad Q' = \begin{tikzcd} & 2 \arrow[dl, "a^*"'] & \\ 1 \arrow[rr, "\rho^*"] & & 4 \arrow[dl, dotted, no head] \\ & 3 \arrow[ul, "c^*"] \arrow[ur, "d"'] & \end{tikzcd}$$

$$W' = [\rho c]c^*\rho^* \qquad \langle R'\rangle = \langle c^*\rho^*\rangle.$$

Similarly from the left-hand side algebra Λ, we obtain the quiver and the set of relations giving $\mathrm{End}_\Lambda(T_1)$, which is given by right-hand side picture.

(1)

$$\begin{tikzcd} 1 \arrow[rrr, dash, dotted] \arrow[d, shift right, "a_2"'] \arrow[d, shift left, "a_1"] & & & 4 \\ 2 \arrow[rrr, "b"] & & & 3 \arrow[u, "c"'] \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} 1 \arrow[rrr, shift left, "\rho_2^*"] \arrow[rrr, shift right, "\rho_1^*"'] & & & 4 \\ 2 \arrow[u, shift left, "a_2^*"] \arrow[u, shift right, "a_1^*"'] \arrow[rrru, dotted, dash] \arrow[rrru, dotted, dash, bend left] \arrow[rrr, "b"] & & & 3 \arrow[u, "c"'] \end{tikzcd}$$

$\langle R\rangle = \langle a_1bc, a_2bc\rangle$ $\qquad$ $\langle R'\rangle = \langle a_1^*\rho_1^*+bc, a_2^*\rho_2^*+bc, a_1^*\rho_2^*, a_2^*\rho_1^*\rangle.$

(2) (i)

$$\begin{tikzcd} 1 \arrow[r, "a_1"] \arrow[d, "b_1"'] \arrow[rrd, dotted, dash] & 2 \arrow[r, "a_1"] & 3 \arrow[d, "a_3"] \\ 4 \arrow[r, "b_2"'] & 5 \arrow[r, "b_3"'] & 6 \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} 1 \arrow[rrd] & 2 \arrow[l, "{a_1}^*"'] \arrow[r, "a_2"] \arrow[rd, dotted, dash] & 3 \arrow[d, "a_3"] \\ 4 \arrow[u, "{b_1}^*"] \arrow[r, "b_2"'] \arrow[rr, dotted, dash, bend left, "\rho^*"] & 5 \arrow[r, "b_3"'] & 6 \end{tikzcd}$$

$\langle R\rangle = \langle a_1a_2a_3\rangle$ $\qquad$ $\langle R'\rangle = \langle a_2a_3 + {a_1}^*\rho^*, {b_1}^*\rho^*\rangle.$

(ii)

$$\begin{tikzcd} 1 \arrow[r, "a_1"] \arrow[d, "b_1"'] \arrow[rrd, dotted, dash] & 2 \arrow[r, "a_1"] & 3 \arrow[d, "a_3"] \\ 4 \arrow[r, "b_2"'] & 5 \arrow[r, "b_3"'] & 6 \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} 1 \arrow[rrd] & 2 \arrow[l, "{a_1}^*"'] \arrow[r, "a_2"] \arrow[rd, dotted, dash] & 3 \arrow[d, "a_3"] \\ 4 \arrow[u, "{b_1}^*"] \arrow[r, "b_2"'] \arrow[rr, dotted, dash, bend left, "\rho^*"] & 5 \arrow[r, "b_3"'] & 6 \end{tikzcd}$$

$\langle R\rangle = \langle a_1a_2a_3 = b_1b_2b_3\rangle$ $\qquad$ $\langle R'\rangle = \langle a_2a_3 + {a_1}^*\rho^*, b_2b_3 + {b_1}^*\rho^*\rangle.$

As examples show, we interpret the degree 1 arrows as relations.

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THE EXAMPLE BY STEPHENS

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Abstract. Concerning the Feit-Thompson Conjecture, Stephens found the serious example. Using Artin map (see [9]), we shall show that numbers 17 and 3313 in the example by Stephen are common index divisors of some subfields of a cyclotomic field $\mathbb{Q}(\zeta_r)$ where $r = 112643$ and $\zeta_r = e^{\frac{2\pi i}{r}}$, and some results in [7, 8] shall be again proved.

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Let $p < q$ be primes and we set

$$f := \frac{q^p - 1}{q - 1} \text{ and } t := \frac{p^q - 1}{p - 1}.$$

Feit and Thompson [3] conjectured that f never divides t. If it would be proved, the proof of their odd order theorem [4] would be greatly simplified (see [1] and [5]).

Throughout this note, we assume that r *is a common prime divisor of* f *and* t. Using computer, Stephens [10] found the example about r as follows: for $p = 17$ and $q = 3313$, $r = 112643 = 2pq + 1$ is the greatest common divisor of f and t. This example is so far the only one.

In this note, using the Artin map, we shall show that both 17 and 3313 are common index divisors (gemeinsamer ausserwesentlicher Discriminantenteiler) of some subfields of a cyclotomic field $\mathbb{Q}(\zeta_r)$ where $r = 112643$ and $\zeta_r = e^{\frac{2\pi i}{r}}$, and some results in [7, 8] shall be again proved from our Theorem.

The assumption on r yields from [7, Lemma, (1) and (3)] that p and q are orders of $q \bmod r$ and $p \bmod r$, respectively. Thus $r \equiv 1 \bmod 2pq$ since r is odd.

We set $q^*q := r - 1$ and $\zeta = e^{\frac{2\pi i}{r}}$. Let n be a divisor of q^*, let L_n be a subfield of $K = \mathbb{Q}(\zeta)$ with $[L_n : \mathbb{Q}] = n$ and let $\mathbb{O}_n$ be the algebraic integer ring of L_n. Using the exact sequence by the Artin map (see [9, p.99 and section 2.16]) and Kummer's theorem.

We have $d(\mu) = I(\mu)^2 d(L_n)$ for $\mu \in \mathbb{O}_n$ where $I(\mu) \in \mathbb{Z}$, $d(\mu)$ and $d(L_n)$ are discriminants of μ and of the field L_n, respectively.

The example by Stephens shows from the next Theorem that $p = 17$ and $q = 3313$ are common index divisors of L_{34} and of L_{6626}, respectively, since we can exchange p for q.

The detailed version of this paper will be submitted for publication elsewhere.

Theorem. *Assume r is a common prime divisor of f and t, and n is a divisor of q^*, where $q^*q = r-1$. Then p splits completely in $\mathbb{O}_n$ and if there exists $\mu \in \mathbb{O}_n$ such that p does not divide $I(\mu)$, then $n \leqq p$. In particular, for $n > p$, p is a common index divisor of $\mathbb{O}_n$ namely, p divides $I(\gamma)$ for all $\gamma \in \mathbb{O}_n$.*

Let c be a primitive root for r, let χ be a character of order n defined by $\chi(c) = \omega$ where $\omega = e^{\frac{2\pi i}{n}}$ and let $g(\chi) = \sum_{a \in \mathbb{F}_r} \chi(a)\zeta^a$ be the Gauss sum of χ where $\mathbb{F}_r$ is a finite field of order r. Let $\sigma(\zeta) = \zeta^c$ be a generator of the Galois group G of K over $\mathbb{Q}$ and set $T_n := \langle \sigma^n \rangle$.

For simplicity, we set $g_0 = -1,\ g_k = g(\chi^k)$ for $n > k > 0$ and $\theta_k = \theta^{\sigma^k}$ for $n > k \geqq 0$ where $\theta = \sum_{\tau \in T_n} \zeta^\tau$ is a trace of ζ.

It is known that $L_n = \mathbb{Q}(\theta)$ and θ is a normal basis element of $\mathbb{O}_n$ over $\mathbb{Z}$ (see [9, p.61, p.74])

The next Lemma is useful to our object. It only needs to assume r is prime and n is a divisor of $r-1$ in this Lemma. This proof is essentially in the first equation of (1) due to [9, p.62]. This idea of classifying primitive roots goes back to Gauss; the regular 17 polygon construction by ruler and compass.

Lemma.

(1) $g_k = \sum_{s=0}^{n-1} \omega^{ks}\theta_s$ *for* $0 \leqq k < n$ *and* $n\theta_k = \sum_{s=0}^{n-1} \bar{\omega}^{ks} g_s$ *for* $0 \leqq k < n$ *where* $\bar{\omega}$ *is the complex conjugate of* ω.

(2) *Using* (1), *determinants of cyclic matrices* A_n, B_n *are given by*

$$|A_n| := \begin{vmatrix} \theta_0 & \theta_1 & \dots & \theta_{n-1} \\ \theta_{n-1} & \theta_0 & \dots & \theta_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_1 & \theta_2 & \dots & \theta_0 \end{vmatrix} = \prod_{k=0}^{n-1} g_k \quad \text{and} \quad |B_n| := \begin{vmatrix} g_0 & g_1 & \cdots & g_{n-1} \\ g_{n-1} & g_0 & \cdots & g_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ g_1 & g_2 & \cdots & g_0 \end{vmatrix} = n^n \prod_{k=0}^{n-1} \theta_k.$$

(3) *We have*

$$d(L_n) = \begin{cases} r^{n-1} & \text{if } n \text{ is odd,} \\ (-1)^{\frac{r-1}{2}} r^{n-1} & \text{if } n \text{ is even.} \end{cases}$$

Some results in [7, 8] are proved again in the next

Corollary. *Let r be a common prime divisor of f and t. Then we have*

(1) $p \equiv 1$ *or* $r \equiv 1 \bmod 4$ (see [7, Lemma, (4)]).

(2) $q \equiv -1 \bmod 9$ *in case* $p = 3$ *and* f *divides* t (see [8, Corollary, (a)]).

Proof of (2). We consider the case $n = p = 3$. If f is composite, then f does not divide t. Thus we may assume f is prime and so $r = f$ (see [7]). f has a primary prime decomposition $f = \eta\bar{\eta}$ in $\mathbb{Z}[\omega]$ where $\omega = e^{\frac{2\pi i}{3}}$ and $\eta = \omega(\omega - q)$, (see [6, 8]). In this case, we set χ is the cubic residue character modulo η. Let $h(x)$ be the minimal polynomial of θ over $\mathbb{Q}$.

$$h(x) := x^3 + a_1x^2 + a_2x + a_3 = (x - \theta_0)(x - \theta_1)(x - \theta_2).$$

where $a_1 = -\theta_0 - \theta_1 - \theta_2 = 1$. If 3 does not divide $I(\theta)$, then $h(x) \equiv x^3 - x \bmod 3$ by Kummer's theorem and our Theorem. This contradicts to $a_1 = 1$. Thus $d(\theta) \equiv 0 \bmod 3$. Using $g_1g_2 = g_1\bar{g_1} = |g_1|^2 = r$, we have

$$f = r = -|A_3| = -(\theta_0 + \theta_1 + \theta_2)\begin{vmatrix} 1 & \theta_1 & \theta_2 \\ 1 & \theta_0 & \theta_1 \\ 1 & \theta_2 & \theta_0 \end{vmatrix} = \theta_0^2 + \theta_1^2 + \theta_2^2 - a_2 = 1 - 3a_2.$$

Thus we obtain $3a_2 = 1 - f = -q(q+1)$. On the other hand, using $g_2 = \bar{g_1}$, $f = \eta\bar{\eta}$ and the Stickelberger relation $g_1^3 = r\eta = f\eta$ (see [6]), we have

$$\begin{aligned} -27a_3 &= 27\theta_0\theta_1\theta_2 = |B_3| = \begin{vmatrix} g_0 & g_1 & g_2 \\ g_2 & g_0 & g_1 \\ g_1 & g_2 & g_0 \end{vmatrix} = g_0^3 + g_1^3 + g_2^3 - 3g_0g_1g_2 \\ &= -1 + f(\eta + \bar{\eta}) + 3f = -1 + f(q-1) + 3f = (q+1)^3. \end{aligned}$$

Thus we have $3^3q^3a_3 = (-q(q+1))^3 = 3^3a_2^3$ and so $a_2 + a_3 \equiv a_2^3 - q^3a_3 = 0 \bmod 3$. Noting $h'(\theta) \equiv a_2 - \theta \bmod 3$ where $h'(x)$ is the derivation of $h(x)$, we obtain

$$0 \equiv -d(\theta) = N_{L_3/\mathbb{Q}}(h'(\theta)) \equiv h(a_2) \equiv a_2 - a_2^2 + a_3 \equiv -a_2^2 \bmod 3.$$

Thus we have $0 \equiv 3a_2 = -q(q+1) \bmod 9$. □

Remark. Using only the quadratic reciprocity law, we can prove

$$q \equiv -1 \bmod 8 \text{ in case } p = 3 \text{ and } f \text{ divides } t.$$

It simplifies the proof of Proposition 3.2 by Lemma 3.3 on p.172 in the paper K. Dilcher and J. Knauer, *On a conjecture of Feit and Thompson,* pp.169-178 in the book, *High primes and misdemeanours,* edited by A. van der Poorten, A. Stein, Fields Institute Communications **41**, Amer. Math. Soc., 2004.

We can understand their proof through the next some results in this order :

- Ex. 11 on p.231, and p.103 in the book, B. C. Berndt, R.J. Evans, K. S. Williams, *Gauss and Jacobi Sums,* Wiley, New York, 1998.
- Proof of Theorem 2 on p.139 in the paper, R. Hudson and K. S. Williams, *Some new residuacity criteria,* Pacific J. of Math. **91**(1980), 135-143.
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HOM-ORTHOGONAL PARTIAL TILTING MODULES FOR DYNKIN QUIVERS

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ABSTRACT. We count the number of the isomorphic classes of basic hom-orthogonal partial tilting modules for an arbitrary Dynkin quiver. This number is independent on the choice of an orientation of arrows, and the number for $\mathbb{A}_n$ or $\mathbb{D}_n$-type can be expressed as a special value of a hypergeometric function. As a consequence of our theorem, we obtain a minimum value of the number of basic relative invariants of corresponding regular prehomogeneous vector spaces.

INTRODUCTION

Let $Q = (Q_0, Q_1)$ be a Dynkin quiver having n vertices (i.e., its base graph is one of Dynkin diagrams of type $\mathbb{A}_n$ with $n \geq 1$, $\mathbb{D}_n$ with $n \geq 4$, or $\mathbb{E}_n$ with $n = 6, 7, 8$), where Q_0, Q_1 is the set of vertices, arrows of Q, respectively. We denote by $\Lambda = \mathbb{K}Q$ its path algebra over an algebraically closed field $\mathbb{K}$ of characteristic zero, and by $\operatorname{mod} \Lambda$ the category of finitely generated right Λ-modules.

Let $X \cong \bigoplus_{k=1}^{s} m_k X_k$ be the decomposition of $X \in \operatorname{mod} \Lambda$ into indecomposable direct summands, where $m_k X_k$ means the direct sum of m_k copies of X_k, and the X_k's are pairwise non-isomorphic. Then X is called *basic* if $m_k = 1$ for all indices k. We call X to be *hom-orthogonal* if $\operatorname{Hom}_\Lambda(X_i, X_j) = 0$ for all $i \neq j$. This notion is equivalent to that X is locally semi-simple in the sense of Shmelkin [8] when Q is a Dynkin quiver. In the case where X is indecomposable, we will say that X itself is hom-orthogonal. Since Λ is hereditary, we say that $X \in \operatorname{mod} \Lambda$ is a *partial tilting module* if it satisfies $\operatorname{Ext}^1_\Lambda(X, X) = 0$.

Each $X \in \operatorname{mod} \Lambda$ with dimension vector $\boldsymbol{d} = \operatorname{\mathbf{dim}} X$ can be regarded as a representation of Q; that is, a point of the variety $\operatorname{Rep}(Q, \boldsymbol{d})$ that consists of representations with dimension vector $\boldsymbol{d} = (d^{(i)})_{i \in Q_0} \in \mathbb{Z}^n_{\geq 0}$. Then the direct product $GL(\boldsymbol{d}) = \prod_{i \in Q_0} GL(d^{(i)})$ acts naturally on $\operatorname{Rep}(Q, \boldsymbol{d})$; see, for example, [3, §2]. Since Λ is representation-finite, $\operatorname{Rep}(Q, \boldsymbol{d})$ has a unique dense $GL(\boldsymbol{d})$-orbit; thus $(GL(\boldsymbol{d}), \operatorname{Rep}(Q, \boldsymbol{d}))$ is a prehomogeneous vector space (abbreviated PV). It follows from the Artin–Voigt theorem [3, Theorem 4.3] that the condition that X is a partial tilting module can be interpreted to that the $GL(\boldsymbol{d})$-orbit containing X is dense in $\operatorname{Rep}(Q, \boldsymbol{d})$; On the other hand, the condition that X is hom-orthogonal corresponds to that the isotropy subgroup (or, stabilizer) at $X \in \operatorname{Rep}(Q, \boldsymbol{d})$ is reductive. Therefore we are interested in hom-orthogonal partial tilting Λ-modules, because they correspond to generic points of *regular PVs* associated with Q; see [5, Theorem 2.28].

In this paper, we count up the number of the isomorphic classes of basic hom-orthogonal partial tilting Λ-modules for an arbitrary Dynkin quiver Q. In other words, this is nothing

The detailed version of this paper has been submitted for publication elsewhere.

$e(n,s)$	$n=6$	7	8
$s=1$	36	63	120
2	108	315	945
3	72	336	1575
4	0	63	675

$e^0(n,s)$	$n=6$	7	8
$s=1$	7	16	44
2	35	120	462
3	35	170	924
4	0	40	462

TABLE 0.1. The values of $e(n,s)$ and $e^0(n,s)$

but essentially counting the number of regular PVs associated with. Our main theorem is the following:

Theorem 0.1. *Let Q be a quiver of type $\mathbb{A}_n$ with $n \geq 1$ (resp. $\mathbb{D}_n$ with $n \geq 4$, $\mathbb{E}_n$ with $n = 6, 7, 8$). Then the number $a(n,s)$ (resp. $d(n,s)$, $e(n,s)$) of the isomorphic classes of basic hom-orthogonal tilting $\mathbb{K}Q$-modules having s pairwise non-isomorphic indecomposable direct summands is given explicitely by the following:*

$$a(n,s) = \frac{(n+1)!}{s!\,(s+1)!\,(n+1-2s)!} \tag{0.1}$$

$$= C_s \cdot \binom{n+1}{2s} \tag{0.2}$$

if $1 \leq s \leq (n+1)/2$, and $a(n,s) = 0$ if otherwise. Here $C_s = \binom{2s}{s}/(s+1)$ denotes the s-th Catalan number.

$$d(n,s) = \frac{(n-1)!}{(s!)^2\,(n+2-2s)!} \cdot \left\{ s^2(s-1) + n(n+1-2s)(n+2-2s) \right\}$$

if $1 \leq s \leq (n+2)/2$, and $d(n,s) = 0$ if otherwise. The values of $e(n,s)$ for $1 \leq s \leq (n+1)/2$ are given in Table 0.1, and we have $e(n,s) = 0$ if otherwise.

Our approach to this theorem, which was inspired by Seidel's paper [7], is based on an observation of perpendicular categories introduced by Schofield [6]. Here we point out that the totality of $a(n,s)$ or $d(n,s)$ for fixed n can be expressed as a special value of a hypergeometric function. As mentioned in Remark 2.4, the formula (0.2) has a combinatorial interpretation.

According to Happel [4], if a Λ-module corresponding to a point contained in the dense orbit of a PV $(GL(\boldsymbol{d}), \operatorname{Rep}(Q, \boldsymbol{d}))$ has s pairwise non-isomorphic indecomposable direct summands, then the PV has exactly $n - s$ basic relative invariants. Thus we obtain a consequence of Theorem 0.1.

Corollary 0.2. *Each regular PV associated with a quiver of type $\mathbb{A}_n$ (resp. $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$, and $\mathbb{E}_8$) has at least $(n-1)/2$ (resp. $(n-2)/2$, 3, 3, and 4) basic relative invariants.*

We say that $X \in \operatorname{mod} \Lambda$ is *sincere* if its dimension vector $\mathbf{dim}\, X$ does not have zero entry. Sincere modules are fairly interesting to the theory of PVs, because $(GL(\boldsymbol{d}), \operatorname{Rep}(Q, \boldsymbol{d}))$ with non-sincere dimension can be regarded as a direct sum of at least two PVs associated with proper subgraphs of Q. So we have counted them:

Theorem 0.3. *Let Q be a quiver of type $\mathbb{A}_n$ with $n \geq 1$ (resp. $\mathbb{D}_n$ with $n \geq 4$, $\mathbb{E}_n$ with $n = 6, 7, 8$). Then the number $a^0(n,s)$ (resp. $d^0(n,s)$, $e^0(n,s)$) of the isomorphic classes* *of basic sincere hom-orthogonal tilting $\mathbb{K}Q$-modules having s pairwise non-isomorphic indecomposables is given explicitely by the following:*

$$a^0(n,s) = \frac{(n-1)!}{s!\,(s-1)!\,(n+1-2s)!} = C_{s-1} \cdot \tbinom{n-1}{2s-2} \tag{0.3}$$

if $1 \le s \le (n+1)/2$, *and* $a^0(n,s) = 0$ *if otherwise.*

$$\begin{aligned} d^0(n,s) = {} & \frac{(n-2)!}{s!\,(s-1)!\,(n+2-2s)!} \\ & \times \bigl\{n(n-1-2s)(n-2s) + 2n(n-2) + (s-1)(s^2-9s+4)\bigr\} \end{aligned}$$

if $1 \le s \le (n+2)/2$, *and* $d^0(n,s) = 0$ *if otherwise. The values of* $e^0(n,s)$ *for* $1 \le s \le (n+1)/2$ *are given in Table* 0.1, *and we have* $e^0(n,s) = 0$ *if otherwise.*

Now we will exceptionally define some values of $a(m,t)$ for simplicity:

$$a(m,\,-1) = 0,\ a(m,\,0) = 1,\ \text{and } a(l,\,t) = 0 \text{ for } l \le 0 \text{ and } t \ne 0.$$

Then we can express $d(n,s)$, $a^0(n,s)$, and $d^0(n,s)$ as the following simpler forms:

$$d(n,s) = (n-1)\cdot a(n-3,\,s-2) + (s+1)\cdot a(n-1,\,s), \tag{0.4}$$

$$a^0(n,s) = a(n-2,\,s-1),$$

$$d^0(n,s) = (s-1)\cdot a(n-3,\,s-2) + (n-2)\cdot a(n-3,\,s-1). \tag{0.5}$$

As will be mentioned in §1, the numbers presented in Theorems 0.1 and 0.3 are independent on the choice of an orientation of arrows of Q. Thus we may assume that its arrows are conveniently oriented.

1. Preliminaries

Let Q be a Dynkin quiver having n vertices, $\Lambda = \mathbb{K}Q$ its path algebra. For an indecomposable Λ-module M, its right perpendicular category $M^\perp$ is defined by

$$M^\perp = \{X \in \operatorname{mod}\Lambda;\ \operatorname{Hom}_\Lambda(M,X) = 0 \text{ and } \operatorname{Ext}^1_\Lambda(M,X) = 0\}.$$

The left perpendicular category ${}^\perp M$ is also defined similarly. To investigate hom-orthogonal partial tilting modules (or, regular PVs), we are interested in their intersection $\operatorname{Per} M = {}^\perp M \cap M^\perp$; we will simply call it the *perpendicular category* of M. Now we recall the Ringel form, which is defied on the Grothendieck group $K_0(\Lambda) \cong \mathbb{Z}^n$:

$$\begin{aligned} \langle \mathbf{dim}\,X,\, \mathbf{dim}\,Y \rangle &= \dim \operatorname{Hom}_\Lambda(X,Y) - \dim \operatorname{Ext}^1_\Lambda(X,Y) \\ &= {}^t(\mathbf{dim}\,X)\cdot R_Q \cdot (\mathbf{dim}\,Y) \end{aligned}$$

for $X, Y \in \operatorname{mod}\Lambda$, where $R_Q = (r_{ij})_{i,j\in Q_0}$ is the representation matrix with respect to the basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ of $K_0(\Lambda) \cong \mathbb{Z}^n$ (here we put $\boldsymbol{e}_k = \mathbf{dim}\,S(k)$, which is the dimension vector of a simple module corresponding to a vertex $k \in Q_0$). This is defined as $r_{ii} = 1$ for all $i \in Q_0$; $r_{ij} = -1$ if there exists an arrow $i \to j$ in Q; and $r_{ij} = 0$ if otherwise.

Lemma 1.1. *For indecomposable Λ-modules X and Y, we have* $\langle \mathbf{dim}\,X,\, \mathbf{dim}\,Y \rangle = 0$ *if and only if* $\operatorname{Hom}_\Lambda(X,Y) = 0$ *and* $\operatorname{Ext}^1_\Lambda(X,Y) = 0$.

Now we will show that the numbers that are presented in our theorems do not depend on the choice of an orientation of arrows of Q. To do this, we need the following lemma:

Lemma 1.2. *For any sink $a \in Q_0$ and any Λ-module M, if $\mathrm{Hom}_\Lambda(S(a), M) = 0$ and $\mathrm{Ext}^1_\Lambda(M, S(a)) = 0$, then we have $\mathrm{Hom}_\Lambda(P(t\alpha), M) = 0$ for any arrow $\alpha : t\alpha \to a$ in Q.*

Let $\sigma = \sigma_a$ be the reflection functor (with the APR-tilting module T, see [2, VII Theorem 5.3]) at a sink $a \in Q_0$, and Q' the quiver obtained by reversing all arrows connecting with a in Q. For a basic hom-orthogonal partial tilting Λ-module $X \cong \bigoplus_{k=1}^s X_k$, we define a Λ'-module as follows (here we put $\Lambda' = \mathbb{K}Q'$):

$$\sigma X := S(a)_{\Lambda'} \oplus \sigma X_2 \oplus \cdots \oplus \sigma X_s$$

if X has a direct summand (say, X_1) isomorphic to the simple module $S(a)_\Lambda$; and

$$\sigma X := \sigma X_1 \oplus \sigma X_2 \oplus \cdots \oplus \sigma X_s$$

if X does not, where we put $\sigma X_k = \mathrm{Hom}_\Lambda(T, X_k)$ for each indecomposable X_k. Let $\mathcal{R}$, $\mathcal{R}'$ be the set of the isomorphic classes of basic hom-orthogonal partial tilting Λ-modules, Λ'-modules, having exactly s indecomposable direct summands, respectively. Then we have the following:

Proposition 1.3. *For a basic hom-orthogonal partial tilting Λ-module X having s indecomposable direct summands, so is Λ'-module σX. The correspondence $[X] \mapsto [\sigma X]$ gives a bijection from $\mathcal{R}$ to $\mathcal{R}'$. In particular, the numbers that are presented in Theorem 0.1 do not depend on the choice of an orientation of arrows.*

Proof. Let R_Q, $R_{Q'}$ be the representation matrix of the Ringel form of Λ, Λ', respectively. Let $r = r_a$ be the simple reflection on $\mathbb{Z}^n$ corresponding to the vertex a (we also denote by the same r its representation matrix). Then we have $R_{Q'} = {}^t r \cdot R_Q \cdot r$. On the other hand, we have $\mathbf{dim}\, \sigma X_k = r \cdot (\mathbf{dim}\, X_k)$ for X_k that is not isomorphic to $S(a)_\Lambda$, and $r(\boldsymbol{e}_a) = -\boldsymbol{e}_a$. Hence, by calculating with the Ringel form (recall Lemma 1.1), we see that σX is also a basic hom-orthogonal partial tilting Λ'-module. This correspondence $[X] \mapsto [\sigma X]$ is obviously a bijection. $\square$

Next we define two subsets of $\mathcal{R}$ as follows:

$$\mathcal{R}_1 = \big\{[X] \in \mathcal{R};\ X \text{ is sincere, but } \sigma X \text{ is not sincere}\big\},$$
$$\mathcal{R}_2 = \big\{[X] \in \mathcal{R};\ X \text{ is not sincere, but } \sigma X \text{ is sincere}\big\}.$$

It follows from Lemma 1.2 that the condition "sincere" implies that any representative of each class of $\mathcal{R}_1$ or $\mathcal{R}_2$ does not have a direct summand isomorphic to the simple module $S(a)_\Lambda$.

Proposition 1.4. *We have $\sharp\mathcal{R}_1 = \sharp\mathcal{R}_2$. In particular, the numbers for sincere modules that are presented in Theorem 0.3 do not depend on the choice of an orientation of arrows.*

Proof. Take the isomorphic class $[X] \in \mathcal{R}_1$ and let $X \cong \bigoplus_{k=1}^s X_k$ be its indecomposable decomposition. Then, since σX is not sincere, only the a-th entry of $\mathbf{dim}\, \sigma X = r \cdot (\mathbf{dim}\, X)$ is zero. Hence so is the a-th entry of each $r(\boldsymbol{\alpha}_k)$, where we put $\boldsymbol{\alpha}_k = \mathbf{dim}\, X_k$. On the other hand, since σX is a basic hom-orthogonal partial tilting Λ'-module, we have ${}^t r(\boldsymbol{\alpha}_i) \cdot R_{Q'} \cdot r(\boldsymbol{\alpha}_j) = 0$ for any pair of distinct indices. Then we see that ${}^t r(\boldsymbol{\alpha}_i) \cdot R_Q \cdot r(\boldsymbol{\alpha}_j) =$

0, because R_Q and $R_{Q'}$ are identical other than the a-th row and the a-th column. Let $\widetilde{X}$ be a Λ-module corresponding to the sum of positive roots $\sum_{k=1}^{s} r(\boldsymbol{\alpha}_k)$; this is not sincere, but $\sigma\widetilde{X}$ is sincere. Thus we see that the correspondence $[X] \mapsto [\widetilde{X}]$ gives a bijection from $\mathcal{R}_1$ to $\mathcal{R}_2$. □

2. $\mathbb{A}_n$-TYPE

Let Q be the equi-oriented quiver $\overset{1}{\circ} \longrightarrow \overset{2}{\circ} \longrightarrow \cdots \longrightarrow \overset{n}{\circ}$ of $\mathbb{A}_n$-type. In the following, we will sometimes consider the corresponding things of "$\mathbb{A}_0$-type" or "$\mathbb{A}_{-1}$-type" to be trivial for simplicity; for example, "$\mathbb{A}_{n-2} \times \mathbb{A}_{-1}$-type" means just "$\mathbb{A}_{n-2}$-type", and so on.

Proposition 2.1. *For each $k = 1, 2, \ldots, n$, the perpendicular category* $\operatorname{Per} I(k)$ *is equivalent to the module category of a path algebra of type $\mathbb{A}_{k-2} \times \mathbb{A}_{n-1-k}$.*

Proposition 2.2. *Let n and s be positive integers. The number $a(n, s)$ satisfies the following recurrence formula:*

$$a(n,\, s) = a(n-1,\, s) + \sum_{t=0}^{s-1} \sum_{m=-1}^{n-2} a(m,\, t) \cdot a(n-3-m,\, s-1-t). \tag{2.1}$$

Proof. Let $X = \bigoplus_{j=1}^{s} X_j$ be a basic hom-orthogonal partial tilting Λ-module having s distinct indecomposable summands. Note that X has at most one injective direct summand. If X does not have any injective, then the first entry of $\mathbf{dim}\, X$ is zero; that is, it is a sum of positive roots that come from $\mathbb{A}_{n-1}$-type. So the number for such modules is equal to $a(n-1,\, s)$. Assume that X has just one injective summand, say $I(k)$. Then, according to Proposition 2.1, X has t and $s-1-t$ direct summands that come from $\mathbb{A}_{k-2}$-type and $\mathbb{A}_{n-1-k}$-type, respectively. Thus we see that there exist $\sum_{t=0}^{s-1} a(k-2,\, t) \cdot a(n-1-k,\, s-1-t)$ such modules. Since k runs from 1 to n, we obtain our assertion. □

By using the recurrence formula above, we prove Theorem 0.1 for $\mathbb{A}_n$-type. Here we notice that the generating function of $a(n,s) = C_s \cdot \binom{n+1}{2s}$ can be immediately obtained from the generalized binomial expansion.

Lemma 2.3. *The generating function $F_s(x) = \sum_{n=0}^{\infty} a(n,s)x^n$ of $a(n,s)$ for fixed s is given by*

$$F_s(x) = \frac{C_s \cdot x^{2s-1}}{(1-x)^{2s+1}}.$$

Proof of Theorem 0.1 *for $\mathbb{A}_n$-type.* First we note that $a(n,1)$ is nothing but the number of positive roots of $\mathbb{A}_n$-type, which is equal to $n(n+1)/2 = C_1 \cdot \binom{n+1}{2}$. In the case of $n = 1$, our assertion is trivial. So we assume that the assertion (0.2) holds for all positive integers less than n (≥ 2). In the recurrence formula (2.1), we note that $a(m,t)$ (resp. $a(n-3-m,\, s-1-t)$) is the coefficient of degree m (resp. $n-3-m$) of $F_t(x)$ (resp. $F_{s-1-t}(x)$). The coefficient of degree $n-3$ of the Taylor expansion at the origin ($x = 0$) of

$$F_t(x) \times F_{s-1-t}(x) = C_t \cdot C_{s-1-t} \cdot \frac{x^{2s-4}}{(1-x)^{2s}}$$

is equal to $\binom{n}{2s-1}$; hence we have

$$\sum_{m=-1}^{n-2} a(m,\,t)\cdot a(n-3-m,\,s-1-t) = C_t \cdot C_{s-1-t}\cdot \binom{n}{2s-1}. \tag{2.2}$$

By the recurrence formula (2.1) and the assumption of induction, we have

$$\begin{aligned} a(n,s) &= a(n-1,\,s) + \binom{n}{2s-1}\sum_{t=0}^{s-1} C_t\cdot C_{s-1-t} \\ &= C_s\cdot\binom{n}{2s} + \binom{n}{2s-1}\cdot C_s = C_s\cdot\binom{n+1}{2s}. \end{aligned}$$

Next we prove that $a(n,s) = 0$ if $s > (n+1)/2$. Let s be such an integer. Then we have $a(n-1,\,s) = 0$ by the assumption of induction because $s > n/2$. Suppose that $t \le (m+1)/2$ and $s-1-t \le (n-3-m+1)/2$ for fixed t. Then we have $s-1 \le (n-1)/2$; a contradiction. Hence $t > (m+1)/2$ or $s-1-t > (n-3-m+1)/2$, and so that $a(m,t) = 0$ or $a(n-3-m,\,s-1-t) = 0$. Thus we conclude $a(n,s) = 0$ by the recurrence formula (2.1). Therefore we obtain our assertion for $\mathbb{A}_n$-type. □

Remark 2.4. The formula (0.2) has a combinatorial interpretation. According to Araya [1, Lemma 3.2], for distinct indecomposables $X, Y \in \operatorname{mod}\Lambda$, their direct sum $X\oplus Y$ is a hom-orthogonal partial tilting module (or, both (X,Y) and (Y,X) are exceptional pairs) if and only if the corresponding codes of a circle with $n+1$ points do not meet each other. It follows from a well-known combinatorics on codes that the number of such codes is equal to $C_2\cdot\binom{n+1}{4} = a(n,2)$. The formula for general $s \ge 2$ can be similarly obtained.

Proposition 2.5. *Let X be a basic sincere hom-orthogonal partial tilting Λ-module. Then X has exactly one direct summand isomorphic to $I(n)$.*

Proof of Theorem 0.3 for $\mathbb{A}_n$-type. Let X be a basic sincere hom-orthogonal partial tilting Λ-module. In the case of $s = 1$ (that is, X itself is indecomposable), it must be isomorphic to $I(n)$. Hence we have $a^0(n,1) = 1$ for any n. If $n = 1$ or $n = 2$, our assertion can be proved directly. So let $n \ge 3$ and $s \ge 2$. By Propositions 2.2 and 2.5, the other summands of X should be taken from a module category of $\mathbb{A}_{n-2}$-type. The number of such candidates is equal to $a(n-2,\,s-1)$. We can prove $a^0(n,s) = 0$ for $s > (n+1)/2$ by a similar manner to the proof of Theorem 0.1. □

Theorems for $\mathbb{D}_n$-type and $\mathbb{E}_n$-type are shown in a similar way. The detailed proof is given in our paper which has been submitted for publication elsewhere

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THE NOETHERIAN PROPERTIES OF THE RINGS OF DIFFERENTIAL OPERATORS ON CENTRAL 2-ARRANGEMENTS

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Abstract. P. Holm began to study the ring of differential operators of the coordinate ring of a hyperplane arrangement. In this paper, we introduce Noetherian properties of the ring differential operators of the coordinate ring of a central 2-arrangement and its graded ring associated to the order filtration.

Key Words: Ring of differential operators, Noetherian property, Hyperplane arrangement.

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1. Introduction

For a commutative algebra R over a field K of characteristic zero, define vector spaces inductively by

$$\mathscr{D}^0(R) := \{\theta \in \mathrm{End}_K(R) \mid a \in R, \theta a - a\theta = 0\},$$
$$\mathscr{D}^m(R) := \left\{\theta \in \mathrm{End}_K(R) \mid a \in R, \theta a - a\theta \in \mathscr{D}^{m-1}(R)\right\} \quad (m \geq 1).$$

We define the ring $\mathscr{D}(R) := \bigcup_{m\geq 0} \mathscr{D}^m(R)$ of differential operators of R.

Let $S := K[x_1, \ldots, x_n]$ be the polynomial ring. The ring $\mathscr{D}(S)$ is the n-th Weyl algebra $K[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n\rangle$ where $\partial_i := \frac{\partial}{\partial x_i}$ (see for example [3]). We use the multi-index notetions, for example, $\partial^{\boldsymbol{\alpha}} := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ and $|\boldsymbol{\alpha}| := \alpha_1 + \cdots + \alpha_n$ for $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. Define $\mathscr{D}^{(m)}(S) := \bigoplus_{|\boldsymbol{\alpha}|=m}$. Then $\mathscr{D}(S) = \bigoplus_{m\geq 0} \mathscr{D}^{(m)}(S)$. It is well known $\mathscr{D}(R)$ that $\mathscr{D}(R)$ is Noetherian, if R is a regular domain (see [3]).

Holm [2] showed that $\mathscr{D}(R)$ is finitely generated as a K-algebra when R is a coordinate ring of a generic hyperplane arrangement. Holm [1] also proved that the ring of differential operators of a central 2-arrangement is a free S-module, and gave a basis of it. We can write any ellement in $\mathscr{D}(R)$ as a linearly combination of this basis ellements.

In this paper, we introduce the Noetherian property of $\mathscr{D}(R)$ when R is the coordinate ring of a central arrangement. In particular, the case $n = 2$, $\mathscr{D}(R)$ is a Noetherian ring . We give an example of a finitely generated ideal in the end of this paper.

The details of this note are in [4].

2. Hyperplane arrangement

In this section, we fix some notation, and we introduce some properties of the ring of differential operators of a central arrangement. Let $\mathscr{A} = \{H_i \mid i = 1, \ldots, r\}$ be a central (hyperplane) arrangement (i.e., every hyperplane in $\mathscr{A}$ contains the origin) in K^n. Fix a

The detailed version of this paper will be submitted for publication elsewhere.

polynomial p_i with $\ker(p_i) = H_i$, and put $Q := p_1 \cdots p_r$. Thus Q is a product of certain homogeneous polynomials of degree 1. Let I denote the principal ideal of S generated by Q. Then S/I is the coordinate ring of the hyperplane arrangement defined by Q.

For any ideal J of S, we define an S-submodule $\mathscr{D}^{(m)}(J)$ of $\mathscr{D}^{(m)}(S)$ and a subring $\mathscr{D}(J)$ of $\mathscr{D}(S)$ by

$$\mathscr{D}^{(m)}(J) := \{\theta \in \mathscr{D}^{(m)}(S) \mid \theta(J) \subseteq J\},$$
$$\mathscr{D}(J) := \{\theta \in \mathscr{D}(S) \mid \theta(J) \subseteq J\}.$$

Holm [2] proved the following proposition.

Proposition 1 (Proposition 4.3 in [2])**.**

$$\mathscr{D}(I) = \bigoplus_{m \geq 0} \mathscr{D}^{(m)}(I).$$

There is a ring isomorphism $\mathscr{D}(S/J) \simeq \mathscr{D}(J)/J\mathscr{D}(S)$ (see [3, Theorem 15.5.13]). Thus we can express $\mathscr{D}(S/J)$ as a subquotient of Weyl algebra.

We can prove that $\mathscr{D}(J)/J\mathscr{D}(S)$ is right Noetherian if and only if $\mathscr{D}(J)/J\mathscr{D}(S)$ is left Noetherian when $J \neq 0$ is a principal ideal. Therefore we conclude that $\mathscr{D}(S/I)$ is right Noetherian if and only if $\mathscr{D}(S/I)$ is left Noetherian.

Theorem 2. *Let $h \neq 0$ be a polynomial, and let $J = hS$. Then the ring $\mathscr{D}(J)/J\mathscr{D}(S)$ is right Noetherian if and only if $\mathscr{D}(J)/J\mathscr{D}(S)$ is left Noetherian.*

Corollary 3. *Let I be the defining ideal of a central arrangement. Then the ring $\mathscr{D}(S/I)$ is right Noetherian if and only if $\mathscr{D}(S/I)$ is left Noetherian.*

To prove that $\mathscr{D}(S/I)$ is a Noetherian ring, we only need to prove that $\mathscr{D}(S/I)$ is a right Noetherian ring.

The operator

$$\varepsilon_m := \sum_{|\boldsymbol{\alpha}|=m} \frac{m!}{\boldsymbol{\alpha}!} x^{\boldsymbol{\alpha}} \partial^{\boldsymbol{\alpha}}$$

is called the Euler operator of order m where $\boldsymbol{\alpha}! = (\alpha_1!) \cdots (\alpha_n!)$ for $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$. Then $\varepsilon_m = \varepsilon_1(\varepsilon_1 - 1) \cdots (\varepsilon_1 - m + 1)$ [2, Lemma 4.9].

3. $n = 2$

In this section, we assume $n = 2$ and $S = K[x, y]$. We introduce the Noetherian property of the ring $\mathscr{D}(S/I) \simeq \mathscr{D}(I)/I\mathscr{D}(S)$. In contrast, the graded ring $\operatorname{Gr} \mathscr{D}(S/I)$ associated to the order filtration is not Noetherian when $r \geq 2$.

Put $P_i := \frac{Q}{p_i}$ for $i = 1, \ldots, r$, and define

$$\delta_i := \begin{cases} \partial_y & \text{if } p_i = ax \quad (a \in K^\times) \\ \partial_x + a_i \partial_y & \text{if } p_i = a(y - a_i x) \quad (a \in K^\times). \end{cases}$$

Then $\delta_i(p_j) = 0$ if and only if $i = j$.

Proposition 4 (Paper III, Proposition 6.7 in [1], Proposition 4.14 in [6])**.** *For any $m \geq 1$, $\mathscr{D}^{(m)}(I)$ is a free left S-module with a basis*

$$\begin{aligned}
&\{\varepsilon_m, P_1\delta_1^m, \dots, P_m\delta_m^m\} \text{ if } m < r-1,\\
&\{P_1\delta_1^m, \dots, P_r\delta_r^m\} \text{ if } m = r-1,\\
&\{P_1\delta_1^m, \dots, P_r\delta_r^m, Q\eta_{r+1}^{(m)}, \dots, Q\eta_{m+1}^{(m)}\} \text{ if } m > r-1,
\end{aligned}$$

where the set $\{\delta_1^m, \dots, \delta_r^m, \eta_{r+1}^{(m)}, \dots, \eta_{m+1}^{(m)}\}$ forms a K-basis for $\sum_{|\boldsymbol{\alpha}|=m} K\partial^{\boldsymbol{\alpha}}$ if $m > r-1$.

By Proposition 1, we have

$$\begin{aligned}
\mathscr{D}(I) = S \oplus &\left(\bigoplus_{m=1}^{r-2} \left(S\varepsilon_m \oplus SP_1\delta_1^m \oplus \cdots \oplus SP_m\delta_m^m\right)\right)\\
&\oplus \left(\bigoplus_{m \geq r-1} \left(SP_1\delta_1^m \oplus \cdots \oplus SP_r\delta_r^m \oplus SQ\eta_{r+1}^{(m)} \oplus \cdots \oplus SQ\eta_{m+1}^{(m)}\right)\right).
\end{aligned}$$

For $i = 1, \dots, r$, we define an additive group

$$L_i := \mathscr{D}(I) \cap (p_1 \cdots p_i)\mathscr{D}(S).$$

Proposition 5. *For $i = 1, \dots, r$, the additive group L_i is a two-sided ideal of $\mathscr{D}(I)$.*

We consider a sequence

$$I\mathscr{D}(S) = L_r \subseteq L_{r-1} \subseteq \cdots \subseteq L_1 \subseteq L_0 = \mathscr{D}(I) \tag{3.1}$$

of two-sided ideals of $\mathscr{D}(I)$. If L_{i-1}/L_i is a right Noetherian $\mathscr{D}(I)$-module for any i, then $\mathscr{D}(I)/I\mathscr{D}(S)$ is a right Noetherian ring. By proving that L_{i-1}/L_i is right Noetherian for all i, we obtain the following main theorem.

Theorem 6. *The ring $\mathscr{D}(S/I) \simeq \mathscr{D}(I)/I\mathscr{D}(S)$ of differential operators of the coordinate ring of a central 2-arrangement is Noetherian (i.e., $\mathscr{D}(S/I)$ is right Noetherian and left Noetherian).*

In contrast, $\operatorname{Gr}\mathscr{D}(S/I)$ is not Noetherian when $r \geq 2$.

Remark 7. The graded ring $\operatorname{Gr}\mathscr{D}(S/I)$ associated to the order filtration is a commutative ring. We consider the ideal $M := \langle \overline{P_1\delta_1^m} \mid m \geq 1\rangle$ of $\operatorname{Gr}\mathscr{D}(S/I)$.

Assume that M is finitely generated with generators $\eta_1, \dots, \eta_\ell$. Then there exists a positive integer m such that

$$M = \langle \eta_1, \dots, \eta_\ell \rangle \subseteq \langle \overline{P_1\delta_1}, \dots, \overline{P_1\delta_1^{m-1}} \rangle.$$

Since $\overline{P_1\delta_1^m} \in M$, we can write

$$\overline{P_1\delta_1^m} = \overline{P_1\delta_1} \cdot \overline{\theta_1} + \cdots + \overline{P_1\delta_1^{m-1}} \cdot \overline{\theta_{m-1}} \tag{3.2}$$

for some $\theta_1, \dots, \theta_{m-1} \in \mathscr{D}(I)$.

If $\theta \in \mathscr{D}(I)$ with $\operatorname{ord}(\theta) \leq 1$, then the polynomial degree of θ is greater than or equal to 1 by Proposition 4. Since the order of the LHS of (3.2) is m, there exists at least one θ_j such that the order of θ_j is greater than or equal to 1. Thus the polynomial degree of the RHS of (3.2) is greater than $r-1$. However, the polynomial degree of the LHS of (3.2) is exactly $r-1$. This is a contradiction.

Hence M is not finitely generated, and thus we have proved that $\mathrm{Gr}\,\mathscr{D}(S/I)$ is not Noetherian.

4. Exaple

Let $n=2$ and $S=K[x,y]$. Let $Q=xy(x-y)$ and $I=QS$. Put $p_1=x, p_2=y, p_3=x-y$. Then $P_1=y(x-y)$ and $\delta_1=\partial_y$. We consider the right ideal $\langle y(x-y)\partial_y^m \mid m\geq 1\rangle$ of $\mathscr{D}(I)$.

For $\ell \geq 4$, we have

$$y(x-y)\partial_y \cdot y(x-y)\partial_y^{\ell+1} = y^2(x-y)^2\partial_y^{\ell+2} + y(x-2y)\partial_y^{\ell+1},$$
$$y(x-y)\partial_y^2 \cdot y(x-y)\partial_y^{\ell} = y^2(x-y)^2\partial_y^{\ell+2} + 2y(x-2y)\partial_y^{\ell+1} - 2y(x-y)\partial_y^{\ell},$$
$$y(x-y)\partial_y^3 \cdot y(x-y)\partial_y^{\ell-1} = y^2(x-y)^2\partial_y^{\ell+2} + 3y(x-2y)\partial_y^{\ell+1} - 6y(x-y)\partial_y^{\ell}.$$

Thus we obtain

$$y(x-y)\partial_y\cdot y(x-y)\partial_y^{\ell+1} - 2y(x-y)\partial_y^2\cdot y(x-y)\partial_y^{\ell} + y(x-y)\partial_y^3\cdot y(x-y)\partial_y^{\ell-1} = -2y(x-y)\partial_y^{\ell}.$$

This leads to

$$y(x-y)\partial_y^{\ell} \in \langle y(x-y)\partial_y^m \mid m=1,2,3\rangle$$

since $y(x-y)\partial_y^m \in \mathscr{D}(I)$ for any $m\geq 1$. We have the identity

$$\langle y(x-y)\partial_y^m \mid m\geq 1\rangle = \langle y(x-y)\partial_y^m \mid m=1,2,3\rangle$$

as right ideals. Hence the right ideal $\langle y(x-y)\partial_y^m \mid m\geq 1\rangle$ is finitely generated.

In contrast, the right ideal $\langle \overline{y(x-y)\partial_y^m} \mid m\geq 1\rangle$ of $\mathrm{Gr}\,\mathscr{D}(S/I)$ is not finitely generated by Remark 7.

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HOCHSCHILD COHOMOLOGY OF QUIVER ALGEBRAS DEFINED BY TWO CYCLES AND A QUANTUM-LIKE RELATION

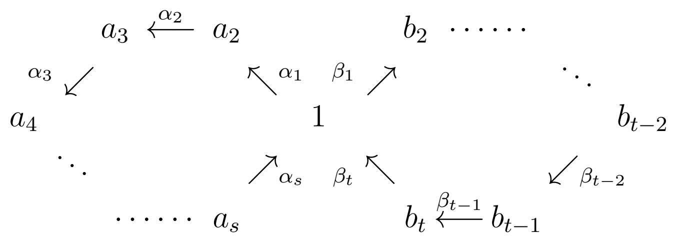
DAIKI OBARA

Abstract. This paper is based on my talk given at the Symposium on Ring Theory and Representation Theory held at Okayama University, Japan, 25–27 September 2011. In this paper, we consider quiver algebras A_q over a field k defined by two cycles and a quantum-like relation depending on a non-zero element q in k. We determine the ring structure of the Hochschild cohomology ring of A_q modulo nilpotence and give a necessary and sufficient condition for A_q to satisfy the finiteness condition given in [19].

1. Introduction

Let A be an indecomposable finite dimensional algebra over a field k. We denote by A^e the enveloping algebra $A \otimes_k A^{op}$ of A, so that left A^e-modules correspond to A-bimodules. The Hochschild cohomology ring is given by $\mathrm{HH}^*(A) = \mathrm{Ext}^*_{A^e}(A, A) = \oplus_{n\geq 0}\mathrm{Ext}^n_{A^e}(A, A)$ with Yoneda product. It is well-known that $\mathrm{HH}^*(A)$ is a graded commutative ring, that is, for homogeneous elements $\eta \in \mathrm{HH}^m(A)$ and $\theta \in \mathrm{HH}^n(A)$, we have $\eta\theta = (-1)^{mn}\theta\eta$. Let $\mathcal{N}$ denote the ideal of $\mathrm{HH}^*(A)$ which is generated by all homogeneous nilpotent elements. Then $\mathcal{N}$ is contained in every maximal ideal of $\mathrm{HH}^*(A)$, so that the maximal ideals of $\mathrm{HH}^*(A)$ are in 1-1 correspondence with those in the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(A)/\mathcal{N}$.

Let q be a non-zero element in k and s, t integers with $s, t \geq 1$. We consider the quiver algebra $A_q = kQ/I_q$ defined by the two cycles Q with $s+t-1$ vertices and $s+t$ arrows as follows:



and the ideal I_q of kQ generated by

$$X^{sa}, X^sY^t - qY^tX^s, Y^{tb}$$

for $a, b \geq 2$ where we set $X := \alpha_1 + \alpha_2 + \cdots + \alpha_s$ and $Y := \beta_1 + \beta_2 + \cdots + \beta_t$. We denote the trivial path at the vertex $a(i)$ and at the vertex $b(j)$ by $e_{a(i)}$ and by $e_{b(j)}$ respectively. We regard the numbers i in the subscripts of $e_{a(i)}$ modulo s and j in the subscripts of $e_{b(j)}$ modulo t. In this paper, we describe the ring structure of $\mathrm{HH}^*(A_q)/\mathcal{N}$.

In [17], Snashall and Solberg used the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(A)/\mathcal{N}$ to define a support variety for any finitely generated module over A. This led us to consider the structure of $\mathrm{HH}^*(A)/\mathcal{N}$. In [17], Snashall and Solberg conjectured that $\mathrm{HH}^*(A)/\mathcal{N}$ is always finitely generated as a k-algebra. But a counterexample to this conjecture was given by Snashall [16] and Xu [21]. This example makes us consider whether we can give necessary and sufficient conditions on a finite dimensional algebra A for $\mathrm{HH}^*(A)/\mathcal{N}$ to be finitely generated as a k-algebra.

On the other hand, in the theory of support varieties, it is interesting to know when the variety of a module is trivial. In [4], Erdmann, Holloway, Snashall, Solberg and Taillefer gave the necessary and sufficient conditions on a module for it to have trivial variety under some finiteness conditions on A. In [19], Solberg gave a condition which is equivalent to the finiteness conditions. In the paper, we show that A_q satisfies the finiteness condition given in [19] if and only if q is a root of unity.

The content of the paper is organized as follows. In Section 1 we deal with the definition of the support variety given in [17] and precedent results about the Hochschild cohomology ring modulo nilpotence. In Section 2, we describe the finiteness condition given in [19] and introduce precedent results about this condition. In Section 3, we determine the Hochschild cohomology ring of A_q modulo nilpotence and show that A_q satisfies the finiteness condition if and only if q is a root of unity.

2. Support variety

In [17], Snasall and Solberg defined the support variety of a finitely generated A-module M over a noetherian commutative graded subalgebra H of $\mathrm{HH}^*(A)$ with $H^0 = \mathrm{HH}^0(A)$. In this paper, we consider the case $H = \mathrm{HH}^*(A)$.

Definition 1 ([17])**.** The support variety of M is given by

$$V(M) = \{m \in \operatorname{MaxSpec} \mathrm{HH}^*(A)/\mathcal{N} \mid \operatorname{AnnExt}^*_A(M,M) \subseteq m'\}$$

where $\operatorname{AnnExt}^*_A(M,M)$ is the annihilator of $\operatorname{Ext}^*_A(M,M)$, m' is the pre-image of m for the natural epimorphism and the $\mathrm{HH}^*(A)$-action on $\operatorname{Ext}^*_A(A,A)$ is given by the graded algebra homomorphism $\mathrm{HH}^*(A) \xrightarrow{-\otimes M} \operatorname{Ext}^*_A(M,M)$.

Since A is indecomposable, we have that $\mathrm{HH}^0(A)$ is a local ring. Thus $\mathrm{HH}^*(A)/\mathcal{N}$ has a unique maximal graded ideal $m_{gr} = \langle \operatorname{rad} \mathrm{HH}^*(A), \mathrm{HH}^{\geq 1}(A)\rangle/\mathcal{N}$. We say that the variety of M is trivial if $V(M) = \{m_{gr}\}$.

In [16], Snashall gave the following question.
Question ([16]). Whether we can give necessary and sufficient conditions on a finite dimensional algebra for the Hochschild cohomology ring modulo nilpotence to be finitely generated as a k-algebra.

With respect to sufficient condition, it is shown that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated as a k-algebra for various classes of algebras by many authors as follows:

(1) In [6], [20], Evens and Venkov showed that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated for any block of a group ring of a finite group.
(2) In [7], Friedlander and Suslin showed that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated for any block of a finite dimensional cocommutative Hopf algebra.
(3) In [9], Green, Snashall and Solberg showed that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated for finite dimensional self-injective algebras of finite representation type over an algebraically closed field.

(4) In [10], Green, Snashall and Solberg showed that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated for finite dimensional monomial algebras.
(5) In [11], Happel showed that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated for finite dimensional algebras of finite global dimension.
(6) In [15], Schroll and Snashall showed that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated for the principal block of the Heche algebra $H_q(S_5)$ with $q = -1$ defined by the quiver

$$\begin{tikzcd} 1 \arrow[loop left, "\varepsilon"] \arrow[r, shift left, "a"] & 2 \arrow[l, shift left, "\bar{a}"] \arrow[loop right, "\bar{\varepsilon}"] \end{tikzcd}$$

and the ideal I of kQ generated by

$$\alpha\bar{\varepsilon}, \bar{\alpha}\varepsilon, \bar{\varepsilon}\bar{\alpha}, \varepsilon^2 - \alpha\bar{\alpha}, \bar{\varepsilon}^2 - \bar{\alpha}\alpha.$$

(7) In [18], Snashall and Taillefer showed that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated for a class of special biserial algebras.
(8) In [12], Koenig and Nagase produced many examples of finite dimensional algebras with a stratifying ideal for which $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated as a k-algebra.
(9) In [16] and [21], Snashall and Xu gave the example of a finite dimensional algebra for which $\mathrm{HH}^*(A)/\mathcal{N}$ is not a finitely generated k-algebra.

Example 2. ([16, Example 4.1]) Let $A = kQ/I$ where Q is the quiver

$$\begin{tikzcd} 1 \arrow[loop above, "a"] \arrow[loop below, "b"] \arrow[r, "c"] & 2 \end{tikzcd}$$

and $I = \langle a^2, b^2, ab - ba, ac \rangle$. Then $\mathrm{HH}^*(A)/\mathcal{N}$ is not finitely generated as a k-algebra.

Xu showed this in the case $\operatorname{char} k = 2$ in [21].

3. Finiteness condition

In [4], Erdmann, Holloway, Snashall, Solberg and Taillefer gave the following two conditions **(Fg1)** and **(Fg2)** for an algebra A and a graded subalgebra H of $\mathrm{HH}^*(A)$.

(Fg1) H is a commutative Noetherian algebra with $H^0 = \mathrm{HH}^0(A)$.
(Fg2) $\mathrm{Ext}^*_A(A/\operatorname{rad} A, A/\operatorname{rad} A)$ is a finitely generated H-module.

In [19], Solberg showed that the finiteness conditions are equivalent to the following condition.

(Fg) $\mathrm{HH}^*(A)$ is Noetherian and $\mathrm{Ext}^*_A(A/\mathrm{rad}A, A/\mathrm{rad}A)$ is a finitely generated $\mathrm{HH}^*(A)$-module.

In [4], under the finiteness condition **(Fg)**, some geometric properties of the support variety and some representation theoretic properties are related. In particular, the following theorem hold.

Theorem 3 ([4, Theorem 2.5]). *Suppose that A satisfies* **(Fg)**.
(a) *A is Gorenstein, that is, A has finite injective dimension both as a left A-module and as a right A-module.*

(b) *The following are equivalent for an A-module M.*
   (i) *The variety of M is trivial.*
   (ii) *The projective dimension of M is finite.*
   (iii) *The injective dimension of M is finite.*

There are some papers which deal with the finiteness condition **(Fg)** as follows.

(1) In [2], Bergh and Oppermann show that a codimension n quantum complete intersection satisfies **(Fg)** if and only if all the commutators q_{ij} are roots of unity.

   Definition 4. Let n be integer with $n \geq 1$, a_i integer with $a_i \geq 2$ for $1 \leq i \leq n$, and q_{ij} a non-zero element in k for every $1 \leq i < j \leq n$. A codimension n quantum complete intersection is defined by

$$k\langle x_1, \dots, x_n \rangle / I$$

   where I generated by

$$x_i^{a_i}, x_j x_i - q_{ij} x_i x_j \quad \text{for } 1 \leq i < j \leq n.$$

(2) In [5], Erdmann and Solberg gave the necessary and sufficient conditions on a Koszul algebra for it to satisfy **(Fg)**.

   Theorem 5 ([5, Theorem 1.3])**.** *Let A be a finite dimensional Koszul algebra over an algebraically closed field, and let* $E(A) = \operatorname{Ext}^*_A(A/\operatorname{rad} A, A/\operatorname{rad} A)$. *A satisfies* **(Fg)** *if and only if* $Z_{gr}(E(A))$ *is Noetherian and* $E(A)$ *is a finitely generated* $Z_{gr}(E(A))$*-module.*

(3) In [8], Furuya and Snashall provided examples of (D, A)-stacked monomial algebras which are not self-injective but satisfy **(Fg)**.

   Example 6. ([8, Example 3.2]) Let Q be the quiver

$$\begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[d, "\beta"] \\ 4 \arrow[u, "\delta"] & 3 \arrow[l, "\gamma"] \end{tikzcd}$$

   and I the ideal of kQ generated by

$$\alpha\beta\gamma\delta\alpha\beta, \gamma\delta\alpha\beta\gamma\delta.$$

   Then, $A = kQ/\mathcal{I}$ is not self-injective but satisfies **(Fg)**.

(4) In [15], Schroll and Snashall show that **(Fg)** hold for the principal block of the Heche algebra $H_q(S_5)$ with $q = -1$.

4. Quiver algebras defined by two cycles and a quantum-like relation

In this section, we consider the quiver algebras $A_q = kQ/I_q$ defined by the quiver Q as follows:

$$\begin{array}{ccccccc}
 & & a_3 \xleftarrow{\alpha_2} a_2 & & b_2 \cdots\cdots & & \\
 & \alpha_3 \swarrow & & \nwarrow \alpha_1 \quad \beta_1 \nearrow & & \ddots & \\
a_4 & & & 1 & & & b_{t-2} \\
 & \ddots & & \nearrow \alpha_s \quad \beta_t \nwarrow & & \swarrow \beta_{t-2} & \\
 & & \cdots\cdots a_s & & b_t \xleftarrow{\beta_{t-1}} b_{t-1} & &
\end{array}$$

and the ideal I_q of kQ generated by

$$X^{sa}, X^sY^t - qY^tX^s, Y^{tb}$$

for $a, b \geq 2$ where we set $X := \alpha_1 + \alpha_2 + \cdots + \alpha_s$ and $Y := \beta_1 + \beta_2 + \cdots + \beta_t$, and q is non-zero element in k. Paths are written from right to left.

In the case $s = t = 1$, A_q is called a quantum complete intersection (cf. [1]). In this case, when $a = b = 2$, the Hochschild cohomology ring $\mathrm{HH}^*(A_q)$ of A_q was described by Buchweitz, Green, Madsen and Solberg [3] for any $q \in k$. Moreover, in the case where $s = t = 1$, $a, b \geq 2$, Bergh and Erdmann [1] determined $\mathrm{HH}^*(A_q)$ if q is not a root of unity. And in the same case, Bergh and Oppermann [2] show that A_q satisfies **(Fg)** if and only if q is a root of unity. In [4], Erdmann, Holloway, Snashall, Solberg and Taillefer describe that if an algebra A satisfies **(Fg)** then $\mathrm{HH}^*(A)$ is a finitely generated k-algebra. Therefore, we consider the case where $s \geq 2$ or $t \geq 2$.

In this paper, we determine the Hochschild cohomology ring of A_q modulo nilpotence $\mathrm{HH}^*(A_q)/\mathcal{N}$ and show that A_q satisfies **(Fg)** if and only if q is a root of unity.

In [13] and [14], we determined the ring structure of $\mathrm{HH}^*(A_q)$ by means of generators and Yoneda product. By this ring structure of $\mathrm{HH}^*(A_q)$, we have the following results.

Theorem 7. *In the case where q is a root of unity, $\mathrm{HH}^*(A_q)$ is finitely generated as a k-algebra.*

Theorem 8. *In the case where q is a root of unity, $\mathrm{HH}^*(A_q)/\mathcal{N}$ is isomorphic to the polynomial ring of two variables. In the case $s, t \geq 2$, $r \geq 1$, we have*

$$\mathrm{HH}^*(A_q)/\mathcal{N} \cong \begin{cases} k[W^{2r}_{0,0,0}, W^{2r}_{2r,0,0}] & \text{if } s,t \geq 2, \bar{a} \neq 0, \bar{b} \neq 0, \\ k[W^{2r}_{0,0,0}, W^{2}_{2,0,0}] & \text{if } s,t \geq 2, \bar{a} = 0, \bar{b} \neq 0, \\ k[W^{2}_{0,0,0}, W^{2r}_{2r,0,0}] & \text{if } s,t \geq 2, \bar{a} \neq 0, \bar{b} = 0, \\ k[W^{2}_{0,0,0}, W^{2}_{2,0,0}] & \text{if } s,t \geq 2, \bar{a} = \bar{b} = 0, \end{cases}$$

where for any integer z, $\bar{z}$ is the remainder when we divide z by r, and for $n \geq 1$,

$$W^{2n}_{0,l,l'} := X^{sl}Y^{tl'}e^{2n}_{b(1)} + \sum_{j=2}^{t} Y^{j-1}X^{sl}Y^{t(l'-1)+t-j+1}e^{2n}_{b(j)} \text{ for } 0 \leq l \leq a-1 \text{ and } 0 \leq l' \leq b,$$

$$W^{2n}_{2n,l,l'} := X^{sl}Y^{tl'}e^{2n}_{a(1)} + \sum_{i=2}^{s} X^{s(l-1)+i-1}Y^{tl'}X^{s-i+1}e^{2n}_{a(i)} \text{ for } 0 \leq l \leq a \text{ and } 0 \leq l' \leq b-1.$$

In the case where $s = 1$ or $t = 1$, we have similar results.

Theorem 9. *In the case where q is not a root of unity,* $\mathrm{HH}^*(A_q)$ *is not a finitely generated k-algebra.*

Theorem 10. *In the case where q is not a root of unity,* $\mathrm{HH}^*(A_q)/\mathcal{N} \cong k$.

There exists an example of our algebra A_q which is not self-injective, monomial or Koszul. Moreover this example of A_q have no stratifying ideal.

Example 11. In the case where $s = 2$, $t = 1$ and $a = b = 2$, A_q is not self-injective, monomial or Koszul. Moreover A_q have no stratifying ideal.

Therefore A_q is new example of a class of algebras for which the Hochschild cohomology ring modulo nilpotence is finitely generated as a k-algebra.

Next, we give the necessary and sufficient condition for A to satisfy **(Fg)**. Now, we consider the case where q is an r-th root of unity for $r \geq 1$, $s, t \geq 2$ and $\bar{a}, \bar{b} \neq 0$.

Let φ: $\mathrm{HH}^*(A_q) \to E(A_q) := \oplus_{n\geq 0}\mathrm{Ext}^n_{A_q}(A_q/\mathrm{rad}\,A_q, A_q/\mathrm{rad}\,A_q)$ be a homomorphism of graded rings given by $\varphi(\eta) = \eta \otimes_{A_q} A_q/\mathrm{rad}\,A_q$. Then it is easy to see that $E(A_q)^n :=$ $\mathrm{Ext}^n_{A_q}(A_q/\mathrm{rad}\,A_q, A_q/\mathrm{rad}\,A_q) \simeq \coprod_{l=0}^n ke_1^n \oplus \coprod_{j=2}^t ke^n_{b(j)} \oplus \coprod_{i=2}^s ke^n_{a(i)}$, and that the image of φ is precisely the graded ring $k[x, y]$ where $x := \sum_{j=1}^t e^{2r}_{b(j)}$ and $y := \sum_{i=1}^s e^{2r}_{a(i)}$ in degree $2r$. Then, we have the following proposition.

Proposition 12. $E(A_q)$ *is a finitely generated left $k[x, y]$-module.*

In the other cases, we have same results as Proposition 12. Then we have the following immediate consequence of Proposition 12.

Theorem 13. *In the case where $s \geq 2$ or $t \geq 2$, if q is a root of unity then A_q satisfies* **(Fg)**.

By [2], Theorem 9 and 13, we have the necessary and sufficient condition for A_q to satisfy **(Fg)**.

Theorem 14. A_q *satisfies* **(Fg)** *if and only if q is a root of unity.*

Remark 15. By Theorem 2.5 in [4] and Theorem 14, in the case where q is a root of unity, we have the following properties

(1) A_q is Gorenstein.
(2) The support variety of an A_q-module M is trivial if and only if the projective dimension of M is finite.

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ALTERNATIVE POLARIZATIONS OF BOREL FIXED IDEALS AND ELIAHOU-KERVAIRE TYPE RESOLUTION

RYOTA OKAZAKI AND KOHJI YANAGAWA

1. Introduction

Let $S := \Bbbk[x_1, \dots, x_n]$ be a polynomial ring over a field $\Bbbk$. For a monomial ideal $I \subset S$, $G(I)$ denotes the set of minimal (monomial) generators of I. We say a monomial ideal $I \subset S$ is *Borel fixed* (or *strongly stable*), if $\mathsf{m} \in G(I)$, $x_i | \mathsf{m}$ and $j < i$ imply $(x_j/x_i) \cdot \mathsf{m} \in I$. Borel fixed ideals are important, since they appear as the *generic initial ideals* of homogeneous ideals (if $\mathrm{char}(\Bbbk) = 0$).

A squarefree monomial ideal I is said to be *squarefree strongly stable*, if $\mathsf{m} \in G(I)$, $x_i | \mathsf{m}$, $x_j \not| \mathsf{m}$ and $j < i$ imply $(x_j/x_i) \cdot \mathsf{m} \in I$. Any monomial $\mathsf{m} \in S$ with $\deg(\mathsf{m}) = e$ has a unique expression

$$\mathsf{m} = \prod_{i=1}^{e} x_{\alpha_i} \quad \text{with} \quad 1 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_e \le n. \tag{1.1}$$

Now we can consider the squarefree monomial

$$\mathsf{m}^{\mathsf{sq}} = \prod_{i=1}^{e} x_{\alpha_i + i - 1}$$

in the "larger" polynomial ring $T = \Bbbk[x_1, \dots, x_N]$ with $N \gg 0$. If $I \subset S$ is Borel fixed, then $I^{\mathsf{sq}} := (\, \mathsf{m}^{\mathsf{sq}} \mid \mathsf{m} \in G(I)\,) \subset T$ is squarefree strongly stable. Moreover, for a Borel fixed ideal I and all i, j, we have $\beta^S_{i,j}(I) = \beta^T_{i,j}(I^{\mathsf{sq}})$. This operation plays a role in the *shifting theory* for simplicial complexes (see [1]).

A minimal free resolution of a Borel fixed ideal I has been constructed by Eliahou and Kervaire [7]. While the minimal free resolution is unique up to isomorphism, its "description" depends on the choice of a free basis, and further analysis of the minimal free resolution is still an interesting problem. See, for example, [2, 9, 10, 11, 13]. In this paper, we will give a new approach which is applicable to both I and I^{sq}. Our main tool is the "alternative" polarization $\mathsf{b\text{-}pol}(I)$ of I.

Let

$$\widetilde{S} := \Bbbk[\, x_{i,j} \mid 1 \le i \le n,\, 1 \le j \le d\,]$$

be the polynomial ring, and set

$$\Theta := \{x_{i,1} - x_{i,j} \mid 1 \le i \le n,\, 2 \le j \le d\} \subset \widetilde{S}.$$

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Then there is an isomorphism $\widetilde{S}/(\Theta) \cong S$ induced by $\widetilde{S} \ni x_{i,j} \longmapsto x_i \in S$. Throughout this paper, $\widetilde{S}$ and Θ are used in this meaning.

Assume that $\mathsf{m} \in G(I)$ has the expression (1.1). If $\deg(\mathsf{m})\,(=e) \leq d$, we set

$$\mathsf{b\text{-}pol}(\mathsf{m}) = \prod_{i=1}^{e} x_{\alpha_i,i} \in \widetilde{S}. \tag{1.2}$$

Note that $\mathsf{b\text{-}pol}(\mathsf{m})$ is a squarefree monomial. If there is no danger of confusion, $\mathsf{b\text{-}pol}(\mathsf{m})$ is denoted by $\widetilde{\mathsf{m}}$. If $\mathsf{m} = \prod_{i=1}^{n} x_i^{a_i}$, then we have

$$\widetilde{\mathsf{m}}\ (= \mathsf{b\text{-}pol}(\mathsf{m}))\ = \prod_{\substack{1 \leq i \leq n \\ b_{i-1}+1 \leq j \leq b_i}} x_{i,j} \in \widetilde{S}, \quad \text{where} \quad b_i := \sum_{l=1}^{i} a_l.$$

If $\deg(\mathsf{m}) \leq d$ for all $\mathsf{m} \in G(I)$, we set

$$\mathsf{b\text{-}pol}(I) := (\mathsf{b\text{-}pol}(\mathsf{m}) \mid \mathsf{m} \in G(I)) \subset \widetilde{S}.$$

The second author ([16]) showed that if I is Borel fixed, then $\widetilde{I} := \mathsf{b\text{-}pol}(I)$ is a "polarization" of I, that is, Θ forms an $\widetilde{S}/\widetilde{I}$-regular sequence with the natural isomorphism

$$\widetilde{S}/(\widetilde{I} + (\Theta)) \cong S/I.$$

Note that $\mathsf{b\text{-}pol}(-)$ does not give a polarization for a general monomial ideal, and is essentially different from the standard polarization. Moreover,

$$\Theta' = \{\, x_{i,j} - x_{i+1,j-1} \mid 1 \leq i < n, 1 < j \leq d \,\} \subset \widetilde{S}$$

forms an $\widetilde{S}/\widetilde{I}$-regular sequence too, and we have $\widetilde{S}/(\widetilde{I} + (\Theta')) \cong T/I^{\mathsf{sq}}$ through $\widetilde{S} \ni x_{i,j} \longmapsto x_{i+j-1} \in T$ (if we adjust the value of $N = \dim T$). The equation $\beta^S_{i,j}(I) = \beta^T_{i,j}(I^{\mathsf{sq}})$ mentioned above easily follows from this observation.

In this paper, we will construct a minimal $\widetilde{S}$-free resolution $\widetilde{P}_\bullet$ of $\widetilde{S}/\widetilde{I}$, which is analogous to the Eliahou-Kervaire resolution of S/I. However, their description can *not* be lifted to $\widetilde{I}$, and we need modification. Clearly, $\widetilde{P}_\bullet \otimes_{\widetilde{S}} \widetilde{S}/(\Theta)$ and $\widetilde{P}_\bullet \otimes_{\widetilde{S}} \widetilde{S}/(\Theta')$ give the minimal free resolutions of S/I and T/I^{sq} respectively.

Under the assumption that a Borel fixed ideal I is generated in one degree (i.e., all elements of $G(I)$ have the same degree), Nagel and Reiner [13] constructed $\widetilde{I} = \mathsf{b\text{-}pol}(I)$, and described a minimal $\widetilde{S}$-free resolution of $\widetilde{I}$ explicitly. Their resolution is equivalent to our description. In this sense, our results are generalizations of those in [13].

In [2], Batzies and Welker tried to construct a minimal free resolutions of monomial ideals J using Forman's *discrete Morse theory* ([8]). If J is *shellable* (i.e., has *linear quotients*, in the sense of [9]), their method works, and we have a *Batzies-Welker type* minimal free resolution. However, it is very hard to compute their resolution explicitly.

A Borel fixed ideal I and its polarization $\widetilde{I} = \mathsf{b\text{-}pol}(I)$ is shellable. We will show that our resolution $\widetilde{P}_\bullet$ of $\widetilde{S}/\widetilde{I}$ and the induced resolutions of S/I and T/I^{sq} are Batzies-Welker type. In particular, these resolutions are cellular. As far as the authors know, an *explicit* description of a Batzies-Welker type resolution of a general Borel fixed ideal has never been obtained before. Finally, we show that the CW complex supporting $\widetilde{P}_\bullet$ is *regular*.

2. The Eliahou-Kervaire type resolution of $\widetilde{S}/\,\mathsf{b}\text{-}\mathsf{pol}(I)$

Throughout the rest of the paper, I is a Borel fixed monomial ideal with $\deg \mathsf{m} \le d$ for all $\mathsf{m} \in G(I)$. For the definitions of the alternative polarization $\mathsf{b}\text{-}\mathsf{pol}(I)$ of I and related concepts, consult the previous section. For a monomial $\mathsf{m} = \prod_{i=1}^n x_i^{a_i} \in S$, set $\mu(\mathsf{m}) := \min\{\, i \mid a_i > 0\,\}$ and $\nu(\mathsf{m}) := \max\{\, i \mid a_i > 0\,\}$. In [7], it is shown that any monomial $\mathsf{m} \in I$ has a unique expression $\mathsf{m} = \mathsf{m}_1 \cdot \mathsf{m}_2$ with $\nu(\mathsf{m}_1) \le \mu(\mathsf{m}_2)$ and $\mathsf{m}_1 \in G(I)$. Following [7], we set $g(\mathsf{m}) := \mathsf{m}_1$. For i with $i < \nu(\mathsf{m})$, let

$$\mathfrak{b}_i(\mathsf{m}) = (x_i/x_k) \cdot \mathsf{m}, \text{ where } k := \min\{\, j \mid a_j > 0,\ j > i\}.$$

Since I is Borel fixed, $\mathsf{m} \in I$ implies $\mathfrak{b}_i(\mathsf{m}) \in I$.

Definition 1 ([14, Definition 2.1])**.** For a finite subset $\widetilde{F} = \{\,(i_1, j_1), (i_2, j_2), \dots, (i_q, j_q)\,\}$ of $\mathbb{N} \times \mathbb{N}$ and a monomial $\mathsf{m} = \prod_{i=1}^e x_{\alpha_i} = \prod_{i=1}^n x_i^{a_i} \in G(I)$ with $1 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_e \le n$, we say the pair $(\widetilde{F}, \widetilde{\mathsf{m}})$ is *admissible* (for $\mathsf{b}\text{-}\mathsf{pol}(I)$), if the following are satisfied:

(a) $1 \le i_1 < i_2 < \cdots < i_q < \nu(\mathsf{m})$,

(b) $j_r = \max\{\, l \mid \alpha_l \le i_r\,\} + 1$ (equivalently, $j_r = 1 + \sum_{l=1}^{i_r} a_l$) for all r.

For $\mathsf{m} \in G(I)$, the pair $(\emptyset, \widetilde{\mathsf{m}})$ is also admissible.

The following are fundamental properties of admissible pairs.

Lemma 2. *Let $(\widetilde{F}, \widetilde{\mathsf{m}})$ be an admissible pair with $\widetilde{F} = \{\,(i_1, j_1), \dots, (i_q, j_q)\,\}$ and $\mathsf{m} = \prod x_i^{a_i} \in G(I)$. Then we have the following.*

(i) $j_1 \le j_2 \le \cdots \le j_q$.

(ii) $x_{k,j_r} \cdot \mathsf{b}\text{-}\mathsf{pol}(\mathfrak{b}_{i_r}(\mathsf{m})) = x_{i_r,j_r} \cdot \mathsf{b}\text{-}\mathsf{pol}(\mathsf{m})$*, where* $k = \min\{\, l \mid l > i_r, a_l > 0\,\}$.

For $\mathsf{m} \in G(I)$ and an integer i with $1 \le i < \nu(\mathsf{m})$, set $\mathsf{m}_{\langle i\rangle} := g(\mathfrak{b}_i(\mathsf{m}))$ and $\widetilde{\mathsf{m}}_{\langle i\rangle} := \mathsf{b}\text{-}\mathsf{pol}(\mathsf{m}_{\langle i\rangle})$. If $i \ge \nu(\mathsf{m})$, we set $\mathsf{m}_{\langle i\rangle} := \mathsf{m}$ for the convenience. In the situation of Lemma 2, $\widetilde{\mathsf{m}}_{\langle i_r\rangle}$ divides $x_{i_r,j_r} \cdot \widetilde{\mathsf{m}}$ for all $1 \le r \le q$.

For $\widetilde{F} = \{\,(i_1, j_1), \dots, (i_q, j_q)\,\}$ and r with $1 \le r \le q$, set $\widetilde{F}_r := \widetilde{F} \setminus \{\,(i_r, j_r)\,\}$, and for an admissible pair $(\widetilde{F}, \widetilde{\mathsf{m}})$ for $\mathsf{b}\text{-}\mathsf{pol}(I)$,

$$B(\widetilde{F}, \widetilde{\mathsf{m}}) := \{\, r \mid (\widetilde{F}_r, \widetilde{\mathsf{m}}_{\langle i_r\rangle}) \text{ is admissible}\,\}.$$

Lemma 3. *Let $(\widetilde{F}, \widetilde{\mathsf{m}})$ be as in Lemma 2.*

(i) *For all r with $1 \le r \le q$, $(\widetilde{F}_r, \widetilde{\mathsf{m}})$ is admissible.*

(ii) *We always have $q \in B(\widetilde{F}, \widetilde{\mathsf{m}})$.*

(iii) *Assume that $(\widetilde{F}_r, \widetilde{\mathsf{m}}_{\langle i_r\rangle})$ satisfies the condition (a) of Definition 1. Then $r \in B(\widetilde{F}, \widetilde{\mathsf{m}})$ if and only if either $j_r < j_{r+1}$ or $r = q$.*

(iv) *For r, s with $1 \le r < s \le q$ and $j_r < j_s$, we have $\mathfrak{b}_{i_r}(\mathfrak{b}_{i_s}(\mathsf{m})) = \mathfrak{b}_{i_s}(\mathfrak{b}_{i_r}(\mathsf{m}))$ and hence $(\widetilde{\mathsf{m}}_{\langle i_r\rangle})_{\langle i_s\rangle} = (\widetilde{\mathsf{m}}_{\langle i_s\rangle})_{\langle i_r\rangle}$.*

(v) *For r, s with $1 \le r < s \le q$ and $j_r = j_s$, we have $\mathfrak{b}_{i_r}(\mathsf{m}) = \mathfrak{b}_{i_r}(\mathfrak{b}_{i_s}(\mathsf{m}))$ and hence $\widetilde{\mathsf{m}}_{\langle i_r\rangle} = (\widetilde{\mathsf{m}}_{\langle i_s\rangle})_{\langle i_r\rangle}$.*

Example 4. Let $I \subset S = \Bbbk[x_1, x_2, x_3, x_4]$ be the smallest Borel fixed ideal containing $\mathsf{m} = (x_1)^2 x_3 x_4$. In this case, $\mathsf{m}'_{\langle i\rangle} = g(\mathfrak{b}_i(\mathsf{m}'))$ for all $\mathsf{m}' \in G(I)$. Hence, we have $\mathsf{m}_{\langle 1\rangle} = (x_1)^3 x_4$, $\mathsf{m}_{\langle 2\rangle} = (x_1)^2 x_2 x_4$ and $\mathsf{m}_{\langle 3\rangle} = (x_1)^2 (x_3)^2$. The following 3 pairs are all admissible.

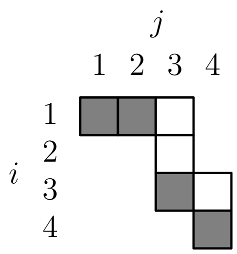
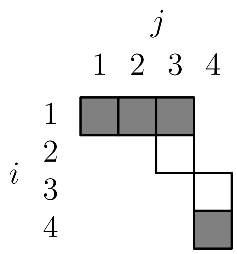
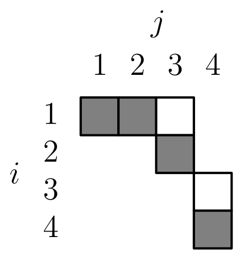
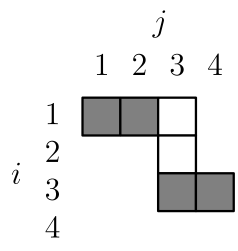
- $(\widetilde{F}, \widetilde{\mathsf{m}}) = (\{\, (1,3), (2,3), (3,4)\,\}, x_{1,1}\, x_{1,2}\, x_{3,3}\, x_{4,4})$
- $(\widetilde{F}_2, \widetilde{\mathsf{m}}_{\langle 2\rangle}) = (\{\, (1,3), (3,4)\,\}, x_{1,1}\, x_{1,2}\, x_{2,3}\, x_{4,4})$
- $(\widetilde{F}_3, \widetilde{\mathsf{m}}_{\langle 3\rangle}) = (\{\, (1,3), (2,3)\,\}, x_{1,1}\, x_{1,2}\, x_{3,3}\, x_{3,4})$

(For this $\widetilde{F}$, $i_r = r$ holds and the reader should be careful). However, $(\widetilde{F}_1, \widetilde{\mathsf{m}}_{\langle 1\rangle}) = (\{\, (2,3), (3,4)\,\}, x_{1,1}\, x_{1,2}\, x_{1,3}\, x_{4,4})$ does not satisfy the condition (b) of Definition 1. Hence $B(\widetilde{F}, \widetilde{\mathsf{m}}) = \{2, 3\}$.

The diagrams of (admissible) pairs are very useful for better understanding. To draw a diagram of $(\widetilde{F}, \widetilde{\mathsf{m}})$, we put a white square in the (i,j)-th position if $(i,j) \in \widetilde{F}$ and a black square there if $x_{i,j}$ divides $\widetilde{\mathsf{m}}$. If $\widetilde{F}$ is maximal among $\widetilde{F}'$ such that $(\widetilde{F}', \widetilde{\mathsf{m}})$ is admissible, then the diagram of $(\widetilde{F}, \widetilde{\mathsf{m}})$ forms a "right side down stairs" (see the leftmost and rightmost diagrams of the table below). If $(\widetilde{F}, \widetilde{\mathsf{m}})$ is admissible but $\widetilde{F}$ is not maximal, then some white squares are removed from the diagram for the maximal case. If the pair is admissible, there is a unique black square in each column and this is the "lowest" of the squares in the column.

If $(\widetilde{F}, \widetilde{\mathsf{m}})$ is admissible and $r \in B(\widetilde{F}, \widetilde{\mathsf{m}})$, then we can get the diagram of $(\widetilde{F}_r, \widetilde{\mathsf{m}}_{\langle i_r\rangle})$ from that of $(\widetilde{F}, \widetilde{\mathsf{m}})$ by the following procedure.

(i) Remove the (sole) black square in the j_r-th column.
(ii) Replace the white square in the (i_r, j_r)-th position by a black one.
(iii) If $\mathsf{m}_{\langle i_r\rangle} \neq \mathfrak{b}_{i_r}(\mathsf{m})$, erase some squares from the lower-right of the diagram. (This step does not occur in the next table.)

			
$(\widetilde{F}, \widetilde{\mathsf{m}})$	$(\widetilde{F}_1, \widetilde{\mathsf{m}}_{\langle 1\rangle})$	$(\widetilde{F}_2, \widetilde{\mathsf{m}}_{\langle 2\rangle})$	$(\widetilde{F}_3, \widetilde{\mathsf{m}}_{\langle 3\rangle})$
admissible	not admissible	admissible	admissible

Next let I' be the smallest Borel fixed ideal containing $\mathsf{m} = (x_1)^2 x_3 x_4$ and $(x_1)^2 x_2$. For $\widetilde{F} = \{\, (1,3), (2,3), (3,4)\,\}$, $(\widetilde{F}, \widetilde{\mathsf{m}})$ is admissible again. However $\widetilde{\mathsf{m}}_{\langle 2\rangle} = (x_1)^2 x_2$ in this time, and $(\widetilde{F}_2, \widetilde{\mathsf{m}}_{\langle 2\rangle}) = (\{\, (1,3), (3,4)\,\}, x_{1,1}\, x_{1,2}\, x_{2,3})$ is no longer admissible. In fact, it does not satisfy (a) of Definition 1. Hence $B(\widetilde{F}, \widetilde{\mathsf{m}}) = \{3\}$ for $\mathsf{b\text{-}pol}(I')$.

For $F = \{i_1, \ldots, i_q\} \subset \mathbb{N}$ with $i_1 < \cdots < i_q$ and $\mathsf{m} \in G(I)$, Eliahou-Kervaire call the pair (F, m) admissible for I, if $i_q < \nu(\mathsf{m})$. In this case, there is a unique sequence $j_1, \ldots, j_q$ such that $(\widetilde{F}, \widetilde{\mathsf{m}})$ is admissible for $\widetilde{I}$, where $\widetilde{F} = \{\, (i_1, j_1), \ldots, (i_q, j_q)\,\}$. In this way, there is a one-to-one correspondence between the admissible pairs for I and those of $\widetilde{I}$. As the free summands of the Eliahou-Kervaire resolution of I are indexed by the admissible pairs for I, our resolution of $\widetilde{I}$ are indexed by the admissible pairs for $\widetilde{I}$.

We will define a $\mathbb{Z}^{n\times d}$-graded chain complex $\widetilde{P}_\bullet$ of free $\widetilde{S}$-modules as follows. First, set $\widetilde{P}_0 := \widetilde{S}$. For each $q \geq 1$, we set

$$A_q := \text{the set of admissible pairs } (\widetilde{F}, \widetilde{\mathsf{m}}) \text{ for } \mathsf{b\text{-}pol}(I) \text{ with } \#\widetilde{F} = q,$$

and

$$\widetilde{P}_q := \bigoplus_{(\widetilde{F},\widetilde{\mathsf{m}})\in A_{q-1}} \widetilde{S}\, e(\widetilde{F}, \widetilde{\mathsf{m}}),$$

where $e(\widetilde{F}, \widetilde{\mathsf{m}})$ is a basis element with

$$\deg\left(e(\widetilde{F}, \widetilde{\mathsf{m}})\right) = \deg\left(\widetilde{\mathsf{m}} \times \prod_{(i_r,j_r)\in\widetilde{F}} x_{i_r,j_r}\right) \in \mathbb{Z}^{n\times d}.$$

We define the $\widetilde{S}$-homomorphism $\partial : \widetilde{P}_q \to \widetilde{P}_{q-1}$ for $q \geq 2$ so that $e(\widetilde{F}, \widetilde{\mathsf{m}})$ with $\widetilde{F} = \{(i_1, j_1), \ldots, (i_q, j_q)\}$ is sent to

$$\sum_{1\leq r\leq q} (-1)^r \cdot x_{i_r,j_r} \cdot e(\widetilde{F}_r, \widetilde{\mathsf{m}}) - \sum_{r\in B(\widetilde{F},\widetilde{\mathsf{m}})} (-1)^r \cdot \frac{x_{i_r,j_r}\cdot\widetilde{\mathsf{m}}}{\widetilde{\mathsf{m}}_{\langle i_r\rangle}} \cdot e(\widetilde{F}_r, \widetilde{\mathsf{m}}_{\langle i_r\rangle}),$$

and $\partial : \widetilde{P}_1 \to \widetilde{P}_0$ by $e(\emptyset, \widetilde{\mathsf{m}}) \longmapsto \widetilde{\mathsf{m}} \in \widetilde{S} = \widetilde{P}_0$. Clearly, ∂ is a $\mathbb{Z}^{n\times d}$-graded homomorphism.

Set

$$\widetilde{P}_\bullet : \cdots \xrightarrow{\partial} \widetilde{P}_i \xrightarrow{\partial} \cdots \xrightarrow{\partial} \widetilde{P}_1 \xrightarrow{\partial} \widetilde{P}_0 \longrightarrow 0.$$

Theorem 5 ([14, Theorem 2.6])**.** *The complex $\widetilde{P}_\bullet$ is a $\mathbb{Z}^{n\times d}$-graded minimal $\widetilde{S}$-free resolution for $\widetilde{S}/\,\mathsf{b\text{-}pol}(I)$.*

Sketch of Proof. Calculation using Lemma 3 shows that $\partial\circ\partial(e(\widetilde{F}, \widetilde{\mathsf{m}})) = 0$ for each admissible pair $(\widetilde{F}, \widetilde{\mathsf{m}})$. That is, $\widetilde{P}_\bullet$ is a chain complex.

Let $I = (\mathsf{m}_1, \ldots, \mathsf{m}_t)$ with $\mathsf{m}_1 \succ \cdots \succ \mathsf{m}_t$, and set $I_r := (\mathsf{m}_1, \ldots, \mathsf{m}_r)$. Here $\succ$ is the lexicographic order with $x_1 \succ x_2 \succ \cdots \succ x_n$. Then I_r are also Borel fixed. The acyclicity of the complex $\widetilde{P}$ can be shown inductively by means of mapping cones. □

Remark 6. Herzog and Takayama [9] explicitly gave a minimal free resolution of a monomial ideal with *linear quotients* admitting a *regular decomposition function.* A Borel fixed ideal I satisfies this property. However, while $\widetilde{I}$ has linear quotients, the decomposition function can not be regular. Hence the method of [9] is not applicable to our case.

3. Applications and Remarks

Let $I \subset S$ be a Borel fixed ideal, and $\Theta \subset \widetilde{S}$ the sequence defined in Introduction. As remarked before, there is a one-to-one correspondence between the admissible pairs for $\widetilde{I}$ and those for I, and if $(\widetilde{F}, \widetilde{\mathsf{m}})$ corresponds to (F, m) then $\#\widetilde{F} = \#F$. Hence we have

$$\beta_{i,j}^{\widetilde{S}}(\widetilde{I}) = \beta_{i,j}^{S}(I) \tag{3.1}$$

for all i, j, where S and $\widetilde{S}$ are considered to be $\mathbb{Z}$-graded. Of course, this equation is clear, if one knows the fact that $\widetilde{I}$ is a polarization of I ([16, Theorem 3.4]). Conversely, we can show this fact by the equation (3.1) and [13, Lemma 6.9].

Corollary 7 ([16, Theorem 3.4])**.** *The ideal $\widetilde{I}$ is a polarization of I.*

The next result also follows from [13, Lemma 6.9].

Corollary 8. *$\widetilde{P}_\bullet \otimes_{\widetilde{S}} \widetilde{S}/(\Theta)$ is a minimal S-free resolution of S/I.*

Remark 9. (1) The correspondence between the admissible pairs for I and those for $\widetilde{I}$, does *not* give a chain map between the Eliahou-Kervaire resolution and our $\widetilde{P}_\bullet \otimes_{\widetilde{S}} \widetilde{S}/(\Theta)$. In this sense, two resolutions are not the same. See Example 21 below.

(2) The *lcm lattice* of I and that of $\widetilde{I}$ are not isomorphic in general. Recall that the lcm-lattice of a monomial ideal J is the set $\mathrm{LCM}(J) := \{ \mathrm{lcm}\{ \mathsf{m} \mid \mathsf{m} \in \sigma \} \mid \sigma \subset G(J) \}$ with the order given by divisibility. Clearly, $\mathrm{LCM}(J)$ is a lattice. For the Borel fixed ideal $I = (x^2, xy, xz, y^2, yz)$, we have $xy \vee xz = xy \vee yz = xz \vee yz = xyz$ in $\mathrm{LCM}(I)$. However, $\widetilde{xy} \vee \widetilde{xz} = x_1 y_2 z_2$, $\widetilde{xy} \vee \widetilde{yz} = x_1 y_1 y_2 z_2$ and $\widetilde{xz} \vee \widetilde{yz} = x_1 y_1 z_2$ are all distinct in $\mathrm{LCM}(\widetilde{I})$.

(3) Eliahou and Kervaire ([7]) constructed minimal free resolutions of *stable monomial ideals*, which form a wider class than Borel fixed ideals. However, $\mathsf{b\text{-}pol}(J)$ is not a polarization for a stable monomial ideal J in general, and our construction does not work.

Let $a = \{a_0, a_1, a_2, \dots\}$ be a non-decreasing sequence of non-negative integers with $a_0 = 0$, and $T = \Bbbk[x_1, \dots, x_N]$ a polynomial ring with $N \gg 0$. In his paper [12], Murai defined an operator $(-)^{\gamma(a)}$ acting on monomials and monomial ideals of S. For a monomial $\mathsf{m} \in S$ with the expression $\mathsf{m} = \prod_{i=1}^{e} x_{\alpha_i}$ as (1.1), set

$$\mathsf{m}^{\gamma(a)} := \prod_{i=1}^{e} x_{\alpha_i + a_{i-1}} \in T,$$

and for a monomial ideal $I \subset S$,

$$I^{\gamma(a)} := (\mathsf{m}^{\gamma(a)} \mid \mathsf{m} \in G(I)) \subset T.$$

If $a_{i+1} > a_i$ for all i, then $I^{\gamma(a)}$ is a squarefree monomial ideal. Particularly in the case $a_i = i$ for all i, $(-)^{\gamma(a)}$ is just $(-)^{\mathsf{sq}}$ mentioned in Introduction.

The operator $(-)^{\gamma(a)}$ also can be described by $\mathsf{b\text{-}pol}(-)$ as is shown in [16]. Let L_a be the $\Bbbk$-subspace of $\widetilde{S}$ spanned by $\{x_{i,j} - x_{i',j'} \mid i + a_{j-1} = i' + a_{j'-1}\}$, and Θ_a a basis of L_a. For example, we can take $\{x_{i,j} - x_{i+1,j-1} \mid 1 \le i < n,\ 1 < j \le d\}$ as Θ_a in the case $a_i = i$ for all i. With a suitable choice of the number N, the ring homomorphism $\widetilde{S} \to T$ with $x_{i,j} \mapsto x_{i+a_{j-1}}$ induces the isomorphism $\widetilde{S}/(\Theta_a) \cong T$.

Proposition 10 ([16, Proposition 4,1])**.** *With the above notation, Θ_a forms an $\widetilde{S}/\widetilde{I}$-regular sequence, and we have $(\widetilde{S}/(\Theta_a)) \otimes_{\widetilde{S}} (\widetilde{S}/\widetilde{I}) \cong T/I^{\gamma(a)}$.*

Applying Proposition 10 and [5, Proposition 1.1.5], we have the following.

Corollary 11. *The complex $\widetilde{P}_\bullet \otimes_{\widetilde{S}} \widetilde{S}/(\Theta_a)$ is a minimal T-free resolution of $T/I^{\gamma(a)}$. In particular, a minimal free resolution of T/I^{sq} is given in this way.*

For a Borel fixed ideal I generated in one degree, Nagel and Reiner [13] constructed a CW complex, which supports a minimal free resolution of $\widetilde{I}$ (or I, I^{sq}).

Proposition 12 ([14, Proposition 4.9]). *Let I be a Borel fixed ideal generated in one degree. Then Nagel-Reiner description of a free resolution of $\widetilde{I}$ coincides with our $\widetilde{P}_\bullet$.*

We do not give a proof of the above proposition here, but just remark that if I is generated in one degree then $\mathsf{m}_{\langle i \rangle} = \mathfrak{b}_i(\mathsf{m})$ for all $\mathsf{m} \in G(I)$ and $\widetilde{P}_\bullet$ becomes simpler.

4. Relation to Batzies-Welker theory

In [2], Batzies and Welker connected the theory of *cellular resolutions* of monomial ideals with Forman's discrete Morse theory ([8]).

Definition 13. A monomial ideal J is called *shellable* if there is a total order $\sqsubset$ on $G(J)$ satisfying the following condition.

(∗) For any $\mathsf{m}, \mathsf{m}' \in G(J)$ with $\mathsf{m} \sqsupset \mathsf{m}'$, there is an $\mathsf{m}'' \in G(J)$ such that $\mathsf{m} \sqsupseteq \mathsf{m}''$, $\deg\left(\frac{\operatorname{lcm}(\mathsf{m},\mathsf{m}'')}{\mathsf{m}}\right) = 1$ and $\operatorname{lcm}(\mathsf{m}, \mathsf{m}'')$ divides $\operatorname{lcm}(\mathsf{m}, \mathsf{m}')$.

For a Borel fixed ideal I, let $\sqsubset$ be the total order on $G(\widetilde{I}) = \{\, \widetilde{\mathsf{m}} \mid \mathsf{m} \in G(I) \,\}$ such that $\widetilde{\mathsf{m}}' \sqsubset \widetilde{\mathsf{m}}$ if and only if $\mathsf{m}' \succ \mathsf{m}$ in the lexicographic order on S with $x_1 \succ x_2 \succ \cdots \succ x_n$. In the rest of this section, $\sqsubset$ means this order.

Lemma 14. *The order $\sqsubset$ makes $\widetilde{I}$ shellable.*

The following construction is taken from [2, Theorems 3.2 and 4.3]. For the background of their theory, the reader is recommended to consult the original paper.

For $\emptyset \neq \sigma \subset G(\widetilde{I})$, let $\widetilde{\mathsf{m}}_\sigma$ denote the largest element of σ with respect to the order $\sqsubset$, and set $\operatorname{lcm}(\sigma) := \operatorname{lcm}\{\, \widetilde{\mathsf{m}} \mid \widetilde{\mathsf{m}} \in \sigma \,\}$.

Definition 15. We define a total order $\prec_\sigma$ on $G(\widetilde{I})$ as follows. Set

$$N_\sigma := \{\, (\widetilde{\mathsf{m}}_\sigma)_{\langle i \rangle} \mid 1 \leq i < \nu(\mathsf{m}_\sigma),\ (\widetilde{\mathsf{m}}_\sigma)_{\langle i \rangle} \text{ divides } \operatorname{lcm}(\sigma) \,\}.$$

For all $\widetilde{\mathsf{m}} \in N_\sigma$ and $\widetilde{\mathsf{m}}' \in G(\widetilde{I}) \setminus N_\sigma$, define $\widetilde{\mathsf{m}} \prec_\sigma \widetilde{\mathsf{m}}'$. The restriction of $\prec_\sigma$ to N_σ is set to be $\sqsubset$, and the same is true for the restriction to $G(\widetilde{I}) \setminus N_\sigma$.

Let X be the $(\#G(\widetilde{I}) - 1)$-simplex associated with $2^{G(\widetilde{I})}$ (more precisely, $2^{G(\widetilde{I})} \setminus \{\emptyset\}$). Hence we freely identify $\sigma \subset G(\widetilde{I})$ with the corresponding cell of the simplex X. Let G_X be the directed graph defined as follows. The vertex set of G_X is $2^{G(\widetilde{I})} \setminus \{\emptyset\}$. For $\emptyset \neq \sigma, \sigma' \subset G(\widetilde{I})$, there is an arrow $\sigma \to \sigma'$ if and only if $\sigma \supset \sigma'$ and $\#\sigma = \#\sigma' + 1$. For $\sigma = \{\, \widetilde{\mathsf{m}}_1, \widetilde{\mathsf{m}}_2, \dots, \widetilde{\mathsf{m}}_k \,\}$ with $\widetilde{\mathsf{m}}_1 \prec_\sigma \widetilde{\mathsf{m}}_2 \prec_\sigma \cdots \prec_\sigma \widetilde{\mathsf{m}}_k \,(= \widetilde{\mathsf{m}}_\sigma)$ and $l \in \mathbb{N}$ with $1 \leq l < k$, set $\sigma_l := \{\, \widetilde{\mathsf{m}}_{k-l}, \widetilde{\mathsf{m}}_{k-l+1}, \dots, \widetilde{\mathsf{m}}_k \,\}$ and

$$u(\sigma) := \sup\{\, l \mid \exists \widetilde{\mathsf{m}} \in G(\widetilde{I}) \text{ s.t. } \widetilde{\mathsf{m}} \prec_\sigma \widetilde{\mathsf{m}}_{k-l} \text{ and } \widetilde{\mathsf{m}} | \operatorname{lcm}(\sigma_l) \,\}.$$

If $u := u(\sigma) \neq -\infty$, we can define $\widetilde{\mathsf{n}}_\sigma := \min_{\prec_\sigma}\{\, \widetilde{\mathsf{m}} \mid \widetilde{\mathsf{m}} \text{ divides } \operatorname{lcm}(\sigma_u) \,\}$. Let E_X be the set of edges of G_X. We define a subset A of E_X by

$$A := \{\, \sigma \cup \{\widetilde{\mathsf{n}}_\sigma\} \to \sigma \mid u(\sigma) \neq -\infty, \widetilde{\mathsf{n}}_\sigma \notin \sigma \,\}.$$

It is easy to see that A is a *matching*, that is, every σ occurs in at most one edges of A. We say $\emptyset \neq \sigma \subset G(\widetilde{I})$ is *critical*, if it does not occurs in any edge of A.

We have the directed graph G_X^A with the vertex set $2^{G(\widetilde{I})} \setminus \{\emptyset\}$ (i.e., same as G_X) and the set of edges $(E_X \setminus A) \cup \{\, \sigma \to \tau \mid (\tau \to \sigma) \in A \,\}$. By the proof of [2, Theorem 3.2], we see that the matching A is *acyclic*, that is, G_X^A has no directed cycle. A directed path in G_X^A is called a *gradient path*.

Forman's discrete Morse theory [8] guarantees the existence of a CW complex X_A with the following conditions.

- There is a one-to-one correspondence between the i-cells of X_A and the *critical* i-cells of X (equivalently, the critical subsets of $G(\widetilde{I})$ consisting of $i+1$ elements).
- X_A is contractible, that is, homotopy equivalent to X.

The cell of X_A corresponding to a critical cell σ of X is denoted by σ_A. By [2, Proposition 7.3], the closure of σ_A contains τ_A if and only if there is a gradient path from σ to τ. See also Proposition 18 below and the argument before it.

Assume that $\emptyset \neq \sigma \subset G(\widetilde{I})$ is critical. Recall that $\widetilde{\mathsf{m}}_\sigma$ denotes the largest element of σ with respect to $\sqsubset$. Take $\mathsf{m}_\sigma = \prod_{l=1}^n x_l^{a_l} \in G(I)$ with $\widetilde{\mathsf{m}}_\sigma = \mathsf{b\text{-}pol}(\mathsf{m}_\sigma)$, and set $q := \#\sigma - 1$. Then there are integers $i_1, \ldots, i_q$ with $1 \leq i_1 < \ldots < i_q < \nu(\mathsf{m}_\sigma)$ and

$$\sigma = \{\, (\widetilde{\mathsf{m}}_\sigma)_{\langle i_r \rangle} \mid 1 \leq r \leq q \,\} \cup \{\widetilde{\mathsf{m}}_\sigma\} \tag{4.1}$$

(see the proof of [2, Proposition 4.3]). Equivalently, we have $\sigma = N_\sigma \cup \{\widetilde{\mathsf{m}}_\sigma\}$. Set $j_r := 1 + \sum_{l=1}^{i_r} a_l$ for each $1 \leq r \leq q$, and $\widetilde{F}_\sigma := \{\, (i_1, j_1), \ldots, (i_q, j_q) \,\}$. Then $(\widetilde{F}_\sigma, \widetilde{\mathsf{m}}_\sigma)$ is an admissible pair for $\widetilde{I}$. Conversely, any admissible pair comes from a critical cell $\sigma \subset G(\widetilde{I})$ in this way. Hence there is a one-to-one correspondence between critical cells and admissible pairs.

Let X_A^i denote the set of all the critical subset $\sigma \subset G(\widetilde{I})$ with $\#\sigma = i+1$, and for (not necessarily critical) subsets σ, τ of $G(\widetilde{I})$, let $\mathcal{P}_{\sigma,\tau}$ denote the set of all the gradient paths from σ to τ. For $\sigma \in X_A^q$ of the form (4.1), $e(\sigma)$ denotes a basis element with degree $\deg(\mathrm{lcm}(\sigma)) \in \mathbb{Z}^{n \times d}$. Set

$$\widetilde{Q}_q = \bigoplus_{\sigma \in X_A^q} \widetilde{S}\, e(\sigma) \qquad (q \geq 0).$$

The differential map $\widetilde{Q}_q \to \widetilde{Q}_{q-1}$ sends $e(\sigma)$ to

$$\sum_{r=1}^{q} (-1)^r x_{i_r, j_r} \cdot e(\sigma \setminus \{(\widetilde{\mathsf{m}}_\sigma)_{\langle i_r \rangle}\}) - (-1)^q \sum_{\substack{\tau \in X_A^{q-1} \\ \mathcal{P} \in \mathcal{P}_{\sigma \setminus \{\widetilde{\mathsf{m}}_\sigma\}, \tau}}} m(\mathcal{P}) \cdot \frac{\mathrm{lcm}(\sigma)}{\mathrm{lcm}(\tau)} \cdot e(\tau), \tag{4.2}$$

where $m(\mathcal{P}) = \pm 1$ is the one defined in [2, p.166].

The following is a direct consequence of [2, Theorem 4.3] (and [2, Remark 4.4]).

Proposition 16 (Batzies-Welker, [2]). *$\widetilde{Q}_\bullet$ is a minimal free resolution of $\widetilde{I}$, and has a cellular structure supported by X_A.*

Theorem 17 ([14, Theorem 5.11]). *Our description of $\widetilde{P}_\bullet$ (more precisely, the truncation $\widetilde{P}_{\geq 1}$) coincides with the Batzies-Welker resolution $\widetilde{Q}_\bullet$. That is, $\widetilde{P}_\bullet$ is a cellular resolution supported by a CW complex X_A, which is obtained by discrete Morse theory.*

First, note that the following hold.

(1) If σ is critical, so is $\sigma \setminus \{ (\widetilde{\mathsf{m}}_\sigma)_{\langle i_r \rangle} \}$ for $1 \le r \le q$.
(2) Let σ and τ be (not necessarily critical) cells with $\mathcal{P}_{\sigma,\tau} \neq \varnothing$. Then $\operatorname{lcm}(\tau)$ divides $\operatorname{lcm}(\sigma)$.
(3) Let $\sigma \in X_A^q$, $\tau \in X_A^{q-1}$ and assume that there is a gradient path $\sigma \to \sigma \setminus \{\widetilde{\mathsf{m}}\} = \sigma_0 \to \sigma_1 \to \cdots \to \sigma_l = \tau$. Then $\#\sigma_{l-1} = \#\tau + 1 = q+1$, $\#\sigma_i = q$ or $q+1$ for each i, and σ_i is not critical for all $0 \le i < l$. Hence, if $l > 1$, then $\widetilde{\mathsf{m}}$ must be $\widetilde{\mathsf{m}}_\sigma$.

Next, we will show the following.

Proposition 18. *Let σ, τ be critical cells with $\#\sigma = \#\tau + 1$, and $(\widetilde{F}_\sigma, \widetilde{\mathsf{m}}_\sigma)$ and $(\widetilde{F}_\tau, \widetilde{\mathsf{m}}_\tau)$ the admissible pairs corresponding to σ and τ respectively. Set $\widetilde{F}_\sigma = \{ (i_1, j_1), \dots, (i_q, j_q) \}$ with $i_1 < \cdots < i_q$. Then $\mathcal{P}_{\sigma \setminus \{\widetilde{\mathsf{m}}_\sigma\}, \tau} \neq \varnothing$ if and only if there is some $r \in B(\widetilde{F}_\sigma, \widetilde{\mathsf{m}}_\sigma)$ with $(\widetilde{F}_\tau, \widetilde{\mathsf{m}}_\tau) = ((\widetilde{F}_\sigma)_r, (\widetilde{\mathsf{m}}_\sigma)_{\langle i_r \rangle})$. If this is the case, we have $\#\mathcal{P}_{\sigma \setminus \{\widetilde{\mathsf{m}}_\sigma\}, \tau} = 1$.*

Sketch of Proof. Only if part follows from the above remark. Note that the second index j of each $x_{i,j} \in \widetilde{S}$ restricts the choice of paths and it makes the proof easier.

Next, assuming $\widetilde{F}_\tau = (\widetilde{F}_\sigma)_r$ and $\widetilde{\mathsf{m}}_\tau = (\widetilde{\mathsf{m}}_\sigma)_{\langle i_r \rangle}$ for some $r \in B(\widetilde{F}_\sigma, \widetilde{\mathsf{m}}_\sigma)$, we will construct a gradient path from $\sigma \setminus \{\widetilde{\mathsf{m}}_\sigma\}$ to τ. For short notation, set $\widetilde{\mathsf{m}}_{[s]} := (\widetilde{\mathsf{m}}_\sigma)_{\langle i_s \rangle}$ and $\widetilde{\mathsf{m}}_{[s,t]} := ((\widetilde{\mathsf{m}}_\sigma)_{\langle i_s \rangle})_{\langle i_t \rangle}$. By (4.1), we have $\sigma_0 := (\sigma \setminus \{\widetilde{\mathsf{m}}_\sigma\}) = \{ \widetilde{\mathsf{m}}_{[s]} \mid 1 \le s \le q \}$ and $\tau = \{ \widetilde{\mathsf{m}}_{[r,s]} \mid 1 \le s \le q, s \neq r \} \cup \{\widetilde{\mathsf{m}}_{[r]}\}$. We can inductively construct a gradient path $\sigma_0 \to \sigma_1 \to \cdots \to \sigma_t \to \cdots \sigma_{2(q-r+1)r-2}$ as follows. Write $t = 2pr + \lambda$ with $t \neq 0$, $0 \le p \le q-r$, and $0 \le \lambda < 2r$. For $0 < t \le 2(q-r)$, we set

$$\sigma_t = \begin{cases} \sigma_{t-1} \cup \{ \widetilde{\mathsf{m}}_{[q-p,s]} \} & \text{if } \lambda = 2s-1 \text{ for some } 1 \le s \le r; \\ \sigma_{t-1} \setminus \{ \widetilde{\mathsf{m}}_{[q-p+1,s]} \} & \text{if } \lambda = 2s \text{ for some } 0 < s < r; \\ \sigma_t \setminus \{ \widetilde{\mathsf{m}}_{[q-p+1]} \} & \text{if } \lambda = 0, \end{cases}$$

where we set $\widetilde{\mathsf{m}}_{[q+1,s]} = \widetilde{\mathsf{m}}_{[s]}$ for all s. In the case $\widetilde{\mathsf{m}}_{[s,t]} = \widetilde{\mathsf{m}}_{[s+1,t]}$, it seems to cause a problem, but skipping the corresponding part of path, we can avoid the problem. Since $r \in B(\widetilde{F}_\sigma, \widetilde{\mathsf{m}}_\sigma)$, we have $\widetilde{\mathsf{m}}_{[s,r]} = \widetilde{\mathsf{m}}_{[r,s]}$ for all $s > r$ by Lemma 3 (iv). Hence

$$\sigma_{2(q-r)} = \{ \widetilde{\mathsf{m}}_{[r+1,s]} \mid 1 \le s < r \} \cup \{ \widetilde{\mathsf{m}}_{[r]} \} \cup \{ \widetilde{\mathsf{m}}_{[r,s]} \mid r < s \le q \}.$$

Now for s with $0 < s \le r-1$, set σ_t with $2(q-r)r < t \le 2(q-r+1)r-2$ to be $\sigma_{t-1} \cup \{ \widetilde{\mathsf{m}}_{[r,s]} \}$ if s is odd and otherwise $\sigma_{t-1} \setminus \{ \widetilde{\mathsf{m}}_{[r+1,s]} \}$. Then we have $\sigma_{2(q-r+1)r-2} = \tau$, and the gradient path $\sigma \rightsquigarrow \tau$.

The uniqueness of the path follows from elementally (but lengthy) argument. □

Sketch of Proof of Theorem 17. Recall that there is the one-to-one correspondence between the critical cells $\sigma \subset G(\widetilde{I})$ and the admissible pairs $(\widetilde{F}_\sigma, \widetilde{\mathsf{m}}_\sigma)$. Hence, for each q, we have the isomorphism $\widetilde{Q}_q \to \widetilde{P}_q$ induced by $e(\sigma) \longmapsto e(\widetilde{F}_\sigma, \widetilde{\mathsf{m}}_\sigma)$.

By Proposition 18, if we forget "coefficients", the differential map of $\widetilde{Q}_\bullet$ and that of $\widetilde{P}_\bullet$ are compatible with the maps $e(\sigma) \longmapsto e(\widetilde{F}_\sigma, \widetilde{\mathsf{m}}_\sigma)$. So it is enough to check the equality of the coefficients. But it follows from direct computation. □

Corollary 19 ([14, Corollary 5.12]). *The free resolution $\widetilde{P}_\bullet \otimes_{\widetilde{S}} \widetilde{S}/(\Theta)$ (resp. $\widetilde{P}_\bullet \otimes_{\widetilde{S}} \widetilde{S}/(\Theta_a)$) of S/I (resp. $T/I^{\gamma(a)}$) is also a cellular resolution supported by X_A. In particular, these resolutions are Batzies-Welker type.*

We say a CW complex is *regular*, if for all i the closure $\overline{\sigma}$ of any i-cell σ is homeomorphic to an i-dimensional closed ball, and $\overline{\sigma} \setminus \sigma$ is the closure of the union of some $(i-1)$-cells. This is a natural condition especially in combinatorics.

Mermin [11] (see also Clark [6]) showed that the Eliahou-Kervaire resolution is cellular and supported by a regular CW complex. Hence it is a natural question whether the CW complex X_A supporting our $\widetilde{P}_\bullet$ is regular. (Since the discrete Morse theory is an "existence theorem" and X_A might not be unique, the *correct* statement is "*can be* regular". This is a non-trivial point, but here we do not show how to avoid it).

Theorem 20 ([15])**.** *The CW complex X_A of Theorem 17 is regular. In particular, our resolution $\widetilde{P}_\bullet$ is supported by a regular CW complex.*

Sketch of Proof. We basically follow Clark [6], which proves the corresponding statement for the Eliahou-Kervaire resolution.

We define a finite poset P_A as follows:

(i) As the underlying set, $P_A =$ (the set of the cells of X_A) $\cup \{\hat{0}\}$. Here $\hat{0}$ is the least element.

(ii) For cells σ and τ of X_A, $\sigma \succeq \tau$ in P_A if and only if the closure of σ contains τ.

It suffices to show that P_A is a *CW poset* in the sense of [4], and we can use [4, Proposition 5.5]. By the behavior of the differential map of $\widetilde{P}_\bullet$, we can check that P_A satisfies the following condition.

- For $\sigma, \tau \in P_A$ with $\sigma \succ \tau$ and $\operatorname{rank}(\sigma) = \operatorname{rank}(\tau) + 2$, there are exactly two elements between σ and τ.

Now it remains to show that the interval $[\,\hat{0}, \sigma\,]$ is shellable for all σ, but we can imitate the argument of Clark [6]. In fact, $[\,\hat{0}, \sigma\,]$ is *EL shellable* in the sense of [3]. □

Example 21.

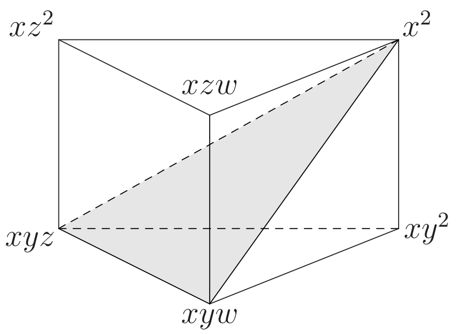


FIGURE 1

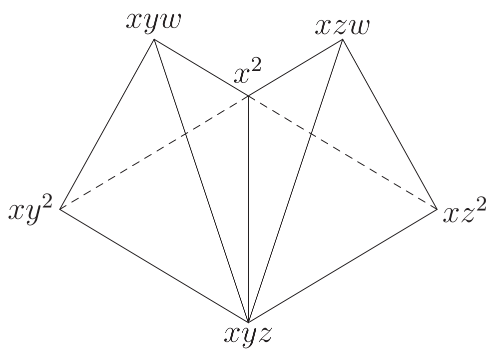


FIGURE 2

Consider the Borel fixed ideal $I = (x^2, xy^2, xyz, xyw, xz^2, xzw)$. Then $\mathsf{b\text{-}pol}(I) = (x_1x_2, x_1y_2y_3, x_1y_2z_3, x_1y_2w_3, x_1z_2z_3, x_1z_3w_3)$, and easy computation shows that the CW complex X_A, which supports our resolutions $\widetilde{P}_\bullet$ of $\widetilde{S}/\widetilde{I}$ and $\widetilde{P}_\bullet \otimes_{\widetilde{S}} \widetilde{S}/(\Theta)$ of S/I, is the one illustrated in Figure 1.

The complex consists of a square pyramid and a tetrahedron glued along trigonal faces of each. For a Borel fixed ideal generated in one degree, any face of the Nagel-Reiner CW complex is a product of several simplices. Hence a square pyramid can not appear in the case of Nagel and Reiner.

We remark that the Eliahou-Kervaire resolution of I is supported by the CW complex illustrated in Figure 2. This complex consists of two tetrahedrons glued along edges of each. These figures show visually that the description of the Eliahou-Kervaire resolution and that of ours are really different.

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SHARP BOUNDS FOR HILBERT COEFFICIENTS OF PARAMETERS

KAZUHO OZEKI

Abstract. Let A be a Noetherian local ring with $d = \dim A > 0$. This paper shows that the Hilbert coefficients $\{\mathrm{e}_Q^i(A)\}_{1 \le i \le d}$ of parameter ideals Q have uniform bounds if and only if A is a generalized Cohen-Macaulay ring. The uniform bounds are huge; the sharp bound for $\mathrm{e}_Q^2(A)$ in the case where A is a generalized Cohen-Macaulay ring with $\dim A \ge 3$ is given.

Key Words: commutative algebra, generalized Cohen-Macaulay local ring, Hilbert coefficient, Castelnuovo-Mumford regularity.

2000 *Mathematics Subject Classification*: Primary 13D40; Secondary 13H10

1. Introduction

This is based on [5] a joint work with Shiro Goto.

The purpose of this paper is to study the problem of when the Hilbert coefficients of parameter ideals in a Noetherian local ring have uniform bounds, and when this is the case, to ask for their sharp bounds.

To state the problem and the results also, let us fix some notation. In what follows, let A be a commutative Noetherian local ring with maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$ denotes the Krull dimension of A. For simplicity, we assume that the residue class field $A/\mathfrak{m}$ of A is infinite. Let $\ell_A(M)$ denote, for an A-module M, the length of M. Then for each $\mathfrak{m}$-primary ideal I in A, we have integers $\{\mathrm{e}_I^i(A)\}_{0 \le i \le d}$ such that the equality

$$\ell_A(A/I^{n+1}) = \mathrm{e}_I^0(A)\binom{n+d}{d} - \mathrm{e}_I^1(A)\binom{n+d-1}{d-1} + \cdots + (-1)^d \mathrm{e}_I^d(A)$$

holds true for all $n \gg 0$, which we call the Hilbert coefficients of A with respect to I.

With this notation our first purpose is to study the problem of when the sets

$$\Lambda_i(A) = \{\mathrm{e}_Q^i(A) \mid Q \text{ is a parameter ideal in } A\}$$

are finite for all $1 \le i \le d$.

Then the first main result is stated as follows. We say that our local ring is a generalized Cohen-Macaulay ring, if the local cohomology modules $\mathrm{H}_{\mathfrak{m}}^i(A)$ are finitely generated for all $i \ne d$.

Theorem 1. *Let A be a commutative Noetherian local ring with $d = \dim A \ge 2$. Then the following conditions are equivalent.*

(1) *A is a generalized Cohen-Macaulay ring.*
(2) *The set $\Lambda_i(A)$ is finite for all $1 \le i \le d$.*

The detailed version of this paper has been submitted for publication elsewhere.

Although the finiteness problem of $\Lambda_i(A)$ is settled affirmatively, we need to ask for the sharp bounds for the values of $\mathrm{e}_Q^i(A)$ of parameter ideals Q, which is our second purpose of the present research. Let $h^i(A) = \ell_A(\mathrm{H}_\mathfrak{m}^i(A))$ for each $i \in \mathbb{Z}$.

When A is a generalized Cohen-Macaulay ring with $d = \dim A \geq 2$, one has the inequalities

$$0 \geq \mathrm{e}_Q^1(A) \geq -\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(A)$$

for every parameter ideal Q in A ([9, Theorem 8], [3, Lemma 2.4]), where the equality $\mathrm{e}_Q^1(A) = -\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(A)$ holds true if and only if Q is a standard parameter ideal in A ([10, Korollar 3.2], [4, Theorem 2.1]), provided $\operatorname{depth} A > 0$. The reader may consult [2] for the characterization of local rings which contain parameter ideals Q with $\mathrm{e}_Q^1(A) = 0$. Thus the behavior of the first Hilbert coefficients $\mathrm{e}_Q^1(A)$ for parameter ideals Q are rather satisfactorily understood.

The second purpose is to study the natural question of how about $\mathrm{e}_Q^2(A)$. First, we will show that in the case where $\dim A = 2$ and $\operatorname{depth} A > 0$, even though A is not necessarily a generalized Cohen-Macaulay ring, the inequality

$$-h^1(A) \leq \mathrm{e}_Q^2(A) \leq 0$$

holds true for every parameter ideal Q in A. We will also show that $\mathrm{e}_Q^2(A) = 0$ if and only if the ideal Q is generated by a system a, b of parameters which forms a d-sequence in A in the sense of C. Huneke [7]. When A is a generalized Cohen-Macaulay ring with $\dim A \geq 3$, we shall show that the inequality

$$-\sum_{j=2}^{d-1} \binom{d-3}{j-2} h^j(A) \leq \mathrm{e}_Q^2(A) \leq \sum_{j=1}^{d-2} \binom{d-3}{j-1} h^j(A)$$

holds true for every parameter ideal Q (Theorem 13). The following theorem which is the second main result of this paper shows that the upper bound $\mathrm{e}_Q^2(A) \leq \sum_{j=1}^{d-2} \binom{d-3}{j-1} h^j(A)$ is sharp, clarifying when the equality $\mathrm{e}_Q^2(A) = \sum_{j=1}^{d-2} \binom{d-3}{j-1} h^j(A)$ holds true.

Theorem 2. *Suppose that A is a generalized Cohen-Macaulay ring with $d = \dim A \geq 3$ and $\operatorname{depth} A > 0$. Let Q be a parameter ideal in A. Then the following two conditions are equivalent.*

(1) $\mathrm{e}_Q^2(A) = \sum_{j=1}^{d-2} \binom{d-3}{j-1} h^j(A)$.

(2) *There exist elements $a_1, a_2, \cdots, a_d \in A$ such that*
  - (a) $Q = (a_1, a_2, \cdots, a_d)$,
  - (b) *the sequence $a_1, a_2, \cdots, a_d$ is a d-sequence in A, and*
  - (c) $Q{\cdot}\mathrm{H}_\mathfrak{m}^j(A/(a_1, a_2, \cdots, a_k)) = (0)$ *for all $j \geq 1$ and $k \geq 0$ with $j + k \leq d - 2$.*

When this is the case, we furthermore have the following :

(i) $(-1)^i{\cdot}\mathrm{e}_Q^i(A) = \sum_{j=1}^{d-i} \binom{d-i-1}{j-1} h^j(A)$ *for $3 \leq i \leq d-1$ and*

(ii) $\mathrm{e}_Q^d(A) = 0$.

At this moment we do not know the sharp uniform bound for $\mathrm{e}_Q^3(A)$ for parameter ideals Q in a generalized Cohen-Macaulay ring A with $\dim A \geq 3$.

Let us briefly note how this paper is organized. We shall prove Theorem 1 in Section 2. Theorem 2 will be proven in Section 4. Section 3 is devoted to some preliminary steps for the proof of Theorem 2. We will closely study in Section 3 the problem of when $\mathrm{e}_Q^2(A) = 0$ in the case where $\dim A = 2$.

In what follows, unless otherwise specified, for each $\mathfrak{m}$-primary ideal I in A, we put

$$\mathrm{R}(I) = A[It], \quad \mathrm{R}'(I) = A[It, t^{-1}], \quad \text{and} \quad \mathrm{G}(I) = \mathrm{R}'(I)/t^{-1}\mathrm{R}'(I),$$

where t is an indeterminate over A. Let $\mathcal{M} = \mathfrak{m}R + R_+$ be the unique graded maximal ideal in $R = \mathrm{R}(I)$. We denote by $\mathrm{H}^i_{\mathcal{M}}(*)$ $(i \in \mathbb{Z})$ the $i^{\underline{th}}$ local cohomology functor of $\mathrm{R}(I)$ with respect to $\mathcal{M}$. Let L be a graded R-module. For each $n \in \mathbb{Z}$ let $[\mathrm{H}^i_{\mathcal{M}}(L)]_n$ stand for the homogeneous component of $\mathrm{H}^i_{\mathcal{M}}(L)$ with degree n. We denote by $L(\alpha)$, for each $\alpha \in \mathbb{Z}$, the graded R-module whose grading is given by $[L(\alpha)]_n = L_{\alpha+n}$ for all $n \in \mathbb{Z}$.

2. Proof of Theorem 1

In this section, we shall prove Theorem 1.

The heart of the proof of the implication (1) $\Rightarrow$ (2) is, in the case where A is a generalized Cohen-Macaulay ring, the existence of uniform bounds of the Castelnuovo-Mumford regularity $\operatorname{reg}\mathrm{G}(Q)$ of the associated graded rings $\mathrm{G}(Q)$ of parameter ideals Q. So, let us briefly recall the definition of the Castelnuovo-Mumford regularity.

Let Q be a parameter ideal in A and let

$$\mathrm{R}(Q) = A[Qt], \quad \mathrm{R}'(Q) = A[Qt, t^{-1}], \quad \text{and} \quad \mathrm{G}(Q) = \mathrm{R}'(Q)/t^{-1}\mathrm{R}'(Q)$$

respectively, denote the Rees algebra, the extended Rees algebra, and the associated graded ring of Q. Let $\mathcal{M} = \mathfrak{m}R + R_+$ be the unique graded maximal ideal in $R = \mathrm{R}(Q)$. For each $i \in \mathbb{Z}$ let

$$\mathrm{a}_i(\mathrm{G}(Q)) = \max\{n \in \mathbb{Z} \mid [\mathrm{H}^i_{\mathcal{M}}(\mathrm{G}(Q))]_n \neq (0)\}$$

and put

$$\mathrm{reg}\mathrm{G}(Q) = \max\{\mathrm{a}_i(\mathrm{G}(Q)) + i \mid i \in \mathbb{Z}\},$$

which we call the Castelnuovo-Mumford regularity of the graded ring $\mathrm{G}(Q)$.

Let us now note the following result of Linh and Trung [8], which gives a uniform bound for $\operatorname{reg}\mathrm{G}(Q)$ for parameter ideals Q in a generalized Cohen-Macaulay ring.

Theorem 3 ([8], Theorem 2.3)**.** *Suppose that A is a generalized Cohen-Macaulay ring and let Q be a parameter ideal in A. Then*

(1) $\operatorname{reg}\mathrm{G}(Q) \leq \max\{\mathrm{I}(A) - 1, 0\}$*, if* $d = 1$.
(2) $\operatorname{reg}\mathrm{G}(Q) \leq \max\{(4\mathrm{I}(A))^{(d-1)!} - \mathrm{I}(A) - 1, 0\}$*, if* $d \geq 2$.

Thus, the following result is the key for our proof of the implication (1) $\Rightarrow$ (2) in Theorem 1, where $h_i(A) = \ell_A(\mathrm{H}^i_{\mathfrak{m}}(A))$ and $\mathrm{I}(A) = \sum_{j=0}^{d-1}\binom{d-1}{j}h^j(A)$.

Theorem 4. *Suppose that A is a generalized Cohen-Macaulay ring. Let Q be a parameter ideal in A and put $r = \operatorname{reg}\mathrm{G}(Q)$. Then*

(1) $|\mathrm{e}^1_Q(A)| \leq \mathrm{I}(A)$.
(2) $|\mathrm{e}^i_Q(A)| \leq 3 \cdot 2^{i-2}(r+1)^{i-1}\mathrm{I}(A)$ *for* $2 \leq i \leq d$.

Proof. See [5, Section 2]. □

Therefore, thanks to the uniform bounds [8, Theorem 2.3] of $\operatorname{reg} \mathrm{G}(Q)$ for parameter ideals Q in a generalized Cohen-Macaulay ring A, we readily get the finiteness in the set $\Lambda_i(A)$ for all $1 \le i \le d$.

We are now in a position to finish the proof of Theorem 1.

Proof of Theorem 1. We may assume that A is complete. Also we may assume A is not unmixed, because $\Lambda_1(A)$ is a finite set (cf. [2, Proposition 4.2]). Let U denote the unmixed component of the ideal (0) in A. We put $B = A/U$ and $t = \dim_A U$ $(\le d-1)$. We must show that B is a generalized Cohen-Macaulay ring and $t = 0$.

Let Q be a parameter ideal in A. We then have

$$\ell_A(A/Q^{n+1}) = \ell_A(B/Q^{n+1}B) + \ell_A(U/Q^{n+1} \cap U)$$

for all integers $n \ge 0$. Therefore, the function $\ell_A(U/Q^{n+1} \cap U)$ is a polynomial in $n \gg 0$ with degree t and there exist integers $\{\mathrm{s}_Q^i(U)\}_{0 \le i \le t}$ with $\mathrm{s}_Q^0(U) = \mathrm{e}_Q^0(U)$ such that

$$\ell_A(U/Q^{n+1} \cap U) = \sum_{i=0}^{t} (-1)^i \mathrm{s}_Q^i(U) \binom{n+t-i}{t-i}$$

for all $n \gg 0$, whence

$$\ell_A(A/Q^{n+1}) = \sum_{i=0}^{d} (-1)^i \mathrm{e}_Q^i(B) \binom{n+d-i}{d-i} + \sum_{i=0}^{t} (-1)^i \mathrm{s}_Q^i(U) \binom{n+t-i}{t-i}.$$

Consequently

$$(-1)^{d-i}\mathrm{e}_Q^{d-i}(A) = \begin{cases} (-1)^{d-i}\mathrm{e}_Q^{d-i}(B) + (-1)^{t-i}\mathrm{s}_Q^{t-i}(U) & \text{if } \ 0 \le i \le t, \\ (-1)^{d-i}\mathrm{e}_Q^{d-i}(B) & \text{if } \ t+1 \le i \le d. \end{cases}$$

Therefore, if $t < d-1$, we have $\mathrm{e}_Q^1(A) = \mathrm{e}_Q^1(B)$, so that $\Lambda_1(B) = \Lambda_1(A)$ is a finite set. If $t = d-1$, we get $-\mathrm{e}_Q^1(A) = -\mathrm{e}_Q^1(B) + \mathrm{s}_Q^0(U)$. Since $\mathrm{e}_Q^1(A), \mathrm{e}_Q^1(B) \le 0$ and $\mathrm{s}_Q^0(U) = \mathrm{e}_Q^0(U) \ge 1$, $\Lambda_1(B)$ is a finite set also in this case. Thus the set $\Lambda_1(B)$ is finite in any case, so that the ring B a generalized Cohen-Macaulay ring.

We now assume that $t \ge 1$ and choose a system $a_1, a_2, \cdots, a_d$ of parameters in A so that $(a_{t+1}, a_{t+2}, \cdots, a_d)U = (0)$. Let $\ell \ge 1$ be an integer such that $\mathfrak{m}^\ell$ is standard for the ring B and choose integers $n \ge \ell$. We look at parameter ideals $Q = (a_1^n, a_2^n, \cdots, a_d^n)$ of A. Then

$$(-1)^{d-t}\mathrm{e}_Q^{d-t}(B) = \sum_{j=1}^{t} \binom{t-1}{j-1} h^j(B)$$

by [10, Korollar 3.2], which is independent of the integers $n \ge \ell$. Therefore, since

$$\mathrm{s}_Q^0(U) = \mathrm{e}^0_{(a_1^n, a_2^n, \cdots, a_t^n)}(U) = n^t {\cdot} \mathrm{e}^0_{(a_1, a_2, \cdots, a_t)}(U) \ge n^t,$$

we see

$$\begin{aligned} (-1)^{d-t}\mathrm{e}_Q^{d-t}(A) &= (-1)^{d-t}\mathrm{e}_Q^{d-t}(B) + \mathrm{s}_Q^0(U) \\ &= \sum_{j=1}^{t} \binom{t-1}{j-1} h^j(B) + n^t {\cdot} \mathrm{e}^0_{(a_1, a_2, \cdots, a_t)}(U) \ge n^t, \end{aligned}$$

whence the set $\Lambda_{d-t}(A)$ cannot be finite. Thus $t = 0$ and A a generalized Cohen-Macaulay ring. $\square$

3. The second Hilbert coefficients $\mathrm{e}_Q^2(A)$ of parameters

In this section we study the second Hilbert coefficients $\mathrm{e}_Q^2(A)$ of parameter ideals Q. The purpose is to find the sharp bound for $\mathrm{e}_Q^2(A)$. The bound $|\mathrm{e}_Q^2(A)| \leq 3(r+1)\mathrm{I}(A)$ given by Theorem 1 is too huge in general and far from the sharp bound.

Let us begin with the following.

Lemma 5. *Suppose that $d = 2$ and $\operatorname{depth} A > 0$. Let $Q = (x, y)$ be a parameter ideal in A and assume that x is superficial with respect to Q. Then*

$$\mathrm{e}_Q^2(A) = -\ell_A\left(\frac{[(x^\ell) : y^\ell] \cap Q^\ell}{(x^\ell)}\right) \leq 0$$

for all $\ell \gg 0$.

Proof. Let $\ell \gg 0$ be an integer which is sufficiently large and put $I = Q^\ell$. Let $G = \mathrm{G}(I)$ and $R = \mathrm{R}(I)$ be the associated graded ring and the Rees algebra of I, respectively. We put $\mathcal{M} = \mathfrak{m}R + R_+$. Then $[\mathrm{H}^i_{\mathcal{M}}(G)]_n = (0)$ for all integers $i \in \mathbb{Z}$ and $n > 0$, thanks to [6, Lemma 2.4]. We put $a = x^\ell$ and $b = y^\ell$. Then the element a remains superficial with respect to I and the equality $I^2 = (a, b)I$ holds true, whence $\mathrm{a}_2(G) < 0$.

We furthermore have the following.

Claim 6. $[\mathrm{H}^i_{\mathcal{M}}(R)]_0 \cong [\mathrm{H}^i_{\mathcal{M}}(G)]_0$ *as A-modules for all $i \in \mathbb{Z}$. Hence $\mathrm{H}^0_{\mathcal{M}}(G) = (0)$, so that $f = at \in R$ is G-regular.*

Proof of Claim 6. Let $L = R_+$ and apply the functors $\mathrm{H}^i_{\mathcal{M}}(*)$ to the following canonical exact sequences

$$0 \to L \to R \xrightarrow{p} A \to 0 \quad \text{and} \quad 0 \to L(1) \to R \to G \to 0,$$

where p denotes the projection, and get the exact sequences

$$(1) \quad \cdots \to \mathrm{H}^{i-1}_{\mathfrak{m}}(A) \to \mathrm{H}^i_{\mathcal{M}}(L) \to \mathrm{H}^i_{\mathcal{M}}(R) \to \mathrm{H}^i_{\mathfrak{m}}(A) \to \cdots \quad \text{and}$$

$$(2) \quad \cdots \to \mathrm{H}^{i-1}_{\mathcal{M}}(G) \to \mathrm{H}^i_{\mathcal{M}}(L)(1) \to \mathrm{H}^i_{\mathcal{M}}(R) \to \mathrm{H}^i_{\mathcal{M}}(G) \to \mathrm{H}^{i+1}_{\mathcal{M}}(L)(1) \to \cdots$$

of local cohomology modules. Then by exact sequence (2) we get the isomorphism

$$[\mathrm{H}^i_{\mathcal{M}}(L)]_{n+1} \cong [\mathrm{H}^i_{\mathcal{M}}(R)]_n$$

for $n \geq 1$, because $[\mathrm{H}^{i-1}_{\mathcal{M}}(G)]_n = [\mathrm{H}^i_{\mathcal{M}}(G)]_n = (0)$ for $n \geq 1$, while we have the isomorphism

$$[\mathrm{H}^i_{\mathcal{M}}(L)]_{n+1} \cong [\mathrm{H}^i_{\mathcal{M}}(R)]_{n+1}$$

for $n \geq 1$, thanks to exact sequence (1). Hence $[\mathrm{H}^i_{\mathcal{M}}(R)]_n \cong [\mathrm{H}^i_{\mathcal{M}}(R)]_{n+1}$ for $n \geq 1$, which implies $[\mathrm{H}^i_{\mathcal{M}}(R)]_n = (0)$ for all $i \in \mathbb{Z}$ and $n \geq 1$, because $[\mathrm{H}^i_{\mathcal{M}}(R)]_n = (0)$ for $n \gg 0$. Thus by exact sequence (1) we get $[\mathrm{H}^i_{\mathcal{M}}(L)(1)]_n = (0)$ for all $i \in \mathbb{Z}$ and $n \geq 0$, so that by exact sequence (2) we see $[\mathrm{H}^i_{\mathcal{M}}(R)]_0 \cong [\mathrm{H}^i_{\mathcal{M}}(G)]_0$ as A-modules for all $i \in \mathbb{Z}$. Considering the case where $i = 1$ in exact sequence (2), we have the embedding

$$0 \to \mathrm{H}^0_{\mathcal{M}}(G) \to \mathrm{H}^1_{\mathcal{M}}(L)(1),$$

so that $[\mathrm{H}^0_{\mathcal{M}}(G)]_0 = (0)$, because $[\mathrm{H}^1_{\mathcal{M}}(L)(1)]_0 = [\mathrm{H}^0_{\mathcal{M}}(L)]_1 = (0)$. Hence $\mathrm{H}^0_{\mathcal{M}}(G) = (0)$, so that f is G-regular, because $(0) :_G f$ is finitely graded. $\square$

Thanks to Serre's formula (cf. [1, Theorem 4.4.3]), Claim 6 shows that

$$\mathrm{e}^2_Q(A) = \sum_{i=0}^{2} (-1)^i \ell_A([\mathrm{H}^i_{\mathcal{M}}(G)]_0) = -\ell_A([\mathrm{H}^1_{\mathcal{M}}(G)]_0),$$

since $a_2(G) < 0$. Therefore to prove

$$\mathrm{e}^2_Q(A) = -\ell_A\left(\frac{[(x^\ell) : y^\ell] \cap Q^\ell}{(x^\ell)}\right),$$

it is suffices to check that

$$[\mathrm{H}^1_{\mathcal{M}}(G)]_0 \cong \frac{[(a) : b] \cap I}{(a)}$$

as A-modules.

Let $\overline{A} = A/(a)$ and $\overline{I} = I\overline{A}$. Then $G/fG \cong \mathrm{G}(\overline{I})$, because $f = at$ is G-regular (cf. Claim 6). We now look at the exact sequence

$$0 \to \mathrm{H}^0_{\mathcal{M}}(\mathrm{G}(\overline{I})) \to \mathrm{H}^1_{\mathcal{M}}(G)(-1) \xrightarrow{f} \mathrm{H}^1_{\mathcal{M}}(G)$$

of local cohomology modules which is induced from the exact sequence

$$0 \to G(-1) \xrightarrow{f} G \to \mathrm{G}(\overline{I}) \to 0$$

of graded G-modules. Then, since $[\mathrm{H}^1_{\mathcal{M}}(G)]_n = (0)$ for all $n \geq 1$, we have an isomorphism

$$[\mathrm{H}^0_{\mathcal{M}}(\mathrm{G}(\overline{I}))]_1 \cong [\mathrm{H}^1_{\mathcal{M}}(G)]_0$$

of A-modules and the vanishing $[\mathrm{H}^0_{\mathcal{M}}(\mathrm{G}(\overline{I})]_n = (0)$ for $n \geq 2$.

Look now at the homomorphism

$$\rho : \frac{[(a) : b] \cap I}{(a)} \to [\mathrm{H}^0_{\mathcal{M}}(\mathrm{G}(\overline{I}))]_1$$

of A-modules defined by $\rho(\overline{x}) = \overline{\overline{x}t}$ for each $x \in [(a) : b] \cap I$, where $\overline{x}$ and $\overline{\overline{x}t}$ denote the images of x in $\overline{A}$ and $\overline{x}t \in [\mathrm{R}(\overline{I})]_1$ in $\mathrm{G}(\overline{I})$, respectively. We will show that the map ρ is an isomorphism. Take $\varphi \in [\mathrm{H}^0_{\mathcal{M}}(\mathrm{G}(\overline{I}))]_1$ and write $\varphi = \overline{\overline{x}t}$ with $x \in I$. Since $[\mathrm{H}^0_{\mathcal{M}}(\mathrm{G}(I))]_2 = (0)$, we have $bt \cdot \overline{\overline{x}t} = \overline{\overline{bx}t^2} = 0$ in $\mathrm{G}(\overline{I})$, whence $bx \in [(a) + I^3] \cap I^2 = [(a) \cap I^2] + I^3 = aI + bI^2$ (recall that $I^2 = (a, b)I$ and that a is super-regular with respect to I). So, we write $bx = ai + bj$ with $i \in I$ and $j \in I^2$. Then, since $b(x - j) = ai \in (a)$, we have $x - j \in [(a) : b] \cap I$, whence $\varphi = \overline{\overline{x}t} = \overline{\overline{(x-j)}t}$. Thus the map ρ is surjective.

To show that the map ρ is injective, take $x \in [(a) : b] \cap I$ and suppose that $\rho(\overline{x}) = \overline{\overline{x}t} = 0$ in $\mathrm{G}(\overline{I})$. Then

$$x \in [(a) : b] \cap [(a) + I^2] = (a) + [((a) : b) \cap I^2].$$

To conclude that $x \in (a)$, we need the following.

Claim 7. *Let $n \geq 2$ be an integer. Then $[(a) : b] \cap I^n \subseteq (a) + [((a) : b) \cap I^{n+1}]$.*

Proof of Claim 7. Take $y \in [(a):b] \cap I^n$. Then, since $by \in (a)$, we see $bt \cdot \overline{\overline{y}t^n} = \overline{\overline{by}t^{n+1}} = 0$ in $\mathrm{G}(\overline{I})$. Hence $\overline{\overline{y}t^n} \in [\mathrm{H}^0_{\mathcal{M}}(\mathrm{G}(\overline{I}))]_n$, because bt is a homogeneous parameter for the graded ring $\mathrm{G}(\overline{I})$. Recall now that $n \geq 2$, whence $[\mathrm{H}^0_{\mathcal{M}}(\mathrm{G}(\overline{I}))]_n = (0)$, so that $\overline{\overline{y}t^n} = 0$. Thus $y \in (a) + I^{n+1}$, whence $y \in (a) + [((a):b) \cap I^{n+1}]$, as claimed. $\square$

Since $x \in (a) + [((a):b) \cap I^2]$, thanks to Claim 7, we get $x \in (a) + I^{n+1}$ for all $n \geq 1$, whence $x \in (a)$, so that the map ρ is injective. Thus

$$[\mathrm{H}^1_{\mathcal{M}}(G)]_0 \cong \frac{[(a):b] \cap I}{(a)}$$

as A-modules. $\square$

Theorem 8. *Suppose that $d = 2$ and $\operatorname{depth} A > 0$. Let $Q = (x, y)$ be a parameter ideal in A and assume that x is superficial with respect to Q. Then*

$$-h^1(A) \leq \mathrm{e}^2_Q(A) \leq 0$$

and the following three conditions are equivalent.

(1) $\mathrm{e}^2_Q(A) = 0$.
(2) *x, y forms a d-sequence in A.*
(3) *x^ℓ, y^ℓ forms a d-sequence in A for all integers $\ell \geq 1$.*

Proof. By Lemma 5 we have

$$\mathrm{e}^2_Q(A) = -\ell_A\left(\frac{[(x^\ell):y^\ell] \cap (x,y)^\ell}{(x^\ell)}\right) \leq 0$$

for all integers $\ell \gg 0$. To show that $-h^1(A) \leq \mathrm{e}^2_Q(A)$, we may assume that $\mathrm{H}^1_{\mathfrak{m}}(A)$ is finitely generated. Take the integer $\ell \gg 0$ so that the system $a = x^\ell$, $b = y^\ell$ of parameters of A is standard. Then since

$$\frac{[(a):b] \cap Q^\ell}{(a)} \subseteq \frac{(a):b}{(a)} \cong \mathrm{H}^0_{\mathfrak{m}}(A/(a)) \cong \mathrm{H}^1_{\mathfrak{m}}(A),$$

we get $-h^1(A) \leq \mathrm{e}^2_Q(A)$.

Let us consider the second assertion.

(1) $\Rightarrow$ (3). Take an integer $N \geq 1$ so that

$$\mathrm{e}^2_Q(A) = -\ell_A\left(\frac{[(x^\ell):y^\ell] \cap (x,y)^\ell}{(x^\ell)}\right)$$

for all $\ell \geq N$ (cf. Lemma 5); hence

$$[(x^\ell):y^\ell] \cap (x,y)^\ell = (x^\ell).$$

Claim 9. $[(x^\ell):y^\ell] \cap (x,y)^\ell = (x^\ell)$ *for all $\ell \geq 1$.*

Proof of Claim 9. We may assume that $1 \leq \ell < N$. Take $\tau \in [(x^\ell):y^\ell] \cap (x,y)^\ell$. Then, since $y^N(x^{N-\ell}\tau) = y^{N-\ell}x^{N-\ell}(y^\ell\tau) \in (x^N)$, we have $x^{N-\ell}\tau \in [(x^N):y^N] \cap (x,y)^N = (x^N)$. Thus $\tau \in (x^\ell)$, because x is A-regular (recall that $\operatorname{depth} A > 0$ and x is superficial with respect to Q). $\square$

Since x^ℓ is A-regular and $[(x^\ell) : y^\ell] \cap (x^\ell, y^\ell) = (x^\ell)$ by Claim 9, we readily see that x^ℓ, y^ℓ is a d-sequence in A.

(3) $\Rightarrow$ (2) This is clear.

(2) $\Rightarrow$ (1) It is well-known that $\mathrm{e}^2_{(x,y)}(A) = 0$, if $\operatorname{depth} A > 0$ and the system x, y of parameters forms a d-sequence in A; see Proposition 11 below. $\square$

Passing to the ring $A/\mathrm{H}^0_\mathfrak{m}(A)$, thanks to Theorem 8, we readily get the following.

Corollary 10. *Suppose that $d = 2$ and let Q be a parameter ideal in A. Then*

$$h^0(A) - h^1(A) \le \mathrm{e}^2_Q(A) \le h^0(A).$$

The results in the following proposition are, more or less, known.

Proposition 11. ([5, Proposition 3.4]) *Suppose that $d > 0$ and let $Q = (a_1, a_2, \cdots, a_d)$ be a parameter ideal in A. Let $G = \mathrm{G}(Q)$ and $R = \mathrm{R}(Q)$. Let $f_i = a_i t \in R$ for $1 \le i \le d$. Assume that the sequence $a_1, a_2, \cdots, a_d$ forms a d-sequence in A. Then we have the following, where $Q_i = (a_1, a_2, \cdots, a_i)$ for $0 \le i \le d$.*

(1) $\mathrm{e}^0_Q(A) = \ell_A(A/Q) - \ell_A([Q_{d-1} : a_d]/Q_{d-1})$.
(2) $(-1)^i \mathrm{e}^i_Q(A) = h^0(A/Q_{d-i}) - h^0(A/Q_{d-i-1})$ *for* $1 \le i \le d-1$ *and* $(-1)^d \mathrm{e}^d_Q(A) = h^0(A)$.
(3) $\ell_A(A/Q^{n+1}) = \sum_{i=0}^{d} (-1)^i \mathrm{e}^i_Q(A) \binom{n+d-i}{d-i}$ *for all* $n \ge 0$, *whence* $\ell_A(A/Q) = \sum_{i=0}^{d} (-1)^i \mathrm{e}^i_Q(A)$.
(4) $f_1, f_2, \cdots, f_d$ *forms a d-sequence in G.*
(5) $\mathrm{H}^0_\mathcal{M}(G) = [\mathrm{H}^0_\mathcal{M}(G)]_0 \cong \mathrm{H}^0_\mathfrak{m}(A)$, *where* $\mathcal{M} = \mathfrak{m}R + R_+$
(6) $[\mathrm{H}^i_\mathcal{M}(G)]_n = (0)$ *for all* $n > -i$ *and* $i \in \mathbb{Z}$, *whence* $\operatorname{reg} G = 0$.

Let us note one example of local rings A which are not generalized Cohen-Macaulay rings but every parameter ideal in A is generated by a system of parameters that forms a d-sequence in A.

Example 12. Let R be a regular local ring with the maximal ideal $\mathfrak{n}$ and $d = \dim R \ge 2$. Let $X_1, X_2, \cdots, X_d$ be a regular system of parameters of R. We put $\mathfrak{p} = (X_1, X_2, \cdots, X_{d-1})$ and $D = R/\mathfrak{p}$. Then D is a DVR. Let $A = R \ltimes D$ denote the idealization of D over R. Then A is a Noetherian local ring with the maximal ideal $\mathfrak{m} = \mathfrak{n} \times D$, $\dim A = d$, and $\operatorname{depth} A = 1$. We furthermore have the following.

(1) $\Lambda_i(A) = \{0\}$ for all $1 \le i \le d$ such that $i \ne d-1$.
(2) $\Lambda_0(A) = \{n \mid 0 < n \in \mathbb{Z}\}$ and $\Lambda_{d-1}(A) = \{(-1)^{d-1} n \mid 0 < n \in \mathbb{Z}\}$.
(3) After renumbering, every system of parameters in A forms a d-sequence.

The ring A is not a generalized Cohen-Macaulay ring, because $\mathrm{H}^1_\mathfrak{m}(A)$ ($\cong \mathrm{H}^1_\mathfrak{n}(D)$) is not a finitely generated A-module.

In the rest of Section 3 let us consider the bound for $\mathrm{e}^2_Q(Q)$ in higher dimensional cases. In the case where $\dim A \ge 3$ we have the following.

Theorem 13. *Suppose that A is a generalized Cohen-Macaulay ring with $d = \dim A \ge 3$. Let $Q = (a_1, a_2, \cdots, a_d)$ be a parameter ideal in A. Then*

$$-\sum_{j=2}^{d-1} \binom{d-3}{j-2} h^j(A) \le \mathrm{e}^2_Q(A) \le \sum_{j=1}^{d-2} \binom{d-3}{j-1} h^j(A).$$

We have $Q{\cdot}\mathrm{H}^j_{\mathfrak{m}}(A/(a_1,a_2,\cdots,a_k))=(0)$ *for all* $k\geq 0$ *and* $j\geq 1$ *with* $j+k\leq d-2$, *if* $\mathrm{e}^2_Q(A)=\sum_{j=1}^{d-2}\binom{d-3}{j-1}h^j(A)$ *and if* $a_1,a_2,\cdots,a_d$ *forms a superficial sequence with respect to* Q.

Proof. See [5, Theorem 3.6]. □

The following result guarantees the implication (2) ⇒ (1) and the last assertion in Theorem 2.

Proposition 14. *Suppose that* A *is a generalized Cohen-Macaulay ring with* $d=\dim A\geq 3$ *and let* $Q=(a_1,a_2,\cdots,a_d)$ *be a parameter ideal in* A. *Assume that the sequence* $a_1,a_2,\cdots,a_d$ *forms a d-sequence in* A *and* $Q{\cdot}\mathrm{H}^j_{\mathfrak{m}}(A/(a_1,a_2,\cdots,a_k))=(0)$ *for all* $k\geq 0$ *and* $j\geq 1$ *with* $j+k\leq d-2$. *Then*

$$(-1)^i\mathrm{e}^i_Q(A)=\sum_{j=1}^{d-i}\binom{d-i-1}{j-1}h^j(A)$$

for $2\leq i\leq d-1$ *and* $(-1)^d\mathrm{e}^d_Q(A)=h^0(A)$.

Proof. See [5, Proposition 3.7]. □

4. Proof of Theorem 2

The purpose of this section is to prove Theorem 2. Thanks to Proposition 11 and 14, we have only to show the following.

Theorem 15. *Suppose that* A *is a generalized Cohen-Macaulay ring with* $d=\dim A\geq 3$ *and* $\operatorname{depth} A>0$. *Let* Q *be a parameter ideal in* A *and assume that* $\mathrm{e}^2_Q(A)=\sum_{j=1}^{d-2}\binom{d-3}{j-1}h^j(A)$. *Then* Q *is generated by a system of parameters which forms a d-sequence in* A.

For each ideal $\mathfrak{a}$ in A ($\mathfrak{a}\neq A$) let $\mathrm{U}(\mathfrak{a})$ denote the unmixed component of $\mathfrak{a}$. When $\mathfrak{a}=(a)$ with $a\in A$, we write $\mathrm{U}(\mathfrak{a})$ simply by $\mathrm{U}(a)$. We have

$$\mathrm{U}(a)=\bigcup_{n\geq 0}\left[(a):_A\mathfrak{m}^n\right],$$

if A is a generalized Cohen-Macaulay ring with $\dim A\geq 2$ and a is a part of a system of parameters in A (cf. [11, Section 2]). The following result is the key in our proof of Theorem 15.

Proposition 16. *Suppose that* A *is a generalized Cohen-Macaulay ring with* $d=\dim A\geq 2$ *and* $\operatorname{depth} A>0$. *Let* $Q=(a_1,a_2,\cdots,a_d)$ *be a parameter ideal in* A. *Assume that* $a_d\mathrm{H}^1_{\mathfrak{m}}(A)=(0)$ *and that the sequence* $a_1,a_2,\cdots,a_{d-1}$ *forms a d-sequence in the generalized Cohen-Macaulay ring* $A/\mathrm{U}(a_d)$. *Then*

$$\mathrm{U}(a_1)\cap[Q+\mathrm{U}(a_d)]=(a_1).$$

Proof. See [5, Proposition 4.2]. □

We are now ready to prove Theorem 15.

Proof of the Theorem 15. We proceed by induction on d. Choose $a_1, a_2, \cdots, a_d \in A$ so that $Q = (a_1, a_2, \cdots, a_d)$ and for each $1 \leq i \leq d-2$, the $i+2$ elements $a_1, a_2, \cdots, a_i, a_{d-1}, a_d$ form a superficial sequence with respect to Q. We will show that there exist $b_2, b_3, \cdots, b_d \in A$ such that $b_1 = a_{d-1}, b_2, b_3, \cdots, b_d$ forms a d-sequence in A and $Q = (b_1, b_2, \cdots, b_d)$. We put $\overline{A} = A/(a_1)$, $\overline{Q} = Q\overline{A}$, and $C = \overline{A}/\mathrm{H}^0_{\mathfrak{m}}(\overline{A})$ $(= A/\mathrm{U}(a_1))$.

Suppose that $d = 3$. Then

$$\mathrm{e}^2_{QC}(C) = \mathrm{e}^2_{\overline{Q}}(\overline{A}) - h^0(\overline{A}) = \mathrm{e}^2_Q(A) - h^0(\overline{A}) = h^1(A) - h^0(\overline{A}) = 0,$$

because $h^1(A) = h^0(\overline{A})$ (recall that $Q\mathrm{H}^1_{\mathfrak{m}}(A) = (0)$ by Proposition 13). Hence, thanks to Proposition 8, a_2, a_3 forms a d-sequence in C, because a_2 is superficial for the ideal $QC = (a_2, a_3)C$. Therefore, since $a_1\mathrm{H}^1_{\mathfrak{m}}(A) = (0)$, we have

$$\mathrm{U}(a_2) \cap [Q + \mathrm{U}(a_1)] = (a_2),$$

by Proposition 16. Let $Q = (a_2, a_3, b_3)$ and $B = A/\mathrm{U}(a_2)$. Then since $\mathrm{e}^2_{QB}(B) = 0$, by Proposition 8 the sequence $b_2 = a_3, b_3$ forms a d-sequence in B, because b_2 is superficial for QB. Therefore, since $\mathrm{U}(a_2) \cap Q \subseteq \mathrm{U}(a_2) \cap [Q + \mathrm{U}(a_1)] = (a_2)$, the sequence b_2, b_3 forms a d-sequence in $A/(a_2)$, so that $b_1 = a_2, b_2, b_3$ forms a d-sequence in A, because b_1 is A-regular.

Assume that $d \geq 4$ and that our assertion holds true for $d-1$. Then, thanks to Theorem 13 and its proof, we have

$$\begin{aligned}
\mathrm{e}^2_Q(A) = \mathrm{e}^2_{\overline{Q}}(\overline{A}) = \mathrm{e}^2_{QC}(C) &\leq \sum_{j=1}^{d-3} \binom{d-4}{j-1} h^j(C) \\
&= \sum_{j=1}^{d-3} \binom{d-4}{j-1} h^j(\overline{A}) \\
&= \sum_{j=1}^{d-2} \binom{d-3}{j-1} h^j(A) = \mathrm{e}^2_Q(A),
\end{aligned}$$

because $Q{\cdot}\mathrm{H}^j_{\mathfrak{m}}(A) = (0)$ for $1 \leq j \leq d-3$. Hence

$$\mathrm{e}^2_{QC}(C) = \sum_{j=1}^{d-3} \binom{d-4}{j-1} h^j(C).$$

Therefore, because $QC = (\overline{a_2}, \overline{a_3}, \cdots, \overline{a_d})C$ and the sequence $\overline{a_2}, \overline{a_3}, \cdots, \overline{a_i}, \overline{a_{d-1}}, \overline{a_d}$ is superficial in the ideal QC for all $1 \leq i \leq d-2$ where $\overline{a_j}$ denotes the image of a_j in C, the hypothesis of induction on d yields that there exist $\gamma_2, \gamma_3, \cdots, \gamma_{d-1} \in C$ such that the sequence $\gamma_1 = \overline{a_{d-1}}, \gamma_2, \gamma_3, \cdots, \gamma_{d-1}$ forms a d-sequence in C and $QC = (\gamma_1, \gamma_2, \cdots, \gamma_{d-1})C$. Let us write $\gamma_j = \overline{c_j}$ for each $2 \leq j \leq d-1$ with $c_j \in Q$, where $\overline{c_j}$ denote the image of c_j in C. We put $\mathfrak{q} = (a_1, a_{d-1}, c_2, c_3, \cdots, c_{d-1})$. Then $\mathfrak{q}$ is a parameter ideal in A, $a_1\mathrm{H}^1_{\mathfrak{m}}(A) = (0)$, and $a_{d-1}, c_2, c_3, \cdots, c_{d-1}$ forms a d-sequence in C. Therefore

$$\mathrm{U}(a_{d-1}) \cap [Q + \mathrm{U}(a_1)] = \mathrm{U}(a_{d-1}) \cap [\mathfrak{q} + \mathrm{U}(a_1)] = (a_{d-1})$$

by Proposition 16, whence $\mathrm{U}(a_{d-1}) \cap Q = (a_{d-1})$.

Let $B = A/\mathrm{U}(a_{d-1})$. We then have

$$\mathrm{e}^2_{QB}(B) = \sum_{j=1}^{d-3} \binom{d-4}{j-1} h^j(B)$$

for the same reason as for the equality $\mathrm{e}^2_{QC}(C) = \sum_{j=1}^{d-3} \binom{d-4}{j-1} h^j(C)$ (in fact, to show $\mathrm{e}^2_{QC}(C) = \sum_{j=1}^{d-3} \binom{d-4}{j-1} h^j(C)$, we only need that a_1 is superficial with respect to Q). Therefore, by the hypothesis of induction on d, we may choose elements $\beta_2, \beta_3, \cdots, \beta_d \in B$ so that $QB = (\beta_2, \beta_3, \cdots, \beta_d)B$ and the sequence $\beta_2, \beta_3, \cdots, \beta_d$ forms a d-sequence in B. We put $b_1 = a_{d-1}$ and write $\beta_j = \overline{b_j}$ with $b_j \in Q$ for $2 \leq j \leq d$, where $\overline{b_j}$ denotes the image of b_j in B. We now put $\mathfrak{q}' = (b_1, b_2, \cdots, b_d)$. Then $\mathfrak{q}'$ is a parameter ideal in A and because $\mathrm{U}(b_1) \cap Q = (b_1)$, we get

$$Q \subseteq [\mathfrak{q}' + \mathrm{U}(b_1)] \cap Q = \mathfrak{q}' + [\mathrm{U}(b_1) \cap Q] \subseteq \mathfrak{q}' + (b_1) = \mathfrak{q}';$$

hence $Q = \mathfrak{q}'$. Thus the sequence $b_2, b_3, \cdots, b_d$ forms a d-sequence in $A/(b_1)$, so that $b_1, b_2, \cdots, b_d$ forms a d-sequence in A, because b_1 is A-regular. This complete the proof of Theorem 15 and that of Theorem 2 as well. $\square$

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PREPOJECTIVE ALGEBRAS AND CRYSTAL BASES OF QUANTUM GROUPS

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Abstract. At the end of the last century, Kashiwara and the author ([10]) made a bride between representation theory of quantum groups and one of quivers. More precisely, consider the variety $X(\mathbf{d})$ of representations of a double quiver, with a fixed dimension vector $\mathbf{d}$. It is known that there is a nice Lagrangian subvariety $\Lambda(\mathbf{d})$ of $X(\mathbf{d})$. In a geometric point of view, $\Lambda(\mathbf{d})$ is defines as the variety of zero points of the moment map for the action of a certain reductive group on $X(\mathbf{d})$. Let $\mathrm{Irr}\Lambda(\mathbf{d})$ be the set of all irreducible components of $\Lambda(\mathbf{d})$. We proved that $\sqcup_{\mathbf{d}}\mathrm{Irr}\Lambda(\mathbf{d})$ is isomorphic to the "crystal basis" $B(\infty)$ of the negative half of a quantum group. This is one of the main results in [10].

On the other hand, in a representation theoretical point of view, the variety $\Lambda(\mathbf{d})$ is nothing but the variety of nilpotent representations of the corresponding preprojective algebra. Namely, the results of [10] tell us that there is a "nice" correspondence between preprojective algebras and crystal basis of quantum groups. In this note, we try to explain what is the meaning of this correspondence. Adding to that, we also discuss resent progress around this area.

1. Introduction

この小論の目的は，Kashiwara-S [10] を解説することである．この論文は 1997 年に出版されたものであり,「なぜ今さらわざわざ古い話をするのか？」と疑問に思われる方も多いと思う．筆者は Lie 代数や量子群等，いわゆる Lie theoretic な表現論の出身で，論文を書いた当時は多元環の表現論については殆ど何も知らなかった．後になって，実は自分のやっていたことが preprojective algebra という多元環の表現論と密接な関わりがあることを知り，大きな衝撃を受けたことを覚えている．そこで今回はこの事実を逆手に取り，[10] を preprojective algebra の表現論の立場から見直すことによって，その再解釈を行いたい．これが「わざわざ古い話を持ち出す理由」である．

さらに付け加えるなら，近年の Geiss-Leclerc-Schöer による一連の研究（[4]〜[7]）等，このような視点に立った研究が現在でも活発に行われている．今回の研究会でも，例えば Demonet さん，木村さんによる講演などは，その一例ある．

そもそも良い数学というものは，黙っていても勝手にいろいろな分野に結びついていくものである．純粋に多元環の表現論的な興味から考案された preprojective algebra という概念が，量子群の表現論という全く違うコンテクストから自然に現れたということは，preprojective algebra の定義の "正しさ" を物語っているのかも知れない．この機会に，両者の不思議な結びつきに少しでも興味を持って頂ければ幸いに思う[1].

• 謝辞　筆者のような門外漢に講演の機会を与えてくださったオーガナイザーの方々に感謝します．特にプログラム責任者の名古屋大学の伊山修さんには準備の段階から相談に

[1]筆者は以前研究集会「環論とその周辺」で，今回とほぼ同じ内容を，別の側面から紹介させて頂いた [16]．手前ミソだが，こちらも併せて参照して頂ければ幸いである．

乗って頂き，貴重なご意見を頂きました．この場を借りて感謝します．

• 記号に関する注意　この小論では quiver を $\Gamma = (I, \Omega)$ なる記号で表す．ここに I は頂点集合，Ω は矢印の集合である．また頂点 $i \in I$ から頂点 $j \in I$ へ向かう矢印 $\tau \in \Omega$ がある時，$i = \mathrm{out}(\tau)$，$j = \mathrm{in}(\tau)$ と表すことにする．また $\tau \in \Omega$ に対し，向きをひっくり返して得られる新たな矢印を $\overline{\tau}$ で表す．

$$
\begin{array}{ccccccc}
i \circ & \xrightarrow{\tau} & \circ\, j & & i \circ & \xleftarrow{\overline{\tau}} & \circ\, j \\
\| & & \| & \Rightarrow & \| & & \| \\
\mathrm{out}(\tau) & & \mathrm{in}(\tau) & & \mathrm{in}(\overline{\tau}) & & \mathrm{out}(\overline{\tau})
\end{array}
$$

3 つの頂点 $i, j, k \in I$ と，i から j への矢印 τ，j から k への矢印 σ があったとする．このとき τ と σ を合成して長さ 2 の path が定義されるが，この path は（$\tau\sigma$ ではなく）$\sigma\tau$ と表すことにする．

$$
\underset{i}{\circ} \xrightarrow{\tau} \underset{j}{\circ} \xrightarrow{\sigma} \underset{k}{\circ} \quad \Rightarrow \quad \underset{i}{\circ} \xrightarrow{\quad\sigma\tau\quad} \underset{k}{\circ}
$$

また，algebra A に対して A-module とは常に left module を表すこととし，以後一切断らない．τ と σ の合成を $\sigma\tau$ と表すのは，このためであると言っても良い．

これらの記法は多くの多元環の文献で用いられているものとは異なるので読みにくいかも知れないが，筆者が慣れている記法を用いないと間違い得てしまいそうなので，今回は御容赦願いたい．

2. Varieties of Representations

2.1. Quiver の表現のなす空間.

K を体，$\Gamma = (I, \Omega)$ を有限 quiver とする．このとき，Γ の（有限次元）表現 $\mathbf{V} = (V, B)$ とは，次のようなものである：

- $V = \oplus_{i \in I} V_i$ は有限次元 I-graded vector space,
- $B = (B_\tau)_{\tau \in \Omega}$ は K-linear maps $B_\tau \in \mathrm{Hom}_K(V_{\mathrm{out}(\tau)}, V_{\mathrm{in}(\tau)})$ の組.

与えられた Γ の表現 $\mathbf{V} = (V, B)$ に対し，

$$\underline{\dim}\mathbf{V} := (\dim_K V_i)_{i \in I} \in \mathbb{Z}_{\geq 0}^I$$

を $\mathbf{V}$ の dimension vector と呼ぶ．また，（B を伴わない）単独の I-graded vector space $V = \oplus_{i \in I} V_i$ に対しても，同様に dimension vector $\underline{\dim}V$ を定義する．

2 つの Γ の表現 $\mathbf{V} = (V, B)$ と $\mathbf{V}' = (V', B')$ に対し，$\mathbf{V}$ から $\mathbf{V}'$ への射 (morphism)$\phi = (\phi_i)_{i \in I}$ とは，K-linear maps $\phi_i : V_i \to V_i'$ $(i \in I)$ の組であって，任意の $\tau \in \Omega$ に対して、

$$\phi_{\mathrm{in}(\tau)} B_\tau = B_\tau' \phi_{\mathrm{out}(\tau)} \tag{2.1.1}$$

が成り立つもののことを言う．特に $\varphi = (\varphi_i)_{i \in I}$ が I-graded vector space の同型写像である時，$\mathbf{V}$ と $\mathbf{V}'$ は同型であるという．

$\mathbf{d}=(d_i)_{i\in I}\in\mathbb{Z}^I_{\geq 0}$ を１つ指定すれば，$\underline{\dim}V=\mathbf{d}$ なる I-graded vector space は一意的に定まる[2]．これを $V(\mathbf{d})$ と書くことにし，次の vector space を考えよう：

$$E_\Omega(\mathbf{d}):=\bigoplus_{\tau\in\Omega}\mathrm{Hom}_K(V(\mathbf{d})_{\mathrm{out}(\tau)},V(\mathbf{d})_{\mathrm{in}(\tau)}).$$

$B\in E_\Omega(\mathbf{d})$ に対し組 $\mathbf{V}=(V(\mathbf{d}),B)$ を考えれば，これは $\underline{\dim}\mathbf{V}=\mathbf{d}$ なる Γ の表現である．逆に $\underline{\dim}\mathbf{V}=\mathbf{d}$ なる Γ の表現は，必ずこの形で書かれる．すなわち，$E_\Omega(\mathbf{d})$ は $\underline{\dim}=\mathbf{d}$ となる Γ の表現を全てかきあつめたものに他ならない．すなわち $E_\Omega(\mathbf{d})$ とは，$\underline{\dim}=\mathbf{d}$ の Γ の表現全体のなす空間（多様体）である．

$E_\Omega(\mathbf{d})$ には群 $G(\mathbf{d}):=\prod_{i\in I}GL(V(\mathbf{d})_i)$ が

$$B=(B_\tau)\mapsto gB=\left(g_{\mathrm{in}(\tau)}B_\tau g^{-1}_{\mathrm{out}(\tau)}\right)\qquad\left(g=(g_i)_{i\in I}\in G(\mathbf{d})\right)$$

で作用する．quiver の表現の射 $\phi=(\phi_i)$ が同型写像であるということは，各 ϕ_i が $GL(V(\mathbf{d})_i)$ の元であることに他ならない．このことと (2.1.1) に注意すれば，

$$\text{２つの }\Gamma\text{ の表現が同型}\quad\Leftrightarrow\quad\begin{array}{c}\text{対応する }E_\Omega(\mathbf{d})\text{ の元が}\\\text{同じ }G(\mathbf{d})\text{-orbit に含まれる}\end{array}$$

となることは明らかであろう．すなわち，次の１対１対応が得られる：

$$\left\{\begin{array}{c}\underline{\dim}=\mathbf{d}\text{ なる}\\\Gamma\text{の表現の同型類}\end{array}\right\}\ \overset{1:1}{\longleftrightarrow}\ \{E_\Omega(\mathbf{d})\text{ の }G(\mathbf{d})\text{-orbit}\}$$

よく知られているように，Γ の表現を考えるということは，対応する path algebra $K[\Gamma]$ 上の module を考えるということに他ならない．したがって

$$\left\{\begin{array}{c}\underline{\dim}=\mathbf{d}\text{ なる}\\K[\Gamma]\text{-module の同型類}\end{array}\right\}\overset{1:1}{\longleftrightarrow}\left\{\begin{array}{c}\underline{\dim}=\mathbf{d}\text{ なる}\\\Gamma\text{の表現の同型類}\end{array}\right\}\overset{1:1}{\longleftrightarrow}\{E_\Omega(\mathbf{d})\text{ の }G(\mathbf{d})\text{-orbit}\}$$

という１対１対応が得られる．その対応は以下の通り：$B=(B_\tau)_{\tau\in\Omega}\in E_\Omega(\mathbf{d})$ が与えられたとする．このとき，I-graded vector space $V(\mathbf{d})$ 上の $K[\Gamma]$-module structure が，$\tau\in K[\Gamma]$ の作用を B_τ で与えることによって，得られる．逆の対応は明らかであろう．

2.2. **Relation 付き quiver の場合**.

前節の議論では $\Gamma=(I,\Omega)$ の表現 $=K[\Gamma]$-modules しか扱うことが出来なかった．これは多元環の表現論としては，いささか適用範囲が狭い．そこで議論を relation 付き quiver の場合に拡張しよう．

$A=K[\Gamma]/J$ なる多元環を考える．ただし J は relations $\rho_1,\cdots,\rho_l$ で生成される両側 ideal とする．V を A-module で $\underline{\dim}V=\mathbf{d}$ なるものとしよう．このとき，A における単位元 1_A の原始ベキ等元分解 $1_A=\sum_{i\in I}e_i$ は，V 上に I-graded vector space の構造を定める．すなわち $V_i:=e_iV$ とおくことで，直和分解 $V=\oplus_{i\in I}V_i$ が得られる．$\tau\in\Omega$ の V への作用は，linear map $B_\tau\in\mathrm{Hom}_K(V_{\mathrm{out}(\tau)},V_{\mathrm{in}(\tau)})$ を定め，こうして Γ の表現 $\mathbf{V}=(V,B)$

[2] 本来は「同型を除いて一意的」と言うべきところだが，あまりこだわると記述がややこしくなるばかりなので，今回はこの程度で止めておく．

が得られる．さらに，今の場合には $B=(B_\tau)$ が relations $\rho_1,\cdots,\rho_l$ を満たしていなければならない．各 ρ_j が具体的に

$$\rho_j=\sum a_{k_1,k_2,\cdots,k_j}\tau_{k_1}\tau_{k_2}\cdots\tau_{k_j}\qquad(1\leq j\leq l, a_{k_1,k_2,\cdots,k_j}\in K)$$

と書かれていたとすれば，$B=(B_\tau)$ は関係式

$$\rho_j(B):=\sum a_{k_1,k_2,\cdots,k_j}B_{\tau_{k_1}}B_{\tau_{k_2}}\cdots B_{\tau_{k_j}}=0\qquad(1\leq j\leq l)\tag{2.2.1}$$

を満たさなければならない．言い換えれば，B は

$$\Lambda_A(\mathbf{d}):=\{B\in E_\Omega(\mathbf{d})\mid B\text{ は (2.2.1) を満たす}\}$$

なる $E_\Omega(\mathbf{d})$ の部分代数多様体の点を与えている，ということになる．この $\Lambda_A(\mathbf{d})$ を variety of A-modules (of dimension vector $\mathbf{d}$) と呼ぶことにしよう．

ここまでいけば，後の議論は前節と同じである．すなわち，１対１対応

$$\left\{\begin{array}{c}\underline{\dim}=\mathbf{d}\text{ なる}\\ A\text{-module の同型類}\end{array}\right\}\overset{1:1}{\longleftrightarrow}\{\Lambda_A(\mathbf{d})\text{ の }G(\mathbf{d})\text{-orbit}\}$$

が得られるわけである．

以上の話を標語的に言えば，

$$\text{多元環 }A\text{ の表現論}\quad\text{“=”}\quad\begin{array}{c}\text{代数多様体 }\Lambda_A(\mathbf{d})\text{ 上の}\\ G(\mathbf{d})\text{-orbit の幾何学}\end{array}$$

ということになるだろう．このような議論は体 K がどんなものであっても考えることができるし，そういうことが可能というのが代数幾何の強みでもあるのだが，簡単のために以下 $K=\mathbb{C}$ と仮定しよう．

例えば A が finite representation type であるとしよう．このとき $\Lambda_A(\mathbf{d})$ は必ず有限個の $G(\mathbf{d})$-orbit を持つ．これは商空間 $G(\mathbf{d})\backslash\Lambda_A(\mathbf{d})$ が必ず有限個の点集合になってしまうことを意味し，非常に扱いやすい．もちろん，一般には A は wild になってしまうわけで，こんな話はほとんどの場合成り立たない．それどころか，商空間 $G(\mathbf{d})\backslash\Lambda_A(\mathbf{d})$ は一般には代数多様体の構造を持たず，その扱いは非常に難しい[3]．

ではどうしたらいいのだろうか？ このような場合，表現論の世界では「考える module の範囲を制限する」ということをしばしば行う．与えられた多元環 A に対し，A-module 全体の category を考えるのではなく，その中からうまい subcategory を取り出して，そちらを調べようという考え方である．このことは，対応する幾何の言葉では『空間をうまく制限して，orbit の空間を調べやすくしているということ』と言っても良いだろう．制限と言っても「ここからここまで」というタイプのものではなく，むしろ「スカスカにする」と言った方がいいかも知れない．空間をスカスカにしたとしても，２つの表現が同値であるという概念は全く同様に定義されるので，orbit そのものは意味を持つことになる．

[3]商集合をいつでも考えることが出来るのは明らかである．問題は「どうやって空間の構造（例えば位相や多様体の構造など）を入れるか？」ということで，これをしないと幾何学的に考える意味が無い．

$G(\mathbf{d})\backslash\Lambda_A(\mathbf{d})$ を空間として扱う道具として，stack と呼ばれる概念がある.「空間としてきちんと定式化する」ということにはもちろんそれなりのメリットもあるが，定式化出来たからといって話が簡単になるわけではない．wild な表現論を扱う難しさは，当然対応する stack の空間としての難しさに遺伝することになる．

他方，本小論で紹介したいのは，これとは別の考え方である．上に述べた variety of A-module $\Lambda_A(\mathbf{d})$ は代数多様体であり，これを点集合とみなした場合に比べ，はるかに多くの情報を持っている．例えば $\Lambda_A(\mathbf{d})$ には位相が入っている．このことは実は非常に大きい．位相が入っていることで，$\Lambda_A(\mathbf{d})$ 内の２点（= ２つの表現）に対し，それらが「近い or 遠い」,「つながっている or 離れている」等の議論が可能となる．

$\Lambda_A(\mathbf{d})$ が持っている幾何学的情報のうち，今回は特に

$$\mathrm{Irr}\Lambda_A(\mathbf{d}) := \Lambda_A(\mathbf{d}) \text{ の（代数多様体としての）既約成分の全体の集合}$$

に着目したい．各既約成分は $G(\mathbf{d})$ の作用に関して不変ではあるけれども，単独の orbit にはなっていない．つまり１つの既約成分には同型でないような表現がたくさん（一般には無限個）含まれてしまうわけで，既約成分を考えるということは，通常の表現論で行われている仕分け（= 同値類による分類）よりも．はるかに目の粗いことをやっていることになる．しかし，今考えているのは『A-module の作る代数多様体』なのだから,『その既約成分も A の表現に関する某かの情報を担っているはず』と考えても，そんなに的外れではないのではないだろうか？

さらに，我々が興味があるのは特定の dimension vector を持つ module ではなく，module 全体の持つ構造である．したがって，調べるべきは個々の $\mathrm{Irr}\Lambda_A(\mathbf{d})$ ではなく，dimension vector の可能性を全て走らせた

$$\bigsqcup_{\mathbf{d}\in\mathbb{Z}^I_{\geq 0}} \mathrm{Irr}\Lambda_A(\mathbf{d})$$

ということになる．

一般の A に対してこのような考え方がどの程度有効なものなのかは，実は筆者は知らない．しかし A が preprojective algebra の場合には，この考え方は有効である．詳しいことは次節に譲るが，preprojective algebra はごく一部の例外を除いて，ほとんどの場合 wild である．したがって，A-module の同型類全体 =orbit 全体の全体像を把握することは，まず不可能である．それにも関わらず，既約成分全体 $\sqcup_{\mathbf{d}\in\mathbb{Z}^I_{\geq 0}}\mathrm{Irr}\Lambda_A(\mathbf{d})$ には常に “crystal” と呼ばれる特殊な構造が入り，それによって $\sqcup_{\mathbf{d}\in\mathbb{Z}^I_{\geq 0}}\mathrm{Irr}\Lambda_A(\mathbf{d})$ はコントロール可能となるのである．これが，Introduction に述べた「筆者と柏原による共著論文 [10] を preprojective algebra の表現論を用いて再解釈した結果」である．具体的な内容については，次節以降で詳しく述べていく予定である．

さらには，単に全体像が把握出来るだけでなく，本来の問題であった preprojective algebra A の表現論に関する情報もそれなりに引き出すことが出来る．多元環の表現論の立場からすれば，この部分が最も興味がある部分であろうとも思うが，かなりの紙数を必要とするので，今回の小論で解説することはしない．詳しくは Geiss-Leclerc-Schröer による一連の論文 [4],[5],[6],[7] や，Kimura の論文[4][11]，およびそれらの中の参考文献を参照して頂きたい．

3. Preprojective algebras

3.1. **定義と性質**.

以下 $\Gamma = (I,\Omega)$ は loop を持たないと仮定する．さらに $H := \Omega \sqcup \overline{\Omega}$ とし，double quiver $\widetilde{\Gamma} := (I,H)$ を考える．

[4]おそらく本報告集にも解説が掲載されることと思う．

Definition 1. double quiver $\widetilde{\Gamma}$ の path algebra $\mathbb{C}[\widetilde{\Gamma}]$ の中で，$|I| = n$ 個の relations

$$\mu_i := \sum_{\substack{\tau \in H \\ \mathrm{out}(\tau)=i}} \varepsilon(\tau)\overline{\tau}\tau \qquad (i \in I)$$

で生成される両側イデアルを J とする．ただし，$\varepsilon(\tau) = \begin{cases} 1 & (\tau \in \Omega) \\ -1 & (\tau \in \overline{\Omega}) \end{cases}$ である．このとき，

$$P(\Gamma) := \mathbb{C}[\widetilde{\Gamma}]/J$$

を Γ に付随する Preprojective algebra と呼ぶ．また，J を生成する μ_i たちを preprojective relations と呼ぶ．

$P(\Gamma)$ について知られていることを列挙しておく．

Proposition 2.
(1) $P(\Gamma)$ が $\mathbb{C}$ 上の有限次元代数 $\Leftrightarrow$ Γ は *Dynkin quiver*.
(2) $P(\Gamma)$ が *finite representation type* $\Leftrightarrow$ Γ は A_n $(n = 1, 2, 3, 4)$ 型.
(3) $P(\Gamma)$ が *tame representation type* $\Leftrightarrow$ Γ は A_5 *or* D_4 型.

つまり，Γ が上記以外の場合には $P(\Gamma)$ は wild になってしまうわけである．

3.2. **Varieties of nilpotent representations.**

Proposition 2 (1) から，Γ が non-Dynkin の場合は，$P(\Gamma)$ は無限次元になってしまう．この場合には，有限次元表現のなす代数多様体を考えるのではなく，以下に述べる表現のベキ零性を仮定することにする．

Definition 3. $B \in E_{\mathbf{d},\Omega}$ がベキ零 (nilpotent) であるとは，非負整数 N が存在して，長さが N 以上の任意の path σ に対して，$B_\sigma = 0$ なることをいう．

これは module の言葉で言えば「B に対応する有限次元 $P(\Gamma)$-module を V_B とする時，長さが N 以上の任意の path σ に対して，$\sigma V = \{0\}$ が成り立つ」ということになる．

さて，今の場合に考える多様体の定義を与えよう．$\mathbf{d} \in \mathbb{Z}^I_{\geq 0}$ に対して，

$$X(\mathbf{d}) := \bigoplus_{\tau \in H} \mathrm{Hom}_{\mathbb{C}}(V(\mathbf{d})_{\mathrm{out}(\tau)}, V(\mathbf{d})_{\mathrm{in}(\tau)})$$

とし，

$$\Lambda(\mathbf{d}) := \{B \in X(\mathbf{d}) \mid \mu_i(B) = 0 \ (\forall\, i \in I) \text{ かつ } B \text{ は nilpotent}\}$$

なる $X(\mathbf{d})$ の部分代数多様体を考える．これは dimension vector が $\mathbf{d}$ の，nilpotent $P(\Gamma)$-modules 全体のなす代数多様体に他ならない．

Remark 4. (1) Γ が Dynkin quiver の場合には，$B \in E_\Omega(\mathbf{d})$ が $\mu_i(B) = 0$ $(\forall\, i \in I)$ を満たせば，自動的に B は nilpotent となることが知られている ([9])．したがって，この場合には $\Lambda(\mathbf{d})$ は dimension vector $= \mathbf{d}$ の $P(\Gamma)$-module 全体を考えていることになる．
(2) 上では，始めから "preprojective algebra ありき" という立場で代数多様体 $\Lambda(\mathbf{d})$ を導入した．ところが，実は $\Lambda(\mathbf{d})$ は（preprojective algebra を全く知らなくても）quiver の幾何

学だけから自動的に現れる，非常に自然な対象である．別の言い方をすれば『preprojective relations は，もともと幾何が知っているもの』ということになる[5]

$P(\Gamma)$-**nilp** を nilpotent な有限次元 $P(\Gamma)$-module 全体のなす category とする．前節の『表現の同値類 = orbit』の対応を今の場合に書けば，

$$\left\{ \begin{array}{c} \underline{\dim} = \mathbf{d} \text{ なる} \\ P(\Gamma)\text{-}\mathbf{nilp} \text{ の object の同型類} \end{array} \right\} \overset{1:1}{\longleftrightarrow} \{\Lambda(\mathbf{d}) \text{ の } G(\mathbf{d})\text{-orbit}\}$$

ということになる．

Proposition 2 で述べたように，ごく少数の例外を除いてほぼ全ての場合に $P(\Gamma)$ は wild になってしまう．したがって，“$G(\mathbf{d})$-orbit の幾何” を考えるのは非常に難しい．そこで，前述のように $\Gamma(\mathbf{d})$ の既約成分を考えることにする．すなわち

$$\mathrm{Irr}\Lambda(\mathbf{d}) := \Lambda(\mathbf{d}) \text{ の（代数多様体としての）既約成分全体のなす集合}$$

とし，可能な dimension vector を全て走らせたもの

$$\mathbb{B} := \bigsqcup_{\mathbf{d} \in \mathbb{Z}^I_{\geq 0}} \mathrm{Irr}\Lambda(\mathbf{d})$$

を考える．『この $\mathbb{B}$ に “crystal” と呼ばれる構造が入り，それによって $\mathbb{B}$ を詳しく調べることが出来るようになる』というのが，[10] の主結果（の一つ）である．

4. A crystal structure on $\mathbb{B}$

4.1. **Root datum.**

crystal を定義するには，その前提として root datum と呼ばれるある種のデータを fix する必要がある．今の場合，それは『dimension vectors が住んでいる lattice $\mathbb{Z}^I$ 上に，quiver から定まる Cartan matrix を使って bilinear form を定める』ということになる．

ところが，この “Cartan matrix” というのがちょっと曲者で，多元環の表現論と Lie theory では，その意味が異なる[6]．ただ，肝心なのは $\mathbb{Z}^I$ 上に定まる bilinear form の方であり，こちらは Cartan matrix の意味をどちらに取ったとしても同じものを定める．したがって問題が無いと言えば無いのだが，何も知らずに文献を読むと誤解を招く恐れがある．そこで本節では，無用の混乱を避けるために両者の違いをはっきりさせておく．

[5] $\Lambda(\mathbf{d})$ の幾何学的な意味は次の通り．一般に，symplectic 多様体 (X,ω) 上に Lie 群 G が symplectic form ω を保つように作用していると，運動量写像 (moment map) と呼ばれる写像 $\mu : X \to \mathrm{Lie}(G)^*$ が定義出来る．ここに $\mathrm{Lie}(G)$ は G の Lie algebra，$\mathrm{Lie}(G)^*$ はその dual space である．これらは symplectic 幾何学と呼ばれる分野の用語である．symplectic 幾何学では moment map による 0 の逆像 $\Lambda := \mu^{-1}(0)$ がしばしば重要な役割を果たす．“運動量” という用語からもわかるように，これらの概念は物理学と非常に関係が深いが，この $\Lambda = \mu^{-1}(0)$ は物理的にも重要な意味を持つことが知られている．

話を $\Lambda(\mathbf{d})$ に戻そう．$X(\mathbf{d}) = E_\Omega(\mathbf{d}) \oplus E_{\overline{\Omega}}(\mathbf{d})$ なる分解を 1 つ固定し，$X(\mathbf{d})$ 上の bilinear form ω を $\omega(B,B') := \sum_{\tau \in H} \varepsilon(\tau)\mathrm{tr}(B_{\overline{\tau}}B'_\tau)$ によって定める．このとき，ω は $X(\mathbf{d})$ 上の非退化な skew-symmetric bilinear form (symplectic form) を定める．さらに $G(\mathbf{d})$ の $X(\mathbf{d})$ への作用はこの symplectic form を保つ．そうすると，上に述べた symplectic 幾何学の一般論から moment map $\mu : X(\mathbf{d}) \to \mathrm{Lie}(G(\mathbf{d}))^*$ を考えることが出来る．この setting で μ による 0 の逆像 $\Lambda = \mu^{-1}(0)$ を考えると，丁度 variety of nilpotent representations $\Lambda(\mathbf{d})$ と一致する．

[6] 筆者は多元環に関しては門外漢なので断言してしまうのは危険だが，少なくとも調べた限り (Ringel [13] や Assem et. al. [1] など) では，「違う」と言って良いようである．

○ 多元環 side

本節に限って，しばらく $\Gamma=(I,\Omega)$ は cycle を持たない (acyclic) と仮定する．このとき $\mathbb{C}[\Gamma]$ は有限次元代数となる．$S(i)$ を頂点 $i\in I$ に対応する simple $\mathbb{C}[\Gamma]$-module, $P(i)$ をその projective cover とする．また $I=\{1,2,\cdots,n\}$ とする．

Definition 5. n 次正方行列 $C_\Gamma=(c_{i,j})_{i,j\in I}$ を次のように定める：

$$c_{i,j}:=\dim_{\mathbb{C}[\Gamma]}(P(i),P(j))\qquad (i,j\in I).$$

この C_Γ を path algebra $\mathbb{C}[\Gamma]$ の Cartan matrix と呼ぶ.

次のことは大事なので，強調しておく.

(i) 一般には Cartan matrix C_Γ は非対称行列である.

(ii) Cartan matrix C_Γ の行列成分 $c_{i,j}$ は常に非負整数である.

(iii) Cartan matrix C_Γ は常に正則行列であり，しかもその逆行列も整係数となる.

筆者のような Lie theory 出身の人間から見ると，これらはどれも「え？」と思ってしまう性質である．上述の通り，誤解の原因は「Cartan matrix の定義が違う」ということにある．(i) と (ii) は定義から明らかなので，(iii) のみ復習しておこう.

Ringel [13] にならって，$\mathbb{Z}^I$ をヨコベクトルの空間と見なす．このとき $\mathbf{s}(i):=\underline{\dim}S(i)$, $\mathbf{p}(i):=\underline{\dim}P(i)$ とすれば,

$$\mathbf{p}(i)=\mathbf{s}(i)^tC_\Gamma \tag{4.1.1}$$

が成り立つ．$\mathbf{s}(i)$ は，第 i 成分のみ 1 で残りが 0 であるような（ヨコ）ベクトルだから, (4.1.1) は「$\mathbf{p}(i)$ は行列 ${}^tC_\Gamma$ の第 i 行である」と言っていることになる.

$\mathbb{C}[\Gamma]$-**mod** を $\mathbb{C}[\Gamma]$-modules のなす abelian category，$K(\mathbb{C}[\Gamma]):=K(\mathbb{C}[\Gamma]\text{-}\mathbf{mod})$ をその Grothendieck 群とする．このとき，自然な対応

$$K(\mathbb{C}[\Gamma])\ni[V]\mapsto\underline{\dim}V\in\mathbb{Z}^I$$

は $\mathbb{Z}$-module の同型

$$\Phi_\Gamma:K(\mathbb{C}[\Gamma])\xrightarrow{\sim}\mathbb{Z}^I$$

を誘導する．ここに $[V]$ は $\mathbb{C}[\Gamma]$-module V の $K(\mathbb{C}[\Gamma])$ における同値類を表す.

$\mathbb{C}[\Gamma]$ は hereditary だから，各 $S(i)$ は高々長さ 1 の projective resolution を持つ．したがって $K(\mathbb{C}[\Gamma])$ の中で $[S(i)]$ は $[P(j)]$ $(j\in I)$ たちの 1 次結合で書ける．同型 Φ_Γ で話を $\mathbb{Z}^I$ の中に移行すれば，これは整係数 n 次正則行列 P' が存在して,

$$E_n=\begin{pmatrix}\mathbf{s}(1)\\ \mathbf{s}(2)\\ \vdots\\ \mathbf{s}(n)\end{pmatrix}=\begin{pmatrix}\mathbf{p}(1)\\ \mathbf{p}(2)\\ \vdots\\ \mathbf{p}(n)\end{pmatrix}P'\qquad (E_n\text{は } n \text{ 次単位行列})$$

と書ける，ということに他ならない．(4.1.1) より $P'={}^tC_\Gamma^{-1}$ であり，(iii) が従う.

C_Γ を用いて $\mathbb{Z}^I$ 上の bilinear form を

$$\langle\langle\mathbf{x},\mathbf{y}\rangle\rangle:=\mathbf{x}\big({}^tC_\Gamma^{-1}\big)^t\mathbf{y}\qquad (\mathbf{x},\mathbf{y}\in\mathbb{Z}^I)$$

で定め，これを Euler form と呼ぶ[7]．上記 (i),(ii),(iii) の帰結として次が従う：

- Euler form $\langle\langle\cdot,\cdot\rangle\rangle$ は対称ではない．
- Euler form $\langle\langle\cdot,\cdot\rangle\rangle$ は $\mathbb{Z}$ に値を持ち，しかも常に非退化である．

Euler form は一般の有限次元代数 A に対しても，同様の方法で定義出来る（ただし非退化性には，A に対する某かの制約が必要となる[8]）．V を projective dimension が有限の A-module, W を injective dimension が有限の A-module とすれば，

$$\langle\langle\underline{\dim}V,\underline{\dim}W\rangle\rangle=\sum_{l\geq 0}\dim_{\mathbb{C}}\mathrm{Ext}^l_A(V,W)$$

となる．特に $A=\mathbb{C}[\Gamma]$ の場合には，

$$\langle\langle\underline{\dim}V,\underline{\dim}W\rangle\rangle=\dim_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}[\Gamma]}(V,W)-\dim_{\mathbb{C}}\mathrm{Ext}_{\mathbb{C}[\Gamma]}(V,W)$$

となる．さらに V,W を simple module にとると

$$\begin{aligned}\langle\langle\mathbf{s}(i),\mathbf{s}(j)\rangle\rangle&=\delta_{i,j}-\dim_{\mathbb{C}}\mathrm{Ext}_{\mathbb{C}[\Gamma]}(S(i),S(j))\\&=\begin{cases}1 & (i=j),\\ -(\Omega\text{ の中で }i\text{ から }j\text{ に向かう arrow の本数}) & (i\neq j)\end{cases}\end{aligned}$$

という，“おなじみの” 式が得られる．

Euler form を対称化して，

$$(\mathbf{x},\mathbf{y})_{alg}:=\frac{1}{2}\mathbf{x}\left(C_\Gamma^{-1}+{}^tC_\Gamma^{-1}\right){}^t\mathbf{y}\qquad(\mathbf{x},\mathbf{y}\in\mathbb{Z}^I)$$

なる $\mathbb{Z}^I$ 上の symmetric bilinear form を考える．$\Gamma=(I,\Omega)$ の arrow をひっくり返して得られる quiver を $\overline{\Gamma}:=(I,\overline{\Omega})$ と書こう．このとき，

$$C_{\overline{\Gamma}}={}^tC_\Gamma$$

であるので，symmetric bilinear form $(\cdot,\cdot)_{alg}$ は，

$$\text{対称行列 }\frac{1}{2}\left({}^tC_{\overline{\Gamma}}^{-1}+{}^tC_\Gamma^{-1}\right)\text{ から定まる symmetric bilinear form}$$

と言って良い．上の計算と併せれば，

$$(\mathbf{s}(i),\mathbf{s}(j))_{alg}=\begin{cases}1 & (i=j),\\ -(H=\Omega\cup\overline{\Omega}\text{ の中で }i\text{ から }j\text{ に向かう arrow の本数})/2 & (i\neq j)\end{cases}$$

となることは明らかであろう．

○ Lie theory side

以下，quiver $\Gamma=(I,\Omega)$ は cycle を持っても良いけれど loop は持たないこととし，Γ から arrow の向きを無視して得られる有限グラフ Γ_{Dyn} を考える．Γ_{Dyn} を quiver Γ の (underlying) Dynkin diagram と呼ぶ[9]．

[7]普通なら $\langle\cdot,\cdot\rangle$ と書くところだが，今回は $\langle\cdot,\cdot\rangle$ は別の意味に使いたいので記号を替えた．

[8]例えば A の global dimension が有限であることなど．

[9]“Dynkin” という言葉の使い方も，多元環と Lie theory で異なっている．多元環では $\mathbb{C}[\Gamma]$ が有限表現型を持つ場合を “Dynkin case”，そうでないときを “non-Dynkin case” と呼ぶが，Lie theory では $\mathbb{C}[\Gamma]$ が有限表現型を持つかどうかに関わらず，グラフ Γ_{Dyn} のことを “Dynkin diagram” と呼ぶ．ただし，この言い

Definition 6. $n \times n$ 行列 $A(\Gamma_{Dyn}) = (a_{i,j})_{1 \le i,j \le n}$ を次のように定める：

$$a_{i,j} := \begin{cases} 2 & (i = j), \\ -(\Gamma_{Dyn} \text{ の中で } i \text{ と } j \text{ を結ぶ辺の本数}) & (i \neq j). \end{cases}$$

この $A(\Gamma_{Dyn}) = (a_{i,j})$ を（Dynkin diagram Γ_{Dyn} に付随する）Cartan matrix と呼ぶ[10].

（Lie theoretic な）Cartan matrix $A(\Gamma_{Dyn})$ の性質を列挙しておこう．

(i)' Cartan matrix $A(\Gamma_{Dyn})$ は常に対称行列である．

(ii)' Cartan matrix $A(\Gamma_{Dyn})$ の行列成分 $a_{i,j}$ は常に整数であるが，非負であるとは限らない（対角成分は常に 2 だが，非対角成分は必ず非正）．

(iii)' Cartan matrix $A(\Gamma_{Dyn})$ は正則行列とは限らない[11].

多元環 side の Cartan matrix C_Γ の性質 (i)∼(iii) と比較すれば，両者の違いを理解して頂けると思う．つまり，全く違うものを同じ名前で呼んでいるわけである[12]．ただし，両者の間に全く関係が無いわけではない．

$A = A(\Gamma_{Dyn})$ に対し，$\mathbb{Z}^I$ 上の bilinear form を

$$(\mathbf{x}, \mathbf{y})_{Lie} := \mathbf{x} A^t \mathbf{y} \qquad (\mathbf{x}, \mathbf{y} \in \mathbb{Z}^I)$$

で定める．$A(\Gamma_{Dyn})$ は対称行列だから，これは symmetric bilinear form である．さらに $A(\Gamma_{Dyn})$ の定義から，次は明らかであろう：

$$(\mathbf{s}(i), \mathbf{s}(j))_{Lie} = \begin{cases} 2 & (i = j), \\ -(\Gamma_{Dyn} \text{ の中で } i \text{ と } j \text{ を結ぶ辺の本数}) & (i \neq j) \end{cases}$$

「$H = \Omega \cup \overline{\Omega}$ の中で i から j に向かう arrow の本数」と「Γ_{Dyn} の中で i と j を結ぶ辺の本数」は等しいので，

$$(\mathbf{s}(i), \mathbf{s}(j))_{Lie} = 2(\mathbf{s}(i), \mathbf{s}(j))_{alg}$$

となり，定数倍の差しかない．この関係を Cartan matrix の言葉で書き直せば，

$$A(\Gamma_{Dyn}) = 2 \cdot \frac{1}{2} \left({}^t C_{\overline{\Gamma}}^{-1} + {}^t C_{\Gamma}^{-1} \right)$$

ということになる．2 つの Cartan matrix はこの関係で結ばれているわけである．

以下，このノートでは $\mathbb{Z}^I$ 上の symmetric bilinear form として $(\cdot, \cdot)_{Lie}$ を採用することとし，簡単のために添字を省略して $(\cdot, \cdot)$ と書く[13].

Remark 7. Lie theory では，与えられた Cartan matrix $A(\Gamma_{Dyn})$ に対して Lie algebra を構成する．こうして出来る Lie algebra は Kac-Moody Lie algebra と呼ばれるものの一部

方は正確ではない．Lie theory で "Dynkin diagram" と呼んでいるものは，実はもっと広い概念で，今の方法で得られる Γ_{Dyn} は Dynkin diagram の中で symmetric と呼ばれる特別な場合になっている．

[10] この定義だと，loop さえ無ければ，Γ に cycle があっても何の問題もない．

[11] 例えば Γ が extended Dynkin（Lie theory では affine 型という）の場合には，$A(\Gamma_{Dyn})$ は corank 1 になる．

[12] 手前ミソになってしまうし，多元環の方々にケンカを売るつもりも毛頭無いのだが，歴史的には Lie theoretic な定義の方が先であると思う．

[13] 「たかが 2 倍の差じゃないか」と思われるかもしれないが，Lie theoretic にはこの "2" に大きな意味があって，ちょっと譲れない．申し訳ないが，この点は御容赦頂きたい．

(symmetric Kac-Moody Lie algebra）になっている[14]．出来上がる Lie algebra は一般には無限次元になるが，これが有限次元になることと，Γ_{Dyn} が A, D, E 型（多元環でいうところの Dynkin case）になることは同値になる．

Definition 8. $\Gamma = (I, \Omega)$ を loop のない有限 quiver とする．$(\mathbb{Z}^I) \otimes \mathbb{Q} = \mathbb{Q}^I$ とその基底 $\{\mathbf{s}(i) | i \in I\}$, およびその上の bilinear form $(\cdot, \cdot)$ の組 $\big(\mathbb{Q}^I, \{\mathbf{s}(i)\}, (\cdot, \cdot)\big)$ を Dynkin diagram Γ_{Dyn} に付随する root datum と呼ぶ．ただし，$(\cdot, \cdot)$ は上で構成した $\mathbb{Z}^I$ 上の bilinear form を自明な方法で $\mathbb{Q}^I$ 上に拡大したものである．

Remark 9. この定義は，Lie theory における通常の root datum の定義とは異なっているので，注意が必要．あくまで『このノートだけのもの』と思っておいて頂きたい[15]．Lie theory における root datum が別の形の定義を採用する理由の１つには『Cartan matrix が symmetric でない場合も扱いたい』というものがある．他方，我々の場合には話を quiver から出発させているので，Cartan matrix は symmetric なものしか出てこない．Cartan matrix が symmetric な場合に限定すれば，この定義でも差し支えない．

4.2. **Crystal の定義（暫定版）.**

そもそも「crystal とは何か？」がはっきりしないと話を始めづらいので，まず最初に crystal の定義を与えてしまうことにする．定義だけ見ても「意味不明」に思えるに違いないが，とりあえずは “そんな感じのもの” という程度に認識して頂ければ十分である．

前節のように root datum $\big(\mathbb{Q}^I, \{\mathbf{s}(i)\}, (\cdot, \cdot)\big)$ を fix しよう．基底 $\{\mathbf{s}(i)\}$ で張られる $\mathbf{Q}^I$ の $\mathbb{Z}$-submodule を

$$Q := \bigoplus_{i \in I} \mathbb{Z}\mathbf{s}(i)$$

と書き，これを root lattice と呼ぶ．もちろん $Q \cong \mathbb{Z}^I$ である．さらに次を満たす P を１つ固定する:

(a) P は $\mathbf{Q}^I$ の rank $n = |I|$ の $\mathbb{Z}$-submodule である．

(b) $(\mathbf{z}, Q) \subset \mathbb{Z}$ for every $\mathbf{z} \in P$.

(c) $\mathbf{s}(i) \in P$ for every $i \in I$.

(c) から，常に $Q \subset P$ であることに注意されたい．

Remark 10. (1) bilinear form $(\cdot, \cdot)$ が非退化である場合（例えば Γ が A, D, E の場合）には，上記 (a), (b), (c) を満たすものの中で一番大きなものが canonical に存在するので，これを P とおくことにする．すなわち，

$$P := \{\mathbf{z} \in \mathbb{Q}^I \mid (\mathbf{z}, Q) \subset \mathbb{Z}\}$$

と定める．これは lattice（格子）の理論で言うところの,（Q の）dual lattice と呼ばれるものに他ならない．

一方，$(\cdot, \cdot)$ が退化している場合（例えば Γ が extended Dynkin の場合）に上のように定義をすると，P が $\mathbb{Z}$ 上有限生成でなくなってしまい (a) を満たさない．この場合には canonical な P の choice というのは存在せず，不定性がある．「１つ固定する」と言ってい

[14] symmetric と呼ばれるのは，$A(\Gamma_{Dyn})$ が必ず対称行列になるからである．一般の場合の Kac-Moody Lie algebra は，対称とは限らない Cartan matrix から出発して構成される．

[15] これが定義だと思って Lie theory の文献を見ると，おそらく混乱する．

るのは，このためである．
(2) P は，Lie theory で weight lattice と呼ばれているもの．

随分準備に時間が掛かってしまったが，以下 crystal の定義（暫定版）を与えよう[16]．

Definition 11. 集合 $\mathcal{B}$ と写像たち

$$\mathrm{wt} : \mathcal{B} \to P, \quad \varepsilon_i : \mathcal{B} \to \mathbb{Z} \sqcup \{-\infty\}, \quad \varphi_i : \mathcal{B} \to \mathbb{Z} \sqcup \{-\infty\},$$
$$\widetilde{e}_i : \mathcal{B} \to \mathcal{B} \sqcup \{0\}, \quad \widetilde{f}_i : \mathcal{B} \to \mathcal{B} \sqcup \{0\} \qquad (i \in I)$$

が以下の公理を満たすとき，組 $\big(\mathcal{B}; \mathrm{wt}, \varepsilon_i, \varphi_i, \widetilde{e}_i, \widetilde{f}_i\big)$ を root datum $\big(\mathbb{Q}^I, \{\mathbf{s}(i)\}, (\cdot,\cdot)\big)$ に付随する crystal と呼ぶ．

(C1) 任意の $i \in I$, $b \in \mathcal{B}$ に対し，

$$\varphi_i(b) = \varepsilon_i(b) + (\mathbf{s}(i), \mathrm{wt}(b)).$$

(C2) $b \in \mathcal{B}$ かつ $\widetilde{e}_i b \in \mathcal{B}$ ならば，

$$\mathrm{wt}(\widetilde{e}_i b) = \mathrm{wt}(b) + \mathbf{s}(i), \quad \varepsilon_i(\widetilde{e}_i b) = \varepsilon_i(b) - 1, \quad \varphi_i(\widetilde{e}_i b) = \varphi_i(b) + 1.$$

(C2)' $b \in \mathcal{B}$ かつ $\widetilde{f}_i b \in \mathcal{B}$ ならば，

$$\mathrm{wt}(\widetilde{f}_i b) = \mathrm{wt}(b) - \mathbf{s}(i), \quad \varepsilon_i(\widetilde{f}_i b) = \varepsilon_i(b) + 1, \quad \varphi_i(\widetilde{f}_i b) = \varphi_i(b) - 1.$$

(C3) $b, b' \in \mathcal{B}$ に対して，

$$b' = \widetilde{e}_i b \quad \Leftrightarrow \quad b = \widetilde{f}_i b'.$$

(C4) $b \in \mathcal{B}$ に対して，$\varphi_i(b) = -\infty$ ならば，

$$\widetilde{e}_i b = \widetilde{f}_i b = 0.$$

少し説明が必要だろう．写像の定義に現れる "$-\infty$" や "0" は，それぞれ「$\mathbb{Z}$ に含まれない extra な元」,「$\mathcal{B}$ に含まれない extra な元」と言っているだけで，基本的にそれ以上の意味はない．ただし，$-\infty$ に関しては，$\mathbb{Z}$ の元との "加法" が

$$(-\infty) + a = a + (-\infty) = -\infty \qquad (a \in \mathbb{Z})$$

によって定まっているものとする[17]．このことから，例えば次が従う：

$$\varphi_i(b) = -\infty \quad \Leftrightarrow \quad \varepsilon_i(b) = -\infty.$$

実際，$b \in \mathcal{B}$ に対して，$\mathrm{wt}(b) \in P$ なのであった．よって $(\mathbf{s}(i), \mathrm{wt}(b)) \in \mathbb{Z}$ である．そうすると，公理 (C1) と "加法" の定義によって，$\varphi_i(b) = -\infty$ と $\varepsilon_i(b) = -\infty$ は同値であることが従う．

正確には crystal というのは組 $\big(\mathcal{B}; \mathrm{wt}, \varepsilon_i, \varphi_i, \widetilde{e}_i, \widetilde{f}_i\big)$ のことであるのだが，いちいち書くと大変なので，以後写像を省略して「crystal $\mathcal{B}$」のように書くことにする．

crystal を考えるとき，その構造を図示したもの（crystal graph）を考えると便利である．

[16] "暫定版" と言っているのは，root datum の与え方が（全ての話を quiver から出発させているために）通常のものとは異なってしまっているからである．root datum 以外の部分はこれで良い．

[17] 1 年生の微積分でこんなことをやったら先生に怒られそうだが，ナイーブには「$-\infty$ という数がある」と思っているわけである．

Definition 12. $\mathcal{B}$ を crystal とする．このとき $\mathcal{B}$ を頂点集合とする色付き有向グラフを以下のルールで定め，これを $\mathcal{B}$ の crystal graph と呼ぶ．

ルール：$b' = \widetilde{f}_i b$ であるとき，b から b' に向かって $i \in I$ で色付けされた矢印を引く．

$$b \xrightarrow{\ i\ } b'$$

Example 13. Γ を A_n 型の quiver とし（orientation はどうでも良い），$\omega_1 \in P$ を

$$(\omega_1, \mathbf{s}(i)) = \delta_{1,i}$$

で定める．また $n+1$ 個の元からなる集合

$$\mathcal{B}(\omega_1) := \{b_0, b_1, \cdots, b_n\}$$

に対して，写像 $\mathrm{wt}, \varepsilon_i, \varphi_i, \widetilde{e}_i, \widetilde{f}_i$ を次のように定める．

$$\mathrm{wt}(b_k) := \omega_1 - \sum_{i=1}^{k} \mathbf{s}(i) \quad (0 \leq k \leq n),$$

$$\varepsilon_i(b_k) := \begin{cases} 1 & (i = k), \\ 0 & (\text{otherwise}), \end{cases} \qquad \varphi_i(b_k) := \begin{cases} 1 & (i = k+1), \\ 0 & (\text{otherwise}), \end{cases}$$

$$\widetilde{e}_i b_k := \begin{cases} b_{k-1} & (i = k), \\ 0 & (\text{otherwise}), \end{cases} \qquad \widetilde{f}_i b_k := \begin{cases} b_{k+1} & (i = k+1), \\ 0 & (\text{otherwise}). \end{cases}$$

このとき $\mathcal{B}(\omega_1)$ は crystal である．その crystal graph は以下のようになる．

$$b_0 \xrightarrow{\ 1\ } b_1 \xrightarrow{\ 2\ } b_2 \xrightarrow{\ 3\ } \ \cdots \ \xrightarrow{\ n\ } b_n$$

この $\mathcal{B}(\omega_1)$ は「A_n 型量子群の vector 表現の crystal basis」というもので，crystal の理論では基本的である．ただし，その意味を述べようとすると量子群の表現論が必要になってしまうので，詳しい説明は割愛する．

Example 14. Γ を loop がない勝手な quiver とする．また $j \in I$ を 1 つ固定する．$\mathbb{Z}$ でパラメトライズされる集合

$$\mathcal{B}_j := \{b_j(k) \mid k \in \mathbb{Z}\}$$

に対して，写像 $\mathrm{wt}, \varepsilon_i, \varphi_i, \widetilde{e}_i, \widetilde{f}_i$ を次のように定める．

$$\mathrm{wt}(b_j(k)) := k\mathbf{s}(j) \quad (k \in \mathbb{Z}),$$

$$\varepsilon_i(b_j(k)) := \begin{cases} -k & (i = j), \\ -\infty & (i \neq j), \end{cases} \qquad \varphi_i(b_j(k)) := \begin{cases} k & (i = j), \\ -\infty & (i \neq j), \end{cases}$$

$$\widetilde{e}_i b_j(k) := \begin{cases} b_j(k+1) & (i = j), \\ 0 & (i \neq j), \end{cases} \qquad \widetilde{f}_i b_j(k) := \begin{cases} b_j(k-1) & (i = j), \\ 0 & (i \neq j). \end{cases}$$

このとき $\mathcal{B}_j$ は crystal となる．その crystal graph は以下のようになる．

$$\cdots \xrightarrow{\ j\ } \underset{b_j(k-1)}{\circ} \xrightarrow{\ j\ } \underset{b_j(k)}{\circ} \xrightarrow{\ j\ } \underset{b_j(k+1)}{\circ} \xrightarrow{\ j\ } \underset{b_j(k+2)}{\circ} \xrightarrow{\ j\ } \cdots$$

Example 15. Γ を loop がない勝手な quiver とし，$\lambda \in P$ を固定する．１点からなる集合

$$\mathcal{T}_\lambda := \{t_\lambda\}$$

写像 $\mathrm{wt}, \varepsilon_i, \varphi_i, \widetilde{e}_i, \widetilde{f}_i$ を次のように定める．

$$\mathrm{wt}(t_\lambda) := \lambda, \quad \varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) := -\infty, \quad \widetilde{e}_i t_\lambda = \widetilde{f}_i t_\lambda := 0 \quad (\forall\, i \in I).$$

このとき $\mathcal{T}_\lambda$ は crystal である．crystal graph は１点のみからなるグラフ（矢印無し）になる．

4.3. $\mathbb{B}$ **の場合.**

$\mathbb{B} = \bigsqcup \mathrm{Irr}\Lambda(\mathbf{d})$ 上に $\big(\mathbb{Q}^I, \{\mathbf{s}(i)\}, (\cdot,\cdot)\big)$ root datum とする crystal の構造を定めよう．

○ wt の定義：$\Lambda \in \mathrm{Irr}\Lambda(\mathbf{d})$ に対し，

$$\mathrm{wt}(\Lambda) := -\mathbf{d}.$$

○ ε_i の定義：$B \in \Lambda(\mathbf{d})$ に対して，$\varepsilon_i(B) \in \mathbb{Z}_{\geq 0}$ を次のように定める：

$$\varepsilon_i(B) := \dim_{\mathbb{C}} \mathrm{Coker}\left(\bigoplus_{\tau \in H; \mathrm{in}(\tau)=i} V(\mathbf{d})_{\mathrm{out}(\tau)} \xrightarrow{\oplus B_\tau} V(\mathbf{d})_i \right).$$

$B = (B_\tau)_{\tau \in H} \in \Lambda(\mathbf{d})$ によって，I-graded vector space $V(\mathbf{d})$ を $P(\Gamma)$-module とみなすことが出来ることに注意しよう．このとき，上で与えた $\varepsilon_i(B)$ は，頂点 $i \in I$ に関する $P(\Gamma)$-module $V(\mathbf{d})$ の top (i-th top) の次元に他ならない．

さらに $\Lambda \in \mathrm{Irr}\Lambda(\mathbf{d})$ に対して，Λ の generic な点 $B \in \Lambda$ を取り，

$$\varepsilon_i(\Lambda) := \varepsilon_i(B)$$

と定義する．

○ φ_i の定義：wt と ε_i が決まると，公理 (C1) によって φ_i は自動的に定まる：

$$\varphi_i(\Lambda) := \varepsilon_i(\Lambda) + (\mathbf{s}(i), \mathrm{wt}(\Lambda)).$$

○ $\widetilde{e}_i$ の定義：まず

$$\big(\mathrm{Irr}\Lambda(\mathbf{d})\big)_{i,p} := \big\{\Lambda \in \mathrm{Irr}\Lambda(\mathbf{d}) \mid \varepsilon_i(\Lambda) = p\big\}$$

とおく．このとき $\mathrm{Irr}\Lambda(\mathbf{d})$ は $\big(\mathrm{Irr}\Lambda(\mathbf{d})\big)_{i,p}$ たちの有限個 disjoint union に分かれる：

$$\mathrm{Irr}\Lambda(\mathbf{d}) = \big(\mathrm{Irr}\Lambda(\mathbf{d})\big)_{i,0} \sqcup \big(\mathrm{Irr}\Lambda(\mathbf{d})\big)_{i,1} \sqcup \big(\mathrm{Irr}\Lambda(\mathbf{d})\big)_{i,2} \sqcup \cdots \sqcup \big(\mathrm{Irr}\Lambda(\mathbf{d})\big)_{i,\dim_{\mathbb{C}} V(\mathbf{d})_i}.$$

もちろん，$\big(\mathrm{Irr}\Lambda(\mathbf{d})\big)_{i,p}$ $(0 \leq p \leq \dim_{\mathbb{C}} V(\mathbf{d})_i)$ は空集合になり得ることを注意しておく．

さて，$\Lambda \in \big(\mathrm{Irr}\Lambda(\mathbf{d})\big)_{i,l}$ とし，Λ の中から generic な点 B を取ろう．このとき，

$$l = \varepsilon_i(B) = P(\Gamma)\text{-module } V(\mathbf{d}) \text{ の } i\text{-th top の次元}$$

なのであった．したがって

$V(\mathbf{d})$ の $P(\Gamma)$-submodule V'' が存在して，$V(\mathbf{d})/V'' \cong S(i)^{\oplus l}$ (i-th top of $V(\mathbf{d})$)

となる．すなわち，V'' は i-th radical ということになる．

I-graded vector space としては，$V'' \cong V(\mathbf{d}'')$ である．ただし $\mathbf{d}'' := \mathbf{d} - l\mathbf{s}(i)$ とした．上で述べたことは,『I-graded vector space $V(\mathbf{d}'')$ 上に $P(\Gamma)$-module structure を定めて，

$$0 \to V(\mathbf{d}'') \xrightarrow{\phi''} V(\mathbf{d}) \xrightarrow{\phi'} S(i)^{\oplus l} \to 0 \tag{4.3.1}$$

が $P(\Gamma)$-module の完全列となるように出来る』ということに他ならない．

$P(\Gamma)$-module $V(\mathbf{d}'')$ に対応する $\Lambda(\mathbf{d}'')$ の元を B'' と書こう．こうして対応

$$\Lambda(\mathbf{d}) \ni B \to B'' \in \Lambda(\mathbf{d}'')$$

が得られる．繰り返しになるが，この対応の意味するところは，ナイーブには

『$V(\mathbf{d})$ に対して，その i-th radical を取れ』

ということである．

ただし，対応 $B \to B''$ は写像にはなっていない．実際，$V'' \cong V(\mathbf{d}'')$ は i-th radical だから，それは同型を除いて一意的に定まる．しかし，$V(\mathbf{d}'')$ 上の $P(\Gamma)$-module structure を定める $B'' \in \Lambda(\mathbf{d}'')$ の選び方には不定性があり，一意的には決まらない．

そこで，もう少しまじめに考えてみよう．まず (4.3.1) を単なる I-graded vector space の完全列と思うことにする．$V(\mathbf{d})$ 上に $P(\Gamma)$-module structure を定めるということは，$B \in \Lambda(\mathbf{d})$ を決めることに他ならないので，これを１つ指定しよう．

同様に，$V(\mathbf{d}'')$ 上に $P(\Gamma)$-module structure を定めるためには，$B'' \in \Lambda(\mathbf{d}'')$ を指定すれば良いわけだが，いい加減に取ってしまうと $\phi'' : V(\mathbf{d}'') \to V(\mathbf{d})$ が $P(\Gamma)$-module の射にならない．$P(\Gamma)$-module の射になるためには，$\mathrm{Im}\phi''$ が B-stable になっていることが必要十分である[18]．このとき，商空間 $V(\mathbf{d})/V(\mathbf{d}'') \cong S(i)^{\oplus l}$ には自然に $P(\Gamma)$-module の構造が入る．すなわち，これだけデータを与えておけば，$\phi' : V(\mathbf{d}) \to S(i)^{\oplus l}$ は自動的に $P(\Gamma)$-module の射になる．

話を整理しよう．まずデータとして組 (B, ϕ', ϕ'') であって，$B \in \Lambda(\mathbf{d})$ かつ $\mathrm{Im}\phi''$ が B-stable となっているようなものを取る．このような組 (B, ϕ', ϕ'') の全体を $\Lambda(\mathbf{d}; \mathbf{d}'')$ と書こう．$\Lambda(\mathbf{d}; \mathbf{d}'')$ から $\Lambda(\mathbf{d})$，$\Lambda(\mathbf{d}'')$ にはそれぞれ

$$q_1 : \Lambda(\mathbf{d}; \mathbf{d}'') \ni (B, \phi', \phi'') \mapsto B \in \Lambda(\mathbf{d}), \quad q_2 : \Lambda(\mathbf{d}; \mathbf{d}'') \ni (B, \phi', \phi'') \mapsto B'' \in \Lambda(\mathbf{d}'')$$

なる写像が定義される．こうして図式

$$\Lambda(\mathbf{d}'') \xleftarrow{q_2} \Lambda(\mathbf{d}; \mathbf{d}'') \xrightarrow{q_1} \Lambda(\mathbf{d}) \tag{4.3.2}$$

が得られる．q_1, q_2 は単射ではなく，したがって対応 $B \to B''$ は写像ではない．

このとき次が成り立つ．

Proposition 16 (Kashiwara-S [10])**.** 図式 (4.3.2) は，全単射

$$\left(\mathrm{Irr}\Lambda(\mathbf{d})\right)_{i,l} \xrightarrow{\sim} \left(\mathrm{Irr}\Lambda(\mathbf{d}'')\right)_{i,0}$$

[18] ϕ'' は I-graded vector space の射だったので，$\mathrm{Im}\phi''$ は $V(\mathbf{d})$ の I-graded vector subspace である．すなわち $\mathrm{Im}\phi'' = \oplus_{i \in I} \left(\mathrm{Im}\phi''\right)_i$ なる分解を持っている．任意の $\tau \in H$ に対して，

$$B_\tau \left(\mathrm{Im}\phi''\right)_{\mathrm{out}(\tau)} \subset \left(\mathrm{Im}\phi''\right)_{\mathrm{in}(\tau)}$$

となるとき，$\mathrm{Im}\phi''$ は B-stable であるという．

を誘導する．

つまり,『与えられた $P(\Gamma)$-module に対して，その i-th radical を取れ』という写像は, $B \to B''$ という意味では well-defined にならないが，既約成分間の対応という意味では well-defined であり，しかも全単射を与える，というわけである．

証明は q_1, q_2 の幾何学的な性質によっており，自明ではない．詳しくは原論文 ([10]) を参照されたい．

上の全単射を

$$\widetilde{e}_i^{max} : \big(\mathrm{Irr}\Lambda(\mathbf{d})\big)_{i,l} \xrightarrow{\sim} \big(\mathrm{Irr}\Lambda(\mathbf{d}'')\big)_{i,0}$$

と書こう．この記法は，あたかも『$\widetilde{e}_i$ なる写像がある』ことを想定しているかのように読めるが，実際その通りで，写像 $\widetilde{e}_i$ は以下のように定義される：

$$\widetilde{e}_i : \begin{cases} \big(\Lambda(\mathbf{d})\big)_{i,l} \xrightarrow{\widetilde{e}_i^{max}} \big(\Lambda(\mathbf{d} - l\mathbf{s}(i))\big)_{i,0} \xrightarrow{(\widetilde{e}_i^{max})^{-1}} \big(\Lambda(\mathbf{d} - \mathbf{s}(i))\big)_{i,l-1} & \text{if } l > 0, \\ \big(\Lambda(\mathbf{d})\big)_{i,0} \longrightarrow \{0\} & \text{if } l = 0. \end{cases}$$

これまでの説明から $\widetilde{e}_i$ の “意味” は明らかであろう．つまり，$\widetilde{e}_i$ とは

『与えられた $P(\Gamma)$-module の i-th top を generic に１次元削れ』

という操作に他ならない[19]．特に２行目は『i-th top が０次元のとき（$l = 0$ のとき）は１次元削ることは出来ないから，その場合には行き先を０にせよ』 というわけである．

$\mathbb{B} = \bigsqcup_{\mathbf{d} \in \mathbb{Z}_{\geq 0}^I} \mathrm{Irr}\Lambda(\mathbf{d})$ であったことを思い出そう．各 component 毎に定義された $\widetilde{e}_i$ たちを束ねて，

$$\widetilde{e}_i : \mathbb{B} \to \mathbb{B} \sqcup \{0\}$$

が well-defined に定まるのは，言うまでもない．

○ $\widetilde{f}_i$ の定義：これも前出の $\widetilde{e}_i^{max}$ を使って構成する：

$$\widetilde{f}_i : \big(\Lambda(\mathbf{d})\big)_{i,l} \xrightarrow{\widetilde{e}_i^{max}} \big(\Lambda(\mathbf{d} - l\mathbf{s}(i))\big)_{i,0} \xrightarrow{(\widetilde{e}_i^{max})^{-1}} \big(\Lambda(\mathbf{d} + \mathbf{s}(i))\big)_{i,l+1}.$$

今度の場合は l が０になるかどうかの場合わけは必要ない．$\widetilde{f}_i$ の “意味” は,

『与えられた $P(\Gamma)$-module の i-th top を generic に１次元増やせ』

ということなので，l の値に関わらず操作を行うことが出来るからである．

したがって，今度の場合は『$\widetilde{f}_i$ を施せない』ということはなく，全てを束ねて得られる写像は

$$\widetilde{f}_i : \mathbb{B} \to \mathbb{B}$$

となる．

[19]「だったら面倒なことはせずに，最初から『与えられた $P(\Gamma)$-module の i-th top を generic に１次元削る』ということを $\widetilde{e}_i$ の定義にすればいいんじゃないか？」と思われるかもしれない．確かにその通りなのであるが，このことを幾何的にちゃんと言おうとすると，全単射

$$\big(\mathrm{Irr}\Lambda(\mathbf{d})\big)_{i,l} \xrightarrow{\sim} \big(\mathrm{Irr}\Lambda(\mathbf{d} - \mathbf{s}(i))\big)_{i,l-1}$$

の存在を示さなければならない．結論としてはもちろん正しいのだけれど，少なくとも我々は $\big(\mathrm{Irr}\Lambda(\mathbf{d}'')\big)_{i,0}$ を経由する方法しか，その証明を知らない．

以上で，必要な写像が全て定義出来た．このとき主張は以下の通り．

Proposition 17 ([10])**.** 組 $\left(\mathbb{B};\mathrm{wt},\varepsilon_i,\varphi_i,\widetilde{e}_i,\widetilde{f}_i\right)$ は，*root datum* $\left(\mathbb{Q}^I,\{\mathbf{s}(i)\},(\cdot,\cdot)\right)$ に付随する *crystal* である．

Proof. 公理 (C1)～(C4) を確かめれば良いが，定義から (C1)～(C3) は明らかである．また (C4) についても，$\varphi_i(\Lambda)=-\infty$ となる $\Lambda\in\mathbb{B}$ は存在しないので，OK. □

4.4. **その特徴付け.**

前節で $\mathbb{B}=\bigsqcup_{\mathbf{d}\in\mathbb{Z}_{\geq 0}^I}\mathrm{Irr}\Lambda(\mathbf{d})$ 上に crystal structure が入ること，およびその preprojective algebra の表現論的な意味を説明した.「定義に現れる各写像が，ちゃんと意味を持っている」ということについては，多少は納得してもらえたのではないかと思う．ただし，

Q：crystal structure が入ったからといって，何がうれしいのか？

という問いに対しては，何も答えてはいない．

この問いを preprojective algebra の表現論だけから説明するのは非常に難しいのだが，crystal の一般論の立場からは次のように説明される．

Theorem 18 (Kashiwara-S [10])**.** *crystal* として $\mathbb{B}$ は $B(\infty)$ と同型である[20].

ここに $B(\infty)$ とは，“量子群のベキ零部分の crystal basis” と呼ばれる crystal で，その構造が非常によく調べられているものである．このノートでは「量子群の表現論に関する事実は一切使わない」ことにしているので，“量子群のベキ零部分の crystal basis” が何か？ということには触れないが，強調しておきたいのは『Theorem 18 が上の問いに対する１つの解答を与えている』ということである．すなわち，

A：crystal の一般論で知られている結果を用いることによって，preprojective algebra の nilpotent な表現全体のなす variety の構造を調べることが出来る

ということである．これさえ頭に留めて頂ければ，このノートを書いた意味は半分以上達成されていると言って良い．

Γ が Dynkin case の場合（特に A 型の場合）については，5 節に詳しい解説を書いたので，そちらを参照して頂きたい．

4.5. **Another crystal structure on $\mathbb{B}$.**

4.3 節で導入した crystal structure は，多元環の立場で言えば “表現の top をいじる” という操作によって実現されたものであった．多元環の表現論においては,「top でうまくいく」話というのは「socle でもうまくいく」ことが多い．では今の場合はどうか？というと，実は OK で，top と socle の役割を入れ替えても $\mathbb{B}$ 上に crystal structure を定義することが出来る．ただし，こうして定義される $\mathbb{B}$ 上の crystal structure は，4.3 節で導入したものとは別物であることに注意されたい．

具体的に書いていこう．まず wt は 4.3 節と共通に取る．次に ε_i だが，これは 4.3 節とは違うものを考えるので，区別のために右上に $*$ を付けて ε_i^* と書くことにしよう．

[20]「crystal の同型」を定義していないのでこの主張は意味をなさないけれども，感じはわかって頂けると思う．

$B \in \Lambda(\mathbf{d})$ に対して，

$$\varepsilon_i^*(B) := \dim_{\mathbb{C}} \operatorname{Ker}\left(V(\mathbf{d})_i \xrightarrow{\oplus B_\tau} \bigoplus_{\tau\in H; \mathrm{out}(\tau)=i} V(\mathbf{d})_{\mathrm{in}(\tau)} \right)$$

とおく．これは対応する $P(\Gamma)$-module の i-th socle の次元に他ならない．

そこで，$\Lambda \in \mathrm{Irr}\Lambda(\mathbf{d})$ に対して Λ の generic な点 $B \in \Lambda$ を取り，

$$\varepsilon_i^*(\Lambda) := \varepsilon_i^*(B)$$

と定義する．また

$$\varphi_i^*(\Lambda) := \varepsilon_i^*(\Lambda) + (\mathbf{s}(i), \mathrm{wt}(\Lambda))$$

と定める．

4.3 節では $\varepsilon_i = i$-th top の次元で $\mathrm{Irr}\Lambda(\mathbf{d})$ を分割したが，今度は $\varepsilon_i^* = i$-th socle の次元を使って同じことをやる．

$$\mathrm{Irr}\Lambda(\mathbf{d}) = \bigsqcup_{p\geq 0} \big(\mathrm{Irr}\Lambda(\mathbf{d})\big)_i^p, \quad \text{where } \big(\mathrm{Irr}\Lambda(\mathbf{d})\big)_i^p := \big\{\Lambda \in \mathrm{Irr}\Lambda(\mathbf{d}) \mid \varepsilon_i^*(\Lambda) = p\big\}.$$

そこで，4.3 節と同様のアイデアで全単射

$$(\widetilde{e}_i^*)^{max} : \big(\mathrm{Irr}\Lambda(\mathbf{d})\big)_i^l \xrightarrow{\sim} \big(\mathrm{Irr}\Lambda(\mathbf{d}')\big)_i^0$$

を構成する．ただし $\mathbf{d}' := \mathbf{d} - l\mathbf{s}(i)$ である[21]．やり方はほぼ同じなので繰り返さないがちょっとだけ注意をしておくと，今回は top と socle の役割を取り替えているので，(4.3.1) に当たる図式は map の向きが反対になる：

$$0 \to S(i)^{\oplus l} \xrightarrow{\phi''} V(\mathbf{d}) \xrightarrow{\phi'} V(\mathbf{d}') \to 0. \tag{4.5.1}$$

この図式を元に (4.3.2) に相当する図式を考えることが出来て，それが全単射 $(\widetilde{e}_i^*)^{max}$ を誘導する，というストーリーになっている．$(\widetilde{e}_i^*)^{max}$ の意味は『i-th cosocle を取れ』ということである．

全単射 $(\widetilde{e}_i^*)^{max}$ を用いて，写像

$$\widetilde{e}_i^* : \mathbb{B} \to \mathbb{B} \sqcup \{0\}, \qquad \widetilde{f}_i^* : \mathbb{B} \to \mathbb{B}$$

を同様の方法で定める．このとき主張は以下の通り．

Theorem 19 ([10]). (1) 組 $\big(\mathbb{B}; \mathrm{wt}, \varepsilon_i^*, \varphi_i^*, \widetilde{e}_i^*, \widetilde{f}_i^*\big)$ は，*root datum* $\big(\mathbb{Q}^I, \{\mathbf{s}(i)\}, (\cdot,\cdot)\big)$ に付随する *crystal* である．

(2) *crystal* として $\big(\mathbb{B}; \mathrm{wt}, \varepsilon_i^*, \varphi_i^*, \widetilde{e}_i^*, \widetilde{f}_i^*\big)$ と $B(\infty)$ は同型である．

(2) は注釈が必要だろう．Theorem 18 と Theorem 19 を併せると

$$\big(\mathbb{B}; \mathrm{wt}, \varepsilon_i, \varphi_i, \widetilde{e}_i, \widetilde{f}_i\big) \cong B(\infty) \cong \big(\mathbb{B}; \mathrm{wt}, \varepsilon_i^*, \varphi_i^*, \widetilde{e}_i^*, \widetilde{f}_i^*\big)$$

ということになってしまい「全く違う定義をしたはずなのに，なぜ？」と思うかもしれない．しかし，よく考えると矛盾はしていない．上の同型はあくまで「抽象的な crystal としての同型」を与えているだけなので，例えば “$\varepsilon_i = \varepsilon_i^*$” ということを主張しているわけではない．上の同型が主張するところは

『top を用いて定義した構造と，socle を用いて定義した構造が，抽象的には一致する』

[21]ベクトルとしては $\mathbf{d}' = \mathbf{d}''$ であるけれども，役割が違うので記号を替えた．

ということだけであり,「top について成り立つことは，socle に対しても成り立つ場合がある」という経験則からすれば，ある意味 “自然” かも知れない.

5. Dynkin case

5.1. 一般論.

本節では話を $\Gamma = (I, \Omega)$ が Dynkin quiver（A, D, E）の場合に限定する．多元環 side から見た場合に，この条件は $P(\Gamma)$ が有限次元代数であることと同値であることは既に述べた．実は幾何学的にも，この条件は大きな意味を持つ.

Proposition 20 (Lusztig [9])**.** $\Gamma = (I, \Omega)$ を *Dynkin quiver* とする．このとき勝手に与えられた $\Lambda \in \mathrm{Irr}\Lambda(\mathbf{d})$ に対して，$E_\Omega(\mathbf{d})$ の $G(\mathbf{d})$*-orbit* $\mathcal{O}_\Omega$ が一意的に存在して，

$$\Lambda = \overline{T^*_{\mathcal{O}_\Omega} E_\Omega(\mathbf{d})}$$

と書ける．逆に，与えられた $E_\Omega(\mathbf{d})$ の $G(\mathbf{d})$*-orbit* $\mathcal{O}_\Omega$ に対し，$\overline{T^*_{\mathcal{O}_\Omega} E_\Omega(\mathbf{d})} \in \mathrm{Irr}\Lambda(\mathbf{d})$ である．したがって，次の１対１対応が得られる：

$$\left\{E_\Omega(\mathbf{d}) \text{ の } G(\mathbf{d})\text{-}orbit\right\} \ni \mathcal{O}_\Omega \ \overset{\sim}{\longleftrightarrow}\ \overline{T^*_{\mathcal{O}_\Omega} E_\Omega(\mathbf{d})} \in \mathrm{Irr}\Lambda(\mathbf{d}). \tag{5.1.1}$$

またしてもヘンな記号が現れたが，$T^*_{\mathcal{O}_\Omega} E_\Omega(\mathbf{d})$ は “$\mathcal{O}_\Omega$ の conormal bundle”，$\overline{T^*_{\mathcal{O}_\Omega} E_\Omega(\mathbf{d})}$ は “その closure” という意味である．conormal bundle というのは，より一般的な setting で定義される，幾何学的には基本的な概念であるが，幾何の言葉に不慣れな方も多いと思う．話を今回の場合に限ってしまえば，以下のように考えてもらっても差し支えない.

まず

$$X(\mathbf{d}) = \bigoplus_{\tau \in H} \mathrm{Hom}_{\mathbb{C}}(V(\mathbf{d})_{\mathrm{out}(\tau)}, V(\mathbf{d})_{\mathrm{in}(\tau)}) = E_\Omega(\mathbf{d}) \oplus E_{\overline{\Omega}}(\mathbf{d})$$

に注意しよう．第１成分への射影 $\pi_{\mathbf{d},\Omega} : X(\mathbf{d}) \to E_\Omega(\mathbf{d})$ とし，$\pi_{\mathbf{d},\Omega}$ の $\Lambda(\mathbf{d})$ への制限を $\pi_{\Lambda(\mathbf{d}),\Omega} := \pi_{\mathbf{d},\Omega}|_{\Lambda(\mathbf{d})}$ と記す．このとき $\pi_{\Lambda(\mathbf{d}),\Omega}$ は全射である．したがって $\Lambda(\mathbf{d})$ の分割

$$\Lambda(\mathbf{d}) = \bigsqcup_{\mathcal{O}_\Omega; G(\mathbf{d})\text{-orbit}} \pi^{-1}_{\Lambda(\mathbf{d}),\Omega}(\mathcal{O}_\Omega)$$

が得られる．Γ は Dynkin type なので，右辺は有限個の disjoint union である．$\pi^{-1}_{\Lambda(\mathbf{d}),\Omega}(\mathcal{O}_\Omega)$ が $\Lambda(\mathbf{d})$ の $G(\mathbf{d})$ 不変な部分集合となることは定義からすぐに解るけれども，単独の $G(\mathbf{d})$-orbit になるとは限らない（一般にはなっていない）.

Proposition 21 ([9])**.** Γ が *Dynkin type* ならば，

$$\pi^{-1}_{\Lambda(\mathbf{d}),\Omega}(\mathcal{O}_\Omega) = T^*_{\mathcal{O}_\Omega} E_\Omega(\mathbf{d})$$

である．したがって次が成立する：

$$\overline{\pi^{-1}_{\Lambda(\mathbf{d}),\Omega}(\mathcal{O}_\Omega)} = \overline{T^*_{\mathcal{O}_\Omega} E_\Omega(\mathbf{d})}.$$

つまり，一般的な conormal bundle の定義を知らなくても，今の場合は $\pi^{-1}_{\Lambda(\mathbf{d}),\Omega}(\mathcal{O}_\Omega)$ をその定義と思ってもいいわけである.

Proposition 21 の “意味” を考えてみよう．まず $\Lambda \in \mathrm{Irr}\Lambda(\mathbf{d})$ とし，Λ の中から generic に $(B_\tau)_{\tau \in H}$ をとる．このとき

『I-graded vector space $V(\mathbf{d})$ を，$(B_\tau)_{\tau\in H}$ によって
nilpotent $P(\Gamma)$-module と思う』

ことが出来る．また，射影 $\pi_{\Lambda(\mathbf{d}),\Omega}((B_\tau)_{\tau\in H})=(B_\tau)_{\tau\in\Omega}$ をとるということは，

『与えられた $P(\Gamma)$-module $V(\mathbf{d})$ を，$\tau\in\overline{\Omega}$ の作用を
忘れることによって，$\mathbb{C}[\Gamma]$-module と思え』

ということに他ならない．したがって Proposition 21 は

$$T^*_{\mathcal{O}_\Omega}E_\Omega(\mathbf{d})=\left\{\begin{array}{c}V(\mathbf{d})\text{ 上の }P(\Gamma)\text{-module structure であって，それを}\\ \mathbb{C}[\Gamma]\text{-module と思ったときに同型になるようなもの}\end{array}\right\}$$

ということを意味していることになる．

話を先に進めよう．いま Γ は Dynkin type だったので，有名な Gabriel の定理により，直既約な $\mathbb{C}[\Gamma]$-module の同型類は dimension vector で一意的に定まり，しかもそれらは対応する Dynkin 図形 Γ_{Dyn} に付随する positive root でパラメトライズされる．したがって，上の手順で得られた $\mathbb{C}[\Gamma]$-module $V(\mathbf{d})$ は

$$V(\mathbf{d})=\bigoplus_{\beta\in\Delta^+}V_\Omega(\beta)^{\oplus a_{\beta,\Omega}}$$

の形に一意的に分解する．ただし $\Delta^+=\Delta^+(\Gamma_{Dyn})$ は positive root 全体の集合，$V_\Omega(\beta)$ は $\beta\in\Delta^+$ に対応する直既約 $\mathbb{C}[\Gamma]$-module である．こうして得られる非負整数の組を

$$\mathbf{a}_\Omega:=(a_{\beta,\Omega})_{\beta\in\Delta^+}\in\mathbb{Z}^N_{\geq 0}\qquad(N:=|\Delta^+|)$$

と書く．

整理すると，

(i) $\Lambda\in\mathrm{Irr}\Lambda(\mathbf{d})$ とし，Λ の中から generic に $(B_\tau)_{\tau\in H}$ をとる；
(ii) $\pi_{\Lambda(\mathbf{d}),\Omega}((B_\tau)_{\tau\in H})=(B_\tau)_{\tau\in\Omega}$ によって，$V(\mathbf{d})$ を $\mathbb{C}[\Gamma]$-module とみなす；
(iii) $\mathbb{C}[\Gamma]$-module $V(\mathbf{d})$ を直既約分解して，各直既約因子の multiplicity を表す非負整数の組 $\mathbf{a}_\Omega=(a_{\beta,\Omega})_{\beta\in\Delta^+}\in\mathbb{Z}^N_{\geq 0}$ を取り出す；

という操作によって，写像 $\Psi_{\mathbf{d},\Omega}:\mathrm{Irr}\Lambda(\mathbf{d})\to\mathbb{Z}^N_{\geq 0}$ が定義されたことになる．(5.1.1) から $\Psi_{\mathbf{d},\Omega}$ は単射であることに注意しよう．dimension vector $\mathbf{d}$ の可能性を全て走らせて，これらを束ねると，全単射

$$\Psi_\Omega:\mathbb{B}\xrightarrow{\sim}\mathbb{Z}^N_{\geq 0}$$

が得られる．また逆写像 $\Psi_\Omega^{-1}:\mathbb{Z}^N_{\geq 0}\xrightarrow{\sim}\mathbb{B}$ は，$\mathbf{a}\mapsto\overline{T^*_{\mathcal{O}_{\mathbf{a},\Omega}}E_\Omega(\mathbf{d})}$ で与えられる．ここに $\mathcal{O}_{\mathbf{a},\Omega}$ は，$\mathbf{a}\in\mathbb{Z}^N_{\geq 0}$ に対応する $G(\mathbf{d})$-orbit である．

「crystal structure を理解するためには crystal graph がわかれば良い」というのは以前述べた通りなのだが，前節で構成した $\mathbb{B}$ 上の crystal structure の情報だけでは，具体的に graph の頂点集合として何を考えたらいいのか，全く解らなかった．ところが Γ が Dynkin type の場合には，『$\mathbb{Z}^N_{\geq 0}$ を頂点集合とするグラフを描けば良い』ということがはっきりしたわけである．あとは全単射 $\Psi_\Omega:\mathbb{B}\xrightarrow{\sim}\mathbb{Z}^N_{\geq 0}$ を通じて $\mathbb{B}$ に入っている crystal structure を $\mathbb{Z}^N_{\geq 0}$ 上に移植させ，各格子点間に矢印を描いていけば良い[22]．

[22]もちろん「$\mathbb{Z}^N_{\geq 0}$ を頂点集合とする graph」なんてものが，本当に絵に描けるわけではないが．

また，全単射 $\Psi_\Omega : \mathbb{B} \xrightarrow{\sim} \mathbb{Z}_{\geq 0}^N$ は orientation Ω の選び方に depend するので，Ω を変えれば $\mathbb{Z}_{\geq 0}^N$ に入る crystal structure も当然変わる．そこで『$\mathbb{Z}_{\geq 0}^N$ 上にどんな crystal structure が入っているか？』を区別したい場合には，これを $\mathcal{B}_\Omega$ と書くことにする．すなわち

$$\mathcal{B}_\Omega := \left\{ \mathbf{a}_\Omega = (a_{\beta,\Omega})_{\beta \in \Delta^+} \;\middle|\; a_{\beta,\Omega} \in \mathbb{Z}_{\geq 0} \text{ for every } \beta \in \Delta^+ \right\}.$$

5.2. A 型の場合（その１）.

本節では $\Gamma = (I, \Omega)$ が A_n 型の Dynkin quiver の場合に限定する．このとき，positive root の集合 $\Delta^+ = \Delta(A_n)^+$ は以下のようになる：

$$\Delta^+ = \left\{ \beta_{i,j} := \sum_{k=i}^{j} \mathbf{s}(i) \;\middle|\; 1 \leq i \leq j \leq n \right\}.$$

したがって，

$$N = |\Delta^+| = \frac{n(n+1)}{2}.$$

同型 $\Psi_\Omega : \mathbb{B} \xrightarrow{\sim} \mathbb{Z}_{\geq 0}^N$ を指定するには orientation Ω を選ばなければならない．ここでは次のようなものを取ろう：

$$\Omega_0 : \quad \underset{1}{\circ} \longleftarrow \underset{2}{\circ} \longleftarrow \underset{3}{\circ} \longleftarrow \quad \cdots \quad \longleftarrow \underset{n-2}{\circ} \longleftarrow \underset{n-1}{\circ} \longleftarrow \underset{n}{\circ}.$$

記述を簡単にするために，

$$a_{i,j} := a_{\beta_{i,j+1},\Omega_0} \ (1 \leq i < j \leq n+1),$$

$$\mathcal{B}_{\Omega_0} = \{ \mathbf{a} := (a_{i,j})_{1 \leq i < j \leq n+1} \mid a_{i,j} \in \mathbb{Z}_{\geq 0} \}$$

と書くことにしよう[23]．このとき $\mathbb{B}$ に入っている２つの crystal structure $\left(\mathbb{B}; \mathrm{wt}, \varepsilon_i, \varphi_i, \widetilde{e}_i, \widetilde{f}_i\right)$ および $\left(\mathbb{B}; \mathrm{wt}, \varepsilon_i^*, \varphi_i^*, \widetilde{e}_i^*, \widetilde{f}_i^*\right)$ を同型 $\Psi_{\Omega_0} : \mathbb{B} \xrightarrow{\sim} \mathcal{B}_{\Omega_0}$ によって $\mathcal{B}_{\Omega_0} \cong \mathbb{Z}_{\geq 0}^N$ 側に移植する．ここでは結果のみ記すこととし，どうしてそうなるか？という理由には一切触れない[24]．

<u>○ wt の定義</u>：$\mathbf{a} \in \mathbb{Z}_{\geq 0}^N$ に対し，

$$\mathrm{wt}(\mathbf{a}) = -\sum_{i \in I} d_i \mathbf{s}(i), \quad \text{where} \quad d_i = \sum_{k=1}^{i} \sum_{l=i+1}^{n+1} a_{k,l} \quad (i \in I).$$

<u>○ ε_i, ε_i^*, φ_i, φ_i^* の定義</u>：$i \in I$ に対して

$$A_k^{(i)}(\mathbf{a}) := \sum_{s=1}^{k} (a_{s,i+1} - a_{s-1,i}) \quad (1 \leq k \leq i),$$

---

[23] 普通に考えれば

$$a_{i,j} := a_{\beta_{i,j}} \ (1 \leq i \leq j \leq n)$$

とするのが自然だろう．ただし，既存の crystal の理論と比較するためには，２つめの添字 j の番号付けを１つずらしておいた方が都合が良い．気持ち悪いかもしれないが，このノートでは「ずらした添字付け」を採用する．

[24] 説明の仕方はいろいろある．参考文献として Reineke [12], Savage [18], 筆者 [17] を挙げておくが，いずれにしてもそれなりに準備が必要で，結果は自明ではない．

$$A_l^{*(i)}(\mathbf{a}) = \sum_{t=l+1}^{n+1} (a_{i,t} - a_{i+1,t+1}) \quad (i \le l \le n)$$

とおく．ただし上式において $a_{0,i} = a_{i+1,n+2} = 0$ と思うことにする．このとき，

$$\varepsilon_i(\mathbf{a}) := \max\left\{A_1^{(i)}(\mathbf{a}), \cdots, A_i^{(i)}(\mathbf{a})\right\}, \quad \varphi_i(\mathbf{a}) := \varepsilon_i(\mathbf{a}) + \big(\mathbf{s}(i), \mathrm{wt}(\mathbf{a})\big),$$

$$\varepsilon_i^*(\mathbf{a}) := \max\left\{A_i^{*(i)}(\mathbf{a}), \cdots, A_n^{*(i)}(\mathbf{a})\right\}, \quad \varphi_i^*(\mathbf{a}) := \varepsilon_i^*(\mathbf{a}) + \big(\mathbf{s}(i), \mathrm{wt}(\mathbf{a})\big).$$

○ $\widetilde{e}_i$, $\widetilde{f}_i$, $\widetilde{e}_i^*$, $\widetilde{f}_i^*$ の定義：

$$k_+ := \min\left\{1 \le k \le i \,\middle|\, \varepsilon_i(\mathbf{a}) = A_k^{(i)}(\mathbf{a})\right\}, \quad k_- := \max\left\{1 \le k \le i \,\middle|\, \varepsilon_i(\mathbf{a}) = A_k^{(i)}(\mathbf{a})\right\},$$

$$l_+ := \max\left\{i \le l \le n \,\middle|\, \varepsilon_i^*(\mathbf{a}) = A_l^{*(i)}(\mathbf{a})\right\}, \ l_- := \min\left\{i \le l \le n \,\middle|\, \varepsilon_i^*(\mathbf{a}) = A_l^{*(i)}(\mathbf{a})\right\}$$

とおく．$\mathbf{a} \in \mathbb{Z}_{\ge 0}^N$ に対し，新たに４つの非負整数の組 $\mathbf{a}^{(p,\natural)} = \left(a_{k,l}^{(p,\natural)}\right) \in \mathbb{Z}_{\ge 0}^N$ $(\natural = \pm, p = 1,2)$ を次で定義する：

$$a_{k,l}^{(1,\pm)} = \begin{cases} a_{k_\pm,i} \pm 1 & (k = k_\pm, l = i), \\ a_{k_\pm,i+1} \mp 1 & (k = k_\pm, l = i+1), \\ a_{k,l} & (\text{otherwise}). \end{cases}$$

$$a_{k,l}^{(2,\pm)} = \begin{cases} a_{i,l_\pm+1} \mp 1 & (k = i,\ l = l_\pm + 1), \\ a_{i+1,l_\pm+1} \pm 1 & (k = i+1,\ l = l_\pm + 1), \\ a_{k,l} & (\text{otherwise}). \end{cases}$$

以上の記法の下に，

$$\widetilde{e}_i\mathbf{a} := \begin{cases} 0 & (\text{if } \varepsilon_i(\mathbf{a}) = 0), \\ \mathbf{a}^{(1,+)} & (\text{if } \varepsilon_i(\mathbf{a}) > 0), \end{cases} \qquad \widetilde{f}_i\mathbf{a} := \mathbf{a}^{(1,+)},$$

$$\widetilde{e}_i^*\mathbf{a} := \begin{cases} 0 & (\text{if } \varepsilon_i^*(\mathbf{a}) = 0), \\ \mathbf{a}^{(2,+)} & (\text{if } \varepsilon_i^*(\mathbf{a}) > 0), \end{cases} \qquad \widetilde{f}_i^*\mathbf{a} := \mathbf{a}^{(2,-)}$$

と定める．

一応定義は書いたものの，これでは何が何やらわからない．もう少し噛み砕いて説明しよう．そのために $\mathbf{a}$ の成分を行列のように並べて，

$$\mathbf{a} = \begin{matrix} a_{1,2} & a_{1,3} & a_{1,4} & \cdots & a_{1,n} & a_{1,n+1} \\ & a_{2,3} & a_{2,4} & \cdots & a_{2,n} & a_{2,n+1} \\ & & a_{3,4} & \cdots & a_{3,n} & a_{3,n+1} \\ & & & \ddots & \vdots & \vdots \\ & & & & a_{n-1,n} & a_{n-1,n+1} \\ & & & & & a_{n,n+1} \end{matrix}$$

と書くことにする．

まず $\widetilde{e}_i$ の作用というのは，基本的に

- $i+1$ 列目[25]に並んでいる数字のどれかから 1 を引いて，i 列目の同じ行にある数字に 1 を足す．
- ただし i 列目の同じ行に数字がないときは，$i+1$ 列目の数字から 1 を引くだけ．

という操作である．問題は

「$i+1$ 列目に並ぶ i 個の数のうち，何行目から 1 を引くか？」

ということだが，これを指定しているのが k_+ であり，それは

「$\varepsilon_i(\mathbf{a}) = A_k^{(i)}$ を満たす $1 \leq k \leq i$ の中で，一番小さいもの」

である．

$\widetilde{f}_i$ の作用は，

- $i+1$ 列目に並んでいる数字のどれかに 1 を足して，i 列目の同じ行にある数字から 1 を引く．
- ただし i 列目の同じ行に数字がないときは，$i+1$ 列目の数字から 1 を足すだけ．

という操作で，

「$i+1$ 列目に並ぶ i 個の数のうち，何行目に 1 を足すか？」

という情報が k_-，すなわち

「$\varepsilon_i(\mathbf{a}) = A_k^{(i)}$ を満たす $1 \leq k \leq i$ の中で，一番大きいもの」

で与えられる．

$\widetilde{e}_i^*, \widetilde{f}_i^*$ については詳しくは書かないが，ほぼ「行と列の役割を入れ替えたもの」になっている．

さて，以上のように $\mathcal{B}_{\Omega_0} \cong \mathbb{Z}_{\geq 0}^N$ 上の写像たちを定義するとき，次が成り立つ．

Proposition 22 ([12],[18],[17]). (1) $\left(\mathcal{B}_{\Omega_0}; \mathrm{wt}, \varepsilon_i, \varphi_i, \widetilde{e}_i, \widetilde{f}_i\right)$ および $\left(\mathcal{B}_{\Omega_0}; \mathrm{wt}, \varepsilon_i^*, \varphi_i^*, \widetilde{e}_i^*, \widetilde{f}_i^*\right)$ はともに *crystal* である．
(2) 集合としての全単射 $\Psi_{\Omega_0} : \mathbb{B} \xrightarrow{\sim} \mathcal{B}_{\Omega_0}$ は，*crystal* としての同型

$$\Psi_{\Omega_0} : \left(\mathbb{B}; \mathrm{wt}, \varepsilon_i, \varphi_i, \widetilde{e}_i, \widetilde{f}_i\right) \xrightarrow{\sim} \left(\mathcal{B}_{\Omega_0}; \mathrm{wt}, \varepsilon_i, \varphi_i, \widetilde{e}_i, \widetilde{f}_i\right),$$

$$\Psi_{\Omega_0} : \left(\mathbb{B}; \mathrm{wt}, \varepsilon_i^*, \varphi_i^*, \widetilde{e}_i^*, \widetilde{f}_i^*\right) \xrightarrow{\sim} \left(\mathcal{B}_{\Omega_0}; \mathrm{wt}, \varepsilon_i^*, \varphi_i^*, \widetilde{e}_i^*, \widetilde{f}_i^*\right)$$

を同時に与える．

5.3. A 型の場合（その２）．

前節で一般的な公式を書き下してみたものの，ややこし過ぎてわかりづらい．そこで話を $n=3$ に限定して，もう一度全てを書き直してみよう[26]．

この場合，$N = 3(3+1)/2 = 6$ だから $\mathbf{a}$ は 6 個の非負整数の組である．これらを行列のように並べて表示すると見やすい．

$$\mathbf{a} = \begin{matrix} a_{1,2} & a_{1,3} & a_{1,4} \\ & a_{2,3} & a_{2,4} \\ & & a_{3,4} \end{matrix}$$

[25]これは成分の添字でカウントしていることに注意．例えば $a_{1,2}$ は「２列目」と思っている．

[26]もちろん $n=1,2$ の方がより簡単であるのだが，単純になり過ぎてかえって話が見えづらい．

定義にしたがって計算すると

$$\mathrm{wt}(\mathbf{a}) = (-d_1, -d_2, -d_3),$$

$$d_1 = a_{1,2} + a_{1,3} + a_{1,4}, \quad d_2 = a_{1,3} + a_{1,4} + a_{2,3} + a_{2,4}, \quad d_3 = a_{1,4} + a_{2,4} + a_{3,4}.$$

念のため $\mathbf{a}$ の意味を復習しておこう．5.1 節によれば，$\mathbf{a}$ は $\mathbb{C}[\Gamma_0]$-module = Γ_0 の表現の直既約成分の multiplicity を表すデータだった（ただし $\Gamma_0 := (I, \Omega_0)$）．具体的には，以下のような Γ_0 の表現を考えていることになる．

$$\mathbf{V}(\mathbf{d}) = \begin{array}{c} (\ \mathbb{C} \ \leftarrow \{0\} \leftarrow \{0\})^{\oplus a_{1,2}} \\ \oplus \\ (\ \mathbb{C} \ \leftarrow \ \mathbb{C} \ \leftarrow \{0\})^{\oplus a_{1,3}} \\ \oplus \\ (\ \mathbb{C} \ \leftarrow \ \mathbb{C} \ \leftarrow \ \mathbb{C}\ )^{\oplus a_{1,4}} \\ \oplus \\ (\{0\} \leftarrow \ \mathbb{C} \ \leftarrow \{0\})^{\oplus a_{2,3}} \\ \oplus \\ (\{0\} \leftarrow \ \mathbb{C} \ \leftarrow \ \mathbb{C}\ )^{\oplus a_{2,4}} \\ \oplus \\ (\{0\} \leftarrow \{0\} \leftarrow \ \mathbb{C}\ )^{\oplus a_{3,4}} \end{array} \qquad (\text{ただし } \mathbf{d} := (d_1, d_2, d_3) = -\mathrm{wt}(\mathbf{a}))$$

さらに，表現 $\mathbf{V}(\mathbf{d})$ の同値類が $E_{\Omega_0}(\mathbf{d})$ の $G(\mathbf{d})$-orbit $\mathcal{O}_{\mathbf{a},\Omega_0}$ であり，対応する既約成分が $\Lambda_{\mathbf{a}} := \Psi_{\Omega_0}^{-1}(\mathbf{a}) = \overline{T^*_{\mathcal{O}_{\mathbf{a},\Omega_0}} E_{\Omega_0}(\mathbf{d})}$ なのであった．

話を元に戻そう．定義通りに計算すると

$$\begin{aligned}
&A_1^{(1)} = a_{1,2}, \\
&A_1^{(2)} = a_{1,3}, \quad A_2^{(2)} = a_{1,3} + (a_{2,3} - a_{1,2}), \\
&A_1^{(3)} = a_{1,4}, \quad A_2^{(3)} = a_{1,4} + (a_{2,4} - a_{1,3}), \quad A_3^{(3)} = a_{1,4} + (a_{2,4} - a_{1,3}) + (a_{3,4} - a_{2,3}), \\
&A_3^{*(1)} = a_{1,4}, \quad A_2^{*(1)} = a_{1,4} + (a_{1,3} - a_{2,4}), \quad A_1^{*(1)} = a_{1,4} + (a_{1,3} - a_{2,4}) + (a_{1,2} - a_{2,3}), \\
&A_3^{*(2)} = a_{2,4}, \quad A_2^{*(2)} = a_{2,4} + (a_{2,3} - a_{3,4}), \\
&A_3^{*(3)} = a_{3,4}.
\end{aligned}$$

したがって

$$\begin{aligned}
\varepsilon_1(\mathbf{a}) &= a_{1,2}, \\
\varepsilon_2(\mathbf{a}) &= \max\{a_{1,3},\ a_{1,3} + (a_{2,3} - a_{1,2})\}, \\
\varepsilon_3(\mathbf{a}) &= \max\{a_{1,4},\ a_{1,4} + (a_{2,4} - a_{1,3}),\ a_{1,4} + (a_{2,4} - a_{1,3}) + (a_{3,4} - a_{2,3})\}, \\
\varepsilon_1^*(\mathbf{a}) &= \max\{a_{1,4},\ a_{1,4} + (a_{1,3} - a_{2,4}),\ a_{1,4} + (a_{1,3} - a_{2,4}) + (a_{1,2} - a_{2,3})\}, \\
\varepsilon_2^*(\mathbf{a}) &= \max\{a_{2,4},\ a_{2,4} + (a_{2,3} - a_{3,4})\}, \\
\varepsilon_3^*(\mathbf{a}) &= a_{3,4}.
\end{aligned}$$

となる．この計算結果は，

$\Lambda_{\mathbf{a}}$ の中から generic に $P(\Gamma_0)$-module を取ったとき，
その i-th top，i-th socle の次元が上の式で与えられる

ということを意味している．

Kashiwara operators（$\widetilde{e}_i,\widetilde{f}_i,\widetilde{e}_i^*,\widetilde{f}_i^*$ たちを総称してこう呼ぶ）の作用については，例を使って説明したい．全部書くと大変なので，$\widetilde{e}_3$ と $\widetilde{f}_3$ の場合だけ詳しく書く．

Example 23. **a** として次のものを考える．

$$\mathbf{a} = \begin{array}{ccc} 4 & 1 & 1 \\ & 2 & 3 \\ & & 2 \end{array}$$

○ $\widetilde{e}_3$ の作用

今考えたいのは $i=3$ の場合なので，

$$A_1^{(3)} = 1, \quad A_2^{(3)} = 1 + (3-1) = 2, \quad A_3^{(3)} = 1 + (3-1) + (2-2) = 2.$$

したがって，

$$\varepsilon_3(\mathbf{a}) = \max\{1, 2, 2\} = 2.$$

したがって $\varepsilon_3(\mathbf{a}) = A_k^{(3)}$ となる k は 2 または 3 である．k_+ の定義は「このような k のうち，一番小さいもの」だったので，$k_+ = 2$．ということは『2 行目で 1 が右から左に移動する』ことになる．

$$\begin{array}{ccc} 4 & 1 & 1 \\ & 2 & 3 \\ & & 2 \end{array} \quad \overset{\widetilde{e}_3}{\longmapsto} \quad \begin{array}{ccc} 4 & 1 & 1 \\ & 3 & 2 \\ & & 2 \end{array}$$

○ $\widetilde{f}_3$ の作用

前述の通り $\varepsilon_3(\mathbf{a}) = A_k^{(3)}$ となる k は 2 または 3 であるが，k_- の定義は「このような k のうち，一番大きいもの」だったので，$k_- = 3$．よって今度は『3 行目で 1 が左から右に移動する』ことになる．しかし $a_{3,3}$ という成分はないので，$a_{3,4}$ に 1 が足されるのみとなる．

$$\begin{array}{ccc} 4 & 1 & 1 \\ & 2 & 3 \\ & & 2 \end{array} \quad \overset{\widetilde{f}_3}{\longmapsto} \quad \begin{array}{ccc} 4 & 1 & 1 \\ & 2 & 3 \\ & & 3 \end{array}$$

$\widetilde{e}_i^*,\widetilde{f}_i^*$ については省略するが，一番面倒な $\widetilde{e}_1^*$ と $\widetilde{f}_1^*$ の場合の答だけ書いておくので，興味のある方は自分でチェックされたい．

$$\begin{array}{ccc} 4 & 1 & 1 \\ & 2 & 3 \\ & & 2 \end{array} \quad \overset{\widetilde{e}_1^*}{\longmapsto} \quad \begin{array}{ccc} 4 & 1 & 0 \\ & 2 & 4 \\ & & 3 \end{array} \qquad\qquad \begin{array}{ccc} 4 & 1 & 1 \\ & 2 & 3 \\ & & 2 \end{array} \quad \overset{\widetilde{f}_1^*}{\longmapsto} \quad \begin{array}{ccc} 5 & 1 & 1 \\ & 2 & 3 \\ & & 3 \end{array}$$

$n=3$ の場合を見れば，一般の場合も大体想像がつくことと思う．

5.4. A 型の場合（その3）．

これまでの話は orientation として，

$$\Omega_0 : \underset{1}{\circ} \longleftarrow \underset{2}{\circ} \longleftarrow \underset{3}{\circ} \longleftarrow \cdots \longleftarrow \underset{n-2}{\circ} \longleftarrow \underset{n-1}{\circ} \longleftarrow \underset{n}{\circ}$$

という特別なものを選んだ場合の話であった．当然のことながら,『一般の Ω のときはどうなるのか？』という問題が頭に浮かぶことと思う．すなわち,

Q 1：与えられた Ω に対し，$\left(\mathcal{B}_\Omega; \mathrm{wt}, \varepsilon_i, \varphi_i, \widetilde{e}_i, \widetilde{f}_i\right)$ および $\left(\mathcal{B}_\Omega; \mathrm{wt}, \varepsilon_i^*, \varphi_i^*, \widetilde{e}_i^*, \widetilde{f}_i^*\right)$ の構造を具体的に記述せよ．

集合としては $\mathcal{B}_\Omega \cong \mathbb{Z}_{\geq 0}^N$ であるので，上の問いは

Q 1'：N 個の非負整数の組 $\mathbf{a}_\Omega = (a_{\beta,\Omega})_{\beta \in \Delta^+} \in \mathcal{B}_\Omega$ に対し，各写像を具体的に記述せよ

ということに他ならない．

この問いに対する直接の解答は知られていないように思うが，“ある種の解答” は得られている．

言葉を準備しよう．与えられた 2 つの orientation Ω, Ω' に対し，crystal の同型

$$R_\Omega^{\Omega'} := \Psi_{\Omega'} \circ \Psi_\Omega^{-1} : \mathcal{B}_\Omega \xrightarrow{\sim} \mathbb{B} \xrightarrow{\sim} \mathcal{B}_{\Omega'}$$

を $\mathcal{B}_\Omega$ から $\mathcal{B}_{\Omega'}$ への transition map と呼ぶ．“crystal の同型” というと何やら仰々しいが，これまでの話を組み合わせればいいだけなので，やっていることは決して難しいことではない．この写像の多元環の表現論的意味は,

(i) 非負整数の組 $\mathbf{a}_\Omega$ を考え，それを直既約成分の multiplicity として持つ $\mathbb{C}[\Gamma]$-module の同型類を考える；
(ii) 射影 $\pi_{\Lambda(\mathbf{d}),\Omega}$ の逆像をとり，上の同型類を $P(\Gamma)$-module に持ち上げる；
(iii) さらにその closure をとる；
(iv) 上記の closure の中から generic に点をとり，$\tau \in \overline{\Omega'}$ の作用を無視することで，これを $\mathbb{C}[\Gamma']$-module と思う．ここに $\Gamma' := (I, \Omega')$；
(v) 得られた $\mathbb{C}[\Gamma']$-module を直既約分解して multiplicity のデータを取り出す；

という操作をしているだけである．

Remark 24. とはいえ，(iii) の「closure をとる」と (iv) の「generic に点をとる」という操作は，多元環の表現論ではあまり用いない方法なのでわかりにくいかもしれない．話をややこしくしている 1 つの原因は次の点にある．与えられた $\mathbf{a}_\Omega$ に対し，$\mathbf{b}_{\Omega'} := R_\Omega^{\Omega'}(\mathbf{a}_\Omega)$ としよう．このとき,

$$\overline{\pi^{-1}_{\Lambda(\mathbf{d}),\Omega}(\mathcal{O}_{\mathbf{a},\Omega})} = \overline{\pi^{-1}_{\Lambda(\mathbf{d}),\Omega'}(\mathcal{O}_{\mathbf{b},\Omega'})} \quad \left(\Leftrightarrow \ \overline{T^*_{\mathcal{O}_{\mathbf{a},\Omega}} E_\Omega(\mathbf{d})} = \overline{T^*_{\mathcal{O}_{\mathbf{b},\Omega'}} E_{\Omega'}(\mathbf{d})}\right)$$

であったとしても，一般には

$$\pi^{-1}_{\Lambda(\mathbf{d}),\Omega}(\mathcal{O}_{\mathbf{a},\Omega}) \neq \pi^{-1}_{\Lambda(\mathbf{d}),\Omega'}(\mathcal{O}_{\mathbf{b},\Omega'})$$

である．したがって「closure をとって，さらにその中から generic に点を選ぶ」という操作は，どうしても行わざるを得ない．

特に重要なのは，$R_{\Omega_0}^\Omega$ と $R_\Omega^{\Omega_0} = \left(R_{\Omega_0}^\Omega\right)^{-1}$ である．特殊な orientation Ω_0 に関しては，crystal structure $\left(\mathcal{B}_{\Omega_0}; \mathrm{wt}, \varepsilon_i, \varphi_i, \widetilde{e}_i, \widetilde{f}_i\right)$ が完全にわかっているので，もし $R_{\Omega_0}^\Omega$ と $R_\Omega^{\Omega_0}$ が具

体的に書ければ，$(\mathcal{B}_\Omega; \mathrm{wt}, \varepsilon_i, \varphi_i, \widetilde{e}_i, \widetilde{f}_i)$ を完全に記述することができる[27]．$*$付きの crystal structure についても同様である．したがって，話は

Q 2：与えられた Ω に対し，$R^{\Omega}_{\Omega_0}$ と $R^{\Omega_0}_{\Omega}$ を具体的に記述せよ

という問題に還元される．

A 型の場合，この問題は完全に解けている ([3],[17]) が，答は非常にややこしいので具体的な公式を述べることは止めておく．詳しくは原論文を参照されたい．

5.5. **Open problems.**
本節では，今回の話に関連する未解決問題をいくつか紹介しよう．

Problem 1：D 型，E 型での $\mathbb{Z}^N_{\geq 0}$ の crystal structure の決定
すでに述べたように，$\mathbb{B} = \sqcup_{\mathbf{d}} \mathrm{Irr}\Lambda(\mathbf{d})$ から $\mathbb{Z}^N_{\geq 0}$（ただし $N = |\Delta^+|$）への全単射

$$\Psi : \mathbb{B} \xrightarrow{\sim} \mathbb{Z}^N_{\geq 0}$$

は『$\Gamma = (I, \Omega)$ が Dynkin quiver』との仮定のもとに存在する．すなわち，A 型でなくとも D 型，E 型でも $\mathbb{B}$ の持つ crystal structure を Ψ_Ω を通じて $\mathbb{Z}^N_{\geq 0}$ に移植することが出来るはずである．種々のデータは完全に与えられているので「あとは計算すればいいという状態」と言えなくもないが，本当に実行するのはなかなか大変である．実際，A 型以外の場合では，$\mathbb{Z}^N_{\geq 0}$ 上の crystal structure を具体的に書き下した結果は知られていないと思う．

Problem 2：Tame case：
同型 $\mathbb{B} \cong B(\infty)$ は，任意の loop がない quiver Γ に対して成り立つ．Dynkin case の次に興味があるのは，tame の場合，すなわち Γ が extended Dynkin の場合である．Lie theory side では，これは "affine case" と呼ばれる．
話をこの場合に限定すると，$B(\infty)$ をパラメトライズする方法が数多く知られているが，各種パラメトリゼーションと，$\Lambda(\mathbf{d})$ の既約成分との対応を明示的に与える公式は殆ど知られていない．

Problem 3：Rigid crystals
これまで

『variety of nilpotent representations $\Lambda(\mathbf{d})$ の $G(\mathbf{d})$-軌道
（= dimension vector $\mathbf{d}$ の nilpotent $P(\Gamma)$-module の同型類）
を考える替わりに，
$\Lambda(\mathbf{d})$ の代数多様体としての既約成分全体 $\mathrm{Irr}\Lambda(\mathbf{d})$ を考える』

[27]実際，例えば $\widetilde{e}_i$ の作用に関しては，

$$\widetilde{e}_i \mathbf{a}_\Omega = \left(R^{\Omega}_{\Omega_0} \circ \widetilde{e}_i \circ R^{\Omega_0}_{\Omega}\right)(\mathbf{a}_\Omega) \qquad (\mathbf{a}_\Omega \in \mathcal{B}_\Omega)$$

が成り立つ．したがって左辺を知りたければ右辺がわかればいいわけだが，左辺に現れる $\widetilde{e}_i$ は「$\mathcal{B}_{\Omega_0}$ の $\widetilde{e}_i$」なので，5.2 節で explicit formula を知っている．ゆえに $R^{\Omega}_{\Omega_0}$ と $R^{\Omega_0}_{\Omega}$ の具体形がわかれば，全てを具体的に計算出来る．

という立場で議論を進めてきた．しかしながら，やはり本来の問題意識である "$\Lambda(\mathbf{d})$ の $G(\mathbf{d})$-軌道そのもの" に興味があるのは，言うまでもない．このことに注意して，次の概念を導入しよう．

Definition 25. $\Lambda \in \mathrm{Irr}\Lambda(\mathbf{d})$ が稠密な $G(\mathbf{d})$-軌道 $\mathcal{O}$ を持つとき，Λ は rigid であるという．

$\Lambda \in \mathrm{Irr}\Lambda(\mathbf{d})$ が rigid であれば，Λ は $P(\Gamma)$-module の同型類と対応していると言って良いだろう．多元環論的な特徴付けとして，次が知られている．

Proposition 26 ([5])**.** *$B \in \Lambda(\mathbf{d})$ とし，対応する $P(\Gamma)$-module を V_B と書く．次は同値.*

(a) $\mathrm{Ext}^1_{P(\Gamma)}(V_B, V_B) = 0.$

(b) *B を通る $\Lambda(\mathbf{d})$ の $G(\mathbf{d})$-軌道を $\mathcal{O}$ とするとき，$\overline{\mathcal{O}} \in \mathrm{Irr}\Lambda(\mathbf{d})$．すなわち，$\Lambda := \overline{\mathcal{O}}$ は rigid.*

多元環の専門家の方々には，(a) の条件の方が "rigid" という感じが伝わるかも知れない．また，次も知られている．

- A_n $(1 \leq n \leq 4)$ の場合には，任意の $\Lambda \in \mathbb{B} = \sqcup_{\mathbf{d}} \mathrm{Irr}\Lambda(\mathbf{d})$ は rigid である．
- それ以外の場合には，rigid でない既約成分が必ず存在する．

そこで次の問題を考えよう．

Q3：rigid な既約成分を全てリストアップせよ．

これは問題としては非常に面白いと思う．ただし，話を A_n 型に限定したとしても，現状では難しすぎる．$\mathbf{d}$ が小さいと既約成分は必ず rigid になる．$n \leq 4$ だと全ての既約成分は rigid であり，この状況が最後まで続く．他方，$n \geq 5$ だと途中に rigid でないものが現れ始め，$\mathbf{d}$ が大きくなるにつれて，ほとんどの既約成分は rigid でなくなってしまう．

以下に知られている rigid でない例の中で,（筆者が知る限り）最も次元の小さいもの（A_5 型の場合）を紹介する．この問題は，多元環 side，Lie theory side 双方で，それぞれの重要な問題に関係しているが，現状ではどうしたらいいか，全くわからない．具体例の計算から情報を集めるだけでも意味があると思うので，興味のある方は実際に手を動かしてみるといいだろう[28].

Example 27. A_5 型の場合で，

$$\Omega_0 : \quad \underset{1}{\circ} \xleftarrow{\tau_1} \underset{2}{\circ} \xleftarrow{\tau_2} \underset{3}{\circ} \xleftarrow{\tau_3} \underset{4}{\circ} \xleftarrow{\tau_4} \underset{5}{\circ}$$

なる orientation をとる．また，$\mathbf{a} \in \mathcal{B}_{\Omega_0}$ として，次のものを考える：

$$\mathbf{a} = \begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 1 & 0 \\ & & 1 & 0 & 0 \\ & & & 0 & 1 \\ & & & & 0 \end{array} .$$

このとき，対応する既約成分 $\Lambda_{\mathbf{a}}$ は rigid ではない．以下，このことを詳しく見てみよう．

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[28]しかも『１つ１つの計算は，単なる行列の計算である』というところがミソ．

$\mathbf{a}$ に対応する $\Gamma_0 = (I, \Omega_0)$ の表現は,

$$\mathbf{V_a}(\mathbf{d}_0) = \begin{array}{c} (\ \mathbb{C} \leftarrow \mathbb{C} \leftarrow \{0\} \leftarrow \{0\} \leftarrow \{0\}) \\ \oplus \\ (\{0\} \leftarrow \mathbb{C} \leftarrow \mathbb{C} \leftarrow \mathbb{C} \leftarrow \{0\}) \\ \oplus \\ (\{0\} \leftarrow \{0\} \leftarrow \mathbb{C} \leftarrow \{0\} \leftarrow \{0\}) \\ \oplus \\ (\{0\} \leftarrow \{0\} \leftarrow \{0\} \leftarrow \mathbb{C} \leftarrow \mathbb{C}\ ) \end{array} \qquad (\text{ただし } \mathbf{d}_0 := (1,2,2,2,1) = -\mathrm{wt}(\mathbf{a})).$$

行列を使って書けば, (基底は適宜選ぶこととして) $B_{\mathbf{a}} = (B_{\tau_1}, B_{\tau_2}, B_{\tau_3}, B_{\tau_4})$ として,

$$B_{\tau_1} = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad B_{\tau_2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B_{\tau_3} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_{\tau_4} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5.5.1}$$

となる. $B_{\mathbf{a}}$ を通る $E_{\Omega_0}(\mathbf{d}_0)$ の $G(\mathbf{d}_0)$-orbit を $\mathcal{O}_{\mathbf{a}.\Omega_0}$ と書く. このとき,

$$\Lambda_{\mathbf{a}} = \overline{\pi^{-1}_{\Lambda(\mathbf{d}_0),\Omega_0}(\mathcal{O}_{\mathbf{a},\Omega_0})} = \overline{G(\mathbf{d}_0) \cdot \pi^{-1}_{\Lambda(\mathbf{d}_0),\Omega_0}(B_{\mathbf{a}})}$$

に注意すると,

$$\begin{aligned} \Lambda_{\mathbf{a}}\text{が rigid.} \quad &\Leftrightarrow \quad G(\mathbf{d}_0) \cdot \pi^{-1}_{\Lambda(\mathbf{d}_0),\Omega_0}(B_{\mathbf{a}}) \text{ が dense } G(\mathbf{d}_0)\text{-orbit を持つ.} \\ &\Leftrightarrow \quad \pi^{-1}_{\Lambda(\mathbf{d}_0),\Omega_0}(B_{\mathbf{a}}) \text{ が dense } G(\mathbf{d}_0)_{B_{\mathbf{a}}}\text{-orbit を持つ.} \end{aligned}$$

ただし

$$G(\mathbf{d}_0)_{B_{\mathbf{a}}} := \{g \in G(\mathbf{d}_0) \mid g \cdot B_{\mathbf{a}} = B_{\mathbf{a}}\} \qquad (B_{\mathbf{a}} \text{ の stabilizer})$$

である. $\pi^{-1}_{\Lambda(\mathbf{d}_0),\Omega_0}(B_{\mathbf{a}})$ と $G(\mathbf{d}_0)_{B_{\mathbf{a}}}$ は簡単に求めることが出来る. 実際,

$$\pi^{-1}_{\Lambda(\mathbf{d}_0),\Omega_0}(B_{\mathbf{a}}) = \left\{ (B_{\overline{\tau}_1}, B_{\overline{\tau}_2}, B_{\overline{\tau}_3}, B_{\overline{\tau}_4}) \;\middle|\; \begin{array}{c} B_{\tau_1}B_{\overline{\tau}_1} = 0,\ B_{\overline{\tau}_1}B_{\tau_1} = B_{\tau_2}B_{\overline{\tau}_2}, \\ B_{\overline{\tau}_2}B_{\tau_2} = B_{\tau_3}B_{\overline{\tau}_3},\ B_{\overline{\tau}_3}B_{\tau_3} = B_{\tau_4}B_{\overline{\tau}_4}, \\ B_{\overline{\tau}_4}B_{\tau_4} = 0 \end{array} \right\}.$$

ここに, $B_{\overline{\tau}_i}$ $(i = 1,2,3,4)$ は

$$\mathbb{C} \xleftarrow{B_{\overline{\tau}_1}} \mathbb{C}^2 \xleftarrow{B_{\overline{\tau}_2}} \mathbb{C}^2 \xleftarrow{B_{\overline{\tau}_3}} \mathbb{C}^2 \xleftarrow{B_{\overline{\tau}_4}} \mathbb{C}$$

なる行列で, 右辺は preprojective relations を書き下したものに他ならない. ここに (5.5.1) を代入し, 連立方程式を解けば,

$$\pi^{-1}_{\Lambda(\mathbf{d}_0),\Omega_0}(B_{\mathbf{a}}) = \left\{ \left(\begin{pmatrix} 0 \\ s \end{pmatrix}, \begin{pmatrix} s & 0 \\ t & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ u & v \end{pmatrix}, \begin{pmatrix} u & 0 \end{pmatrix} \right) \;\middle|\; s,t,u,v \in \mathbb{C} \right\} \cong \mathbb{C}^4$$

となる. また, stabilizer も簡単な計算から,

$$\begin{aligned} G(\mathbf{d}_0)_{B_{\mathbf{a}}} &= \left\{ g = (g_1, g_2, g_3, g_4, g_5) \in GL_1(\mathbb{C}) \times (GL_2(\mathbb{C}))^3 \times GL_1(\mathbb{C}) \;\middle|\; g \cdot B_{\mathbf{a}} = B_{\mathbf{a}} \right\} \\ &= \left\{ g = \left(a, \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}, \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}, \begin{pmatrix} c & 0 \\ e & f \end{pmatrix}, f \right) \;\middle|\; a,c,d,f \in \mathbb{C}^{\times},\ b,e \in \mathbb{C} \right\} \end{aligned}$$

となることがわかる．作用の具体形を書き下してみると，

$$\left(\begin{pmatrix}0\\ s\end{pmatrix}, \begin{pmatrix}s & 0\\ t & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ u & v\end{pmatrix}, \begin{pmatrix}u & 0\end{pmatrix}\right)$$
$$\xmapsto{g} \left(\begin{pmatrix}0\\ a^{-1}cs\end{pmatrix}, \begin{pmatrix}a^{-1}cs & 0\\ a^{-1}dt & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ c^{-1}fu & d^{-1}fv\end{pmatrix}, \begin{pmatrix}c^{-1}fu & 0\end{pmatrix}\right)$$

となり，b と e の部分は自明に作用していることがわかる．すなわち，実際に "効く" のは，a, c, d, f の部分のみで，これは $\left(\mathbb{C}^{\times}\right)^4$ に同型．したがって考えるべき状況は，

$$\left(\mathbb{C}^{\times}\right)^4 \curvearrowright \mathbb{C}^4.$$

簡単のため s, t, u, v は generic （どれも 0 ではない）として，比 $\dfrac{su}{tv}$ を考えよう．このとき，

$$\frac{su}{tv} \xmapsto{g} \frac{a^{-1}cs \cdot c^{-1}fu}{a^{-1}dt \cdot d^{-1}fv} = \frac{su}{tv}.$$

すなわち，群 $G(\mathbf{d}_0)_{B_{\mathbf{a}}}$ の $\pi^{-1}_{\Lambda(\mathbf{d}_0),\Omega_0}(B_{\mathbf{a}}) \cong \mathbb{C}^4$ への作用は，$\mathbb{C}^4$ の 3 次元部分多様体

$$\frac{su}{tv} = const$$

を不変に保ってしまう．したがって，orbit の次元は 3 を超えることが出来ず，dense orbit は存在しない[29]．

それ以外だと，例えば，

$$\begin{matrix}0 & 2 & 0 & 0 & 0\\ & 0 & 0 & 2 & 0\\ & & 2 & 0 & 0\\ & & & 0 & 2\\ & & & & 0\end{matrix}\ , \qquad \begin{matrix}0 & 1 & 0 & 1 & 0\\ & 0 & 1 & 0 & 1\\ & & 0 & 1 & 0\\ & & & 0 & 1\\ & & & & 0\end{matrix}$$

などに対応する既約成分も rigid ではない．左の例は，Example 27 の $\mathbf{a}$ の各成分を全て 2 倍したものになっている．その意味でこれを $2\mathbf{a}$ と書くことにしよう．これを一般化して，成分を k 倍したものを $k\mathbf{a}$ と書くことにする．この記法の下に，一般に，

$$\Lambda_{\mathbf{a}} \text{ が non-rigid} \quad \Rightarrow \quad \Lambda_{k\mathbf{a}}\ (k \in \mathbb{Z}_{>0}) \text{ も non-rigid}$$

ということは比較的容易にわかる．しかし，この系列（k 倍していく系列）の一番最初にあるもの（仮に primitive な rigid component とでも呼ぼう）を探し出すのはなかなか難しく，一般的な解答は与えられていない．

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MATRIX FACTORIZATIONS, ORBIFOLD CURVES AND MIRROR SYMMETRY

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Abstract. Mirror symmetry is now understood as a categorical duality between algebraic geometry and symplectic geometry. One of our motivations is to apply some ideas of mirror symmetry to singularity theory in order to understand various mysterious correspondences among isolated singularities, root systems, Weyl groups, Lie algebras, discrete groups, finite dimensional algebras and so on.

In my talk, I explained the homological mirror symmetry conjecture between orbifold curves and cusp singularities via Orlov type semi-orthogonal decompositions. I also gave a summary of our results on categories of maximally-graded matrixfactorizations, in particular, on the existence of full strongly exceptional collections which gives triangulated equivalences to derived categories of finite dimensional modules over finite dimensional algebras.

1. まえがき

ミラー対称性は複素代数幾何学とシンプレクティック幾何学の双対性と考えることができる．ミラー対称性のアイデアを特異点理論に応用することで，特異点・ルート系・ワイル群・リー環・有限次元代数...といった異なる数学的背景を持つ分野を結び付け，新たな知見を得ることができる．ここでは，14 個の例外型特異点に対する Arnold の「奇妙な双対性」を可逆多項式という 3 変数の重み付き斉次多項式のクラスに拡張し，その代数的・幾何学的背景を明確にする．例えば，Arnold の「奇妙な双対性」を述べるためには Gablielov 数という概念が必要であるが，Gablielov 数は 14 個の例外型特異点に対してのみ「実験的に」与えられたものであり，一般の特異点に対する定義は存在していなかった．このことは，重み付き斉次多項式に対して系統的にカスプ特異点を対応させることで解決されることになる．そして，重み付き射影直線とカスプ特異点のホモロジー的ミラー対称性現象が Arnold の「奇妙な双対性」の真の姿であることがわかるのである．

より正確に述べることにしよう．$f(x,y,z)$ を原点 $0\in\mathbb{C}^3$ にのみ孤立特異点を持つ多項式とする．f の Milnor ファイバーにおける消滅 Lagrangian 部分多様体の distinguish basis は，有向深谷圏と呼ばれる A_∞-圏 $\mathrm{Fuk}^{\to}(f)$ に圏化される．とくに，その導来圏 $D^b\mathrm{Fuk}^{\to}(f)$ は，三角圏としては，さまざまな幾何的変形・選択によらないことが知られている．このようにして，f に対してシンプレクティック幾何学に対する不変量が得られる．とくに，定義により $D^b\mathrm{Fuk}^{\to}(f)$ は full exceptional collection を持つことがわかる．

一方で，$f(x,y,z)$ が重み付き斉次多項式ならば，極大次数付き行列因子化の圏と呼ばれる三角圏 $\mathrm{HMF}^{L_f}_S(f)$ を構成することができる．ここで，$S:=\mathbb{C}[x,y,z]$，L_f は（後で定義を述べる）f の極大次数である．なお，定義からでは明らかではないが，この圏 $\mathrm{HMF}^{L_f}_S(f)$ は，滑らかで固有な代数多様体の有界導来圏の重み付き斉次特異点に対する類似である．

Calabi–Yau 多様体の位相的ミラー対は，特異点（物理学における Landau–Ginzburg 軌道体理論）のミラー対称性により，系統的に構成された．そこでは，良い性質を持つ重み

付き斉次多項式に対する Berglund–Hübsch 転置が重要であった．そこで，Calabi–Yau 多様体のホモロジー的ミラー対称性のアイデアを特異点に対して適用し，これらを合わせて考察することで，次の予想が自然に期待されることとなる：

Conjecture 1 ([12][13])**.** $f(x,y,z)$ を可逆多項式とする．

(1) 箙と関係式 (Q,I) で，三角同値

$$\mathrm{HMF}^{L_f}_S(f) \simeq D^b(\text{mod-}\mathbb{C}Q/I) \simeq D^b\mathrm{Fuk}^{\to}(f^t) \tag{1.1}$$

をもたらすものが存在する．

(2) 箙と関係式 (Q',I') で，三角同値

$$D^b\mathrm{coh}(\mathcal{C}_{G_f}) \simeq D^b(\text{mod-}\mathbb{C}Q'/I') \simeq D^b\mathrm{Fuk}^{\to}(T_{\gamma_1,\gamma_2,\gamma_3}) \tag{1.2}$$

をもたらすものが存在する．とくに，これは三角同値 (1.1) と整合的である．ここで，$\mathcal{C}_{G_f}$ は f の極大可換対称性 G_f に付随した重み付き射影直線，$T_{\gamma_1,\gamma_2,\gamma_3}$ は「カスプ特異点」である．

これらの予想に対する多くの証拠が多くの研究者によりすでに発見されている．その中でも最も重要なものは，(Q',I') として後で述べる図形 $T(\gamma_1,\gamma_2,\gamma_3)$ に適切な向き付けと 2 重破線に対する関係式を与えたものがとれるということである．これにより，主定理（Theorem 36）が証明されることになるのである．

2. 可逆多項式

$f(x_1,\dots,x_n)$ を重み付き斉次多項式とする．つまり，正の整数 $w_1,\dots,w_n$ および d で，$\lambda\in\mathbb{C}^*$ に対して $f(\lambda^{w_1}x_1,\dots,\lambda^{w_n}x_n)=\lambda^d f(x_1,\dots,x_n)$ が成り立つものとする．このとき，$(w_1,\dots,w_n;d)$ を**ウェイト系**という．$\gcd(w_1,\dots,w_n,d)=1$ ならば, ウエイト系は**既約**であるという．ここでは，既約でないウエイト系も取り扱う．

Definition 2. 次の条件をみたす重み付き斉次多項式 $f(x_1,\dots,x_n)$ を**可逆多項式**でという：

(1) 変数の数 $(=n)$ が $f(x_1,\dots x_n)$ に現れる単項式の数に一致する，つまり，$a_i\in\mathbb{C}^*$ および非負整数 E_{ij}（$i,j=1,\dots,n$）に対して，

$$f(x_1,\dots,x_n)=\sum_{i=1}^n a_i\prod_{j=1}^n x_j^{E_{ij}}$$

となる．

(2) ウェイト系 $(w_1,\dots,w_n;d)$ は，$f(x_1,\dots,x_n)$ によって（$\gcd(w_1,\dots,w_n;d)$ をのぞき）ただひととおりに決定される．つまり, 行列 $E:=(E_{ij})$ は有理数体 $\mathbb{Q}$ 上可逆である．

(3) $f(x_1,\dots,x_n)$ および

$$f^t(x_1,\dots,x_n):=\sum_{i=1}^n a_i\prod_{j=1}^n x_j^{E_{ji}},$$

として定義される $f(x_1,\dots,x_n)$ の **Berglund–Hübsch 転置** $f^t(x_1,\dots,x_n)$ が原点 $0\in\mathbb{C}^n$ に孤立特異点を持つ．言い換えれば，f,f^t の **Jacobi 環** $Jac(f),Jac(f^t)$

$$Jac(f):=\mathbb{C}[x_1,\dots,x_n]\bigg/\left(\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n}\right)$$

$$Jac(f^t) := \mathbb{C}[x_1, \ldots, x_n] \bigg/ \left(\frac{\partial f^t}{\partial x_1}, \ldots, \frac{\partial f^t}{\partial x_n}\right)$$

は，ともに複素数体 $\mathbb{C}$ 上の有限次元代数となり，$\dim_{\mathbb{C}} Jac(f), \dim_{\mathbb{C}} Jac(f^t) \geq 1$ である．

これらの性質をもつ多項式は，ミラー対称性研究の初期から長期間にわたり熱心に研究されてきた．その代表例は，多項式とその多項式の対称性からなる組のミラー対を考えることによる，位相的ミラー対をなす Calabi–Yau 多様体の大量構成である．なお，**可逆多項式**という名前は Kreuzer [9] によって導入された．

Definition 3. $f(x_1, \ldots, x_n) = \sum_{i=1}^n a_i \prod_{j=1}^n x_j^{E_{ij}}$ を可逆多項式とする．方程式

$$E \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \det(E) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \quad d := \det(E).$$

の解として与えられるウェイト系 $(w_1, \ldots, w_n; d)$ を f の**標準ウェイト系**といい，W_f であらわす．

Remark 4. クラメルの公式から，標準ウェイト系にあらわれる数 $w_1, \ldots, w_n$ は正の整数であることがすぐにわかる．

Definition 5. $f(x_1, \ldots, x_n)$ を可逆多項式，$W_f = (w_1, \ldots, w_n; d)$ をその標準ウェイト系とする．このとき

$$c_f := \gcd(w_1, \ldots, w_n, d)$$

と定義する．

Definition 6. $f(x_1, \ldots, x_n) = \sum_{i=1}^n a_i \prod_{j=1}^n x_j^{E_{ij}}$ を可逆多項式とする． 各変数 $x_i, i = 1, \ldots, n$ に対する文字 $\vec{x_i}$ および多項式 f に対する文字 $\vec{f}$ によって生成される自由アーベル群 $\oplus_{i=1}^n \mathbb{Z}\vec{x_i} \oplus \mathbb{Z}\vec{f}$ を考える．このとき，可逆多項式 f の**極大次数** L_f を，商

$$L_f := \bigoplus_{i=1}^n \mathbb{Z}\vec{x_i} \oplus \mathbb{Z}\vec{f}/I_f$$

によって定義する．ここで，I_f は元

$$\vec{f} - \sum_{j=1}^n E_{ij}\vec{x_j}, \quad i = 1, \ldots, n$$

により生成される部分群である．

Remark 7. L_f は階数 1 のアーベル群である．ただし，必ずしも自由アーベル群ではない．

Definition 8. $f(x_1, \ldots, x_n)$ を可逆多項式，L_f をその極大次数とする．f の**極大可換対称性** G_f を，

$$G_f := \mathrm{Spec}(\mathbb{C}L_f)$$

で定義されるアーベル群とする．ここで，$\mathbb{C}L_f$ で L_f の群環をあらわす．言い換えれば，

$$G_f = \left\{ (\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n \;\middle|\; \prod_{j=1}^n \lambda_j^{E_{1j}} = \cdots = \prod_{j=1}^n \lambda_j^{E_{nj}} \right\}$$

である．

可逆多項式 f は G_f の変数への自然な作用により斉次である．つまり，$(\lambda_1,\dots,\lambda_n)\in G_f$，$\lambda:=\prod_{j=1}^n \lambda_j^{E_{1j}}=\cdots=\prod_{j=1}^n \lambda_j^{E_{nj}}$ に対して，

$$f(\lambda_1 x_1,\dots,\lambda_n x_n)=\lambda f(x_1,\dots,x_n)$$

が成り立つ．

3. 行列因子化の圏とその性質

f を可逆多項式とする．多項式環 $\mathbb{C}[x_1,\dots,x_n]$ を S であらわし，環 R_f を $R_f:=S/(f)$ で定義する．有限生成 L_f-次数付き R_f-加群の圏を $\mathrm{gr}^{L_f}\text{-}R_f$ で，射影的加群のなす $\mathrm{gr}^{L_f}\text{-}R_f$ の部分圏を $\mathrm{proj}^{L_f}\text{-}R_f$ であらわす．

Definition 9. 三角圏

$$D_{Sg}^{L_f}(R_f):=D^b(\mathrm{gr}^{L_f}\text{-}R_f)/K^b(\mathrm{proj}^{L_f}\text{-}R_f) \tag{3.1}$$

を，**極大次数付き特異点の圏**という．

Remark 10. 環 R_f が正則であるならば，圏同値 $D^b(\mathrm{gr}^{L_f}\text{-}R_f)\simeq K^b(\mathrm{proj}^{L_f}\text{-}R_f)$ が得られるので，商 $D_{Sg}^{L_f}(R_f)$ は特異点 $\{f=0\}$ の複雑さを測っていると思える．f.

Remark 11. 特異点に台を持つ単純加群

$$\mathbb{C}(\vec{l}):=(R_f/\mathfrak{m})(\vec{l})\in D_{Sg}^{L_f}(R_f),\quad \vec{l}\in L_f,$$

は，あとで重要な役割を果たすこととなる．

三角圏 $D_{Sg}^{L_f}(R_f)$ の定義は簡単で分かりやすいものであるが，実際にこの圏における射の空間を計算するのは非常に難しい．また，この圏はミラー対称性の観点からも自然であるとは言い難いので，別の同値な三角圏に置き換えることを考える．

Definition 12. $M\in\mathrm{gr}^{L_f}\text{-}R_f$ が

$$\mathrm{Ext}^i_{R_f}(R_f/\mathfrak{m},M)=0,\quad i<\dim R_f,$$

をみたすとき，が極大次数付き **Cohen-Macauley**R_f-加群であるという．

R_f は極大次数付き **Gorenstein** 環である， つまり，$(\vec{l})$ を $\vec{l}\in L_f$ による shift とするとき，

$$K_{R_f}\simeq R_f(-\vec{\epsilon}_f),\quad \vec{\epsilon}_f:=\sum_{i=1}^n \vec{x_i}-\vec{f}, \tag{3.2}$$

が成立する．ゆえに，次の結果が得られる：

Lemma 13 (Auslander)**.** 極大次数付き *Cohen-Macauley* R_f-加群の圏 $\mathrm{CM}^{L_f}(R_f)\subset\mathrm{gr}^{L_f}\text{-}R_f$ は **Frobenius 圏**である．つまり，十分豊富な射影的対象および入射的対象を持ち，射影的対象と入射的対象が一致するような，完全圏の構造を持つ． □

Definition 14. 圏 $\underline{\mathrm{CM}}^{L_f}(R_f)$ を次のように定義する：

$$Ob(\underline{\mathrm{CM}}^{L_f}(R_f)) = Ob(\mathrm{CM}^{L_f}(R_f)),$$

$$\underline{\mathrm{CM}}^{L_f}(R_f)(M,N) := \mathrm{Hom}_{\mathrm{gr}^{L_f}\text{-}R_f}(M,N)/\mathcal{P}(M,N).$$

ここで，$g \in \mathcal{P}(M,N)$ ということを，射影的対象 P および射 $g' : M \to P$, $g'' : P \to N$ で $g = g'' \circ g'$ となるものが存在すること，として定める．

圏 $\underline{\mathrm{CM}}^{L_f}(R_f)$ を $\mathrm{CM}^{L_f}(R_f)$ の**安定圏**という．

このとき，一般論からつぎのことがわかる：

Proposition 15 (Happel[7]). 圏 $\underline{\mathrm{CM}}^{L_f}(R_f)$ は三角圏の構造を持つ． □

また，f が孤立特異点であることを用いると，次の有限性条件が得られる：

Proposition 16. $\underline{\mathrm{CM}}^{L_f}(R_f)$ は有限である，つまり

$$\sum_i \dim_k \underline{\mathrm{CM}}^{L_f}(R_{f_W})(M, T^i N) < \infty,$$

が成立する．さらに，任意の対象は直既約対象の有限直和と同型となる． □

三角圏 $\underline{\mathrm{CM}}^{L_f}(R_f)$ は特別に良い自己同値函手を持つ：

Proposition 17 (Auslander-Reiten[2]). $\underline{\mathrm{CM}}^{L_f}(R_f)$ 上の函手 $\mathcal{S} = T^{n-2} \circ (-\vec{\epsilon}_f)$ は **Serre 函手**を与える．つまり，双函手的な同型

$$\underline{\mathrm{CM}}^{L_f}(R_{f_W})(M,N) \simeq \mathrm{Hom}_k(\underline{\mathrm{CM}}^{L_f}(R_{f_W})(N, \mathcal{S}M), k)$$

が存在する． □

R_f は超曲面環であり，極大次数付き環として局所環であるので，任意の $M \in \mathrm{CM}^{L_f}(R_f)$ に対して，gr^{L_f}-S における自由分解

$$0 \to F_1 \xrightarrow{f_1} F_0 \to M \to 0$$

が取れることに注意する．一方で，f を M に掛け算するという操作は 0 を掛けることに他ならないので，ホモトピー $f_0 : F_0 \to F_1$ で

$$f_1 f_0 = f \cdot \mathrm{id}_{F_0}, \quad f_0 f_1 = f \cdot \mathrm{id}_{F_1}$$

となるものが存在する．このことに基づいて，Eisenbud は行列因子化（matrix factorization）の概念を導入した．

Definition 18 (Eisenbud[4]). F_0, F_1 を極大次数付き自由加群，$f_0 : F_0 \to F_1$, $f_1 : F_1 \to F_0$ を $f_1 f_0 = f \cdot \mathrm{id}_{F_0}, f_0 f_1 = f \cdot \mathrm{id}_{F_1}$ が成立するような S-準同型とする．このとき，組 (F_0, F_1, f_0, f_1) を f の極大次数付き**行列因子化**といい，

$$\overline{F} := \left(F_0 \underset{f_1}{\overset{f_0}{\rightleftarrows}} F_1 \right)$$

であらわす．

Example 19. 分解

$$f = x_1 f_1 + x_2 f_2 + \cdots + x_n f_n.$$

が成立するような $f_i \in \mathfrak{m}$, $i = 1, \ldots, n$ が取れる．対応する行列因子化は，後で述べる圏同値により，$D^{L_f}_{Sg}(R_f)$ の対象 $\mathbb{C}(\vec{l})$ に写される．

Lemma 20. *f の行列因子化の圏 $\mathrm{MF}^{L_f}_S(f)$ は Frobenius 圏の構造を持つ．とくに，その安定圏*

$$\mathrm{HMF}^{L_f}_S(f) := \underline{\mathrm{MF}}^{L_f}_S(f)$$

は三角圏の構造を持つ． □

Lemma 21. *圏 $\mathrm{HMF}^{L_f}_S(f)$ においては，$T^2 = (\vec{f})$ が成立する．ここで，T は三角圏の平行移動函手である．とくに，$\mathrm{HMF}^{L_f}_S(f)$ は次元 $(n-2) - 2\frac{\epsilon_f}{h_f}$ の分数的 Calabi–Yau 圏となる．ただし，$\epsilon_f := \deg(\vec{\epsilon_f})$ and $h_f := \deg(\vec{f})$ とする．* □

行列因子化 $\overline{F} = \left(F_0 \underset{f_1}{\overset{f_0}{\rightleftarrows}} F_1 \right)$ に対して $\mathrm{Coker}(f_1)$ を取ることにより．$\mathrm{CM}^{L_f}(R_f)$ の対象が得られる．これは先に述べたものの逆構成である．とくに，これは次の有名な三角同値をもたらす．

Theorem 22 (c.f., Buchweitz, Orlov[10])**.** *三角同値*

$$\mathrm{HMF}^{L_f}_S(f) \simeq \underline{\mathrm{CM}^{L_f}}(R_f) \simeq D^{L_f}_{Sg}(R_f)$$

が存在する． □

Orlov 型の半直交分解定理を述べるために，商スタック

$$\mathcal{X}_{L_f} := [\mathrm{Spec}(R_f)\backslash\{0\} / \mathrm{Spec}(\mathbb{C} \cdot L_f)]$$

を導入しておく．このとき，$D^b\mathrm{coh}(\mathcal{X}_{L_f}) \simeq D^b(\mathrm{gr}^{L_f}\text{-}R_f)/D^b(\mathrm{tor}^{L_f}\text{-}R_f)$ が成立している．

Proposition 23 (c.f., Orlov[10])**.** *次の三角同値が成立する：*

(1) $\epsilon_f > 0$ ならば，

$$D^b\mathrm{coh}(\mathcal{X}_{L_f}) \simeq \left\langle D^{L_f}_{Sg}(R_f), \mathcal{A}(0), \ldots, \mathcal{A}(\epsilon_f - 1) \right\rangle$$

である．ここで，$\mathcal{A}(i) := \left\langle \mathcal{O}_{\mathcal{X}_{L_f}}(\vec{l}) \right\rangle_{\deg(\vec{l})=i}$ とする．

(2) $\epsilon_f = 0$ ならば，$D^b\mathrm{coh}(\mathcal{X}_{L_f}) \simeq D^{L_f}_{Sg}(R_f)$ である．.

(3) $\epsilon_f < 0$ ならば，

$$D^{L_f}_{Sg}(R_f) \simeq \left\langle D^b\mathrm{coh}(\mathcal{X}_{L_f}), \mathcal{K}(0), \ldots, \mathcal{K}(-\epsilon_f + 1) \right\rangle$$

である．ここで，$\mathcal{K}(i) := \left\langle \mathbb{C}(\vec{l}) \right\rangle_{\deg(\vec{l})=i}$ とする．

□

この半直交分解に対するシンプレクティック幾何学側の対応物を考えたい．とくに，そのためには商スタック $\mathcal{X}_{L_f}$ およびそのミラー双対の理解が不可欠となる．

4. Dolgachev 数

これからは三変数の可逆多項式に制限して話を進める．そこでは，次の可逆多項式の分類結果が重要な役割を果たす．

Proposition 24 ([1]). *$f(x,y,z)$ を可逆多項式とする．このとき，各変数を適当にスケール変換することにより，f は Table 1 における 5 つのタイプのいずれかの形となる．* □

Type	Class	f	f^t
I	I	$x^{p_1}+y^{p_2}+z^{p_3}$ $(p_1,p_2,p_3\in\mathbb{Z}_{\geq 2})$	$x^{p_1}+y^{p_2}+z^{p_3}$ $(p_1,p_2,p_3\in\mathbb{Z}_{\geq 2})$
II	II	$x^{p_1}+y^{p_2}+yz^{\frac{p_3}{p_2}}$ $(p_1,p_2,\frac{p_3}{p_2}\in\mathbb{Z}_{\geq 2})$	$x^{p_1}+y^{p_2}z+z^{\frac{p_3}{p_2}}$ $(p_1,p_2,\frac{p_3}{p_2}\in\mathbb{Z}_{\geq 2})$
III	IV	$x^{p_1}+zy^{q_2+1}+yz^{q_3+1}$ $(p_1\in\mathbb{Z}_{\geq 2},\ q_2,q_3\in\mathbb{Z}_{\geq 1})$	$x^{p_1}+zy^{q_2+1}+yz^{q_3+1}$ $(p_1\in\mathbb{Z}_{\geq 2},\ q_2,q_3\in\mathbb{Z}_{\geq 1})$
IV	V	$x^{p_1}+xy^{\frac{p_2}{p_1}}+yz^{\frac{p_3}{p_2}}$ $(p_1,\frac{p_3}{p_2}\in\mathbb{Z}_{\geq 2},\frac{p_2}{p_1}\in\mathbb{Z}_{\geq 1})$	$x^{p_1}y+y^{\frac{p_2}{p_1}}z+z^{\frac{p_3}{p_2}}$ $(p_1,\frac{p_3}{p_2}\in\mathbb{Z}_{\geq 2},\frac{p_2}{p_1}\in\mathbb{Z}_{\geq 1})$
V	VII	$x^{q_1}y+y^{q_2}z+z^{q_3}x$ $(q_1,q_2,q_3\in\mathbb{Z}_{\geq 1})$	$zx^{q_1}+xy^{q_2}+yz^{q_3}$ $(q_1,q_2,q_3\in\mathbb{Z}_{\geq 1})$

Table 1. 3 変数の可逆多項式

これから，Table 1 における「Type」という分類表記を用いる．これは [11] における分類に基づいており，[1] においては「Class」という表記で分類されている．

可逆多項式 $f(x,y,z)$ に対して，商スタック

$$\mathcal{C}_{G_f} := \left[f^{-1}(0)\backslash\{0\} / G_f \right] \tag{4.1}$$

を考えることができる．これは先に述べた $\mathcal{X}_{L_f}$ と同じものである．f は原点 $0\in\mathbb{C}^3$ にのみ孤立特異点をもち，G_f は 1 次元複素トーラス $\mathbb{C}^*$ を位数 c_f の有限アーベル群で拡大したものなので，商スタック $\mathcal{C}_{G_f}$ は Deligne–Mumford スタックであり，とくに有限個の固定点を持つ滑らかな射影的曲線であることがわかる．さらに，次のことがわかる：

Theorem 25 ([5]). *$f(x,y,z)$ を可逆多項式とする．このとき，商スタック $\mathcal{C}_{G_f}$ は高々 3 点の固定点を持つ射影直線 $\mathbb{P}^1$ である．各固定点における固定化群の位数は表 2 における $\alpha_1,\alpha_2,\alpha_3$ で与えられる．ただし，固定点の数は $\alpha_i\geq 2$ となる i の数である．* □

Definition 26. Theorem 25 における数 $(\alpha_1,\alpha_2,\alpha_3)$ を組 (f,G_f) に対する **Dolgachev 数**といい，A_{G_f} であらわす．

なお，Theorem 25，Orlov 型半直交分解定理 Theorem 23 と Geigle–Lenzing[6] による重み付き射影直線の導来圏の構造定理により，次のことがわかる：

Corollary 27 ([12][13]). *極大次数付き行列因子化の圏 $\mathrm{HMF}^{L_f}_S(f)$ は full exceptional collection をもつ．* □

さらに強く，次のことが成立する：

Type	$f(x,y,z)$	$(\alpha_1,\alpha_2,\alpha_3)$
I	$x^{p_1}+y^{p_2}+z^{p_3}$	(p_1,p_2,p_3)
II	$x^{p_1}+y^{p_2}+yz^{\frac{p_3}{p_2}}$	$\left(p_1,\frac{p_3}{p_2},(p_2-1)p_1\right)$
III	$x^{p_1}+zy^{q_2+1}+yz^{q_3+1}$	(p_1,p_1q_2,p_1q_3)
IV	$x^{p_1}+xy^{\frac{p_2}{p_1}}+yz^{\frac{p_3}{p_2}}$	$\left(\frac{p_3}{p_2},(p_1-1)\frac{p_3}{p_2},p_2-p_1+1\right)$
V	$x^{q_1}y+y^{q_2}z+z^{q_3}x$	$(q_2q_3-q_3+1,q_3q_1-q_1+1,q_1q_2-q_2+1)$

Table 2. 組 (f,G_f) に対する Dolgachev 数

Theorem 28 ([8][13]). 極大次数付き行列因子化の圏 $\mathrm{HMF}_S^{L_f}(f)$ は full strongly exceptional collection をもつ．つまり，有限箙 Q および経路代数 $\mathbb{C}Q$ の許容的イデアル I で，三角同値 $D_{Sg}^{L_f}(R_f)\simeq D^b(\text{mod-}\mathbb{C}Q/I)$ が成立するものが存在する．とくに，有限次元代数 $\mathbb{C}Q/I$ の大局次元が 3 以下になるような full strongly exceptional collection を選ぶことができる． □

この定理は，他の同種の結果と同様に，次のように示される：

(1) 「良い」行列因子化を必要な数だけ見つける．
(2) これらの行列因子化が strongly exceptional collection をなすことを示す．
(3) 次の **Category Generating Lemma** を用いて，その strongly exceptional collection が full であることを示す．

Theorem 29 (Category Generating Lemma). $\mathrm{HMF}_S^{L_f}(f)$ の充満部分三角圏 $\mathcal{T}'$ が exceptional collection $(E_1,\dots,E_n)$ によって生成されていて，さらに以下の性質をみたすとする：

(1) $\mathcal{T}'$ は $(\vec{l})$, $\vec{l}\in L_f$ によって閉じている．
(2) 対象 $E\in\mathcal{T}'$ で， $D_{Sg}^{L_f}(R_f)$ において，$\mathbb{C}(\vec{0})$ と同型となるものが存在する．

このとき，$\mathcal{T}'\simeq\mathrm{HMF}_S^{L_f}(f)$ が成立する．

証明の概略を述べておこう．まず，$\mathcal{T}'$ が right admissible であることに注意する：

Lemma 30. $X\in\mathrm{HMF}_S^{L_f}(f)$ に対して，完全三角形

$$N\to X\to M\to TN$$

で，$N\in\mathcal{T}'$ および $\mathrm{Hom}(N,M)=0$ が成立するものが存在する． □

ここで,

$$\begin{aligned}
&\mathrm{HMF}_S^{L_f}(f)(E(\vec{l}),T^iM)=0 \quad {}^\forall\vec{l}\in L_f,\quad {}^\forall i\in\mathcal{Z}\\
\Longleftrightarrow\ &\mathrm{Ext}^i_{R_f}(R_f/\mathfrak{m},M)=0\ (i\neq d)\\
\Longleftrightarrow\ &M\in\mathrm{CM}^{L_f}(R_f)\text{ is Gorenstein}\\
\Longleftrightarrow\ &M\in\mathrm{CM}^{L_f}(R_f)\text{ is free}\\
\Longleftrightarrow\ &M\simeq 0\text{ in }\underline{\mathrm{CM}}^{L_f}(R_f)
\end{aligned}$$

となることから，$\mathcal{T}'\simeq\mathrm{HMF}_S^{L_f}(f)$ であることがわかる．

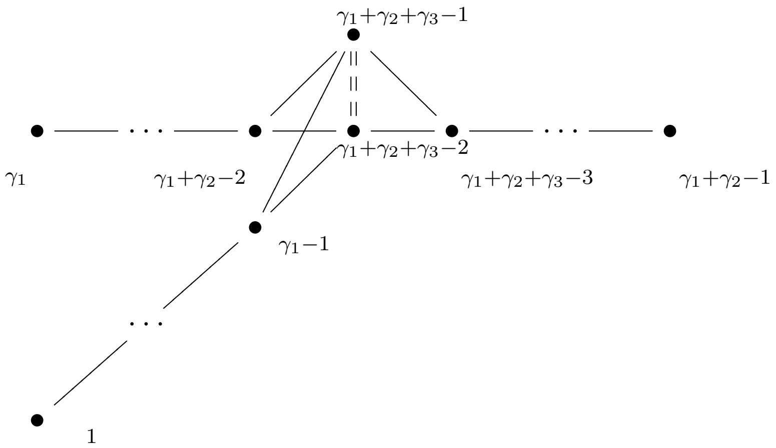

Figure 1. Coxeter–Dynkin 図形 $T(\gamma_1,\gamma_2,\gamma_3)$

5. Gabrielov 数

前節では可逆多項式から代数的不変量としての正の整数の組を構成した．この節では，幾何学的不変量を取り出すことが目標である．

Definition 31. 整数 $\gamma_1,\gamma_2,\gamma_3$ に対して，多項式

$$x^{\gamma_1}+y^{\gamma_2}+z^{\gamma_3}-txyz,\quad t\in\mathbb{C}\backslash\{0\},$$

を $T_{\gamma_1,\gamma_2,\gamma_3}$-型の多項式という．

正の整数の組 (a,b,c) に対して，

$$\Delta(a,b,c):=abc-bc-ac-ab$$

とおく．$\Delta(\gamma_1,\gamma_2,\gamma_3)>0$ ならば，$T_{\gamma_1,\gamma_2,\gamma_3}$-型の多項式はカスプ特異点を定める．ただし，ここでは $\Delta(\gamma_1,\gamma_2,\gamma_3)>0$ に制限せず，一般的な条件のもとで考える．

$T_{\gamma_1,\gamma_2,\gamma_3}$-型の多項式の Coxeter–Dynkin 図式を $T(\gamma_1,\gamma_2,\gamma_3)$ であらわす（図 1）．ここで $T(\gamma_1,\gamma_2,\gamma_3)$ は，$\Delta(\gamma_1,\gamma_2,\gamma_3)\geq 0$ のときは，零点集合 $T_{\gamma_1,\gamma_2,\gamma_3}=0$ を $(\mathbb{C}^3,0)$ の中で考えたときの，$\Delta(\gamma_1,\gamma_2,\gamma_3)<0$ のときは零点集合 $T_{\gamma_1,\gamma_2,\gamma_3}=0$ を $\mathbb{C}^3$ の中で大局的に考えたときの，Milnor ファイバーにおける消滅サイクルの交叉行列を組み合わせ論的に記述したものである．つまり，交叉行列 $I=(I_{ij})$ は，各頂点 $\bullet_i$ に対して $I_{ii}=-2$，2 頂点 $\bullet_i$ および $\bullet_j$ が線分で結ばれていないとき $I_{ij}=0$，さらに

$$I_{ij}=1\Leftrightarrow\ \bullet_i \text{——} \bullet_j\ ,\quad I_{ij}=-2\Leftrightarrow\ \bullet_i = = = \bullet_j$$

として与えられる．

Theorem 32 ([5])**.** $f(x,y,z)$ を可逆多項式とする．*Table 3* に基づき，f に正の整数 $\gamma_1,\gamma_2,\gamma_3$ を対応させる．

(i) $\Delta(\gamma_1,\gamma_2,\gamma_3)<0$ ならば，原点 $0\in\mathbb{C}^3$ における適当な多項式座標変換により，多項式 $f(x,y,z)-xyz$ は $T_{\gamma_1,\gamma_2,\gamma_3}$-型の多項式の単項式変形

$$x^{\gamma_1}+y^{\gamma_2}+z^{\gamma_3}-xyz+\sum_{i=1}^{\gamma_1-1}a_ix^i+\sum_{j=1}^{\gamma_2-1}b_jy^j+\sum_{k=1}^{\gamma_3-1}c_kz^k+c,\ a_i,b_j,c_k,c\in\mathbb{C}$$

となる.
(ii) $\Delta(\gamma_1,\gamma_2,\gamma_3)=0$ ならば，原点 $0\in\mathbb{C}^3$ における適当な正則座標変換により，多項式 $f(x,y,z)-txyz$ はある $a\in\mathbb{C}^*$ に対して $T_{\gamma_1,\gamma_2,\gamma_3}$-型の多項式となる.
(iii) $\Delta(\gamma_1,\gamma_2,\gamma_3)>0$ ならば，原点 $0\in\mathbb{C}^3$ における適当な正則座標変換により，多項式 $f(x,y,z)-xyz$ は $T_{\gamma_1,\gamma_2,\gamma_3}$-型の多項式となる.

Type	$f(x,y,z)$	$(\gamma_1,\gamma_2,\gamma_3)$
I	$x^{p_1}+y^{p_2}+z^{p_3}$	(p_1,p_2,p_3)
II	$x^{p_1}+y^{p_2}+yz^{\frac{p_3}{p_2}}$	$\left(p_1,p_2,(\frac{p_3}{p_2}-1)p_1\right)$
III	$x^{p_1}+zy^{q_2+1}+yz^{q_3+1}$	(p_1,p_1q_2,p_1q_3)
IV	$x^{p_1}+xy^{\frac{p_2}{p_1}}+yz^{\frac{p_3}{p_2}}$	$\left(p_1,(\frac{p_3}{p_2}-1)p_1,\frac{p_3}{p_1}-\frac{p_3}{p_2}+1\right)$
V	$x^{q_1}y+y^{q_2}z+z^{q_3}x$	$(q_2q_3-q_2+1,q_3q_1-q_3+1,q_1q_2-q_1+1)$

TABLE 3. f に対する Gabrielov 数

□

Definition 33. Theorem 32 における正の整数の組 $(\gamma_1,\gamma_2,\gamma_3)$ を f の **Gabrielov 数**といい Γ_f であらわす.

Corollary 34. $f(x,y,z)$ を可逆多項式，$\Gamma_f=(\gamma_1,\gamma_2,\gamma_3)$ をその Gabrielov 数とする.
(i) $\Delta(\Gamma_f)<0$ ならば，f の Milnor ファイバー $f(x,y,z)=1$ は $T_{\gamma_1,\gamma_2,\gamma_3}$-型の多項式の Milnor ファイバーに変形できる.
(ii) $\Delta(\Gamma_f)>0$ ならば，特異点 $f(x,y,z)$ は $T_{\gamma_1,\gamma_2,\gamma_3}$-型のカスプ特異点に変形できる.

特異点 f が特異点 g に変形できるとき，g の Coxeter–Dynkin 図形に頂点と辺を付け加えて f の Coxeter–Dynkin 図形にできる．したがって次のことがわかる：

Corollary 35. $f(x,y,z)$ を可逆多項式，$\Gamma_f=(\gamma_1,\gamma_2,\gamma_3)$ をその Gabrielov 数とする.
(i) $\Delta(\Gamma_f)<0$ ならば，f の Coxeter–Dynkin 図形は $T(\gamma_1,\gamma_2,\gamma_3)$ に含まれる．とくに，f の Coxeter–Dynkin 図形は ADE 型である.
(ii) $\Delta(\Gamma_f)=0$ ならば，f の Coxeter–Dynkin 図形は $T(\gamma_1,\gamma_2,\gamma_3)$ に一致する.
(iii) $\Delta(\Gamma_f)>0$ ならば，$T(\gamma_1,\gamma_2,\gamma_3)$ は f の Coxeter–Dynkin 図形の一部である.

より強く, Corollary 34 は深谷圏 $D^b\mathrm{Fuk}^{\to}(f)$ および $D^b\mathrm{Fuk}^{\to}(T_{\gamma_1,\gamma_2,\gamma_3})$ の間の半直交分解定理を与える．とくに，この半直交分解は特異点の圏 $D_{Sg}^{L_f}(R_f)$ に対する半直交分解定理 (c.f., [10]) のミラー対称性対応物である.

6. 奇妙な双対性

これまでの準備により，主定理を述べることができる．とくに，Arnold の奇妙な双対性（strange duality）はもはや「奇妙」でななく，ミラー対称性として自然に理解されるものであることがわかる.

Theorem 36 ([5]). $f(x,y,z)$ を可逆多項式とする．このとき

(6.1) $$A_{G_f}=\Gamma_{f^t},\quad A_{G_{f^t}}=\Gamma_f$$

が成り立つ．つまり，組 (f, G_f) に対する Dolgachev 数 A_{G_f} は f の Berglund–Hübsch 転置 f^t の Gabrielov 数 Γ_{f^t} に一致し，組 (f^t, G_{f^t}) に対する Dolgachev 数 $A_{G_{f^t}}$ は f の Gabrielov 数 Γ_f に一致する． □

Type	$A_{G_f} = (\alpha_1, \alpha_2, \alpha_3) = \Gamma_{f^t}$	$\Gamma_f = (\gamma_1, \gamma_2, \gamma_3) = A_{G_{f^t}}$
I	(p_1, p_2, p_3)	(p_1, p_2, p_3)
II	$\left(p_1, \frac{p_3}{p_2}, (p_2 - 1)p_1\right)$	$\left(p_1, p_2, (\frac{p_3}{p_2} - 1)p_1\right)$
III	(p_1, p_1q_2, p_1q_3)	(p_1, p_1q_2, p_1q_3)
IV	$\left(\frac{p_3}{p_2}, (p_1 - 1)\frac{p_3}{p_2}, p_2 - p_1 + 1\right)$	$\left(p_1, (\frac{p_3}{p_2} - 1)p_1, \frac{p_3}{p_1} - \frac{p_3}{p_2} + 1\right)$
V	$(q_2q_3 - q_3 + 1, q_3q_1 - q_1 + 1, q_1q_2 - q_2 + 1)$	$(q_2q_3 - q_2 + 1, q_3q_1 - q_3 + 1, q_1q_2 - q_1 + 1)$

Table 4. 奇妙な双対性

なお，冒頭に述べた Conjecture 1 に対して，次の結果も得られている．

Theorem 37. $f(x, y, z)$ を可逆多項式，$\Gamma_f = (\gamma_1, \gamma_2, \gamma_3)$ をその Gabrielov 数とする．$\sum_{i=1}^{3}(1/\gamma_i) > 1$ ならば，三角同値

$$D^b(\mathrm{coh}\mathbb{P}^1_{\gamma_1,\gamma_2,\gamma_3}) \simeq D^b(\text{mod-}\mathbb{C}\Delta_{\gamma_1,\gamma_2,\gamma_3}) \simeq D^b\mathrm{Fuk}^{\to}(T_{\gamma_1,\gamma_2,\gamma_3}) \tag{6.2}$$

が成立する．ここで，$\mathbb{P}^1_{\gamma_1,\gamma_2,\gamma_3} := \mathcal{C}_{G_{f^t}}$ は Dolgachev 数 $(\gamma_1, \gamma_2, \gamma_3)$ を持つ重み付き射影直線，$\Delta_{\gamma_1,\gamma_2,\gamma_3}$ は $(\gamma_1, \gamma_2, \gamma_3)$-型の拡大 Dynkin 箙とする．

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ON A GENERALIZATION OF COSTABLE TORSION THEORY

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Abstract. E. P. Armendariz characterized a stable torsion theory in [1]. R. L. Bernhardt dualised a part of characterizations of stable torsion theory in Theorem1.1 of [3], as follows. Let ($\mathcal{T}$,$\mathcal{F}$) be a hereditary torsion theory for Mod-R such that every torsionfree module has a projective cover. Then the following are equivalent. (1) $\mathcal{F}$ is closed under taking projective covers. (2) every projective module splits. In this paper we generalize and characterize this by using torsion theory. In the remainder of this paper we study a dualization of Eckman and Shopf's Theorem and a generalization of Wu and Jans's Theorem.

1. INTRODUCTION

Throughout this paper R is a right perfect ring with identity. Let Mod-R be the categories of right R-modules. For $M \in$ Mod-R we denote by $[0 \to K(M) \to P(M) \overset{\pi_M}{\to} M \to 0\,]$ the projective cover of M, where $P(M)$ is projective and $\ker\pi_M$ is small in $P(M)$. A subfunctor of the identity functor of Mod-R is called a preradical. For a preradical σ, $\mathcal{T}_\sigma := \{M \in \text{Mod-}R\ ;\ \sigma(M) = M\}$ is the class of σ-torsion right R-modules, and $\mathcal{F}_\sigma := \{M \in \text{Mod-}R\ ;\ \sigma(M) = 0\}$ is the class of σ-torsionfree right R-modules. A right R-module M is called σ-projective if the functor $\mathrm{Hom}_R(M, \quad)$ preserves the exactness for any exact sequence $0 \to A \to B \to C \to 0$ with $A \in \mathcal{F}_\sigma$. A preradical σ is idempotent[radical] if $\sigma(\sigma(M)) = \sigma(M)[\sigma\,(M/\sigma(M)) = 0]$ for a module M, respectively. A preradical σ is called epi-preserving if $\sigma\,(M/N) = (\sigma(M) + N)/N$ holds for any module M and any submodule N of M. For a preradical σ, a short exact sequence $[0 \to K_\sigma(M) \to P_\sigma(M) \overset{\pi^\sigma_M}{\to} M \to 0]$ is called σ-projective cover of a module M if $P_\sigma(M)$ is σ-projective, $K_\sigma(M)$ is σ-torsion free and $K_\sigma(M)$ is small in $P_\sigma(M)$. If σ is an idempotent radical and a module M has a projective cover, then M has a σ-projective cover and it is given $K_\sigma(M) = K(M)/\sigma(K(M)), P_\sigma(M) = P(M)/\sigma(K(M))$. For $X, Y \in$ Mod-R we call an epimorphism $g \in \mathrm{Hom}_R(X, Y)$ a minimal epimorphism if $g(H) \subsetneqq Y$ holds for any proper submodule H of X. It is well known that a minimal epimorphism is an epimorphism having a small kernel. For a preradical σ we say that M is a σ-coessential extension of X if there exists a minimal epimorphism $h : M \twoheadrightarrow X$ with $\ker h \in \mathcal{F}_\sigma$.

For a module M, $P_\sigma(M)$ is a σ-coessential extension of M. We say that a subclass $\mathcal{C}$ of Mod-R is closed under taking σ-coessential extensions if : for any minimal epimorphism $f : M \twoheadrightarrow X$ with $\ker f \in \mathcal{F}_\sigma$ if $X \in \mathcal{C}$ then $M \in \mathcal{C}$. For the sake of simplicity we say that M is a σ-coessential extension of M/N if N is a σ-torsionfree small submodule of M. We say that a subclass $\mathcal{C}$ of Mod-R is closed under taking σ-coessential extensions if : if $M/N \in \mathcal{C}$ then $M \in \mathcal{C}$ for any σ-torsion free small submodule N of any module M.

The final version of this paper will be submitted for publication elsewhere.

We say that a subclass $\mathcal{C}$ of Mod-R is closed under taking $\mathcal{F}_\sigma$-factor modules if : if $M \in \mathcal{C}$ and N is a σ-torsionfree submodule of M then $M/N \in \mathcal{C}$.

2. COSTABLE TORSION THEORY

Lemma 1. *Let σ be an idempotent radical. For a module M and its submodule N, consider the following diagram with exact rows.*

$$\begin{array}{ccccccc} 0 \to K_\sigma(M) & \to & P_\sigma(M) & \overset{f}{\to} & M & \to & 0 \\ & & & & \downarrow_j & & \\ 0 \to K_\sigma(M/N) & \to & P_\sigma(M/N) & \underset{g}{\to} & M/N & \to & 0, \end{array}$$

where f and g are epimorphisms associated with the σ-projective covers and j is the canonical epimorphism. Since g is a minimal epimorphism, there exists an epimorphism $h : P_\sigma(M) \to P_\sigma(M/N)$ induced by the σ-projectivity of $P_\sigma(M)$ such that $jf = gh$. Then the following conditions hold.

(1) If M is a σ-coessential extension of M/N, then $h : P_\sigma(M) \to P_\sigma(M/N)$ is an isomorphism.

(2) Moreover if σ is epi-preserving and $h : P_\sigma(M) \to P_\sigma(M/N)$ is an isomorphism, then M is a σ-coessential extension of M/N.

Proof. (1): Let $N \in \mathcal{F}_\sigma$ be a small submodule of a module M. Since jf is an epimorphism and g is a minimal epimorphism, h is also an epimorphism. Since $j(f(\ker h)) = g(h(\ker h)) = g(0) = 0$, it follows that $f(\ker h) \subseteq \ker j = N \in \mathcal{F}_\sigma$, and so $f(\ker h) \in \mathcal{F}_\sigma$. Let $f|_{\ker h}$ be the restriction of f to $\ker h$. Then it follows that $\ker(f|_{\ker h}) = \ker h \cap \ker f = \ker h \cap K_\sigma(M) \subseteq K_\sigma(M) \in \mathcal{F}_\sigma$. Consider the exact sequence $0 \to \ker f|_{\ker h} \to \ker h \to f(\ker h) \to 0$. Since $\mathcal{F}_\sigma$ is closed under taking extensions, it follows that $\ker h \in \mathcal{F}_\sigma$. As $P_\sigma(M/N)$ is σ-projective, the exact sequence $0 \to \ker h \to P_\sigma(M) \to P_\sigma(M/N) \to 0$ splits, and so there exists a submodule L of $P_\sigma(M)$ such that $P_\sigma(M) = L \oplus \ker h$. So it follows that $f(P_\sigma(M)) = f(L) + f(\ker h)$. As $f(\ker h) \subseteq N$ and $f(P_\sigma(M)) = M$, $M = f(L) + N$. Since N is small in M, it follows that $M = f(L)$. As f is a minimal epimorphism, it follows that $P_\sigma(M) = L$ and $\ker h = 0$, and so $h : P_\sigma(M) \simeq P_\sigma(M/N)$, as desired.

(2): Suppose that $h : P_\sigma(M) \simeq P_\sigma(M/N)$. By the commutativity of the above diagram with h, it follows that $h(f^{-1}(N)) \subseteq K_\sigma(M/N) \in \mathcal{F}_\sigma$. Since h is an isomorphism, $f^{-1}(N) \in \mathcal{F}_\sigma$. As $f|_{f^{-1}(N)} : f^{-1}(N) \to N \to 0$ and σ is an epi-preserving preradical, it follows that $N \in \mathcal{F}_\sigma$. Next we will show that N is small in M. Let K be a submodule of M such that $M = N + K$. If $f^{-1}(K) \subsetneqq P_\sigma(M)$, then $h(f^{-1}(K)) \subsetneqq P_\sigma(M/N)$ as h is an isomorphism. Since $g(h(f^{-1}(K))) = j(f(f^{-1}(K))) = j(K) = (K + N)/N = M/N$ and g is a minimal epimorphism, this is a contradiction. Thus it holds that $f^{-1}(K) = P_\sigma(M)$, and so $K = f(f^{-1}(K)) = f(P_\sigma(M)) = M$. Thus it follows that N is small in M. □

We call a preradical t σ-costable if $\mathcal{F}_t$ is closed under taking σ-projective covers. Now we characterize σ-costable preradicals.

Theorem 2. *Let t be a radical and σ be an idempotent radical. Consider the following conditions.*

(1) t is σ-costable.

(2) $P/t(P)$ is σ-projective for any σ-projective module P.

(3) For any module M consider the following commutative diagram, then $t(P_\sigma(M))$ is contained in $kerf$.

$$\begin{array}{ccc} P_\sigma(M) & \overset{h}{\to} & M \to 0 \\ \downarrow_f & & \downarrow_j \\ P_\sigma(M/t(M)) & \underset{g}{\to} & M/t(M) \to 0, \end{array}$$

where j is a canonical epimorphism, h and g are epimorphisms associated with their projective covers and f is a morphism induced by σ-projectivity of $P_\sigma(M)$.

(4) $\mathcal{F}_t$ is closed under taking σ-coessential extensions.

(5) For any σ-projective module P such that $t(P) \in \mathcal{F}_\sigma$, $t(P)$ is a direct sumand of P.

Then $(1) \Leftarrow (5) \Longleftarrow (2) \Longleftrightarrow (1) \Longleftrightarrow (3), (4) \Longrightarrow (1)$ *hold. Moreover if $\mathcal{F}_t$ is closed under taking $\mathcal{F}_\sigma$-*factor *modules, then all conditions are equivalent.*

Proof. $(1) \to (2)$: Let P be a σ-projective module. Since $P/t(P) \in \mathcal{F}_t$, it follows that $P_\sigma(P/t(P)) \in \mathcal{F}_t$ by the assumption. Consider the following commutative diagram.

$$\begin{array}{ccccc} & & & & P \\ & & & f \swarrow & \downarrow h \\ 0 \to K_\sigma(P/t(P)) & \to & P_\sigma(P/t(P)) & \underset{g}{\to} & P/t(P) \to 0, \end{array}$$

where h is a canonical epimorphism, g is an epimorphism associated with the σ-projective cover of $P/t(P)$ and f is a morphism induced by σ-projectivity of $P_\sigma(P/t(P))$. Since $f(t(P)) \subseteq t(P_\sigma(P/t(P))) = 0$, f induces $f' : P/t(P) \to P_\sigma(P/t(P))$ $(x + t(P) \longmapsto f(x))$. Thus for $x \in P$, $h(x) = gf(x) = gf'h(x)$. So the above exact sequence splits. Therefore $P/t(P)$ is a direct summand of σ-projective module $P_\sigma(P/t(P))$, and so $P/t(P)$ is also a σ-projective module, as desired.

$(2) \to (5)$: Let P be σ-projective and $t(P) \in \mathcal{F}_\sigma$. By the assumption $P/t(P)$ is σ-projective. Thus the sequence $(0 \to t(P) \to P \to P/t(P) \to 0)$ splits, and so $t(P)$ is a direct summand of P.

$(5) \to (1)$: Let M be in $\mathcal{F}_t$. Consider the exact sequence $0 \to K_\sigma(M) \to P_\sigma(M) \underset{f}{\to} M \to 0$. Since $f(t(P_\sigma(M))) \subseteq t(M) = 0$, $K_\sigma(M) = \ker f \supseteq t(P_\sigma(M))$. As $K_\sigma(M) \in \mathcal{F}_\sigma$, $t(P_\sigma(M)) \in \mathcal{F}_\sigma$. Since $P_\sigma(M)$ is σ-projective, $t(P_\sigma(M))$ is a direct summand of $P_\sigma(M)$ by the assumption. Thus there exists a submodule K of $P_\sigma(M)$ such that $P_\sigma(M) = t(P_\sigma(M)) \oplus K$. Since $K_\sigma(M) = \ker f \supseteq t(P_\sigma(M))$, $P_\sigma(M) = K_\sigma(M) + K$. As $K_\sigma(M)$ is small in $P_\sigma(M)$, $P_\sigma(M) = K$. Thus $t(P_\sigma(M)) = 0$, as desired.

$(1) \to (3)$: Consider the following commutative diagram.

$$\begin{array}{ccc} P_\sigma(M) & \overset{h}{\to} & M \to 0 \\ f \downarrow & & \downarrow j \\ P_\sigma(M/t(M)) & \overset{g}{\to} & M/t(M) \to 0, \end{array}$$

where j is a canonical epimorphism, h and g are epimorphisms associated with their projective covers and f is a morphism induced by σ-projectivity of $P_\sigma(M)$. As g is a minimal epimorphism, f is an epimorphism. By the assumption $P_\sigma(M/t(M)) \in \mathcal{F}_t$, and so $f(t(P_\sigma(M))) \subseteq t(P_\sigma(M/t(M))) = 0$. Hence $t(P_\sigma(M)) \subseteq \ker f$.

(3) $\rightarrow$ (1) : Let M be in $\mathcal{F}_t$. By the above commutative diagram, f is an identity. Thus by the assumption $t(P_\sigma(M)) \subseteq \ker f = 0$, as desired.

(1) $\rightarrow$ (4) : Let $N \in \mathcal{F}_\sigma$ be a small submodule of a module M such that $M/N \in \mathcal{F}_t$. By the assumption $P_\sigma(M/N) \in \mathcal{F}_t$. By Lemma1, $P_\sigma(M/N) \simeq P_\sigma(M)$, and so $P_\sigma(M) \in \mathcal{F}_t$. Consider the sequence $0 \rightarrow K_\sigma(M) \rightarrow P_\sigma(M) \rightarrow M \rightarrow 0$. Since $\mathcal{F}_t$ is closed under taking $\mathcal{F}_\sigma$-factor modules, it follows that $M \in \mathcal{F}_t$, as desired.

(4) $\rightarrow$ (1) : Since $P_\sigma(M)$ is σ-coessential extension of a module M in $\mathcal{F}_t$, $\mathcal{F}_t$ is closed under taking σ-projective covers. $\square$

Remark 3. It is well known that t is epi-preserving if and only if t is a radical and $\mathcal{F}_t$ is closed under taking factor modules. Therefore if t is epi-preserving and σ be an idempotent radical, then all conditions in Theorem 2 are equivalent.

Next if σ is identity, then the following corollary holds. The following have the another characterization of Theorem1.1 of [3].

Corollary 4. *For a radical t the following conditions except (4) are equivalent. Moreover if t is an epi-preserving preradical, then all conditions are equivalent.*

(1) t is costable, that is, $\mathcal{F}_t$ is closed under taking projective covers.

(2) $P/t(P)$ is projective for any projective module P.

(3)
$$\begin{array}{ccc} P(M) & \xrightarrow{h} & M \rightarrow 0 \\ \downarrow_f & & \downarrow_j \\ P(M/t(M)) & \xrightarrow[g]{} & M/t(M) \rightarrow 0, \end{array}$$

where j is a canonical epimorphism, h and g are epimorphisms associated with their projective covers and f is induced by the projectivity of $P(M)$. Then $t(P(M))$ is contained in $\ker f$.

(4) $\mathcal{F}_t$ is closed under taking coessential extensions.

(5) For any projective module P, $t(P)$ is a direct summand of P.

3. DUALIZATION OF ECKMAN & SHOPF'S THEOREM

In [8] we state a torsion theoretic generalization of Eckman & Shopf's Theorem, as follows. Let σ be a left exact radical and $0 \rightarrow M \rightarrow E$ be a exact sequence of Mod-R. Then the following conditions from (1) to (4) are equivalent. (1) E is σ-injective and σ-essential extension of M. (2) E is minimal in $\{Y \in \text{Mod-}R | M \hookrightarrow Y \text{ and } Y \text{ is } \sigma\text{-injective}\}$. (3) E is maximal in $\{Y \in \text{Mod-}R | M \hookrightarrow Y \text{ and } Y \text{ is } \sigma\text{-essential extension of } M\}$. (4) E is isomorphic to $E_\sigma(M)$, where $\sigma(E(M)/M) = E_\sigma(M)/M$. Here we dualised this.

Lemma 5. *If P is σ-projective, then $P_\sigma(P)$ is isomorphic to P.*

Theorem 6. *Let $P \xrightarrow{f} M \rightarrow 0$ be a exact sequence of Mod-R. Let σ is an idempotent radical. Consider the following conditions, then the implications* (1) $\Longleftrightarrow$ (3) *and* (1) $\Longrightarrow$ (2) *hold. Moreover if σ is an epi-preserving preradical, then all conditions are equivalent.*

(1) P is σ-projective and $P \overset{f}{\twoheadrightarrow} M$ is a σ-coessential extension of M.

(2) P is a minimal σ-projective extension of M(i.e. P is σ-projective and if I is σ-projective and $P \overset{h}{\twoheadrightarrow} I, I \twoheadrightarrow M$, then h is an isomorphism.).

(3) P is a maximal σ-coessential extension of M (i.e. $P \overset{f}{\twoheadrightarrow} M$ is σ-coessential extension of M and if there exists an epimorphism $I \overset{h}{\twoheadrightarrow} P$ and $I \overset{h}{\twoheadrightarrow} P \twoheadrightarrow M$ is σ-coessential of M, then h is an isomorphism.).

(4) P is isomorphic to $P_\sigma(M)$.

Proof. (1)$\to$(2): Let P be σ-projective and $P \overset{f}{\twoheadrightarrow} M$ be a σ-coessential extension of M. Consider the following diagram.

$$\begin{array}{ccccccc} 0 \to & \ker h & \to & P & \overset{h}{\to} & I & \to 0 \\ & & & & \searrow_f & \downarrow_g & \\ & & & & & M, & \end{array}$$

where I is σ-projective, g and h are epimorphisms such that $gh = f$.

Since $\mathcal{F}_\sigma \ni f^{-1}(0) = h^{-1}(g^{-1}(0)) \supseteq h^{-1}(0)$, it follows that $\mathcal{F}_\sigma \ni h^{-1}(0) = \ker h$. As f is a minimal epimorphism and g is an epimorphism, h is also a minimal epimorphism. Since I is σ-projective, there exists a submodule L of P such that $P = \ker h \oplus L$ and $L \cong I$. As $\ker h$ is small in P, $P = L$, and so $P \cong I$.

(2)$\to$(1): Let σ be an epi-preserving idempotent radical and P be a minimal σ-projective extension of M. Consider the following commutative diagram.

$$\begin{array}{ccc} P_\sigma(P) & \overset{j}{\to} & P \to 0 \\ g\downarrow & & \downarrow f \\ P_\sigma(M) & \overset{h}{\to} & M \to 0, \end{array}$$

where h and j are epimorphisms associated with the projective covers of M and P respectively and g is an induced epimorphism by the σ-projectivity of $P_\sigma(P)$. Since P is σ-projective, j is an isomorphism by Lemma 4. As $P_\sigma(P)$ and $P_\sigma(M)$ are σ-projective, g is an isomorphism by the assumption. By Lemma 1, it follows that $P \overset{f}{\to} M \to 0$ is a σ-coessential extension of M.

(1)$\to$(3): Let $I \overset{g}{\to} P$ be an epimorphism. Let $P \overset{f}{\twoheadrightarrow} M$ and $I \overset{h}{\twoheadrightarrow} M$ be σ-coessential extensions of M such that $fg = h$. Consider the following exact diagram.

$$\begin{array}{ccc} & & I \\ & g \swarrow & \downarrow h \\ P & \underset{f}{\to} & M \to 0 \end{array}$$

Since f is a minimal epimorphism, g is an epimorphim. As h and f are minimal epimorphisms, g is a minimal epimorphim. Since $\mathcal{F}_\sigma \ni h^{-1}(0) = g^{-1}(f^{-1}(0)) \supseteq g^{-1}(0)$, it follows that $\mathcal{F}_\sigma \ni g^{-1}(0)$. Since P is σ-projective, $0 \to \ker g \to I \overset{g}{\to} P \to 0$ splits, and so there exists a submodule H of I such that $H \cong P$ and $I = \ker g \oplus H$. As $\ker g$ is small in I, $I = H \cong P$, as desired.

(3)$\to$(1): We show that P is σ-projective. Since $P \overset{f}{\twoheadrightarrow} M$ is a σ-coessential extension of M by the assumption, an induced morphism $P_\sigma(P) \to P_\sigma(M)$ is an isomorphism by Lemma 1. Consider the following commutative diagram.

$$\begin{array}{ccc} P_\sigma(P) & \to & P \to 0 \\ \downarrow & & \downarrow \\ P_\sigma(M) & \to & M \to 0. \end{array}$$

Since $P_\sigma(P) \simeq P_\sigma(M) \twoheadrightarrow M$ is a σ-coessential extension of M and $P \overset{f}{\twoheadrightarrow} M$ is a σ-coessential extension of M, it follows that $P_\sigma(P) \cong P$ by the assumption, and so P is σ-projective.

(1)$\to$(4): By Lemma 1, $P_\sigma(P) \simeq P_\sigma(M)$. By Lemma 4, $P_\sigma(P) \simeq P$, and so $P \simeq P_\sigma(M)$ as desired.

(4)$\to$(1): It is clear. □

In Theorem 5, if $\sigma = 1$, then the following corollary is obtained.

Corollary 7. *Let $P \xrightarrow{f} M \to 0$ be a exact sequence of Mod-R. Then the following conditions are equivalent.*

(1) P is projective and $P \overset{f}{\twoheadrightarrow} M$ is a coessential extension of M(that is, $\ker f$ is small in M).

(2) P is a minimal projective extension of M(i.e. P is projective and if I is projective and $P \overset{h}{\twoheadrightarrow} I, I \twoheadrightarrow M$, then h is an isomorphism).

(3) P is a maximal coessential extension of M(i.e. $P \overset{f}{\twoheadrightarrow} M$ is coessential extension of M and if there exists an epimorphism $I \overset{h}{\twoheadrightarrow} P$ and $I \overset{h}{\twoheadrightarrow} P \twoheadrightarrow M$ is coessential of M, then h is an isomorphism.).

(4) P is isomorphic to $P(M)$.

4. A GENERALIZATION OF WU, JANS AND MIYASHITA'S THEOREM AND AZUMAYA'S THEOREM

In [8] we state a torsion theoretic generalization of Johnson and Wong's Theorem. Here we study a dualization of this. For a module M and N, we call M σ-N-projective if $\mathrm{Hom}_R(M, \quad)$ preserves the exactness of the short exact sequence $0 \to K \to N \to N/K \to 0$ with $K \in \mathcal{F}_\sigma$.

Theorem 8. *Let M and N be modules. Consider the following conditions for an idempotent radical σ.*

(1) $\gamma(K_\sigma(M)) \subseteq K_\sigma(N)$ holds for any $\gamma \in Hom_R(P_\sigma(M), P_\sigma(N))$.

(2) M is σ-N-projective.

Then the implication (1)$\to$(2) holds. If σ is epi-preserving, then the implication (2)$\to$(1) holds.

Proof. (1)$\to$(2): Let f be in $\mathrm{Hom}_R(M, N/K)$ with $K \in \mathcal{F}_\sigma$. Then there exists $h \in \mathrm{Hom}_R(P_\sigma(M), N)$ such that $f\pi_M^\sigma = nh$, where n is a canonical epimorphism from N to N/K. And there exists $\gamma \in \mathrm{Hom}_R(P_\sigma(M), P_\sigma(N))$ such that $h = \pi_N^\sigma\gamma$. So we have the following commutative diagramm.

$$\begin{tikzcd} & P_\sigma(M) \arrow[r, two heads, "\pi_M^\sigma"] \arrow[dl, "\gamma"'] \arrow[d, "h"] & M \arrow[d, "f"] \\ P_\sigma(N) \arrow[r, two heads, "\pi_N^\sigma"'] & N \arrow[r, "n"'] & N/K \end{tikzcd}$$

By the assumption, γ induces $\gamma' : P_\sigma(M)/K_\sigma(M) \to P_\sigma(N)/K_\sigma(N)$, and so γ' induces $\gamma'' : M \to N$ such that $f = \gamma'' n$, as desired.

(2)→(1): Let σ be epi-preserving and $\gamma \in \mathrm{Hom}_R(P_\sigma(M), P_\sigma(N))$. We will show that $\gamma(K_\sigma(M)) \subseteq K_\sigma(N)$. We put $T = \gamma(K_\sigma(M)) + K_\sigma(N)$. Since $T \supseteq \gamma(K_\sigma(M))$, γ induces $\gamma' : M \simeq P_\sigma(M)/K_\sigma(M) \to P_\sigma(N)/\gamma(K_\sigma(M)) \to P_\sigma(N)/T \to N/\pi_N^\sigma(T)$ $(\pi_M^\sigma(x) \longleftrightarrow x + K_\sigma(M) \to \gamma(x) + \gamma(K_\sigma(M)) \to \gamma(x) + T \to \pi_N^\sigma(\gamma(x)) + \pi_N^\sigma(T))$. Let n_N be a canonical epimorphism from N to $N/\pi_N^\sigma(T)$. Since $\pi_N^\sigma(T) = \pi_N^\sigma(\gamma(K_\sigma(M)) + K_\sigma(N)) = \pi_N^\sigma(\gamma(K_\sigma(M)), K_\sigma(M) \in \mathcal{F}_\sigma$ and $\mathcal{F}_\sigma$ is closed under taking factor modules, it follows that $\pi_N^\sigma(T) \in \mathcal{F}_\sigma$. Since M is σ-N-projective, there exists $\beta : M \to N$ such that $\gamma' = n_N\beta$. Therefore we have the following commutative diagramm.

$$\begin{array}{ccccccc} & & & & M & & \\ & & & \beta \swarrow & \downarrow \gamma' & & \\ 0 \to \pi_N^\sigma(T) & \to & N & \underset{n_N}{\to} & N/\pi_N^\sigma(T) & \to & 0 \end{array}$$

·By the σ-projectivity of $P_\sigma(M)$, there exists $\alpha : P_\sigma(M) \to P_\sigma(N)$ such that $\pi_N^\sigma\alpha = \beta\pi_M^\sigma$. Thus we have the following commutative diagramm.

$$\begin{array}{ccccccc} 0 \to K_\sigma(M) & \to & P_\sigma(M) & \overset{\pi_M^\sigma}{\to} & M & \to & 0 \\ & & \downarrow_\alpha & & \downarrow_\beta & & \\ 0 \to K_\sigma(N) & \to & P_\sigma(N) & \underset{\pi_N^\sigma}{\to} & N & \to & 0 \end{array}$$

Thus by the commutativity of the above diagram, we have $\alpha(K_\sigma(M)) \subseteq K_\sigma(N)$.

We put $X = \{x \in P_\sigma(M) | \gamma(x) - \alpha(x) \in K_\sigma(N)\}$. We will show that $X + K_\sigma(M) = P_\sigma(M)$. For any $x \in P_\sigma(M)$ it follows that $\gamma'(\pi_M^\sigma(x)) = \pi_N^\sigma(\gamma(x)) + \pi_N^\sigma(T)$, $(n_N\beta)(\pi_M^\sigma(x)) = \beta(\pi_M^\sigma x) + \pi_N^\sigma(T)$ and $\gamma' = n_N\beta$, it follows that $\pi_N^\sigma(\gamma(x)) + \pi_N^\sigma(T) = \beta(\pi_M^\sigma x) + \pi_N^\sigma(T)$, and so $\pi_N^\sigma(\gamma(x)) - \beta(\pi_M^\sigma x) \in \pi_N^\sigma(T)$. Since $\pi_N^\sigma\alpha = \beta\pi_M^\sigma$, it follows that $\pi_N^\sigma(\gamma(x)) - \pi_N^\sigma(\alpha(x)) \in \pi_N^\sigma(T)$, and so $\gamma(x) - \alpha(x) \in T + (\pi_N^\sigma)^{-1}(0) = T + K_\sigma(N) = \gamma(K_\sigma(M)) + K_\sigma(N)$. Thus there exists $m \in K_\sigma(M)$ such that $\gamma(x) - \alpha(x) - \gamma(m) \in K_\sigma(N)$, and so $\gamma(x - m) - \alpha(x - m) \in \alpha(m) + K_\sigma(N) \subseteq \alpha(K_\sigma(M)) + K_\sigma(N) = K_\sigma(N)$. Therefore it follows that $x - m \in X$, and so $x \in K_\sigma(M) + X$. Thus we conclude that $P_\sigma(M) = K_\sigma(M) + X$. Since $K_\sigma(M)$ is small in $P_\sigma(M)$, it holds that $X = P_\sigma(M)$. Thus it follows that $\{x \in P_\sigma(M) | \gamma(x) - \alpha(x) \in K_\sigma(N)\} = P_\sigma(M)$. Thus if $x \in K_\sigma(M) (\subseteq P_\sigma(M))$, then $\gamma(x) - \alpha(X) \in K_\sigma(N)$, and so $\gamma(x) \in \alpha(x) + K_\sigma(N) \subseteq \alpha(K_\sigma(M)) + K_\sigma(N) = K_\sigma(N)$, and so it follows that $\gamma(K_\sigma(M)) \subseteq K_\sigma(N)$. □

In Theorem 7 we put $\sigma = 1$, then we have a generalization of Azumaya's Theorem in [2]. In Theorem 7 we put $M = N$ and $\sigma = 1$, then we have a generalization of Wu, Jans and Miyashita's Theorem in [9] and [5].

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GRADED FROBENIUS ALGEBRAS AND QUANTUM BEILINSON ALGEBRAS

KENTA UEYAMA

Abstract. Frobenius algebras are one of the important classes of algebras studied in representation theory of finite dimensional algebras. In this article, we will study when given graded Frobenius Koszul algebras are graded Morita equivalent. As applications, we apply our results to quantum Beilinson algebras.

Key Words: Frobenius Koszul algebras, quantum Beilinson algebras, graded Morita equivalence.

2010 *Mathematics Subject Classification*: Primary 16W50, 16S37; Secondary 16D90.

1. Introduction

This is based on a joint work with Izuru Mori.

Classification of Frobenius algebras is an active project in representation theory of finite dimensional algebras. This article tries to answer the question when given graded Frobenius Koszul algebras are graded Morita equivalent, that is, they have equivalent graded module categories.

This problem is related to classification of quasi-Fano algebras. It is known that every finite dimensional algebra of global dimension 1 is a path algebra of a finite acyclic quiver up to Morita equivalence, so such algebras can be classified in terms of quivers. As an obvious next step, it is interesting to classify finite dimensional algebras of global dimension 2 or higher. Recently, Minamoto introduced a nice class of finite dimensional algebras of finite global dimension, called (quasi-)Fano algebras [2], which are a very interesting class of algebras to study and classify. It was shown that, for a graded Frobenius Koszul algebra A, we can define another algebra ∇A, called the quantum Beilinson algebra associated to A, and with some additional assumptions, ∇A turns out to be a quasi-Fano algebra. Moreover, it was shown that two graded Frobenius algebras A, A' are graded Morita equivalent if and only if $\nabla A, \nabla A'$ are isomorphic as algebras, so classifying graded Frobenius (Koszul) algebra up to graded Morita equivalence is related to classifying quasi-Fano algebras up to isomorphism (see [3] for details).

In addition, this problem is related to the study of AS-regular algebras which are the most important class of algebras in noncommutative algebraic geometry (see [8]).

Our main theorem (Theorem 9) is as follows. For every co-geometric Frobenius Koszul algebra A, we define another graded algebra $\overline{A}$, and see that if two co-geometric Frobenius Koszul algebras A, A' are graded Morita equivalent, then $\overline{A}, \overline{A'}$ are isomorphic as graded algebras. Unfortunately, the converse does not hold in general. On the other hand, the converse is also true for many co-geometric Frobenius Koszul algebras of Gorenstein parameter -3.

The detailed version of this paper will be submitted for publication elsewhere.

2. Frobenius Koszul algebras

Throughout this paper, we fix an algebraically closed field k of characteristic 0, and we assume that all vector spaces and algebras are over k unless otherwise stated. In this paper, a graded algebra means a connected graded algebra finitely generated in degree 1, that is, every graded algebra can be presented as $A = T(V)/I$ where V is a finite dimensional vector space, $T(V)$ is the tensor algebra on V over k, and I is a homogeneous two-sided ideal of $T(V)$. We denote by $\operatorname{GrMod} A$ the category of graded right A-modules. Morphisms in $\operatorname{GrMod} A$ are right A-module homomorphisms preserving degrees. We say that two graded algebras A and A' are graded Morita equivalent if $\operatorname{GrMod} A \cong \operatorname{GrMod} A'$.

For a graded module $M \in \operatorname{GrMod} A$ and an integer $n \in \mathbb{Z}$, we define the truncation $M_{\geq n} := \bigoplus_{i \geq n} M_i \in \operatorname{GrMod} A$ and the shift $M(n) \in \operatorname{GrMod} A$ by $M(n)_i := M_{n+i}$ for $i \in \mathbb{Z}$. For $M, N \in \operatorname{GrMod} A$, we write

$$\underline{\operatorname{Hom}}_A(M, N) := \bigoplus_{n \in \mathbb{Z}} \operatorname{Hom}_{\operatorname{GrMod} A}(M, N(n)).$$

We denote by V^* the dual vector space of a vector space V. If M is a graded right (resp. left) module over a graded algebra A, then we denote by $M^* := \underline{\operatorname{Hom}}_k(M, k)$ the dual graded vector space of M by abuse of notation, i.e. $(M^*)_i := (M_{-i})^*$. Note that M^* has a graded left (resp. right) A-module structure.

Let A be a graded algebra, and $\tau \in \operatorname{Aut}_k A$ a graded algebra automorphism. For a graded right A-module $M \in \operatorname{GrMod} A$, we define a new graded right A-module $M_\tau \in \operatorname{GrMod} A$ by $M_\tau = M$ as a graded vector space with the new right action $m * a := m\tau(a)$ for $m \in M$ and $a \in A$. If M is a graded A-A bimodule, then M_τ is also a graded A-A bimodule by this new right action. The rule $M \mapsto M_\tau$ is a k-linear autoequivalence for $\operatorname{GrMod} A$.

A graded algebra A is called quadratic if $A \cong T(V)/(R)$ where $R \subseteq V \otimes_k V$ is a subspace and (R) is the ideal of $T(V)$ generated by R. If $A = T(V)/(R)$ is a quadratic algebra, then we define the dual graded algebra by $A^! := T(V^*)/(R^\perp)$ where

$$R^\perp := \{\lambda \in V^* \otimes_k V^* \cong (V \otimes_k V)^* \mid \lambda(r) = 0 \text{ for all } r \in R\}.$$

Clearly, $A^!$ is again a quadratic algebra and $(A^!)^! \cong A$ as graded algebras.

We now recall the definitions of Koszul algebras and graded Frobenius algebras. Frobenius algebras are one of the main classes of algebras of study in representation theory of finite dimensional algebras.

Definition 1. Let A be a connected graded algebra, and suppose $k \in \operatorname{GrMod} A$ has a minimal free resolution of the form

$$\cdots \to \bigoplus_{j=1}^{r_i} A(-s_{ij}) \to \cdots \to \bigoplus_{j=1}^{r_0} A(-s_{0j}) \to k \to 0.$$

The complexity of A is defined by

$$c_A := \inf\{d \in \mathbb{R}^+ \mid r_i \leq c i^{d-1} \text{for some constant } c > 0, i \gg 0\}.$$

We say that A is Koszul if $s_{ij} = i$ for all $1 \leq j \leq r_i$ and all $i \in \mathbb{N}$.

It is known that if A is Koszul, then A is quadratic, and its dual graded algebra $A^!$ is also Koszul, which is called the Koszul dual of A.

Definition 2. A graded algebra A is called a graded Frobenius algebra of Gorenstein parameter ℓ if $A^* \cong {}_\nu A(-\ell)$ as graded A-A bimodules for some graded algebra automorphism $\nu \in \operatorname{Aut}_k A$, called the Nakayama automorphism of A. We say that A is graded symmetric if $A^* \cong A(-\ell)$ as graded A-A bimodules.

A skew exterior algebra

$$A = k\langle x_1, \dots, x_n\rangle/(\alpha_{ij}x_ix_j + x_jx_i, x_i^2)$$

where $\alpha_{ij} \in k$ such that $\alpha_{ij}\alpha_{ji} = \alpha_{ii} = 1$ for $1 \le i, j \le n$ is a typical example of a Frobenius Koszul algebra.

At the end of this section, we give an interesting result about graded Morita equivalence of graded skew exterior algebras. It is known that every (ungraded) Frobenius algebra which is Morita equivalent to symmetric algebra is symmetric. The situation in the graded case is different as the following theorem shows.

Proposition 3. [7] *Every skew exterior algebra is graded Morita equivalent to a graded symmetric skew exterior algebra.*

For example, a 3-dimensional skew exterior algebra

$$A = k\langle x, y, z\rangle/(\alpha yz + zy, \beta zx + xz, \gamma xy + yx, x^2, y^2, z^2)$$

is graded Morita equivalent to a symmetric skew exterior algebra

$$A = k\langle x, y, z\rangle/(\sqrt[3]{\alpha\beta\gamma}yz + zy, \sqrt[3]{\alpha\beta\gamma}zx + xz, \sqrt[3]{\alpha\beta\gamma}xy + yx, x^2, y^2, z^2).$$

3. Co-geometric Frobenius Koszul algebras

In order to state our main result, let us define a co-geometric algebra (see [4] for details).

Definition 4. [4] Let $A = T(V)/I$ be a graded algebra. We say that $N \in \operatorname{GrMod} A$ is a co-point module if N has a free resolution of the form

$$\cdots \to A(-2) \to A(-1) \to A \to N \to 0\,.$$

For a graded algebra $A = T(V)/I$, we can define the pair $\mathcal{P}^!(A) = (E, \sigma)$ consisting of the set $E \subseteq \mathbb{P}(V)$ and the map $\sigma : E \to E$ as follows:

- $E := \{p \in \mathbb{P}(V) \mid N_p := A/pA \in \operatorname{GrMod} A \text{ is a co-point module}\}$, and
- the map $\sigma : E \to E$ is defined by $\Omega N_p(1) = N_{\sigma(p)}$.

Meanwhile, for a geometric pair (E, σ) consists of a closed subscheme $E \subseteq \mathbb{P}(V)$ and an automorphism $\sigma \in \operatorname{Aut}_k E$, we can define the algebra $\mathcal{A}^!(E, \sigma)$ as follows:

$$\mathcal{A}^!(E, \sigma) := (T(V^*)/(R))^! \quad \text{where } R := \{f \in V^* \otimes_k V^* \mid f(p, \sigma(p)) = 0, \forall p \in E\}.$$

Definition 5. [4] A graded algebra $A = T(V)/I$ is called co-geometric if A satisfies the following conditions:

- $\mathcal{P}^!(A)$ consisting of a closed subscheme $E \subseteq \mathbb{P}(V)$ and an automorphism $\sigma \in \operatorname{Aut}_k E$,
- $A^!$ is noetherian, and
- $A \cong \mathcal{A}^!(\mathcal{P}^!(A))$.

Example 6. [4] Let $A = k\langle x,y\rangle/(\alpha xy + yx, x^2, y^2)$ be a 2-dimensional skew exterior algebra. Then for any point $p = (a,b) \in \mathbb{P}(V) = \mathbb{P}^1$, $N_p = A/(ax+by)A$ has a free resolution of the form

$$\cdots \longrightarrow A(-2) \xrightarrow{(\alpha^2 ax+by)\cdot} A(-1) \xrightarrow{(\alpha ax+by)\cdot} A \xrightarrow{(ax+by)\cdot} N_p \longrightarrow 0\ .$$

Since $\Omega N_p(1) = A/(\alpha ax + by)A$, it follow that

$$\mathcal{P}^!(A) = (\mathbb{P}^1, \sigma), \text{ where } \sigma(a,b) := (\alpha a, b).$$

In fact, A is co-geometric.

Example 7. The algebras below are examples of co-geometric algebras.

- A Frobenius Koszul algebra of finite complexity and of Gorenstein parameter -3. For example, if $A = k\langle x,y,z\rangle$ with the defining relations

  $$\begin{aligned} &\alpha x^2 - \gamma yz, \quad \alpha y^2 - \gamma zx, \quad \alpha z^2 - \gamma xy, \\ &\beta yz - \alpha zy, \quad \beta zx - \alpha xz, \quad \beta xy - \alpha yx. \end{aligned}$$

  for a generic choice of $\alpha, \beta, \gamma \in k$, then $A = \mathcal{A}^!(E,\sigma)$ is a Frobenius Koszul algebra of complexity 3 and of Gorenstein parameter -3 such that

  $$E = \mathcal{V}(\alpha\beta\gamma(x^3+y^3+z^3) - (\alpha^3+\beta^3+\gamma^3)xyz) \subset \mathbb{P}^2$$

  is an elliptic curve and $\sigma \in \operatorname{Aut}_k E$ is the translation automorphism by the point $(\alpha,\beta,\gamma) \in E$.
- The skew exterior algebra.

Let $A = \mathcal{A}^!(E,\sigma)$ be a co-geometric Frobenius Koszul algebra of Gorenstein parameter $-\ell$ with the Nakayama automorphism $\nu \in \operatorname{Aut}_k A$. The restriction $\nu|_{A_1} = \tau|_V$ induces an automorphism $\nu \in \operatorname{Aut}_k \mathbb{P}(V)$. Moreover, $\nu \in \operatorname{Aut}_k \mathbb{P}(V)$ restricts to an automorphism $\nu \in \operatorname{Aut}_k E$ by abuse of notation (see [5] for details). We can now define a new graded algebra $\overline{A}$ as follows:

$$\overline{A} := \mathcal{A}^!(E, \nu\sigma^\ell).$$

Example 8. If $A = k\langle x,y,z\rangle$ with the defining relations

$$\begin{aligned} &x^2 + \beta xz, \quad zx + xz, \quad z^2, \\ &y^2 + \alpha yz, \quad zy + yz, \quad xy + yx - (\beta+\gamma)xz - (\alpha+\gamma)yz, \end{aligned}$$

where $\alpha,\beta,\gamma \in k, \alpha+\beta+\gamma \neq 0$, then $A = \mathcal{A}^!(E,\sigma)$ is a Frobenius Koszul algebra of complexity 3 and of Gorenstein parameter -3 such that

$$E = \mathcal{V}(x) \cup \mathcal{V}(y) \cup \mathcal{V}(x-y) \subset \mathbb{P}^2$$

is a union of three lines meeting at one point, and $\sigma \in \operatorname{Aut}_k E$ is given by

$$\begin{aligned} \sigma|_{\mathcal{V}(x)}(0,b,c) &= (0,b,\alpha b + c), \\ \sigma|_{\mathcal{V}(y)}(a,0,c) &= (a,0,\beta a + c), \\ \sigma|_{\mathcal{V}(x-y)}(a,a,c) &= (a,a,-\gamma a + c) \end{aligned}$$

In this case, $\nu \in \operatorname{Aut}_k E$ induced by the Nakayama automorphism $\nu \in \operatorname{Aut}_k A$ is given by

$$\nu(a,b,c) = (a,b,(\alpha+\gamma-2\beta)a + (\beta+\gamma-2\alpha)b + c)$$

It follows that $\overline{A} = \mathcal{A}^!(E, \nu\sigma^3)$ is $k\langle x,y,z\rangle$ with the defining relations

$$x^2 + (\alpha+\beta+\gamma)xz, \quad zx + xz, \quad z^2,$$
$$y^2 + (\alpha+\beta+\gamma)yz, \quad zy + yz, \quad xy + yx - 2(\alpha+\beta+\gamma)xz - 2(\alpha+\beta+\gamma)yz.$$

Our main result is as follows.

Theorem 9. [7] *Let A, A' be co-geometric Frobenius Koszul algebras. Then*

$$\operatorname{GrMod} A \cong \operatorname{GrMod} A' \Longrightarrow \overline{A} \cong \overline{A'} \text{ as graded algebras.}$$

In particular, let $A = \mathcal{A}^!(E,\sigma), A' = \mathcal{A}^!(E',\sigma')$ be Frobenius Koszul algebras of finite complexities and of Gorenstein parameter -3 such that $E \cong E'$. Suppose that $E = \mathbb{P}^2$ or E is a reduced and reducible cubic in $\mathbb{P}^2$, then

$$\operatorname{GrMod} A \cong \operatorname{GrMod} A' \Longleftrightarrow \overline{A} \cong \overline{A'} \text{ as graded algebras.}$$

4. Quantum Beilinson Algebras

Finally, we apply our results to quantum Beilinson algebras.

Definition 10. [1], [6] Let A be a graded Frobenius algebra of Gorenstein parameter $-\ell$. Then the quantum Beilinson algebra of A is defined by

$$\nabla A := \begin{pmatrix} A_0 & A_1 & \cdots & A_{\ell-1} \\ 0 & A_0 & \cdots & A_{\ell-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 \end{pmatrix}.$$

Theorem 11. [3] *Let A, A' be graded Frobenius algebras. Then*

$$\operatorname{GrMod} A \cong \operatorname{GrMod} A' \iff \nabla A \cong \nabla A' \text{ as algebras.}$$

By the above theorem, classifying graded Frobenius algebras up to graded Morita equivalence is the same as classifying quantum Beilinson algebras up to isomorphism.

Quasi-Fano algebras introduced by Minamoto [2] are one of the nice classes of a finite dimensional algebras of finite global dimensions (see [3], [6] for details).

Definition 12. A finite dimensional algebra R is called quasi-Fano of dimension n if $\operatorname{gldim} R = n$ and w_R^{-1} is a quasi-ample two-sided tilting complex, that is, $h^i((w_R^{-1})^{\otimes^{\mathbb{L}}_R j}) = 0$ for all $i \neq 0$ and all $j \geq 0$, where $w_R := R^*[-n]$.

Let A be a graded Frobenius Koszul algebra of Gorenstein parameter $-d$. Assume that A has the Hilbert series

$$H_A(t) := \sum_i (\dim_k A_i)t^i = (1+t)^d$$

and that $A^!$ is noetherian. Then ∇A is a quasi-Fano algebra of dimension $d-1$.

In general, it is not easy to check if two algebras given as path algebras of quivers with relations are isomorphic as algebras by constructing an explicit algebra isomorphism. On the other hand, it is much easier to check if two graded algebras $T(V)/I$ and $T(V')/I'$ generated in degree 1 over k are isomorphic as graded algebras since any such isomorphism is induced by the vector space isomorphism $V \to V'$. In this sense, our main result is useful for the classification of a class of finite dimensional algebras of global dimension 2, namely, quantum Beilinson algebras of global dimension 2.

Fix the Beilinson quiver

$$Q = \bullet \overset{x_1}{\underset{z_1}{\overset{\longrightarrow}{\underset{\longrightarrow}{\xrightarrow{y_1}}}}} \bullet \overset{x_2}{\underset{z_2}{\overset{\longrightarrow}{\underset{\longrightarrow}{\xrightarrow{y_2}}}}} \bullet$$

and let

$$B = kQ/I, \quad B' = kQ/I', \quad B'' = kQ/I''$$

be path algebras with relations

$$\begin{aligned}
I &= (\alpha y_1z_2 + z_1y_2, \beta z_1x_2 + x_1z_2, \gamma x_1y_2 + y_1x_2, x_1x_2, y_1y_2, z_1z_2)\\
I' &= (x_1x_2 + \alpha' y_1z_2, y_1y_2 + \beta' z_1x_2, z_1z_2 + \gamma' x_1y_2, z_1y_2, x_1z_2, y_1x_2)\\
I'' &= (\alpha'' y_1z_2 + z_1y_2, \beta'' z_1x_2 + x_1z_2, \beta'' x_1y_2 + y_1x_2, x_1x_2 + y_1z_2, y_1y_2, z_1z_2)
\end{aligned}$$

where $\alpha\beta\gamma \neq 0, 1, \alpha'\beta'\gamma' \neq 0, 1, \alpha''(\beta'')^2 \neq 0, 1$. Then B, B', B'' are the quantum Beilinson algebras of co-geometric Frobenius Koszul algebras A, A', A'' of Gorenstein parameter -3

$$\begin{aligned}
A &= \mathcal{A}^!(E, \sigma) = k\langle x, y, z\rangle/(\alpha yz + zy, \beta zx + xz, \gamma xy + yx, x^2, y^2, z^2),\\
A' &= \mathcal{A}^!(E', \sigma') = k\langle x, y, z\rangle/(x^2 + \alpha' yz, y^2 + \beta' zx, z^2 + \gamma' xy, zy, xz, yx),\\
A'' &= \mathcal{A}^!(E'', \sigma'') = k\langle x, y, z\rangle/(\alpha'' yz + zy, \beta'' zx + xz, \beta'' xy + yx, x^2 + yz, y^2, z^2),
\end{aligned}$$

where E is a triangle and $\sigma \in \operatorname{Aut}_k E$ stabilizes each component, E' is a triangle and $\sigma' \in \operatorname{Aut}_k E'$ circulates three components, and E'' is a union of a line and a conic meeting at two points and $\sigma'' \in \operatorname{Aut}_k E''$ stabilizes each component and two intersection points. Since

$$E \cong E' \not\cong E'',$$

we see that

$$B \not\cong B'', \quad B' \not\cong B''.$$

Moreover, it is not difficult to compute

$$\begin{aligned}
\overline{A} &= \mathcal{A}^!(E, \nu\sigma^3)\\
&= k\langle x, y, z\rangle/(\alpha\beta\gamma yz + zy, \alpha\beta\gamma zx + xz, \alpha\beta\gamma xy + yx, x^2, y^2, z^2),\\
\overline{A'} &= \mathcal{A}^!(E', \nu'(\sigma')^3)\\
&= k\langle x, y, z\rangle/(yz + \alpha'\beta'\gamma' zy, zx + \alpha'\beta'\gamma' xz, xy + \alpha'\beta'\gamma' yx, x^2, y^2, z^2).
\end{aligned}$$

Since $\overline{A}, \overline{A'}$ are skew exterior algebras, it is easy to check when they are isomorphic as graded algebras. Using theorems, the following are equivalent.

(1) $B \cong B'$ as algebras.
(2) $\operatorname{GrMod} A \cong \operatorname{GrMod} A'$.
(3) $\overline{A} \cong \overline{A'}$ as graded algebras.
(4) $\alpha'\beta'\gamma' = (\alpha\beta\gamma)^{\pm 1}$.

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APPLICATIONS OF FINITE FROBENIUS RINGS TO THE FOUNDATIONS OF ALGEBRAIC CODING THEORY

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Abstract. This article addresses some foundational issues that arise in the study of linear codes defined over finite rings. Linear coding theory is particularly well-behaved over finite Frobenius rings. This follows from the fact that the character module of a finite ring is free if and only if the ring is Frobenius.

Key Words: Frobenius ring, generating character, linear code, extension theorem, MacWilliams identities.

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1. Introduction

At the center of coding theory lies a very practical problem: how to ensure the integrity of a message being transmitted over a noisy channel? Even children are aware of this problem: the game of "telephone" has one child whisper a sentence to a second child, who in turn whispers it to a third child, and the whispering continues. The last child says the sentence out loud. Usually the children burst out laughing, because the final sentence bears little resemblance to the original.

Using electronic devices, messages are transmitted over many different noisy channels: copper wires, fiber optic cables, saving to storage devices, and radio, cell phone, and deep-space communications. In all cases, it is desirable that the message being received is the same as the message being sent. The standard approach to error-correction is to incorporate redundancy in a cleverly designed way (encoding), so that transmission errors can be efficiently detected and corrected (decoding).

Mathematics has played an essential role in coding theory, with the seminal work of Claude Shannon [27] leading the way. Many constructions of encoding and decoding schemes make strong use of algebra and combinatorics, with linear algebra over finite fields often playing a prominent part. The rich interplay of ideas from multiple areas has led to discoveries that are of independent mathematical interest.

This article addresses some of the topics that lie at the mathematical foundations of algebraic coding theory, specifically topics related to linear codes defined over finite rings. This article is not an encyclopedic survey; the mathematical questions addressed are ones in which the author has been actively involved and are ones that apply to broad classes of finite rings, not just to specific examples.

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The topics covered are ring-theoretic analogs of results that go back to one of the early leaders of the field, Florence Jessie MacWilliams (1917–1990). MacWilliams worked for many years at Bell Labs, and she received her doctorate from Harvard University in 1962, under the direction of Andrew Gleason [22]. She is the co-author, with Neil Sloane, of the most famous textbook on coding theory [23].

Two of the topics discussed in this article are found in the doctoral dissertation of MacWilliams [22]. One topic is the famous MacWilliams identities, which relate the Hamming weight enumerator of a linear code to that of its dual code. The MacWilliams identities have wide application, especially in the study of self-dual codes (linear codes that equal their dual code). The MacWilliams identities are discussed in Section 4, and some interesting aspects of self-dual codes due originally to Gleason are discussed in Section 6.

The other topic to be discussed, also found in MacWilliams's dissertation, is the MacWilliams extension theorem. This theorem is not as well known as the MacWilliams identities, but it underlies the notion of equivalence of linear codes. It is easy to show that a monomial transformation defines an isomorphism between linear codes that preserves the Hamming weight. What is not so obvious is the converse: whether every isomorphism between linear codes that preserves the Hamming weight must extend to a monomial transformation. MacWilliams proves that this is indeed the case over finite fields. The MacWilliams extension theorem is a coding-theoretic analog of the extension theorems for isometries of bilinear forms and quadratic forms due to Witt [30] and Arf [1].

This article describes, in large part, how these two results, the MacWilliams identities and the MacWilliams extension theorem, generalize to linear codes defined over finite rings. The punch line is that both theorems are valid for linear codes defined over finite Frobenius rings. Moreover, Frobenius rings are the largest class of finite rings over which the extension theorem is valid.

Why finite Frobenius rings? Over finite fields, both the MacWilliams identities and the MacWilliams extension theorem have proofs that make use of character theory. In particular, finite fields $\mathbb{F}$ have the simple, but crucial, properties that their characters $\widehat{\mathbb{F}}$ form a vector space over $\mathbb{F}$ and $\widehat{\mathbb{F}} \cong \mathbb{F}$ as vector spaces. The same proofs will work over a finite ring R, provided R has the same crucial property that $\widehat{R} \cong R$ as one-sided modules. It turns out that finite Frobenius rings are exactly characterized by this property ([14, Theorem 1] and, independently, [31, Theorem 3.10]). The character theory of finite Frobenius rings is discussed in Section 2, and the extension theorem is discussed in Section 5. Some standard terminology from algebraic coding theory is discussed in Section 3.

While much of this article is drawn from earlier works, especially [31] and [33], some of the treatment of generating characters for Frobenius rings in Section 2 has not appeared before. The new results are marked with a dagger (†).

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2. Finite Frobenius Rings

In an effort to make this article somewhat self-contained, both for ring-theorists and coding-theorists, I include some background material on finite Frobenius rings. The goal of this section is to show that finite Frobenius rings are characterized by having free character modules. Useful references for this material are Lam's books [19] and [20].

All rings will be associative with 1, and all modules will be unitary. While left modules will appear most often, there are comparable results for right modules. Almost all of the rings used in this article will be finite, so that some definitions that are more broadly applicable may be simplified in the finite context.

2.1. **Definitions.** Given a finite ring R, its (Jacobson) *radical* $\operatorname{rad}(R)$ is the intersection of all the maximal left ideals of R; $\operatorname{rad}(R)$ is itself a two-sided ideal of R. A left R-module is *simple* if it has no nonzero proper submodules. Given a left R-module M, its *socle* $\operatorname{soc}(M)$ is the sum of all the simple submodules of M. A ring R has a left socle $\operatorname{soc}({}_RR)$ and a right socle $\operatorname{soc}(R_R)$ (from viewing R as a left R-module or as a right R-module); both socles are two-sided ideals, but they may not be equal. (They are equal if R is semiprime, which, for finite rings, is equivalent to being semisimple.)

Let R be a finite ring. Then the quotient ring $R/\operatorname{rad}(R)$ is semi-simple and is isomorphic to a direct sum of matrix rings over finite fields (Wedderburn-Artin):

$$R/\operatorname{rad}(R) \cong \bigoplus_{i=1}^{k} M_{m_i}(\mathbb{F}_{q_i}), \tag{2.1}$$

where each q_i is a prime power; $\mathbb{F}_q$ denotes a finite field of order q, q a prime power, and $M_m(\mathbb{F}_q)$ denotes the ring of $m \times m$ matrices over $\mathbb{F}_q$.

Definition 1 ([19, Theorem 16.14])**.** A finite ring R is *Frobenius* if ${}_R(R/\operatorname{rad}(R)) \cong \operatorname{soc}({}_RR)$ and $(R/\operatorname{rad}(R))_R \cong \operatorname{soc}(R_R)$.

This definition applies more generally to Artinian rings. It is a theorem of Honold [15, Theorem 2] that, for finite rings, only one of the isomorphisms (left or right) is needed.

Each of the matrix rings $M_{m_i}(\mathbb{F}_{q_i})$ in (2.1) has a simple left module $T_i := M_{m_i \times 1}(\mathbb{F}_{q_i})$, consisting of all $m_i \times 1$ matrices over $\mathbb{F}_{q_i}$, under left matrix multiplication. From (2.1) it follows that, as left R-modules, we have an isomorphism

$${}_R(R/\operatorname{rad}(R)) \cong \bigoplus_{i=1}^{k} m_i T_i. \tag{2.2}$$

It is known that the T_i, $i = 1, \ldots, k$, form a complete list of simple left R-modules, up to isomorphism.

Because the left socle of an R-module is a sum of simple left R-modules, it can be expressed as a sum of the T_i. In particular, the left socle of R itself admits such an expression:

$$\operatorname{soc}({}_R R) \cong \bigoplus_{i=1}^{k} s_i T_i, \tag{2.3}$$

for some nonnegative integers $s_1, \ldots, s_k$. Thus a finite ring is Frobenius if and only if $m_i = s_i$ for all $i = 1, \ldots, k$.

2.2. **Characters.** Let G be a finite abelian group. In this article, a *character* is a group homomorphism $\varpi : G \to \mathbb{Q}/\mathbb{Z}$. The set of all characters of G forms a group called the *character group* $\widehat{G} := \operatorname{Hom}_{\mathbb{Z}}(G, \mathbb{Q}/\mathbb{Z})$. It is well-known that $|\widehat{G}| = |G|$. (Characters with values in the multiplicative group of nonzero complex numbers can be obtained by composing with the complex exponential function $a \mapsto \exp(2\pi i a)$, $a \in \mathbb{Q}/\mathbb{Z}$; this multiplicative form of characters will be needed in later sections.)

If R is a finite ring and A is a finite left R-module, then $\widehat{A}$ consists of the characters of the additive group of A; $\widehat{A}$ is naturally a right R-module via the scalar multiplication $(\varpi r)(a) := \varpi(ra)$, for $\varpi \in \widehat{A}$, $r \in R$, and $a \in A$. The module $\widehat{A}$ will be called the *character module* of A. Similarly, if B is a right R-module, then $\widehat{B}$ is naturally a left R-module.

Example 2. Let $\mathbb{F}_p$ be a finite field of prime order. Define $\vartheta_p : \mathbb{F}_p \to \mathbb{Q}/\mathbb{Z}$ by $\vartheta_p(a) = a/p$, where we view $\mathbb{F}_p$ as $\mathbb{Z}/p\mathbb{Z}$. Then ϑ_p is a character of $\mathbb{F}_p$, and every other character ϖ of $\mathbb{F}_p$ has the form $\varpi = a\vartheta_p$, for some $a \in \mathbb{F}_p$ (because $\widehat{\mathbb{F}}_p$ is a one-dimensional vector space over $\mathbb{F}_p$).

Let $\mathbb{F}_q$ be a finite field with $q = p^\ell$ for some prime p. Let $\operatorname{tr}_{q/p} : \mathbb{F}_q \to \mathbb{F}_p$ be the trace. Define $\vartheta_q : \mathbb{F}_q \to \mathbb{Q}/\mathbb{Z}$ by $\vartheta_q = \vartheta_p \circ \operatorname{tr}_{q/p}$. Then ϑ_q is a character of $\mathbb{F}_q$, and every other character ϖ of $\mathbb{F}_q$ has the form $\varpi = a\vartheta_q$, for some $a \in \mathbb{F}_q$.

Example 3. Let $R = M_m(\mathbb{F}_q)$ be the ring of $m \times m$ matrices over a finite field $\mathbb{F}_q$, and let $A = M_{m\times k}(\mathbb{F}_q)$ be the left R-module consisting of all $m \times k$ matrices over $\mathbb{F}_q$. Then $\widehat{A} \cong M_{k\times m}(\mathbb{F}_q)$ as right R-modules. Indeed, given a matrix $Q \in M_{k\times m}(\mathbb{F}_q)$, define a character ϖ_Q of A by $\varpi_Q(P) = \vartheta_q(\operatorname{tr}(QP))$, for $P \in A$, where tr is the matrix trace and ϑ_q is the character of $\mathbb{F}_q$ defined in Example 2. The map $M_{k\times m}(\mathbb{F}_q) \to \widehat{A}$, $Q \mapsto \varpi_Q$ is the desired isomorphism.

Given a short exact sequence of finite left R-modules $0 \to A \to B \to C \to 0$, there is an induced short exact sequence of right R-modules

$$0 \to \widehat{C} \to \widehat{B} \to \widehat{A} \to 0. \tag{2.4}$$

In particular, if we define the *annihilator* $(\widehat{B} : A) := \{\varpi \in \widehat{B} : \varpi(A) = 0\}$, then

$$(\widehat{B} : A) \cong \widehat{C} \quad \text{and} \quad |(\widehat{B} : A)| = |C| = |B|/|A|. \tag{2.5}$$

2.3. **Generating Characters.** In the special case that $A = R$, R is both a left and a right R-module. A character $\varpi \in \widehat{R}$ induces both a left and a right homomorphism $R \to \widehat{R}$ ($r \mapsto r\varpi$ is a left homomorphism, while $r \mapsto \varpi r$ is a right homomorphism). The character ϖ is called a left (resp., right) *generating character* if $r \mapsto r\varpi$ (resp., $r \mapsto \varpi r$) is a module isomorphism. In this situation, the character ϖ generates the left (resp., right)

R-module $\widehat{R}$. Because $|\widehat{R}| = |R|$, one of these homomorphisms is an isomorphism if and only if it is injective if and only if it is surjective.

Remark 4. The phrase *generating character* ("erzeugenden Charakter") is due to Klemm [17]. Claasen and Goldbach [6] used the adjective *admissible* to describe the same phenomenon, although their use of left and right is the reverse of ours.

The theorem below relates generating characters and finite Frobenius rings. While the theorem is over ten years old, we will give a new proof.

Theorem 5 ([14, Theorem 1], [31, Theorem 3.10])**.** *Let R be a finite ring. Then the following are equivalent:*

(1) *R is Frobenius;*
(2) *R admits a left generating character, i.e., $\widehat{R}$ is a free left R-module;*
(3) *R admits a right generating character, i.e., $\widehat{R}$ is a free right R-module.*

Moreover, when these conditions are satisfied, every left generating character is also a right generating character, and vice versa.

Example 6. Here are several examples of finite Frobenius rings and generating characters (when easy to describe).

(1) Finite field $\mathbb{F}_q$ with generating character ϑ_q of Example 2. Note that ϑ_p is injective, but that for $q > p$, $\ker \vartheta_q = \ker \mathrm{tr}_{q/p}$ is a nonzero $\mathbb{F}_p$-linear subspace of $\mathbb{F}_q$. However, $\ker \vartheta_q$ is not an $\mathbb{F}_q$-linear subspace. (Compare with Proposition 7 below.)
(2) Integer residue ring $\mathbb{Z}/n\mathbb{Z}$ with generating character ϑ_n defined by $\vartheta_n(a) = a/n$, for $a \in \mathbb{Z}/n\mathbb{Z}$.
(3) Finite chain ring R; i.e., a finite ring all of whose left ideals form a chain under inclusion. See Corollary 15 for information about a generating character.
(4) If $R_1, \ldots, R_n$ are Frobenius with generating characters $\varrho_1, \ldots, \varrho_n$, then their direct sum $R = \oplus R_i$ is Frobenius with generating character $\varrho = \sum \varrho_i$. Conversely, if $R = \oplus R_i$ is Frobenius with generating character ϱ, then each R_i is Frobenius, with generating character $\varrho_i = \varrho \circ \iota_i$, where $\iota_i : R_i \to R$ is the inclusion; $\varrho = \sum \varrho_i$.
(5) If R is Frobenius with generating character ϱ, then the matrix ring $M_m(R)$ is Frobenius with generating character $\varrho \circ \mathrm{tr}$, where tr is the matrix trace.
(6) If R is Frobenius with generating character ϱ and G is any finite group, then the group ring $R[G]$ is Frobenius with generating character $\varrho \circ \mathrm{pr}_e$, where $\mathrm{pr}_e : R[G] \to R$ is the projection that associates to every element $a = \sum a_g g \in R[G]$ the coefficient a_e of the identity element of G.

In preparation for the proof of Theorem 5, we prove several propositions concerning generating characters.

Proposition 7 ([6, Corollary 3.6])**.** *Let R be a finite ring. A character ϖ of R is a left* (*resp., right*) *generating character if and only if* $\ker \varpi$ *contains no nonzero left* (*resp., right*) *ideal of R.*

Proof. By the definition and $|\widehat{R}| = |R|$, ϖ is a left generating character if and only if the homomorphism $f : R \to \widehat{R}$, $r \mapsto r\varpi$, is injective. Then $r \in \ker f$ if and only if the principal ideal $Rr \subset \ker \varpi$. Thus, $\ker f = 0$ if and only if $\ker \varpi$ contains no nonzero left ideals. The proof for right generating characters is similar. □

Proposition 8 ([31, Theorem 4.3])**.** *A character ϱ of a finite ring R is a left generating character if and only if it is a right generating character.*

Proof. Suppose ϱ is a left generating character, and suppose that $I \subset \ker \varrho$ is a right ideal. Then for every $r \in R$, $Ir \subset \ker \varrho$, so that $I \subset \ker(r\varrho)$, for all $r \in R$. But every character of R is of the form $r\varrho$, because ϱ is a left generating character. Thus the annihilator $(\widehat{R} : I) = \widehat{R}$, and it follows from (2.5) that $I = 0$. By Proposition 7, ϱ is a right generating character. □

Proposition 9 ([33, Proposition 3.3])**.** *Let A be a finite left R-module. Then* $\operatorname{soc}(\widehat{A}) \cong (A/\operatorname{rad}(R)A)\widehat{\ }$.

Proof. There is a short exact sequence of left R-modules

$$0 \to \operatorname{rad}(R)A \to A \to A/\operatorname{rad}(R)A \to 0.$$

Taking character modules, as in (2.4), yields

$$0 \to (A/\operatorname{rad}(R)A)\widehat{\ } \to \widehat{A} \to (\operatorname{rad}(R)A)\widehat{\ } \to 0.$$

Because $A/\operatorname{rad}(R)A$ is a sum of simple modules, the same is true for $(A/\operatorname{rad}(R)A)\widehat{\ } \cong (\widehat{A} : \operatorname{rad}(R)A)$. Thus $(\widehat{A} : \operatorname{rad}(R)A) \subset \operatorname{soc}(\widehat{A})$.

Conversely, $\operatorname{soc}(\widehat{A}) \operatorname{rad}(R) = 0$, because the radical annihilates simple modules [7, Exercise 25.4]. Thus $\operatorname{soc}(\widehat{A}) \subset (\widehat{A} : \operatorname{rad}(R)A)$, and we have the equality $\operatorname{soc}(\widehat{A}) = (\widehat{A} : \operatorname{rad}(R)A)$. Now remember that $(\widehat{A} : \operatorname{rad}(R)A) \cong (A/\operatorname{rad}(R)A)\widehat{\ }$. □

Using Proposition 7 as a model, we extend the definition of a generating character to modules. Let A be a finite left (resp., right) R-module. A character ϖ of A is a *generating character* of A if $\ker \varpi$ contains no nonzero left (resp., right) R-submodules of A.

Lemma 10 (†)**.** *Let A be a finite left R-module, and let $B \subset A$ be a submodule. If A admits a left generating character, then B admits a left generating character.*

Proof. Simply restrict a generating character of A to B. Any submodule of B inside the kernel of the restriction will also be a submodule of A inside the kernel of the original generating character. □

Lemma 11 (†)**.** *Let R be any finite ring. Define $\varrho : \widehat{R} \to \mathbb{Q}/\mathbb{Z}$ by $\varrho(\varpi) = \varpi(1)$, evaluation at $1 \in R$, for $\varpi \in \widehat{R}$. Then ϱ is a left and right generating character of $\widehat{R}$.*

Proof. Suppose $\varpi_0 \neq 0$ has the property that $R\varpi_0 \subset \ker \varrho$. This means that for every $r \in R$, $0 = \varrho(r\varpi_0) = (r\varpi_0)(1) = \varpi_0(r)$, so that $\varpi_0 = 0$. Thus ϱ is a left generating character by definition. Similarly for ϱ being a right generating character. □

Proposition 12 (†)**.** *Let A be a finite left R-module. Then A admits a left generating character if and only if A can be embedded in $\widehat{R}$.*

Proof. If A embeds in $\widehat{R}$, then A admits a generating character, by Lemmas 10 and 11.

Conversely, let ϱ be a generating character of A. We use ϱ to define $f : A \to \widehat{R}$, as follows. For $a \in A$, define $f(a) \in \widehat{R}$ by $f(a)(r) = \varrho(ra)$, $r \in R$. It is easy to check that $f(a)$ is indeed in $\widehat{R}$, i.e., that $f(a)$ is a character of R. It is also easy to verify that f is a left R-module homomorphism from A to $\widehat{R}$. If $a \in \ker f$, then $\varrho(ra) = 0$ for all $r \in R$. Thus the left R-submodule $Ra \subset \ker \varrho$. Because ϱ is a generating character, we conclude that $Ra = 0$. Thus $a = 0$, and f is injective. □

When $A = R$, Proposition 12 is consistent with the definition of a generating character of a ring. Indeed, if R embeds into $\widehat{R}$, then R and $\widehat{R}$ are isomorphic as one-sided modules, because they have the same number of elements.

Theorem 13 (†)**.** *Let $R = M_m(\mathbb{F}_q)$ be the ring of $m \times m$ matrices over a finite field $\mathbb{F}_q$. Let $A = M_{m\times k}(\mathbb{F}_q)$ be the left R-module of all $m \times k$ matrices over $\mathbb{F}_q$. Then A admits a left generating character if and only if $m \geq k$.*

Proof. If $m \geq k$, then, by appending $m - k$ columns of zeros, A can be embedded inside R as a left ideal. By Example 3 and Lemma 10, A admits a generating character.

Conversely, suppose $m < k$. We will show that no character of A is a generating character of A. To that end, let ϖ be any character of A. By Example 3, ϖ has the form ϖ_Q for some $k \times m$ matrix Q over $\mathbb{F}_q$. Because $k > m$, the rows of Q are linearly dependent over $\mathbb{F}_q$. Let P be any nonzero matrix over $\mathbb{F}_q$ of size $m \times k$ such that $PQ = 0$. Such a P exists because the rows of Q are linearly dependent: use the coefficients of a nonzero dependency relation as the entries for a row of P. We claim that the nonzero left submodule of A generated by P is contained in $\ker \varpi_Q$. Indeed, for any $B \in R$, $\varpi_Q(BP) = \vartheta_q(\mathrm{tr}(Q(BP))) = \vartheta_q(\mathrm{tr}((BP)Q)) = \vartheta_q(\mathrm{tr}(B(PQ))) = 0$, using $PQ = 0$ and the well-known property $\mathrm{tr}(BC) = \mathrm{tr}(CB)$ of the matrix trace. Thus, no character of A is a generating character. □

Proposition 14 (†)**.** *Suppose A is a finite left R-module. Then A admits a left generating character if and only if* $\mathrm{soc}(A)$ *admits a left generating character.*

Proof. If A admits a generating character, then so does $\mathrm{soc}(A)$, by Lemma 10.

Conversely, suppose $\mathrm{soc}(A)$ admits a generating character ϑ. Utilizing the short exact sequence (2.4), let ϱ be any extension of ϑ to a character of A . We claim that ϱ is a generating character of A. To that end, suppose B is a submodule of A such that $B \subset \ker \varrho$. Then $\mathrm{soc}(B) \subset \mathrm{soc}(A) \cap \ker \varrho = \mathrm{soc}(A) \cap \ker \vartheta$, because ϱ is an extension of ϑ. But ϑ is a generating character of $\mathrm{soc}(A)$, so $\mathrm{soc}(B) = 0$. Since B is a finite module, we conclude that $B = 0$. Thus ϱ is a generating character of A. □

Corollary 15 (†)**.** *Let A be a finite left R-module. Suppose* $\mathrm{soc}(A)$ *admits a left generating character ϑ. Then any extension of ϑ to a character of A is a left generating character of A.*

We now (finally) turn to the proof of Theorem 5.

(†) *Proof of Theorem 5.* Statements (2) and (3) are equivalent by Proposition 8. We next show that (3) implies (1).

By Example 3, the right R-module $(R/\operatorname{rad}(R))_R$ equals the character module of the left R-module ${}_R(R/\operatorname{rad}(R))$. By Proposition 9 applied to the left R-module $A = {}_RR$, we have $({}_R(R/\operatorname{rad}(R)))\widehat{\ } \cong \operatorname{soc}(\widehat{R}_R) \cong \operatorname{soc}(R_R)$, because $\widehat{R}$ is assumed to be right free. We thus have an isomorphism $(R/\operatorname{rad}(R))_R \cong \operatorname{soc}(R_R)$ of right R-modules. One can either repeat the argument for a left isomorphism (using (2)) or appeal to the theorem of Honold [15, Theorem 2] mentioned after Definition 1.

Now assume (1). Referring to (2.1), we see that R being Frobenius implies that $\operatorname{soc}(R)$ is a sum of matrix modules of the form $M_{m_i}(\mathbb{F}_{q_i})$. By Theorem 13 and summing, $\operatorname{soc}(R)$ admits a left generating character. By Propositions 7 and 14, R itself admits a left generating character. Thus (2) holds. $\square$

2.4. **Frobenius Algebras.** In this subsection I want to point out the similarity between a general (not necessarily finite) Frobenius algebra and a finite Frobenius ring. I thank Professor Yun Fan for suggesting this short exposition.

Definition 16. A finite-dimensional algebra A over a field F is a *Frobenius algebra* if there exists a linear functional $\lambda : A \to F$ such that $\ker \lambda$ contains no nonzero left ideals of A.

It is apparent that the structure functional λ plays a role for a Frobenius algebra comparable to that played by a left generating character ϱ of a finite Frobenius ring. As one might expect, the connection between λ and ϱ is even stronger when one considers a finite Frobenius algebra. Recall that every finite field $\mathbb{F}_q$ admits a generating character ϑ_q, by Example 2.

Theorem 17 (†). *Let R be a Frobenius algebra over a finite field $\mathbb{F}_q$, with structure functional $\lambda : R \to \mathbb{F}_q$. Then R is a finite Frobenius ring with left generating character $\varrho = \vartheta_q \circ \lambda$.*

Conversely, suppose R is a finite-dimensional algebra over a finite field $\mathbb{F}_q$ and that R is a Frobenius ring with generating character ϱ. Then R is a Frobenius algebra, and there exists a structure functional $\lambda : R \to \mathbb{F}_q$ such that $\varrho = \vartheta_q \circ \lambda$.

Proof. Both $R^* := \operatorname{Hom}_{\mathbb{F}_q}(R, \mathbb{F}_q)$ and $\widehat{R} = \operatorname{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ are (R,R)-bimodules satisfying $|R^*| = |\widehat{R}| = |R|$. A generating character ϑ_q of $\mathbb{F}_q$ induces a bimodule homomorphism $f : R^* \to \widehat{R}$ via $\lambda \mapsto \vartheta_q \circ \lambda$. We claim that f is injective. To that end, suppose $\lambda \in \ker f$. Then $\vartheta_q \circ \lambda = 0$, so that $\lambda(R) \subset \ker \vartheta_q$. Note that $\lambda(R)$ is an $\mathbb{F}_q$-vector subspace contained in $\ker \vartheta_q \subset \mathbb{F}_q$. Because ϑ_q is a generating character of $\mathbb{F}_q$, $\lambda(R) = 0$, by Proposition 7. Thus $\lambda = 0$, and f is injective. Because $|R^*| = |\widehat{R}|$, f is in fact a bimodule isomorphism.

We next claim that the structure functionals in R^* correspond under f to the generating characters in $\widehat{R}$. That is, if $\varpi = f(\lambda)$, where $\lambda \in R^*$ and $\varpi \in \widehat{R}$, then λ satisfies the condition that $\ker \lambda$ contains no nonzero left ideals of R if and only if ϖ is a generating character of R (i.e., $\ker \varpi$ contains no nonzero left ideals of R).

Suppose ϖ is a generating character of R, and suppose that I is a left ideal of R with $I \subset \ker \lambda$. Since $\varpi = \vartheta_q \circ \lambda$, we also have $I \subset \ker \varpi$. Because ϖ is a generating character, Proposition 7 implies $I = 0$, as desired.

Conversely, suppose λ satisfies the condition that $\ker \lambda$ contains no nonzero left ideals of R, and suppose that I is a left ideal of R with $I \subset \ker \varpi$. Then $\lambda(I)$ is an $\mathbb{F}_q$-linear subspace inside $\ker \vartheta_q \subset \mathbb{F}_q$. Because ϑ_q is a generating character of $\mathbb{F}_q$, we have $\lambda(I) = 0$, i.e., $I \subset \ker \lambda$. By the condition on λ, we conclude that $I = 0$, as desired. $\square$

Remark 18. The proof of Theorem 17 shows the equivalence of the Morita duality functors $*$ and $\widehat{\ }$ when R is a finite-dimensional algebra over a finite field $\mathbb{F}$ (cf., [31, Remark 3.12]). For a finite R-module M, observe that $M^* := \mathrm{Hom}_{\mathbb{F}}(M, \mathbb{F}) \cong \mathrm{Hom}_R(M, R^*)$ and $\widehat{M} = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_R(M, \widehat{R})$.

3. The Language of Algebraic Coding Theory

3.1. **Background on Error-Correcting Codes.** Error-correcting codes provide a way to protect messages from corruption during transmission (or storage). This is accomplished by adding redundancies in such a way that, with high probability, the original message can be recovered from the received message.

Let us be a little more precise. Let I be a finite set (of "information") which will be the possible messages that can be transmitted. An example: numbers from 0 to 63 representing gray scales of a photograph. Let A be another finite set (the "alphabet"); $A = \{0, 1\}$ is a typical example. An *encoding* of the information set I is an injection $f : I \to A^n$ for some n. The image $f(I)$ is a *code* in A^n.

For a given message $x \in I$, the string $f(x)$ is transmitted across a channel (which could be copper wire, fiber optic cable, saving to a storage device, or transmission by radio or cell phone). During the transmission process, some of the entries in the string $f(x)$ might be corrupted, so that the string $y \in A^n$ that is received may be different from the string $f(x)$ that was originally sent.

The challenge is this: for a given channel, to choose an encoding f in such a way that it is possible, with high probability, to recover the original message x knowing only the corrupted received message y (and the method of encoding). The process of recovering x is called *decoding*.

The seminal theorem that launched the field of coding theory is due to Claude Shannon [27]. Paraphrasing, it says: up to a limit determined by the channel, it is always possible to find an encoding which will decode with as high a probability as one desires, provided one takes the encoding length n sufficiently large. Shannon's proof is not constructive; it does not build an encoding, nor does it describe how to decode. Much of the research in coding theory since Shannon's theorem has been devoted to finding good codes and developing decoding algorithms for them. Good references for background on coding theory are [16] and [23].

3.2. **Algebraic Coding Theory.** Researchers have more tools at their disposal in constructing codes if they assume that the alphabet A and the codes $C \subset A^n$ are equipped with algebraic structures. The first important case is to assume that A is a finite field and that $C \subset A^n$ is a linear subspace.

Definition 19. Let $\mathbb{F}$ be a finite field. A *linear code* of length n over $\mathbb{F}$ is a linear subspace $C \subset \mathbb{F}^n$. The dimension of the linear code is traditionally denoted by $k = \dim_{\mathbb{F}} C$.

Given two vectors $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{F}^n$, their *Hamming distance* $d(x, y) = |\{i : x_i \neq y_i\}|$ is the number of positions where the vectors differ. The *Hamming* *weight* $\mathrm{wt}(x) = d(x, 0)$ of a vector $x \in \mathbb{F}^n$ equals the number of positions where the vector is nonzero. Note that $d(x, y) = \mathrm{wt}(x - y)$; d is symmetric and satisfies the triangle inequality. The *minimum distance* of a code $C \subset \mathbb{F}^n$ is the smallest value d_C of $d(x, y)$ for $x \neq y$, $x, y \in C$. When C is a linear code, d_C equals the smallest value of $\mathrm{wt}(x)$ for $x \neq 0$, $x \in C$.

The minimum distance of a code C is a measure of the code's error-correcting capability. Let $B(x, r) = \{y \in \mathbb{F}^n : d(x, y) \leq r\}$ be the ball in $\mathbb{F}^n$ centered at x of radius r. Set $r_0 = [(d_C - 1)/2]$, the greatest integer less than or equal to $(d_C - 1)/2$. Then all the balls $B(x, r_0)$ for $x \in C$ are disjoint. Suppose $x \in C$ is transmitted and $y \in \mathbb{F}^n$ is received. Decode y to the nearest element in the code C (and flip a coin if there is a tie). If at most r_0 entries of x are corrupted in the transmission, then this method always decodes correctly. We say that C corrects r_0 errors. The larger d_C is, the more errors that can be corrected.

3.3. **Weight Enumerators.** It is useful to keep track of the weights of all the elements of a code C. The *Hamming weight enumerator* $W_C(X, Y)$ is a polynomial (generating function) defined by

$$W_C(X, Y) = \sum_{x \in C} X^{n - \mathrm{wt}(x)} Y^{\mathrm{wt}(x)} = \sum_{i=0}^{n} A_i X^{n-i} Y^i,$$

where A_i is the number of elements of weight i in C. Only the zero vector has weight 0. In a linear code, $A_0 = 1$, and $A_i = 0$ for $0 < i < d_C$.

Define an $\mathbb{F}$-valued inner product on $\mathbb{F}^n$ by

$$x \cdot y = \sum_{i=1}^{n} x_i y_i, \quad x = (x_1, \ldots, x_n), \quad y = (y_1, \ldots, y_n) \in \mathbb{F}^n.$$

Associated to every linear code $C \subset \mathbb{F}^n$ is its *dual code* $C^\perp$:

$$C^\perp = \{y \in \mathbb{F}^n : x \cdot y = 0, x \in C\}.$$

If $k = \dim C$, then $\dim C^\perp = n - k$.

One of the most famous results in algebraic coding theory relates the Hamming weight enumerator of a linear code C to that of its dual code $C^\perp$: the MacWilliams identities, which is the subject of Section 4.

Theorem 20 (MacWilliams Identities)**.** *Let C be a linear code in $\mathbb{F}_q^n$. Then*

$$W_C(X, Y) = \frac{1}{|C^\perp|} W_{C^\perp}(X + (q - 1)Y, X - Y).$$

Of special interest are self-dual codes. A linear code C is *self-orthogonal* if $C \subset C^\perp$; C is *self-dual* if $C = C^\perp$. Note that a self-dual code C of length n and dimension k satisfies $n = 2k$, so that n must be even.

3.4. **Linear Codes over Rings.** While there had been some early work on linear codes defined over the rings $\mathbb{Z}/k\mathbb{Z}$, a major breakthrough came in 1994 with the paper [13]. (There was similar, independent work in [25].) It had been noticed that there were two families of nonlinear binary codes that behaved as if they were duals; their weight enumerators satisfied the MacWilliams identities. This phenomenon was explained in [13]. The authors discovered two families of linear codes over $\mathbb{Z}/4\mathbb{Z}$ that are duals of each other and, therefore, their weight enumerators satisfy the MacWilliams identities. In addition, by using a so-called *Gray map* $g : \mathbb{Z}/4\mathbb{Z} \to \mathbb{F}_2^2$ defined by $g(0) = 00$, $g(1) = 01$, $g(2) = 11$, and $g(3) = 10$ (g is not a homomorphism), the authors showed that the two families of linear codes over $\mathbb{Z}/4\mathbb{Z}$ are mapped to the original families of nonlinear codes over $\mathbb{F}_2$. The paper [13] launched an interest in linear codes defined over rings that continues to this day.

Definition 21. Let R be a finite ring. A left (right) *linear code* C of length n over R is a left (right) R-submodule $C \subset R^n$.

It will be useful in Section 5 to be even more general and to define linear codes over modules. These ideas were introduced first by Nechaev and his colloborators [18].

Definition 22. Let R be a finite ring, and let A (for *alphabet*) be a finite left R-module. A left *linear code* C over A of length n is a left R-submodule $C \subset A^n$.

The Hamming weight is defined in the same way as for fields. For $x = (x_1, \ldots, x_n) \in R^n$ (or A^n), define $\mathrm{wt}(x) = |\{i : x_i \neq 0\}|$, the number of nonzero entries in the vector x.

4. The MacWilliams Identities

In this section, we present a proof of the MacWilliams identities that is valid over any finite Frobenius ring. The proof, which dates to [31, Theorem 8.3], is essentially the same as one due to Gleason found in [3, §1.12]. While the MacWilliams identities hold in even more general settings (see the later sections in [33], for example), the setting of linear codes over a finite Frobenius ring will show clearly the role of characters in the proof.

Let R be a finite ring. As we did earlier for fields, we define a *dot product* on R^n by

$$x \cdot y = \sum_{i=1}^{n} x_i y_i, \quad x = (x_1, \ldots, x_n), \quad y = (y_1, \ldots, y_n) \in R^n.$$

For a left linear code $C \subset R^n$, define the *right annihilator* $r(C)$ by $r(C) = \{y \in R^n : x \cdot y = 0, x \in C\}$. The right annihilator will play the role of the dual code $C^\perp$. (Because R may be non-commutative, one must choose between a left and a right annihilator.) The Hamming weight enumerator $W_C(X, Y)$ of a left linear code C is defined exactly as for fields.

Theorem 23 (MacWilliams Identities)**.** *Let R be a finite Frobenius ring, and let $C \subset R^n$ be a left linear code. Then*

$$W_C(X, Y) = \frac{1}{|r(C)|} W_{r(C)}(X + (|R| - 1)Y, X - Y).$$

4.1. **Fourier Transform.** Gleason's proof of the MacWilliams identities uses the Fourier transform and the Poisson summation formula, which we describe in this subsection. Let $(G, +)$ be a finite abelian group.

Throughout this section, we will use the multiplicative form of characters; that is, characters are group homomorphisms $\pi : (G, +) \to (\mathbb{C}^\times, \cdot)$ from a finite abelian group to the multiplicative group of nonzero complex numbers. The set $\widehat{G}$ of all characters of G forms an abelian group under pointwise multiplication. The following list of properties of characters is well-known and presented without proof (see [26] or [28]).

Lemma 24. *Characters of a finite abelian group G satisfy the following properties.*

(1) $|\widehat{G}| = |G|$;
(2) $(G_1 \times G_2)\widehat{\ } \cong \widehat{G}_1 \times \widehat{G}_2$;
(3) $\sum_{x\in G} \pi(x) = \begin{cases} |G|, & \pi = 1, \\ 0, & \pi \neq 1; \end{cases}$
(4) $\sum_{\pi\in\widehat{G}} \pi(x) = \begin{cases} |G|, & x = 0, \\ 0, & x \neq 0; \end{cases}$
(5) *The characters form a linearly independent subset of the vector space of complex-valued functions on G. (In fact, the characters form a basis.)* □

Let V be a vector space over the complex numbers. For any function $f : G \to V$, define its *Fourier transform* $\hat{f} : \widehat{G} \to V$ by

$$\hat{f}(\pi) = \sum_{x\in G} \pi(x) f(x), \quad \pi \in \widehat{G}.$$

Given a subgroup $H \subset G$, define the *annihilator* $(\widehat{G} : H) = \{\pi \in \widehat{G} : \pi(H) = 1\}$. As we saw in (2.5), $|(\widehat{G} : H)| = |G|/|H|$.

The Poisson summation formula relates the sum of a function over a subgroup to the sum of its Fourier transform over the annihilator of the subgroup. The proof is an exercise.

Proposition 25 (Poisson Summation Formula). *Let $H \subset G$ be a subgroup, and let $f : G \to V$ be any function from G to a complex vector space V. Then*

$$\sum_{x\in H} f(x) = \frac{1}{|(\widehat{G} : H)|} \sum_{\pi\in(\widehat{G}:H)} \hat{f}(\pi).$$

The next technical result describes the Fourier transform of a function that is the product of functions of one variable. Again, the proof is an exercise for the reader.

Lemma 26. *Suppose V is a commutative algebra over the complex numbers, and suppose $f_i : G \to V$, $i = 1, \ldots, n$, are functions from G to V. Let $f : G^n \to V$ be defined by $f(x_1, \ldots, x_n) = \prod_{i=1}^n f_i(x_i)$. Then*

$$\hat{f}(\pi_1, \ldots, \pi_n) = \prod_{i=1}^{n} \hat{f}_i(\pi_i).$$

4.2. **Gleason's Proof.**

Proof of Theorem 23. Given a left linear code $C \subset R^n$, we apply the Poisson summation formula with $G = R^n$, $H = C$, and $V = \mathbb{C}[X, Y]$, the polynomial ring over $\mathbb{C}$ in two indeterminates. Define $f_i : R \to \mathbb{C}[X, Y]$ by $f_i(x_i) = X^{1-\mathrm{wt}(x_i)} Y^{\mathrm{wt}(x_i)}$, $x_i \in R$, where $\mathrm{wt}(r) = 0$ for $r = 0$, and $\mathrm{wt}(r) = 1$ for $r \neq 0$ in R. Let $f : R^n \to \mathbb{C}[X, Y]$ be the product of the f_i; i.e.,

$$f(x_1, \ldots, x_n) = \prod_{i=1}^{n} X^{1-\mathrm{wt}(x_i)} Y^{\mathrm{wt}(x_i)} = X^{n-\mathrm{wt}(x)} Y^{\mathrm{wt}(x)},$$

where $x = (x_1, \ldots, x_n) \in R^n$. We recognize that $\sum_{x \in H} f(x)$, the left side of the Poisson summation formula, is simply the Hamming weight enumerator $W_C(X, Y)$.

To begin to simplify the right side of the Poisson summation formula, we must calculate $\hat{f}$. By Lemma 26, we first calculate $\hat{f}_i$.

$$\begin{aligned}
\hat{f}_i(\pi_i) &= \sum_{a \in R} \pi_i(a) f_i(a) = \sum_{a \in R} \pi_i(a) X^{1-\mathrm{wt}(a)} Y^{\mathrm{wt}(a)} = X + \sum_{a \neq 0} \pi_i(a) Y \\
&= \begin{cases} X + (|R| - 1)Y, & \pi_i = 1, \\ X - Y, & \pi_i \neq 1. \end{cases}
\end{aligned}$$

At the end of the first line, one evaluates the case $a = 0$ versus the cases where $a \neq 0$. In going to the second line, one uses Lemma 24. Using Lemma 26, we see that

$$\hat{f}(\pi) = (X + (|R| - 1)Y)^{n-\mathrm{wt}(\pi)} (X - Y)^{\mathrm{wt}(\pi)},$$

where $\pi = (\pi_1, \ldots, \pi_n) \in \widehat{R}^n$ and $\mathrm{wt}(\pi)$ counts the number of π_i such that $\pi_i \neq 1$.

The last task is to identify the character-theoretic annihilator $(\widehat{G} : H) = (\widehat{R}^n : C)$ with $r(C)$, which is where R being Frobenius enters the picture. Let ρ be a generating character of R. We use ρ to define a homomorphism $\beta : R \to \widehat{R}$. For $r \in R$, the character $\beta(r) \in \widehat{R}$ has the form $\beta(r)(s) = (r\rho)(s) = \rho(sr)$ for $s \in R$. One can verify that $\beta : R \to \widehat{R}$ is an isomorphism of left R-modules. In particular, $\mathrm{wt}(r) = \mathrm{wt}(\beta(r))$.

Extend β to an isomorphism $\beta : R^n \to \widehat{R}^n$ of left R-modules, via $\beta(x)(y) = \rho(y \cdot x)$, for $x, y \in R^n$. Again, $\mathrm{wt}(x) = \mathrm{wt}(\beta(x))$. For $x \in R^n$, when is $\beta(x) \in (\widehat{R}^n : C)$? This occurs when $\beta(x)(C) = 1$; that is, when $\rho(C \cdot x) = 1$. This means that the left ideal $C \cdot x$ of R is contained in $\ker \rho$. Because ρ is a generating character, Proposition 7 implies that $C \cdot x = 0$. Thus $x \in r(C)$. The converse is obvious. Thus $r(C)$ corresponds to $(\widehat{R}^n : C)$ under the isomorphism β.

The right side of the Poisson summation formula now simplifies as follows:

$$\begin{aligned}
\frac{1}{|(\widehat{G} : H)|} \sum_{\pi \in (\widehat{G}:H)} \hat{f}(\pi) &= \frac{1}{|r(C)|} \sum_{x \in r(C)} (X + (|R| - 1)Y)^{n-\mathrm{wt}(x)} (X - Y)^{\mathrm{wt}(x)} \\
&= \frac{1}{|r(C)|} W_{r(C)}(X + (|R| - 1)Y, X - Y),
\end{aligned}$$

as desired. ∎

5. The Extension Problem

In this section, we will discuss the extension problem, which originated from understanding equivalence of codes. The main result is that a finite ring has the extension property for linear codes with respect to the Hamming weight if and only if the ring is Frobenius.

5.1. **Equivalence of Codes.**

When should two linear codes be considered to be the same? That is, what should it mean for two linear codes to be equivalent? There are two (related) approaches to this question: via monomial transformations and via weight-preserving isomorphisms.

Definition 27. Let R be a finite ring. A (left) *monomial transformation* $T : R^n \to R^n$ is a left R-linear homomorphism of the form

$$T(x_1, \ldots, x_n) = (x_{\sigma(1)}u_1, \ldots, x_{\sigma(n)}u_n), \quad (x_1, \ldots, x_n) \in R^n,$$

for some permutation σ of $\{1, 2, \ldots, n\}$ and units $u_1, \ldots, u_n$ of R.

Two left linear codes $C_1, C_2 \subset R^n$ are *equivalent* if there exists a monomial transformation $T : R^n \to R^n$ such that $T(C_1) = C_2$.

Another possible definition of equivalence of linear codes $C_1, C_2 \subset R^n$ is this: there exists an R-linear isomorphism $f : C_1 \to C_2$ that preserves the Hamming weight, i.e., $\mathrm{wt}(f(x)) = \mathrm{wt}(x)$, for all $x \in C_1$. The next lemma shows that equivalence using monomial transformations implies equivalence using a Hamming weight-preserving isomorphism.

Lemma 28. *If $T : R^n \to R^n$ is a monomial transformation, then T preserves the Hamming weight: $\mathrm{wt}(T(x)) = \mathrm{wt}(x)$, for all $x \in R^n$. If linear codes $C_1, C_2 \subset R^n$ are equivalent via a monomial transformation T, then the restriction f of T to C_1 is an R-linear isomorphism $C_1 \to C_2$ that preserves the Hamming weight.*

Proof. For any $r \in R$ and any unit $u \in R$, $r = 0$ if and only if $ru = 0$. The result follows easily from this. $\square$

Does the converse hold? This is an extension problem: given $C_1, C_2 \subset R^n$ and an R-linear isomorphism $f : C_1 \to C_2$ that preserves the Hamming weight, does f extend to a monomial transformation $T : R^n \to R^n$? We will phrase this in terms of a property.

Definition 29. Let R be a finite ring. The ring R has the *extension property* (EP) with respect to the Hamming weight if, whenever two left linear codes $C_1, C_2 \subset R^n$ admit an R-linear isomorphism $f : C_1 \to C_2$ that preserves the Hamming weight, it follows that f extends to a monomial transformation $T : R^n \to R^n$.

Thus, the two notions of equivalence coincide precisely when the ring R satisfies the extension property. Another important theorem of MacWilliams is that finite fields have the extension property [21], [22].

Theorem 30 (MacWilliams)**.** *Finite fields have the extension property with respect to the Hamming weight.*

Other proofs that finite fields have the extension property with respect to the Hamming weight have been given by Bogart, Goldberg, and Gordon [5] and by Ward and Wood [29]. We will not prove the finite field case separately, because it is a special case of the main theorem of this section:

Theorem 31. *Let R be a finite ring. Then R has the extension property with respect to the Hamming weight if and only if R is Frobenius.*

One direction, that finite Frobenius rings have the extension property, first appeared in [31, Theorem 6.3]. The proof (which will be given in subsection 5.2) is based on the linear independence of characters and is modeled on the proof in [29] of the finite field case. A combinatorial proof appears in work of Greferath and Schmidt [12]. More generally yet, Greferath, Nechaev, and Wisbauer have shown that the character module of any finite ring has the extension property for the homogeneous and the Hamming weights [11]. Ideas from this latter paper greatly influenced the work presented in subsection 5.4.

The other direction, that only finite Frobenius rings have the extension property, first appeared in [32]. That paper carried out a strategy due to Dinh and López-Permouth [8]. Additional relevant material appeared in [33].

The rest of this section will be devoted to the proof of Theorem 31.

5.2. **Frobenius is Sufficient.** In this subsection we prove half of Theorem 31, that a finite Frobenius ring has the extension property, following the treatment in [31, Theorem 6.3].

Assume $C_1, C_2 \subset R^n$ are two left linear codes, and assume $f : C_1 \to C_2$ is an R-linear isomorphism that preserves the Hamming weight. We want to show that f extends to a monomial transformation of R^n. The core idea is to express the weight-preservation property of f as an equation of characters of C_1 and to use the linear independence of characters to match up terms.

Let $\mathrm{pr}_1, \ldots, \mathrm{pr}_n : R^n \to R$ be the coordinate projections, so that $\mathrm{pr}_i(x_1, \ldots, x_n) = x_i$, $(x_1, \ldots, x_n) \in R^n$. Let $\lambda_1, \ldots, \lambda_n$ denote the restrictions of $\mathrm{pr}_1, \ldots, \mathrm{pr}_n$ to $C_1 \subset R^n$. Similarly, let $\mu_1, \ldots, \mu_n : C_1 \to R$ be given by $\mu_i = \mathrm{pr}_i \circ f$. Then $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n \in \mathrm{Hom}_R(C_1, R)$ are left R-linear functionals on C_1. It will suffice to prove the existence of a permutation σ of $\{1, \ldots, n\}$ and units $u_1, \ldots, u_n$ of R such that $\mu_i = \lambda_{\sigma(i)} u_i$, for $i = 1, \ldots, n$.

For any $x \in C_1$, the Hamming weight of x is given by $\mathrm{wt}(x) = \sum_{i=1}^n \mathrm{wt}(\lambda_i(x))$, while the Hamming weight of $f(x)$ is given by $\mathrm{wt}(f(x)) = \sum_{i=1}^n \mathrm{wt}(\mu_i(x))$. Because f preserves the Hamming weight, we have

$$\sum_{i=1}^{n} \mathrm{wt}(\lambda_i(x)) = \sum_{i=1}^{n} \mathrm{wt}(\mu_i(x)). \tag{5.1}$$

Using Lemma 24, observe that $1 - \mathrm{wt}(r) = (1/|R|) \sum_{\pi \in \widehat{R}} \pi(r)$, for any $r \in R$. Apply this observation to (5.1) and simplify:

$$\sum_{i=1}^{n} \sum_{\pi \in \widehat{R}} \pi(\lambda_i(x)) = \sum_{i=1}^{n} \sum_{\pi \in \widehat{R}} \pi(\mu_i(x)), \quad x \in C_1. \tag{5.2}$$

Because R is assumed to be Frobenius, R admits a (left) generating character ρ. Every character $\pi \in \widehat{R}$ thus has the form $\pi = a\rho$, for some $a \in R$. Recall that the scalar multiplication means that $\pi(r) = (a\rho)(r) = \rho(ra)$, for $r \in R$. Use this to simplify (5.2) (and use different indices on each side of the resulting equation):

$$\sum_{i=1}^{n} \sum_{a \in R} \rho \circ (\lambda_i a) = \sum_{j=1}^{n} \sum_{b \in R} \rho \circ (\mu_j b). \tag{5.3}$$

This is an equation of characters of C_1. Because characters are linearly independent, we can match up terms from the left and right sides of (5.3). In order to get unit multiples, some care must be taken.

Because C_1 is a left R-module, $\operatorname{Hom}_R(C_1, R)$ is a right R-module. Define a preorder $\preceq$ on $\operatorname{Hom}_R(C_1, R)$ by $\lambda \preceq \mu$ if $\lambda = \mu r$ for some $r \in R$. By a result of Bass [4, Lemma 6.4], $\lambda \preceq \mu$ and $\mu \preceq \lambda$ imply $\mu = \lambda u$ for some unit u of R.

Among the linear functionals $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n$ (a finite list), choose one that is maximal in the preorder $\preceq$. Without loss of generality, assume μ_1 is maximal in $\preceq$. (This means: if $\mu_1 \preceq \lambda$ for some λ, then $\mu_1 = \lambda u$ for some unit u of R.) In (5.3), consider the term on the right side with $j = 1$ and $b = 1$. By linear independence of characters, there exists i_1, $1 \leq i_1 \leq n$, and $a \in R$ such that $\rho \circ (\lambda_{i_1} a) = \rho \circ \mu_1$. This equation implies that $\operatorname{im}(\mu_1 - \lambda_{i_1} a) \subset \ker \rho$. But $\operatorname{im}(\mu_1 - \lambda_{i_1} a)$ is a left ideal of R, and ρ is a generating character of R. By Proposition 7, $\operatorname{im}(\mu_1 - \lambda_{i_1} a) = 0$, so that $\mu_1 = \lambda_{i_1} a$. This means that $\mu_1 \preceq \lambda_{i_1}$. Because μ_1 was chosen to be maximal, we have $\mu_1 = \lambda_{i_1} u_1$, for some unit u_1 of R. Begin to define a permutation σ by $\sigma(1) = i_1$.

By a reindexing argument, all the terms on the left side of (5.3) with $i = i_1$ match the terms on the right side of (5.3) with $j = 1$. That is, $\sum_{a \in R} \rho \circ (\lambda_{i_1} a) = \sum_{b \in R} \rho \circ (\mu_1 b)$. Subtract these sums from (5.3), thereby reducing the size of the outer summations by one. Proceed by induction, building a permutation σ and finding units $u_1, \ldots, u_n$ of R, as desired.

5.3. **Reformulating the Problem.** The proof that being a finite Frobenius ring is sufficient for having the extension property with respect to the Hamming weight was based on the proof of the extension theorem over finite fields that used the linear independence of characters [29]. In contrast, the proof that Frobenius is necessary will make use of the approach for proving the extension theorem due to Bogart, et al. [5]. This requires a reformulation of the extension problem.

Every left linear code $C \subset R^n$ can be viewed as the image of the inclusion map $C \to R^n$. More generally, every left linear code is the image of an R-linear homomorphism $\Lambda : M \to R^n$, for some finite left R-module M. By composing with the coordinate projections pr_i, the homomorphism Λ can be expressed as an n-tuple $\Lambda = (\lambda_1, \ldots, \lambda_n)$, where each $\lambda_i \in \operatorname{Hom}_R(M, R)$. The λ_i will be called the *coordinate functionals* of the linear code.

Remark 32. It is typical in coding theory to present a linear code $C \subset R^n$ by means of a *generator matrix* G. The matrix G has entries from R, the number of columns of G equals the length n of the code C, and (most importantly) the rows of G generate C as a left submodule of R^n.

The description of a linear code via coordinate functionals is essentially equivalent to that using generator matrices. If one has coordinate functionals $\lambda_1, \dots, \lambda_n$, then one can produce a generator matrix G by choosing a set $v_1, \dots, v_k$ of generators for C as a left module over R and taking as the (i,j)-entry of G the value $\lambda_j(v_i)$. Conversely, given a generator matrix, its columns define coordinate functionals. Thus, using coordinate functionals is a "basis-free" approach to generator matrices. This idea goes back to [2].

We are interested in linear codes up to equivalence. For a linear code given by $\Lambda = (\lambda_1, \dots, \lambda_n) : M \to R^n$, the order of the coordinate functionals $\lambda_1, \dots, \lambda_n$ is irrelevant, as is replacing any λ_i with $\lambda_i u_i$, for some unit u_i of R. We want to encode this information systematically. Let $\mathcal{U}$ be the group of units of the ring R. The group $\mathcal{U}$ acts on the module $\mathrm{Hom}_R(M,R)$ by right scalar multiplication; let $\mathcal{O}^\sharp$ denote the set of orbits of this action: $\mathcal{O}^\sharp = \mathrm{Hom}_R(M,R)/\mathcal{U}$. Then a linear code $M \to R^n$, up to equivalence, is specified by choosing n elements of $\mathcal{O}^\sharp$ (counting with multiplicities). This choice can be encoded by specifying a function (a *multiplicity function*) $\eta : \mathcal{O}^\sharp \to \mathbb{N}$, the nonnegative integers, where $\eta(\lambda)$ is the number of times λ (or a unit multiple of λ) appears as a coordinate functional. The length n of the linear code is given by $\sum_{\lambda \in \mathcal{O}^\sharp} \eta(\lambda)$.

In summary, linear codes $M \to R^n$ (for fixed M, but any n), up to equivalence, are given by multiplicity functions $\eta : \mathcal{O}^\sharp \to \mathbb{N}$. Denote the set of all such functions by $F(\mathcal{O}^\sharp, \mathbb{N}) = \{\eta : \mathcal{O}^\sharp \to \mathbb{N}\}$, and define $F_0(\mathcal{O}^\sharp, \mathbb{N}) = \{\eta \in F(\mathcal{O}^\sharp, \mathbb{N}) : \eta(0) = 0\}$.

We are also interested in the Hamming weight of codewords and in how to describe the Hamming weight in terms of the multiplicity function η. Fix a multiplicity function $\eta : \mathcal{O}^\sharp \to \mathbb{N}$. Define $W_\eta : M \to \mathbb{N}$ by

$$W_\eta(x) = \sum_{\lambda \in \mathcal{O}^\sharp} \mathrm{wt}(\lambda(x))\, \eta(\lambda), \quad x \in M. \tag{5.4}$$

Then $W_\eta(x)$ equals the Hamming weight of the codeword given by $x \in M$. Notice that $W_\eta(0) = 0$.

Lemma 33. *For $x \in M$ and unit $u \in \mathcal{U}$, $W_\eta(ux) = W_\eta(x)$.*

Proof. This follows immediately from the fact that $\mathrm{wt}(ur) = \mathrm{wt}(r)$ for $r \in R$ and unit $u \in \mathcal{U}$; that is, $ur = 0$ if and only if $r = 0$. ☐

Because M is a left R-module, the group of units $\mathcal{U}$ acts on M on the left. Let $\mathcal{O}$ denote the set of orbits of this action. Observe that Lemma 33 implies that W_η is a well-defined function from $\mathcal{O}$ to $\mathbb{N}$. Let $F(\mathcal{O}, \mathbb{N})$ denote the set of all functions from $\mathcal{O}$ to $\mathbb{N}$, and define $F_0(\mathcal{O}, \mathbb{N}) = \{w \in F(\mathcal{O}, \mathbb{N}) : w(0) = 0\}$. Now define $W : F(\mathcal{O}^\sharp, \mathbb{N}) \to F_0(\mathcal{O}, \mathbb{N})$ by $\eta \in F(\mathcal{O}^\sharp, \mathbb{N}) \mapsto W_\eta \in F_0(\mathcal{O}, \mathbb{N})$. (Remember that $W_\eta(0) = 0$.) Thus W associates to every linear code, up to equivalence, a listing of the Hamming weights of all the codewords. The discussion to this point (plus a technical argument on the role of the zero functional, which is relegated to subsection 5.5) proves the following reformulation of the extension property.

Theorem 34. *A finite ring R has the extension property with respect to the Hamming weight if and only if the function*

$$W : F_0(\mathcal{O}^\sharp, \mathbb{N}) \to F_0(\mathcal{O}, \mathbb{N}), \quad \eta \mapsto W_\eta,$$

is injective for every finite left R-module M.

Observe that the function spaces $F_0(\mathcal{O}^\sharp, \mathbb{N}), F_0(\mathcal{O}, \mathbb{N})$ are additive monoids and that $W : F_0(\mathcal{O}^\sharp, \mathbb{N}) \to F_0(\mathcal{O}, \mathbb{N})$ is additive, i.e., a monoid homomorphism. If we tensor with the rational numbers $\mathbb{Q}$ (which means we formally allow coordinate functionals to have multiplicities equal to any rational number), it is straight-forward to generalize Theorem 34 to:

Theorem 35. *A finite ring R has the extension property with respect to the Hamming weight if and only if the $\mathbb{Q}$-linear homomorphism*

$$W : F_0(\mathcal{O}^\sharp, \mathbb{Q}) \to F_0(\mathcal{O}, \mathbb{Q}), \quad \eta \mapsto W_\eta,$$

is injective for every finite left R-module M.

Theorem 35 is very convenient because the function spaces $F_0(\mathcal{O}^\sharp, \mathbb{Q}), F_0(\mathcal{O}, \mathbb{Q})$ are $\mathbb{Q}$-vector spaces, and we can use the tools of linear algebra over fields to analyze the linear homomorphism W. In fact, in [5], Bogart et al. prove the extension theorem over finite fields by showing that the matrix representing W is invertible. The form of that matrix is apparent from (5.4). Greferath generalized that approach in [10].

For use in the next subsection, we will need a version of Theorem 35 for linear codes defined over an alphabet A. Let A be a finite left R-module, with automorphism group $\mathrm{Aut}(A)$. A left R-linear code in A^n is given by the image of an R-linear homomorphism $M \to A^n$, for some finite left R-module M. In this case, the coordinate functionals will belong to $\mathrm{Hom}_R(M, A)$. The group $\mathrm{Aut}(A)$ acts on $\mathrm{Hom}_R(M, A)$ on the right; let $\mathcal{O}^\sharp$ denote the set of orbits of this action. A linear code over A, up to equivalence, is again specified by a multiplicity function $\eta \in F(\mathcal{O}^\sharp, \mathbb{N})$.

Just as before, the group of units $\mathcal{U}$ of R acts on the module M on the left, with set $\mathcal{O}$ of orbits. In the same way as above, we formulate the extension property for the alphabet A as:

Theorem 36. *Let A be a finite left R-module. Then A has the extension property with respect to the Hamming weight if and only if the linear homomorphism*

$$W : F_0(\mathcal{O}^\sharp, \mathbb{Q}) \to F_0(\mathcal{O}, \mathbb{Q}), \quad \eta \mapsto W_\eta,$$

is injective for every finite left R-module M.

5.4. **Frobenius is Necessary.** In this subsection we follow a strategy of Dinh and López-Permouth [8] and use Theorem 35 to prove the other direction of Theorem 31; viz., if a finite ring has the extension property with respect to the Hamming weight, then the ring must be Frobenius.

The strategy of Dinh and López-Permouth [8] can be summarized as follows.

(1) If a finite ring R is not Frobenius, then its left socle contains a left R-module of the form $M_{m \times k}(\mathbb{F}_q)$ with $m < k$, for some q (cf., (2.1) and (2.3)).
(2) Use the matrix module $M_{m \times k}(\mathbb{F}_q)$ as the alphabet A. If $m < k$, show that A does not have the extension property.
(3) Take the counter-examples over A to the extension property, consider them as R-modules, and show that they are also counter-examples to the extension property over R.

The first and last points were already proved in [8]. Here's one way to see the first point. We know from (2.3) that $\mathrm{soc}({}_RR)$ is a sum of matrix modules $M_{m_i \times s_i}(\mathbb{F}_{q_i})$. If $m_i \geq s_i$ for all i, then each of the $M_{m_i \times s_i}(\mathbb{F}_{q_i})$ would admit a generating character, by Theorem 13. By adding these generating characters, one would obtain a generating character for $\mathrm{soc}({}_RR)$ itself. Then, by Proposition 14, R would admit a generating character, and hence would be Frobenius by Theorem 5.

For the third point, consider counter-examples $C_1, C_2 \subset A^n$ to the extension property for the alphabet A with respect to the Hamming weight. Because $A^n \subset \mathrm{soc}({}_RR)^n \subset {}_RR^n$, C_1, C_2 can also be viewed as R-modules via (2.1). The Hamming weight of an element x of A^n equals the Hamming weight of x considered as an element of R^n, because the Hamming weight just depends upon the entries of x being zero or not. In this way, C_1, C_2 will also be counter-examples to the extension property for the alphabet R with respect to the Hamming weight.

Thus, the key step remaining is the second point in the strategy. An explicit construction of counter-examples to the extension property for the alphabet $A = M_{m \times k}(\mathbb{F}_q)$, $m < k$, was given in [32]. Here, we give a short existence proof; more details are available in [32] and [33].

Let $R = M_m(\mathbb{F}_q)$ be the ring of $m \times m$ matrices over $\mathbb{F}_q$. Let $A = M_{m \times k}(\mathbb{F}_q)$, with $m < k$; A is a left R-module. It is clear from Theorem 36 that A will fail to have the extension property with respect to the Hamming weight if we can find a finite left R-module M with $\dim_{\mathbb{Q}} F_0(\mathcal{O}^\sharp, \mathbb{Q}) > \dim_{\mathbb{Q}} F_0(\mathcal{O}, \mathbb{Q})$. It turns out that this inequality will hold for any nonzero M.

Because R is simple, any finite left R-module M has the form $M = M_{m \times \ell}(\mathbb{F}_q)$, for some ℓ. First, let us determine $\mathcal{O}$, which is the set of left $\mathcal{U}$-orbits on M. The group $\mathcal{U}$ is the group of units of R, which is precisely the general linear group $GL_m(\mathbb{F}_q)$. The left orbits of $GL_m(\mathbb{F}_q)$ on $M = M_{m \times \ell}(\mathbb{F}_q)$ are represented by the row reduced echelon matrices[1] over $\mathbb{F}_q$ of size $m \times \ell$.

Now, let us determine $\mathcal{O}^\sharp$, which is the set of right $\mathrm{Aut}(A)$-orbits on $\mathrm{Hom}_R(M, A)$. The automorphism group $\mathrm{Aut}(A)$ equals $GL_k(\mathbb{F}_q)$, acting on $A = M_{m \times k}(\mathbb{F}_q)$ by right matrix multiplication. On the other hand, $\mathrm{Hom}_R(M, A) = M_{\ell \times k}(\mathbb{F}_q)$, again using right matrix multiplication. Thus $\mathcal{O}^\sharp$ consists of the right orbits of $GL_k(\mathbb{F}_q)$ acting on $M_{\ell \times k}(\mathbb{F}_q)$. These orbits are represented by the column reduced echelon matrices over $\mathbb{F}_q$ of size $\ell \times k$.

Because the matrix transpose interchanges row reduced echelon matrices and column reduced echelon matrices, we see that $|\mathcal{O}^\sharp| > |\mathcal{O}|$ if and only if $k > m$ (for any positive ℓ). Finally, notice that $\dim_{\mathbb{Q}} F_0(\mathcal{O}^\sharp, \mathbb{Q}) = |\mathcal{O}^\sharp| - 1$ and $\dim_{\mathbb{Q}} F_0(\mathcal{O}, \mathbb{Q}) = |\mathcal{O}| - 1$. Thus, for any nonzero module M, $\dim_{\mathbb{Q}} F_0(\mathcal{O}^\sharp, \mathbb{Q}) > \dim_{\mathbb{Q}} F_0(\mathcal{O}, \mathbb{Q})$ if and only if $m < k$. Consequently, if $m < k$, then W fails to be injective and A fails to have the extension property with respect to Hamming weight.

5.5. **Technical Remarks.** Here is the technical argument regarding the zero functional needed to justify Theorem 34.

Remark 37. For $\eta \in F(\mathcal{O}^\sharp, \mathbb{N})$, define the *length* of η to be $l(\eta) = \sum_{\lambda \in \mathcal{O}^\sharp} \eta(\lambda)$ and the *essential length* of η to be $l_0(\eta) = \sum_{\lambda \neq 0} \eta(\lambda)$. The length $l(\eta)$ equals the length of the

[1] Prof. Yamagata tells me that the Japanese name for this concept translates literally as "step matrices."

linear code defined by η; the reduced length $l_0(\eta)$ equals the length of the linear code defined by η after any all-zero positions have been removed. (In terms of a generator matrix, one removes all the zero columns.)

Assume the extension property holds with respect to the Hamming weight. This means that if $\eta, \eta' \in F(\mathcal{O}^\sharp, \mathbb{N})$ satisfy $l(\eta) = l(\eta')$ and $W_\eta = W_{\eta'}$, then $\eta = \eta'$. That is, W is injective along the level sets of the length function l. If $l(\eta') < l(\eta)$ and $W_\eta = W_{\eta'}$, then we can append zeros to η' until its length is the same as $l(\eta)$ without changing $W_{\eta'}$. More precisely, define η'' by $\eta''(\lambda) = \eta'(\lambda)$ for $\lambda \neq 0$ and set $\eta''(0) = \eta'(0) + l(\eta) - l(\eta')$. Then $l(\eta'') = l(\eta)$ and $W_{\eta''} = W_\eta$. Then $\eta'' = \eta$, by the extension property. In particular, the reduced lengths are equal: $l_0(\eta) = l_0(\eta') = l_0(\eta'')$.

There is a projection $\mathrm{pr} : F(\mathcal{O}^\sharp, \mathbb{N}) \to F_0(\mathcal{O}^\sharp, \mathbb{N})$ which sets $(\mathrm{pr}\,\eta)(0) = 0$ and leaves the other values unchanged, $(\mathrm{pr}\,\eta)(\lambda) = \eta(\lambda)$, $\lambda \neq 0$. This projection splits the monoid as $F(\mathcal{O}^\sharp, \mathbb{N}) = F_0(\mathcal{O}^\sharp, \mathbb{N}) \oplus \mathbb{N}$. The argument of the previous paragraph shows that if $W_\eta = W_{\eta'}$, then $\mathrm{pr}\,\eta = \mathrm{pr}\,\eta'$ as elements of $F_0(\mathcal{O}^\sharp, \mathbb{N})$.

Conversely, suppose $W : F_0(\mathcal{O}^\sharp, \mathbb{N}) \to F_0(\mathcal{O}, \mathbb{N})$ is injective. Let $\eta, \eta' \in F(\mathcal{O}^\sharp, \mathbb{N})$ satisfy $l(\eta) = l(\eta')$ and $W_\eta = W_{\eta'}$. Because the value of $\eta(0)$ does not affect W_η, we see that $W_{\mathrm{pr}\,\eta} = W_{\mathrm{pr}\,\eta'}$. By assumption, W is injective on $F_0(\mathcal{O}^\sharp, \mathbb{N})$, so that $\mathrm{pr}\,\eta = \mathrm{pr}\,\eta'$. In particular, $l_0(\eta) = l_0(\eta')$. Since $l(\eta) = l(\eta')$, we must also have $\eta(0) = \eta'(0)$, and thus $\eta = \eta'$.

6. Self-Dual Codes

I want to finish this article by touching on a very active research topic: self-dual codes.

As we saw in subsection 3.3, if $C \subset \mathbb{F}^n$ is a linear code of length n over a finite field $\mathbb{F}$, then its dual code $C^\perp$ is defined by $C^\perp = \{y \in \mathbb{F}^n : x \cdot y = 0, x \in C\}$. A linear code C is *self-orthogonal* if $C \subset C^\perp$ and is *self-dual* if $C = C^\perp$. Because $\dim C^\perp = n - \dim C$, a necessary condition for the existence of a self-dual code C over a finite field is that the length n must be even; then $\dim C = n/2$.

The Hamming weight enumerator of a self-dual code appears on both sides of the MacWilliams identities:

$$W_C(X, Y) = \frac{1}{|C|} W_C(X + (q-1)Y, X - Y),$$

where C is self-dual over $\mathbb{F}_q$. As $|C| = q^{n/2}$ and the total degree of the polynomial $W_C(X, Y)$ is n, the MacWilliams identities for a self-dual code can be written in the form

$$W_C(X, Y) = W_C\left(\frac{X + (q-1)Y}{\sqrt{q}}, \frac{X - Y}{\sqrt{q}}\right).$$

Every element x of a self-dual code satisfies $x \cdot x = 0$. In the binary case, $q = 2$, notice that $x \cdot x \equiv \mathrm{wt}(x) \bmod 2$. Thus, every element of a binary self-dual code C has even length. This implies that $W_C(X, -Y) = W_C(X, Y)$.

Restrict to the binary case, $q = 2$. Define two complex 2×2 matrices P, Q by

$$P = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Notice that $P^2 = Q^2 = I$. Let $\mathcal{G}$ be the group generated by P and Q (inside $GL_2(\mathbb{C})$). Define an action of $\mathcal{G}$ on the polynomial ring $\mathbb{C}[X, Y]$ by linear substitution: $(fS)(X, Y) = f((X, Y)S)$ for $S \in \mathcal{G}$. The paragraphs above prove the following

Proposition 38. *Let C be a self-dual binary code. Then its Hamming weight enumerator $W_C(X, Y)$ is invariant under the action of the group $\mathcal{G}$. That is, $W_C(X, Y) \in \mathbb{C}[X, Y]^{\mathcal{G}}$, the ring of $\mathcal{G}$-invariant polynomials.*

Much more is true, in fact. Let $C_2 \subset \mathbb{F}_2^2$ be the linear code $C_2 = \{00, 11\}$. Then C_2 is self-dual, and $W_{C_2}(X, Y) = X^2 + Y^2$. Let $E_8 \subset \mathbb{F}_2^8$ be the linear code generated by the rows of the following binary matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

Then E_8 is also self-dual, with $W_{E_8}(X, Y) = X^8 + 14X^4Y^4 + Y^8$.

Theorem 39 (Gleason (1970))**.** *The ring of $\mathcal{G}$-invariant polynomials is generated as an algebra by W_{C_2} and W_{E_8}. That is,*

$$\mathbb{C}[X, Y]^{\mathcal{G}} = \mathbb{C}[X^2 + Y^2, X^8 + 14X^4Y^4 + Y^8].$$

Gleason proved similar statements in several other contexts (doubly-even self-dual binary codes, self-dual ternary codes, Hermitian self-dual quaternary codes) [9]. The results all have this form: for linear codes of a certain *type* (e.g., binary self-dual), their Hamming weight enumerators are invariant under a certain finite matrix group $\mathcal{G}$, and the ring of $\mathcal{G}$-invariant polynomials is generated as an algebra by the weight enumerators of two explicit linear codes of the given type.

Gleason's Theorem has been generalized greatly by Nebe, Rains, and Sloane [24]. Those authors have a general definition of the *type* of a self-dual linear code defined over an alphabet A, where A is a finite left R-module. Associated to every type is a finite group $\mathcal{G}$, called the Clifford-Weil group, and the (complete) weight enumerator of every self-dual linear code of the given type is $\mathcal{G}$-invariant. Finally, the authors show (under certain hypotheses on the ring R) that the ring of all $\mathcal{G}$-invariant polynomials is spanned by weight enumerators of self-dual codes of the given type.

In order to define self-dual codes over non-commutative rings, Nebe, Rains, and Sloane must cope with the difficulty that the dual code of a left linear code C in A^n is a right linear code of the form $(\widehat{A}^n : C) \subset \widehat{A}^n$ (cf., the proof of Theorem 23 in subsection 4.2). This difficulty can be addressed first by assuming that the ring R admits an anti-isomorphism ε, i.e., an isomorphism $\varepsilon : R \to R$ of the additive group, with $\varepsilon(rs) = \varepsilon(s)\varepsilon(r)$, for $r, s \in R$. Then every left (resp., right) R-module M defines a right (resp., left) R-module $\varepsilon(M)$. The additive group of $\varepsilon(M)$ is the same as that of M, and the right scalar multiplication on $\varepsilon(M)$ is $mr := \varepsilon(r)m$, $m \in M$, $r \in R$, where $\varepsilon(r)m$ uses the left scalar multiplication of M. (And similarly for right modules.)

Secondly, in order to identify the character-theoretic annihilator $(\widehat{A}^n : C) \subset \widehat{A}^n$ with a submodule in A^n, Nebe, Rains, and Sloane assume the existence of an isomorphism

$\psi : \varepsilon(A) \to \widehat{A}$. In this way, $C^\perp := \varepsilon^{-1}\psi^{-1}(\widehat{A}^n : C)$ can be viewed as the dual code of C; $C^\perp$ is a left linear code in A^n if C is. With one additional hypothesis on ψ, $C^\perp$ satisfies all the properties one would want from a dual code, such as $(C^\perp)^\perp = C$ and the MacWilliams identities. (See [34] for an exposition.)

There are several questions that arise immediately from the work of Nebe, Rains, and Sloane that may be of interest to ring theorists.

(1) Which finite rings admit anti-isomorphisms? Involutions?
(2) Assume a finite ring R admits an anti-isomorphism ε. Which finite left R-modules A admit an isomorphism $\psi : \varepsilon(A) \to \widehat{A}$?
(3) Even in the absence of complete answers to the preceding, are there good sources of examples?

There are a few results in [34], but much more is needed. Progress on these questions may prove helpful in understanding the limits and the proper setting for the work of Nebe, Rains, and Sloane.

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REALIZING STABLE CATEGORIES AS DERIVE CATEGORIES

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Abstract. In this paper, we compare two different kinds of triangulated categories. First one is the stable category $\underline{\mathrm{mod}}A$ of the category of $\mathbb{Z}$-graded modules over a positively grade self-injective algebra A. Second one is the derived category $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Lambda)$ of the category of modules over an algebra Λ. Our aim is give the complete answer to the following question. For a positively graded self-injective algebra A, when is $\underline{\mathrm{mod}}A$ triangle-equivalent to $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Lambda)$ for some algebra Λ ? The main result of this paper gives the following very simple answer. $\underline{\mathrm{mod}}A$ is triangle-equivalent to $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Lambda)$ for some algebra Λ if and only if the 0-th subring A_0 of A has finite global dimension.

1. Main Result

There are two kinds of triangulated categories which are important for representation theory for algebras. First one is the derived category $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Lambda)$ of the category $\mathrm{mod}A$ of modules over an algebra Λ. Second one is algebraic triangulated categories, that is the stable categories of Frobenius categories (cf. [5]). A typical example is the stable category $\underline{\mathrm{mod}}A$ of the category $\mathrm{mod}A$ of modules over a self-injective algebra A.

In this paper, our aim is to compare derived categories of algebras and the stable categories of self-injective algebras, and find a "nice" relationship between them. If we find it, then those triangulated categories can be investigated from mutual viewpoints.

There several method to compare derived categories of algebras and the stable categories of self-injective algebras. We focus on the following Happel's result. For any algebra Λ, one can associate a self-injective algebra A which is called the trivial extension of Λ. A admits a natural positively grading such that $A_0 = \Lambda$ where A_0 is the 0-th subring. Therefore A is a positively graded self-injective algebra. So the stable category $\underline{\mathrm{mod}}^{\mathbb{Z}}A$ of the category $\mathrm{mod}^{\mathbb{Z}}A$ of $\mathbb{Z}$-graded A-modules has the structure of triangulated category. In this setting, D. Happel [6] showed that Λ has finite global dimension if and only if there exists a triangle-eqiuvalence

$$\underline{\mathrm{mod}}^{\mathbb{Z}}A \simeq \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Lambda). \tag{1.1}$$

This equivalence gives a "nice" relationship between derived category $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Lambda)$ and the stable categories $\underline{\mathrm{mod}}^{\mathbb{Z}}A$. The above result asserts that sometimes representation theory of Λ and that of A are deeply related.

We consider the drastic generalization of the above Happel's result. Happel started from an algebra Λ, and constructed the special positively graded self-injective algebra of A. In contrast, we start from a positively graded self-injective algebra $A = \bigoplus_{i\geq 0} A_i$, and suggest the following question.

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Question. When is $\underline{\mathrm{mod}}^{\mathbb{Z}}A$ triangle-equivalent to the derived category $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Lambda)$ for some algebra Λ ?

The following result is main theorem of this paper which gives the complete answer to our question.

Theorem 1. *Let A be a positively graded self-injective algebra. Then the following are equivalent.*

(1) *The global dimension of A_0 is finite.*
(2) *There exists an algebra Λ, and a triangle-equivalence*

$$\underline{\mathrm{mod}}^{\mathbb{Z}}A \simeq \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Lambda). \tag{1.2}$$

The aim of the rest of this paper is to give an explanation of the proof of Theorem 1, and some examples. Our plan is as follows.

In Section 2, we give two preliminaries. First we recall that $\mathrm{mod}^{\mathbb{Z}}A$ for a positively graded algebra A is a Frobenius category, and so its stable category $\underline{\mathrm{mod}}^{\mathbb{Z}}A$ is an algebraic triangulated category. Secondly we give an explanation of Keller's tilting theorem. Our approach to the question is using Keller's titling theorem for algebraic triangulated categories. B. Keller [7] introduced and investigated differential graded categories and its derived categories. In his work, it was determine when is an algebraic triangulated category triangle-equivalent to the derived category of some algebra by the existence of tilting objects (tilting theorem). In Section 3, we apply Keller's tilting theorem to our study.

In Section 3, we give an outline of the proof of Theorem 1. We omit the proof of (2) $\Rightarrow$ (1). We give proofs (1) $\Rightarrow$ (2). We start from finding a concrete tilting object in $\underline{\mathrm{mod}}^{\mathbb{Z}}A$ which has "good" properties. After finding it, we show two ways to prove (1) $\Rightarrow$ (2). The first proof is based on Keller's tilting theorem, namely we entrust with constructing the triangle-equivalence (1.2). The second proof is direct more than the first one, namely we construct the triangle-equivalence (1.2) explicitly.

In Section 4, we give some examples of Theorem 1. In particular as an application of our main theorem, we show Happel's result, and its generalization shown by X-W Chen [2].

Throughout this paper, let K be an algebraically closed field. An algebra means a finite dimensional associative algebra over K. We always deal with finitely generated right modules over algebras. For an algebra Λ, we denote by $\mathrm{mod}\Lambda$ the category of Λ-modules, $\mathrm{proj}\Lambda$ the category of projective Λ-modules. The same notations is used for graded case. For an additive category $\mathcal{A}$, we denote by $\mathsf{K}^{\mathrm{b}}(\mathcal{A})$ the homotopy category of bounded complexes of $\mathcal{A}$. For an abelian category $\mathcal{A}$, we denote by $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$ the bounded derived category of $\mathcal{A}$.

2. Preliminaries

In this section, we recall basic facts about representation theory of a positively graded algebras, and tilting theorem for algebraic triangulated categories for the readers convenient.

2.1. **Positively graded self-injective algebras.** In this subsection, our aim is to recall that the stable category of $\mathbb{Z}$-graded modules over positively graded self-injective algebras are algebraic triangulated categories. Most of results stated here are due to Gordon-Green [3, 4]. In details, readers should refer to [3, 4].

We start with setting notations. Let $A = \bigoplus_{i \geq 0} A_i$ be a positively graded self-injective algebra. We say that an A-module is $\mathbb{Z}$-gradable if it can be regarded as a $\mathbb{Z}$-graded A-module. For a $\mathbb{Z}$-graded A-module X, we write X_i the i-degree part of X. We denote by $\mathrm{mod}^{\mathbb{Z}} A$ the category of $\mathbb{Z}$-graded A-modules. For $\mathbb{Z}$-graded A-modules X and Y, we write $\mathrm{Hom}_A(X, Y)_0$ the morphism space in $\mathrm{mod}^{\mathbb{Z}} A$ from X to Y.

We recall that $\mathrm{mod}^{\mathbb{Z}} A$ has two important functors. The first one is the grading shift functor. For $i \in \mathbb{Z}$, we denote by

$$(i) : \mathrm{mod}^{\mathbb{Z}} A \to \mathrm{mod}^{\mathbb{Z}} A$$

the grading shift functor, that is defined as follows. For a $\mathbb{Z}$-graded A-module X,

- $X(i) := X$ as an A-module,
- $\mathbb{Z}$-grading on $X(i)$ is defined by $X(i)_j := X_{j+i}$ for any $j \in \mathbb{Z}$.

This is an autofunctor on $\mathrm{mod}^{\mathbb{Z}} A$ whose inverse is $(-i)$.

The second one is the K-dual. It is already known that there is the standard duality

$$D := \mathrm{Hom}_K(-, K) : \mathrm{mod} A \to \mathrm{mod} A^{\mathrm{op}}.$$

This functor induces the following duality. For a $\mathbb{Z}$-graded A-module X, we regard DX as a $\mathbb{Z}$-graded A^{op}-module by defining $(DX)_i := D(X_{-i})$ for any $i \in \mathbb{Z}$. By this observation, we have the duality

$$D : \mathrm{mod}^{\mathbb{Z}} A \to \mathrm{mod}^{\mathbb{Z}} A^{\mathrm{op}}.$$

Next we recall a few important facts about objects and morphism spaces in $\mathrm{mod}^{\mathbb{Z}} A$. The following results are two of the most basic categorical properties of $\mathrm{mod}^{\mathbb{Z}} A$.

Proposition 2. $\mathrm{mod}^{\mathbb{Z}} A$ *is a Hom-finite Krull-Schmidt category*

Proposition 3. [3, Theorem 3.2. Theorem 3.3.] *The following assertions hold.*

1. *A $\mathbb{Z}$-graded A-module is indecomposable in* $\mathrm{mod}^{\mathbb{Z}} A$ *if and only if it is an indecompsable A-module.*
2. *Any direct summand of a $\mathbb{Z}$-gradable A-module is also $\mathbb{Z}$-gradable.*
3. *Let X and Y be indecomposable $\mathbb{Z}$-graded A-modules. If X and Y are isomorphic to each other in* $\mathrm{mod} A$*, then there exists $i \in \mathbb{Z}$ such that X and $Y(i)$ are isomorphic to each other in* $\mathrm{mod}^{\mathbb{Z}} A$.

Next we recall what are projective objects and injective objects in $\mathrm{mod}^{\mathbb{Z}} A$. A is naturally regarded as a $\mathbb{Z}$-graded A-module. By Proposition 3 (2), any projective A-modules are $\mathbb{Z}$-gradable. Moreover it is easy to check that all projective object in $\mathrm{mod}^{\mathbb{Z}} A$ is given by projective A-modules. By the standard duality, the same argument hold for injective objects in $\mathrm{mod}^{\mathbb{Z}} A$.

Proposition 4. *A complete list of indecomposable projective objects in* $\mathrm{mod}^{\mathbb{Z}} A$ *is given by*

$$\{P(i) \mid i \in \mathbb{Z}, P \textit{ is an indecomposable projective } A\textit{-module}\}.$$

Dually a complete list of indecomposable injective objects in $\mathrm{mod}^{\mathbb{Z}}A$ *is given by*

$$\{I(i) \mid i \in \mathbb{Z}, I \textit{ is an indecomposable injective } A\textit{-module}\}.$$

If A is self-injective, then $\mathrm{mod}^{\mathbb{Z}}A$ is a Frobenius category by Proposition 3 and Proposition 4. So in this case, the stable category $\underline{\mathrm{mod}}^{\mathbb{Z}}A$ has a structure of triangulated category by [5].

Lemma 5. *If A is self-injective, the following assertions hold.*

(1) $\mathrm{mod}^{\mathbb{Z}}A$ *is a Frobenius category.*

(2) $\underline{\mathrm{mod}}^{\mathbb{Z}}A$ *has a structure of triangulated category whose shift functor* $[1]$ *is given by the graded cosyzygy functor* $\Omega^{-1} : \underline{\mathrm{mod}}^{\mathbb{Z}}A \to \underline{\mathrm{mod}}^{\mathbb{Z}}A$.

2.2. **Tilting theorem for algebraic triangulated categories.** In this subsection, we recall tilting theorem for algebraic triangulated categories which is due to Keller [7]. It is a theorem which provides a method for comparison of given triangulated category and homotopy category of bounded complexes of projective modules over some algebra.

First let us recall the definition of algebraic triangulated categories again.

Definition 6. A triangulated category $\mathcal{T}$ is *algebraic* if it is triangle-equivalent to the stable category of some Frobenius category.

A class of algebraic triangulated categories contains the following important examples.

Example 7. (1) Let $\mathbb{Z}$ be an abelian group, and A a $\mathbb{Z}$-graded self-injective algebra. Then $\mathrm{mod}^{\mathbb{Z}}A$ is a Frobenius category, and the stable category $\underline{\mathrm{mod}}^{\mathbb{Z}}A$ is an algebraic triangulated category (Lemma 5).

(2) Let Λ be an algebra. The category $\mathsf{C}^{\mathrm{b}}(\mathrm{proj}\Lambda)$ of bounded complexes of projective Λ-modules can be regarded as a Frobenius category whose stable category is the homotopy category $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)$ of bounded complexes of projective Λ-modules (cf. [5]).

In tilting theory, tilting objects which is defined as follows play an important role.

Definition 8. Let $\mathcal{T}$ be a triangulated category. An object $T \in \mathcal{T}$ is called a *tilting object* in $\mathcal{T}$ if it satisfies the following conditions.

(1) $\mathrm{Hom}_{\mathcal{T}}(T, T[i]) = 0$ for $i \neq 0$.

(2) $\mathcal{T} = \mathrm{thick}T$.

Here $\mathrm{thick}T$ is the smallest triangulated full subcategory of $\mathcal{T}$ which contains T, and is closed under direct summands.

The following is a typical example of tilting objects.

Example 9. Let Λ be a ring. Λ can be regarded as a complex which concentrates in degree 0. So Λ is contained in a triangulated category $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)$. It is a tilting object in $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)$.

The following result is Keller's tilting theorem which determine when is an algebraic triangulated category triangle-equivalent to $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)$ for some algebra Λ,

Theorem 10. [7, Theorem 4.3.] *Let $\mathcal{T}$ be an algebraic triangulated category. If $\mathcal{T}$ has a tilting object T, then there exists a triangle-equivalence up to direct summands*

$$\mathcal{T} \simeq \mathsf{K}^{\mathrm{b}}(\mathrm{proj}\mathrm{End}_{\mathcal{T}}(T)).$$

By the above result, finding tilting objects is a basic problem for the study of a given algebraic triangulated category. We will consider this problem for Example 7 (1) in the next section (Theorem 11).

3. Triangle-equivalences between stable categories and derived categories

Throughout this section, let A be a positively graded self-injective algebra. In this section, we discuss triangle-equivalences between the stable category $\underline{\mathrm{mod}}^{\mathbb{Z}}A$ and derived categories of algebras.

First we prove Theorem 1 in the half of this section. We omit the proof of (2) $\Rightarrow$ (1). We prove (1) $\Rightarrow$ (2). We begin the proof from giving the necessary and sufficient condition for existence of tilting objects in the stable category $\underline{\mathrm{mod}}^{\mathbb{Z}}A$. The necessary and sufficient condition is described by important homological property of the subring A_0 of A which is stated as follows.

Theorem 11. *$\underline{\mathrm{mod}}^{\mathbb{Z}}A$ has a tilting object if and only if A_0 has finite global dimension.*

We omit the proof of only if part of Theorem 11. In the following, we show the proof of if part of Theorem 11 which is given by constructing a tilting object in $\underline{\mathrm{mod}}^{\mathbb{Z}}A$. To construct it, we consider truncation functors

$$(-)_{\geq i} : \mathrm{mod}A \to \mathrm{mod}A$$

and

$$(-)_{\leq i} : \mathrm{mod}A \to \mathrm{mod}A$$

which are defined as follows. For a $\mathbb{Z}$-graded A-module X, $X_{\geq i}$ is a $\mathbb{Z}$-graded sub A-module of X defined by

$$(X_{\geq i})_j := \begin{cases} 0 & (j < i) \\ X_j & (j \geq i), \end{cases}$$

and $X_{\leq i}$ is a $\mathbb{Z}$-graded factor A-module $X/X_{\geq i+1}$ of X.

Now we define

$$T := \bigoplus_{i \geq 0} A(i)_{\leq 0}. \tag{3.1}$$

which is an object in $\mathrm{Mod}^{\mathbb{Z}}A$ but not an object in $\mathrm{mod}^{\mathbb{Z}}A$. However since $A(i)_{\leq 0} = A(i)$ for enough large i, T can be regarded as an object in $\underline{\mathrm{mod}}^{\mathbb{Z}}A$.

Then we have the following result.

Theorem 12. *Under the above setting, the following assertions hold.*

(1) *T is a tilting object in* $\mathrm{thick}T$.
(2) *If A_0 has finite global dimension, then T is a tilting object in $\underline{\mathrm{mod}}^{\mathbb{Z}}A$.*

It is proved that T satisfies the first condition in Definition 8 with no assumptions for A, and T satisfies the second condition in Definition 8 if A_0 has finite global dimension. Then we finish the proof of if part of Theorem 11. $\square$

Now we keep the notation as above and put

$$\Gamma := \underline{\mathrm{End}}_A(T)_0.$$

the endomorphism algebra of T in $\underline{\mathrm{mod}}^{\mathbb{Z}}A$. This endomorphism algebra Γ has a nice homological property if so does A_0.

Theorem 13. *If A_0 has finite global dimension, then so does Γ.*

Now we ready to prove Theorem 1 (1) $\Rightarrow$ (2).

Theorem 14. *Under the above setting, the following assertions hold.*

(1) *There exists a triangle-equivalence*

$$\mathrm{thick}T \longrightarrow \mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Gamma).$$

(2) *If A_0 has finite global dimension, then there exists a triangle-equivalence*

$$\underline{\mathrm{mod}}^{\mathbb{Z}}A \longrightarrow \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Gamma).$$

Proof. (1) By Theorem 10 and Theorem 12 (1), we have the triangle-equivalence $\mathrm{thick}T \longrightarrow \mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Gamma)$.

(2) We assume that A_0 has finite global dimension. First by Theorem 10 and Theorem 12 (2), we have the triangle-equivalence $\underline{\mathrm{mod}}^{\mathbb{Z}}A \longrightarrow \mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Gamma)$. Next by Theorem 13, the natural triangle-functor $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda) \longrightarrow \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Gamma)$ is an equivalence. Finally by composing these equivalences, we have a triangle-equivalence

$$\underline{\mathrm{mod}}^{\mathbb{Z}}A \longrightarrow \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Gamma).$$

□

In the above proof, the triangle-equivalence $\underline{\mathrm{mod}}^{\mathbb{Z}}A \to \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Gamma)$ was given by the existence of tilting object T in $\underline{\mathrm{mod}}^{\mathbb{Z}}A$ and Keller's Theorem 10 automatically. In the rest of this section, we construct a triangle-equivalence $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Gamma) \to \underline{\mathrm{mod}}^{\mathbb{Z}}A$ by derived tensor functor directly.

To construct the triangle-equivalence, first we want to consider the derived tensor functor $- \overset{\mathbb{L}}{\otimes}_\Gamma T : \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Gamma) \to \mathsf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}}A)$. However Γ does not act on T naturally since Γ is defined by the morphism space in the stable category $\underline{\mathrm{mod}}^{\mathbb{Z}}A$. To solve this problem, we give the description of T in $\underline{\mathrm{mod}}^{\mathbb{Z}}A$ below. The description allow us to realize Γ as the morphism space in the category $\mathrm{mod}^{\mathbb{Z}}A$.

Proposition 15. *T is decomposed as $T = \underline{T} \oplus P$ where $\underline{T}$ is a direct sum of all indecomposable non-projective direct summand of T. Then the following assertions hold.*

(1) *$\underline{T}$ is in* $\mathrm{mod}^{\mathbb{Z}}A$.
(2) *T and $\underline{T}$ are isomorphic to each other in* $\underline{\mathrm{mod}}^{\mathbb{Z}}A$.
(3) *There exists an algebra isomorphism* $\Gamma \simeq \mathrm{End}_A(\underline{T})_0$.

Let $T = \underline{T} \oplus P$ be the decomposition which was given in Proposition 15. By Proposition 15 (3), $\underline{T}$ is regarded as a $\mathbb{Z}$-graded $\Gamma^{\mathrm{op}} \otimes_K A$-module naturally. So we have the left derived tensor functor

$$- \overset{\mathbb{L}}{\otimes}_\Gamma \underline{T} : \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Gamma) \to \mathsf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}}A).$$

Next we consider the quotient category $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}}A)/\mathsf{K}^{\mathrm{b}}(\mathrm{proj}^{\mathbb{Z}}A)$ of $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}}A)$, and the quotient functor

$$\mathsf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}}A) \longrightarrow \mathsf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}}A)/\mathsf{K}^{\mathrm{b}}(\mathrm{proj}^{\mathbb{Z}}A).$$

The following triangle-equivalence is the realization of $\underline{\mathrm{mod}}^{\mathbb{Z}}A$ as the quotient category $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}}A)/\mathsf{K}^{\mathrm{b}}(\mathrm{proj}^{\mathbb{Z}}A)$. The ungraded version of this realization was studied by several authors [1], [8] and [9].

Theorem 16. [9, Theorem 2.1.] *The natural embedding* $\mathrm{mod}^{\mathbb{Z}}A \to \mathsf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}}A)$ *induces a triangle-equivalence*

$$\underline{\mathrm{mod}}^{\mathbb{Z}}A \longrightarrow \mathsf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}}A)/\mathsf{K}^{\mathrm{b}}(\mathrm{proj}^{\mathbb{Z}}A)$$

Now we consider the following composition of the above three functors

$$G : \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Gamma) \xrightarrow{-\overset{\mathbb{L}}{\otimes}_{\Gamma}\underline{T}} \mathsf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}}A) \longrightarrow \mathsf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}}A)/\mathsf{K}^{\mathrm{b}}(\mathrm{proj}^{\mathbb{Z}}A) \longrightarrow \underline{\mathrm{mod}}^{\mathbb{Z}}A.$$

where the second one is the quotient functor, and the third one is a quasi-inverse of Theorem 16. This is the triangle-functor which we want.

Theorem 17. *Under the above setting, the following assertions hold.*

(1) *G is fully faithful on $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Gamma)$.*

(2) *A_0 has finite global dimension if and only if G is a triangle-equivalence.*

Proof. (1) It is easy to check that $G(\Gamma)$ is isomorphic to $\underline{T}$, so is isomorphic to T. Moreover by Theorem 12 (1), G induces an isomorphism

$$\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Gamma)}(\Gamma, \Gamma[i]) \simeq \underline{\mathrm{Hom}}_A(G(\Gamma), G(\Gamma)[i])_0$$

for any $i \in \mathbb{Z}$. By this and $\mathrm{thick}\Gamma = \mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Gamma)$, G is fully faithful on $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Gamma)$. Thus G induces a triangle-equivalence $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Gamma) \to \mathrm{thick}T$.

(2) We assume that A_0 has finite global dimension. Then Γ has finite global dimension by Theorem 13. Thus we have $\mathrm{thick}\Gamma = \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Gamma)$, and so G is fully faithful. Again since A_0 has finite global dimension, we have $\mathrm{thick}T = \underline{\mathrm{mod}}^{\mathbb{Z}}A$ by Theorem 12 (2). Thus G is dense.

We omit the proof of converse. $\square$

4. Examples

In this section, we show some examples and applications of results which was shown in previous section.

First example is famous Happel's result [6], which gives a relationship between representation theory of algebras and that of the trivial extensions. We show it as an application of Theorem 1.

Example 18. If an algebra is given, then we can always construct a positively graded self-injective algebra called trivial extension, which contains original algebra as a subalgebra. Let us recall the definition of trivial extensions. Let Λ be an algebra. The trivial extension A of Λ is defined as follows.

- $A := \Lambda \oplus D\Lambda$ as an abelian group.

- The multiplication on A is defined by
$$(x,f)\cdot(y,g) := (xy, xg+fy).$$
for any $x,y\in\Lambda$ and $f,g\in D\Lambda$. Here xg and fy is defined by (Λ,Λ)-bimodule structure on $D\Lambda$.

This A becomes an algebra with respect to the above operations. Moreover it is known that A is self-injective.

Now we introduce a positively grading on A by
$$A_i := \begin{cases} \Lambda & (i=0),\\ D\Lambda & (i=1),\\ 0 & (i\geq 2).\end{cases}$$
Then obviously $A=\bigoplus_{i\geq 0}A_i$ becomes a positively graded self-injective algebra.

Under the above setting, we apply Theorem 17 to the trivial extension A of an algebra Λ. Then we have the following Happel's triangle-equivalence.

Theorem 19. [6, Theorem 2. 3.] *Under the above setting, the following are equivalent.*

(1) Λ *has finite global dimension.*

(2) *There exists an triangle-equivalence*
$$\underline{\mathrm{mod}}^{\mathbb{Z}}A \simeq \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Lambda).$$

Proof. We calculate T constructed in (3.1) for our setting. Then one can check that $T=\Lambda$, and $\underline{\mathrm{End}}_A(T)_0 = \mathrm{End}_A(T)_0 \simeq \Lambda$. Thus the assertion follows from this and Theorem 17. □

Next example is X-W Chen's result [2] which gives a generalization of Happel's result.

Example 20. Chen [2] studied relationship between the stable category $\underline{\mathrm{mod}}^{\mathbb{Z}}A$ of a positively graded self-injective algebra A which has Gorenstein parameter and the derived category $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Gamma)$ of the Beilinson algebra Γ of A. The notion of Gorenstein parameter is defined as follows.

Definition 21. Let A be a positively graded self-injective algebra. We say that A has *Gorenstein parameter* ℓ if $\mathrm{Soc}A$ is contained in A_ℓ.

Let A be a positively graded self-injective algebra of Gorenstein parameter ℓ. The Beilinson algebra Γ of A is defined by
$$\Gamma := \begin{pmatrix} A_0 & A_1 & \cdots & A_{\ell-2} & A_{\ell-1}\\ & A_0 & \cdots & A_{\ell-3} & A_{\ell-2}\\ & & \ddots & \vdots & \vdots\\ & & & A_0 & A_1\\ 0 & & & & A_0\end{pmatrix}.$$
Then Chen showed the following result.

Theorem 22. [2, Corollary 1.2.] *Under the above setting, the following are equivalent.*

(1) A_0 *has finite global dimension.*

(2) *There exists a triangle-equivalence*

$$\underline{\mathrm{mod}}^{\mathbb{Z}} A \simeq \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Gamma).$$

As an application of Theorem 12, we give a proof of the above result. Let T be the object defined in (3.1), and $\underline{T}$ the direct summand of T defined in Proposition 15. We calculate $\underline{T}$ and the endomorphism algebra $\underline{\mathrm{End}}_A(T)_0$. Then since A has Gorenstein parameter ℓ, those can be represented as the following explicit form.

Proposition 23. *Under the above setting, the following assertions hold*

(1) $\underline{T} = \bigoplus_{i=0}^{\ell-1} A(i)_{\leq 0}$.
(2) *There exists an algebra isomorphism* $\underline{\mathrm{End}}_A(T)_0 \simeq \Gamma$.

Proof. Since A has Gorenstein parameter ℓ, we have $\underline{T} = \bigoplus_{i=0}^{\ell-1} A(i)_{\leq 0}$ by the definition of $\underline{T}$. Moreover it is easy to calculate that there is an algebra isomorphism $\underline{\mathrm{End}}_A(T)_0 = \underline{\mathrm{End}}_A\left(\bigoplus_{i=0}^{\ell-1} A(i)_{\leq 0}\right)_0 \simeq \Gamma$. □

Proof of Theorem 22. The assertion follows from Theorem 17 and Proposition 23. □

Remark 24. The trivial extensions of algebras are positively graded self-injective algebras of Gorenstein parameter 1. Thus Theorem 22 contains Theorem 19.

Next we show a concrete examples.

Example 25. We consider $A := K[x]/(x^{n+1})$, and define a grading on A by $\deg x := 1$. Then A is a positively graded self-injective algebra of Gorenstein parameter n.

Since the global dimension of $A_0 = K$ is equal to zero, $\underline{\mathrm{mod}}^{\mathbb{Z}} A$ has a tilting object by Theorem 11. Let T be the object in $\mathrm{mod}^{\mathbb{Z}} A$ which was defined in (3.1). Since A has a unique chain

$$A \supset (x)/(x^{n+1}) \supset (x^2)/(x^{n+1}) \supset \cdots \supset (x^n)/(x^{n+1})$$

of $\mathbb{Z}$-graded A-submodules of A, it is easy to calculate that the endomorphism algebra $\Gamma := \underline{\mathrm{End}}_A(T)_0$ of T is isomorphic to the $n \times n$ upper triangular matrix algebra over K. By Theorem 12, there exists a triangle-equivalence

$$\underline{\mathrm{mod}}^{\mathbb{Z}} A \simeq \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Gamma).$$

We observe the above triangle-equivalences by considering the case that $n = 2$, namely the case that $A = K[x]/(x^3)$. For $i = 1, 2$, we put $X^i := (x^i)/(x^3)$ the $\mathbb{Z}$-graded A-submodule of A. Then we have a chain $A \supset X^1 \supset X^2$ of $\mathbb{Z}$-graded A-submodules of A. It is known that $\{X^i(j) \mid i = 1, 2,\ j \in \mathbb{Z}\}$ is a complete set of indecomposable non-projective $\mathbb{Z}$-graded A-modules.

The Auslander-Reiten quiver of $\underline{\mathrm{mod}}^{\mathbb{Z}} A$ is as follows.

$$\begin{tikzcd}
 & & X^1(-2) \arrow[dr] & & X^1(-1) \arrow[ll, dotted] \arrow[dr] & & X^1 \arrow[ll, dotted] \arrow[dr] & & X^1(1) \arrow[ll, dotted] \arrow[dr] & & X^1(2) \arrow[ll, dotted] & \\
\cdots\cdots & X^2(-2) \arrow[ur] & & X^2(-1) \arrow[ll, dotted] \arrow[ur] & & X^2 \arrow[ll, dotted] \arrow[ur] & & X^2(1) \arrow[ll, dotted] \arrow[ur] & & X^2(2) \arrow[ll, dotted] \arrow[ur] & & \cdots\cdots
\end{tikzcd}$$

Here dotted arrows mean the Auslander-Reiten translation in $\underline{\mathrm{mod}}^{\mathbb{Z}} A$. We can observe that the Auslander-Reiten translation coincides with the graded shift functor (-1).

Next we write the Auslander-Reiten quiver of $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Gamma)$. In this case, $\Gamma = \underline{\mathrm{End}}_A(T)_0$ is isomorphic to 2×2 upper triangular matrix algebra over K. We put $P^1 := (KK)$, $P^2 := (0K)$ and $I^1 := (K0)$. It is known that the set $\{P^1, P^2, I^1\}$ is a complete set of indecomposable Γ-modules, and the Auslander-Reiten quiver of $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Gamma)$ is as follows.

$$
\begin{tikzcd}[column sep=small]
 & & I^1[-1] \arrow[dr] & & P^1 \arrow[ll, dotted] \arrow[dr] & & P^2[1] \arrow[ll, dotted] \arrow[dr] & & I^1[1] \arrow[ll, dotted] \arrow[dr] & & P^1[2] \arrow[ll, dotted] & \\
\cdots\cdots & P^1[-1] \arrow[ur] & & P^2 \arrow[ll, dotted] \arrow[ur] & & I^1 \arrow[ll, dotted] \arrow[ur] & & P^1[1] \arrow[ll, dotted] \arrow[ur] & & P^2[2] \arrow[ll, dotted] \arrow[ur] & & \cdots\cdots
\end{tikzcd}
$$

Here dotted arrows mean the Auslander-Reiten translation in $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Gamma)$.

From shape of the above Auslander-Reiten quivers, one can see that $\underline{\mathrm{mod}}^{\mathbb{Z}} A$ and $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Gamma)$ should be equivalent to each other. In fact, we gave a triangle-equivalence between those.

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ALGEBRAIC STRATIFICATIONS OF DERIVED MODULE CATEGORIES AND DERIVED SIMPLE ALGEBRAS

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Abstract. In this note I will survey on some recent progress in the study of recollements of derived module categories.

Key Words: Recollement, Algebraic stratification, Derived simple algebra.

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The notion of recollement of triangulated categories was introduced in [5] as an analogue of short exact sequence of modules or groups. In representation theory of algebras it provides us with reduction techniques, which have proved very useful, for example, in

- proving conjectures on homological dimensions, see [9];
- computing homological invariants, see [11, 12];
- classifying t-structures, see [14].

In this note I will survey on some recent progress in the study of recollements of derived module categories.

1. Recollements

Let k be a field. For a k-algebra A denote by $\mathcal{D}(A) = \mathcal{D}(\mathsf{Mod}\,A)$ the (unbounded) derived category of the category $\mathsf{Mod}\,A$ of right A-modules. The objects of $\mathcal{D}(A)$ are complexes of right A-modules. The category $\mathcal{D}(A)$ is triangulated with shift functor Σ being the shift of complexes. See [10] for a nice introduction on derived categories.

A *recollement* of derived module categories is a diagram of derived module categories and triangle functors

$$\begin{tikzcd} \mathcal{D}(B) \arrow[r, "i_*=i_!"] & \mathcal{D}(A) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^!=j^*"] & \mathcal{D}(C), \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd} \tag{1.1}$$

where A, B and C are k-algebras, such that

(1) $(i^*, i_* = i_!, i^!)$ and $(j_!, j^! = j^*, j_*)$ are adjoint triples;
(2) $j_!$, i_* and j_* are fully faithful;
(3) $j^*i_* = 0$;

The detailed /final/ version of this paper will be /has been/ submitted for publication elsewhere.

(4) for every object M of $\mathcal{D}(A)$ there are two triangles

$$i_!i^!M \longrightarrow M \longrightarrow j_*j^*M \longrightarrow \Sigma i_!i^!M$$

and

$$j_!j^!M \longrightarrow M \longrightarrow i_*i^*M \longrightarrow \Sigma j_!j^!M \ ,$$

where the four morphisms starting from and ending at M are the units and counits.

Necessary and sufficient conditions under which such a recollement exists were discussed in [13, 16].

Example 1. Let A be the path algebra of the Kronecker quiver

$$1 \rightrightarrows 2 \ .$$

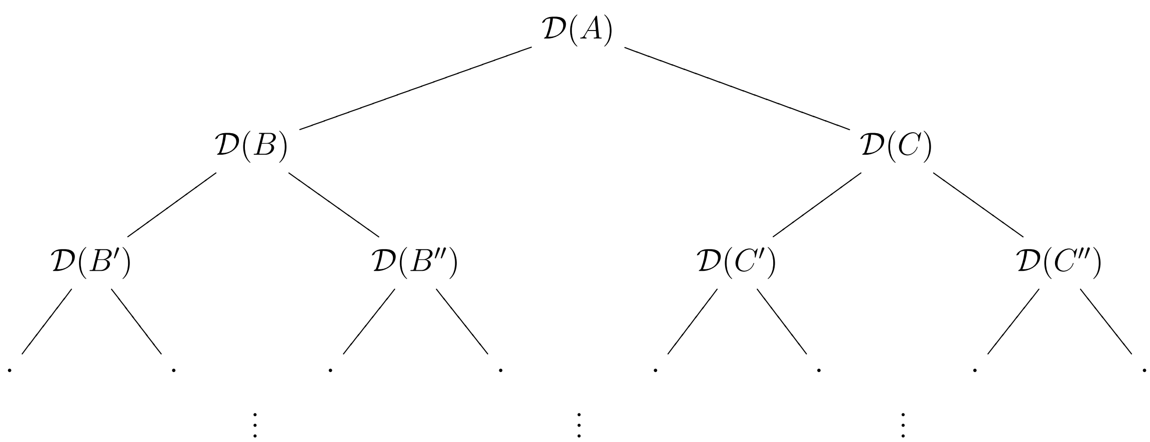
The trivial path e_1 at 1 is an idempotent of A and e_1A is a projective A-module. The following diagram is a recollement

$$\begin{tikzcd}[column sep=huge]
\mathcal{D}(A/Ae_1A) \arrow[r, "? \overset{L}{\otimes}_{A/Ae_1A} A/Ae_1A"] & \mathcal{D}(A) \arrow[l, bend right, "? \overset{L}{\otimes}_A A/Ae_1A"'] \arrow[l, bend left, "\mathsf{RHom}_A(A/Ae_1A,?)"] \arrow[r, "? \overset{L}{\otimes}_A Ae_1"] & \mathcal{D}(e_1Ae_1). \arrow[l, bend right, "? \overset{L}{\otimes}_{e_1Ae_1} e_1A"'] \arrow[l, bend left, "\mathsf{RHom}_{e_1Ae_1}(Ae_1,?)"]
\end{tikzcd}$$

Note that both e_1Ae_1 and A/Ae_1A are isomorphic to k.

2. Algebraic stratifications of derived module categories

Let A be an algebra. An *algebraic stratification* of $\mathcal{D}(A)$ is a sequence of iterated non-trivial recollements of derived module categories. It can be depicted as a binary tree as below, where each edge represents an adjoint triple of triangle functors and each hook represents a recollement



The leaves of the tree are the *simple factors* of the stratification. The following questions are basic:

(a) Does every derived module category admit a finite algebraic stratification?
(b) Do two finite algebraic stratifications of a derived module category have the same number of simple factors? Do they have the same simple factors (up to triangle equivalence and up to reordering)?
(c) Which derived module categories occur as simple factors of some algebraic stratifications?

The question (c) will be discussed in the next section. The questions (a) and (b) ask for a Jordan–Hölder type result for derived module categories. For general (possibly infinite-dimensional) algebras the answers are negative. Below we give some (counter-)examples.

Example 2. ([2]) Let $A = \prod_{\mathbb{N}} k$. Then $\mathcal{D}(A)$ does not admit a finite algebraic stratification.

Example 3. ([6]) Let A be as in Example 1. Let V be a regular simple A-module, namely, V corresponds to one of the following representations of the Kronecker quiver

$$k \underset{\lambda}{\overset{1}{\rightrightarrows}} k \;\; (\lambda \in k), \qquad k \underset{1}{\overset{0}{\rightrightarrows}} k \; .$$

Let $\varphi : A \to A_V$ be the corresponding universal localisation. Then $T = A \oplus A_V/\varphi(A)$ is an (infinitely generated) tilting A-module. We refer to [6] for the unexplained notions.

Let $B = \mathsf{End}_A(T)$. Then there are two algebraic stratifications of $\mathcal{D}(B)$ of length 3 and 2 respectively :

$$\begin{tikzcd}[column sep=small] & \mathcal{D}(B) \arrow[dl, dash] \arrow[dr, dash] & & \\ \mathcal{D}(k(\!(t)\!)) & & \mathcal{D}(A) \arrow[dl, dash] \arrow[dr, dash] & \\ & \mathcal{D}(k) & & \mathcal{D}(k) \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & \mathcal{D}(B) \arrow[dl, dash] \arrow[dr, dash] & \\ \mathcal{D}(k[t]) & & \mathcal{D}(k[\![t]\!]) \end{tikzcd}$$

Examples of this type are systematically studied in [7].

Notice that the algebra B in the preceding example is infinite-dimensional. For finite-dimensional algebras, the questions (a) and (b) are open. For piecewise hereditary algebras the answers to them are positive. Recall that a finite-dimensional algebra is *piecewise hereditary* if it is derived equivalent to a hereditary abelian category.

Theorem 4. ([1, 3]) *Let A be a piecewise hereditary algebra. Then any algebraic stratification of $\mathcal{D}(A)$ has the same set (with multiplicities) of simple factors: they are precisely the derived categories of the endomorphism algebras of the simple A-modules.*

3. Derived simple algebras

An algebra is said to be *derived simple* if its derived category does not admit any non-trivial recollements of derived module categories. For example, the field k is derived simple. Derived simple algebras are precisely those algebras whose derived categories occur as simple factors of some algebraic stratifications.

Example 5. ([17, 4]) Let $n \in \mathbb{N}$. Let A be the algebra given by the quiver

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$$

with relations $(\alpha\beta)^n = 0 = (\beta\alpha)^n$ or with relations $(\alpha\beta)^n\alpha = 0 = \beta(\alpha\beta)^n$. Then A is derived simple.

Example 6. ([8]) There are finite-dimensional derived simple algebras of finite global dimension. In [8], Happel constructed a family of finite-dimensional algebras A_m ($m \in \mathbb{N}$) such that

- the global dimension of A_m is $6m - 3$,
- A_m is derived simple.

All these algebras have exactly two isomorphism classes of simple modules. For example, A_1 is given by the quiver

$$\begin{tikzcd} 1 \arrow[r, shift left=2, "\alpha"] \arrow[r, shift right=2, "\gamma"'] & 2 \arrow[l, "\beta" description] \end{tikzcd}$$

with relations $\beta\alpha = 0 = \gamma\beta$.

The classification of derived simple algebras turns out to be a wild problem. Besides those in the above examples, only a few families of algebras have been shown to be derived simple.

Theorem 7. *The following algebras are derived simple:*

(a) ([2]) *local algebras,*
(b) ([2]) *simple artinian algebras,*
(c) ([4]) *indecomposable commutative algebras,*
(d) ([15]) *blocks of finite group algebras.*

Sketch of the proof for (d): First recall that a block of an algebra is an indecomposable algebra direct summand.

Step 1: Let A, B and C be finite-dimensional algebras such that there is a recollement of the form (1.1). Then $i_*(B)$ and $j_!(C)$ has no self-extensions. Moreover, $i_*(B) \in \mathcal{D}^b(\mathsf{mod}\, A)$, $j_!(C) \in K^b(\mathsf{proj}\, A)$ and $i^*(A) \in K^b(\mathsf{proj}\, B)$. Here $\mathcal{D}^b(\mathsf{mod})$ denotes the bounded derived category of finite-dimensional modules and $K^b(\mathsf{proj})$ denotes the homotopy category of bounded complexes of finite-dimensional projective modules. They can be considered as triangulated subcategories of the (unbounded) derived category.

Step 2: Let A be a finite-dimensional symmetric algebra, *i.e.* $D(A) \cong A$ as A-A-bimodules. Here $D = \mathsf{Hom}_k(?, k)$ is the k-dual. Then for $M, N \in K^b(\mathsf{proj}\, A)$, we have

$$D\,\mathsf{Hom}_A(M, N) \cong \mathsf{Hom}_A(N, M).$$

Step 3: Let A be a finite-dimensional symmetric algebra satisfying the following condition

(#) for any finite-dimensional A-module M, the space $\bigoplus_{i\in\mathbb{Z}} \mathsf{Ext}^i_A(M, M)$ is infinite-dimensional.

Let $M \in \mathcal{D}^b(\mathsf{mod}\, A)$. Then either $M \in K^b(\mathsf{proj}\, A)$ or the space $\bigoplus_{i\in\mathbb{Z}} \mathsf{Hom}_A(M, \Sigma^i M)$ is infinite-dimensional.

Step 4: Let G be a finite group. Then the group algebra kG satisfies the condition (#). So each block of kG is a finite-dimensional indecomposable symmetric algebra satisfying the condition (#).

Step 5: Let A be a finite-dimensional indecomposable symmetric algebra satisfying the condition (#). Then A is derived simple.

To show this, suppose on the contrary that there is a non-trivial recollement of the form (1.1). Then there is a triangle

$$j_!j^!(A) \longrightarrow A \longrightarrow i_*i^*(A) \longrightarrow \Sigma j_!j^!(A). \tag{3.1}$$

By Steps 1 and 3, we know that $i_*(B) \in K^b(\mathsf{proj}\, A)$, which implies that $i_*i^*(A) \in K^b(\mathsf{proj}\, A)$, and hence $j_!j^!(A) \in K^b(\mathsf{proj}\, A)$ as well. For any $n \in \mathbb{Z}$ we have

$$\mathsf{Hom}_A(j_!j^!(A), \Sigma^n i_*i^*(A)) = \mathsf{Hom}_A(j^!(A), \Sigma^n j^*i_*i^*(A)) = 0, \tag{3.2}$$

where the first equality follows from the adjointness of $j_!$ and j^*, and the second one follows from the fact that $j^*i_* = 0$ (the third condition in the definition of a recollement). It then follows from the formula in Step 2 that for any $n \in \mathbb{Z}$

$$\mathsf{Hom}_A(i_*i^*(A), \Sigma^n j_!j^!(A)) = 0. \tag{3.3}$$

Taking $n = 1$, we see that the triangle (3.1) splits, and hence $A = j_!j^!(A) \oplus i_*i^*(A)$. The formulas (3.2) and (3.3) for $n = 0$ say that there are no morphisms between $j_!j^!(A)$ and $i_*i^*(A)$. Thus we have

$$A = \mathsf{End}_A(A) = \mathsf{End}_A(j_!j^!(A) \oplus i_*i^*(A)) = \mathsf{End}_A(j_!j^!(A)) \oplus \mathsf{End}_A(i_*i^*(A)),$$

contradicting the assumption that A is indecomposable. □

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RECOLLEMENTS GENERATED BY IDEMPOTENTS AND APPLICATION TO SINGULARITY CATEGORIES

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Abstract. In this note I report on an ongoing work joint with Martin Kalck, which generalises and improves a construction of Thanhoffer de Völcsey and Van den Bergh.

Key Words: Recollement, Singularity category, Non-commutative resolution.

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In [15] Thanhoffer de Völcsey and Van den Bergh showed that the stable category of maximal Cohen–Macaulay modules over a local complete commutative Gorenstein algebra with isolated singularity can be realized as the triangle quotient of the perfect derived category by the finite-dimensional category of a certain nice dg algebra constructed from the given Gorenstein algebra. We generalises and improves their construction by studying recollements of derived categories generated by idempotents.

1. Recollements generated by idempotents

Let k be a field, let A be a k-algebra and $e \in A$ be an idempotent. Let $\mathcal{D}(A)$ denote the (unbounded) derived category of the category of right modules over A. This is a triangulated category with shift functor Σ being the shift of complexes. Consider the following *standard diagram*

$$\begin{tikzcd} \mathcal{D}(A/AeA) \arrow[rr, "i_*=i_!"] & & \mathcal{D}(A) \arrow[ll, bend right, "i^*"'] \arrow[ll, bend left, "i^!"] \arrow[rr, "j^!=j^*"] & & \mathcal{D}(eAe) \arrow[ll, bend right, "j_!"'] \arrow[ll, bend left, "j_*"] \end{tikzcd} \tag{1.1}$$

where

$$\begin{aligned} i^* &= ? \overset{L}{\otimes}_A A/AeA, & j_! &= ? \overset{L}{\otimes}_{eAe} eA, \\ i_* &= \mathsf{RHom}_{A/AeA}(A/AeA, ?), & j^! &= \mathsf{RHom}_A(eA, ?), \\ i_! &= ? \overset{L}{\otimes}_{A/AeA} A/AeA, & j^* &= ? \overset{L}{\otimes}_A Ae, \\ i^! &= \mathsf{RHom}_A(A/AeA, ?), & j_* &= \mathsf{RHom}_{eAe}(Ae, ?). \end{aligned}$$

One asks when this diagram is a *recollement* ([3]), *i.e.* the following conditions hold

(1) $(i^*, i_* = i_!, i^!)$ and $(j_!, j^! = j^*, j_*)$ are adjoint triples;
(2r) $j_!$ and j_* are fully faithful;
(2l) $i_* = i_!$ is fully faithful;

The detailed version of this paper will be submitted for publication elsewhere.

(3) $j^*i_* = 0$;
(4) for every object M of $\mathcal{D}(A)$ there are two triangles

$$i_!i^!M \longrightarrow M \longrightarrow j_*j^*M \longrightarrow \Sigma i_!i^!M$$

and

$$j_!j^!M \longrightarrow M \longrightarrow i_*i^*M \longrightarrow \Sigma j_!j^!M \ ,$$

where the four morphisms starting from and ending at M are the units and counits.

This type of recollements attracts considerable attention, see for example [6, 8, 7, 14]. The conditions (1) and (3) are easy to check, and it is known that (2r) holds (by applying [11, Proposition 3.2] to eA). However, in general (2l) is not necessarily true, as seen from the next example.

Example 1. Let A be the finite-dimensional algebra given by the quiver

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] & 2 \arrow[l, shift left, "\beta"] \end{tikzcd}$$

with relation $\alpha\beta = 0$. Take the idempotent $e = e_1$, the trivial path at the vertex 1. Then the associated functor $i_* : \mathcal{D}(A/AeA) \to \mathcal{D}(A)$ is not fully faithful. Indeed, $i_*(A/AeA)$ is the simple A-module at vertex 2, which has non-vanishing self-extensions in degree 2, while as an A/AeA-module A/AeA has no self-extensions.

Theorem 2. ([8]) *The following conditions are equivalent*

(i) *the standard diagram* (1.1) *is a recollement,*
(ii) *the homomorphism $A \to A/AeA$ is a* homological epimorphism, i.e. *the functor $i_* : \mathcal{D}(A/AeA) \to \mathcal{D}(A)$ is fully faithful,*
(iii) *the ideal AeA is a* stratifying ideal, i.e. *the counit $Ae \overset{L}{\otimes}_{eAe} eA \to A$ induces an isomorphism $Ae \overset{L}{\otimes}_{eAe} eA \cong AeA$.*

In general, to make the standard diagram (1.1) a recollement, one needs to replace A/AeA by a dg (=differential graded) algebra, which, in some sense, enhances A/AeA. For dg algebras and their derived categories, we refer to [13]. We remark that a k-algebra can be viewed as a dg k-algebra concentrated in degree 0.

Theorem 3. ([12]) *Let A and $e \in A$ be as above. There is a dg k-algebra B with a homomorphism of dg algebras $f : A \to B$ and a recollement of derived categories*

$$\begin{tikzcd} \mathcal{D}(B) \arrow[r, "i_*=i_!"] & \mathcal{D}(A) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^!=j^*"] & \mathcal{D}(eAe) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd} \ ,$$

such that

(a) *the adjoint triples* $(i^*, i_* = i_!, i^!)$ *and* $(j_!, j^! = j^*, j_*)$ *are given by*

$$\begin{array}{ll} i^* = ? \overset{L}{\otimes}_A B, & j_! = ? \overset{L}{\otimes}_{eAe} eA, \\ i_* = \mathsf{RHom}_B(B, ?), & j^! = \mathsf{RHom}_A(eA, ?), \\ i_! = ? \overset{L}{\otimes}_B B, & j^* = ? \overset{L}{\otimes}_A Ae, \\ i^! = \mathsf{RHom}_A(B, ?), & j_* = \mathsf{RHom}_{eAe}(Ae, ?), \end{array}$$

where B *is considered as a left* A*-module and as a right* A*-module via the homomorphism* f*;*

(b) *the degree* i *component* B^i *of* B *vanishes for* $i > 0$*;*

(c) *the* 0*-th cohomology* $H^0(B)$ *of* B *is isomorphic to* A/AeA*.*

As a consequence of the recollement, there is a triangle equivalence

$$\mathsf{per}(B) \cong (K^b(\mathsf{proj}\, A)/\,\mathsf{thick}(eA))^\omega.$$

Here $\mathsf{per}(B)$ *is the smallest triangulated subcategory of* $\mathcal{D}(B)$ *which contains* B *and which is closed under taking direct summands,* $K^b(\mathsf{proj}\, A)$ *is the homotopy category of bounded complexes of finitely generated projective* A*-modules,* $\mathsf{thick}(eA)$ *is the smallest triangulated subcategory of* $K^b(\mathsf{proj}\, A)$ *which contains* eA *and which is closed under taking direct summands, and* $()^\omega$ *denotes the idempotent completion.*

Assume further that A/AeA *is finite-dimensional and that each simple* A/AeA*-module has finite projective dimension over* A*. Then*

(d) $H^i(B)$ *is finite-dimensional over* k *for any* $i \in \mathbb{Z}$*, equivalently,* $\mathsf{per}(B)$ *is Hom-finite,* i.e. $\mathsf{Hom}(M, N)$ *is finite-dimensional over* k *for any* $M, N \in \mathsf{per}(B)$*,*

(e) $\mathcal{D}_{fd}(B) \subseteq \mathsf{per}(B)$*, here* $\mathcal{D}_{fd}(B)$ *denotes the full subcategory of* $\mathcal{D}(B)$ *consisting of those objects whose total cohomology is finite-dimensional over* k*,*

(f) $\mathsf{per}(B)$ *has a* t*-structure whose heart is* $\mathsf{fdmod}-A/AeA$*, the category of finite-dimensional modules over* A/AeA*,*

(g) *if moreover there is a quasi-isomorphism from a dg algebra* $\tilde{A} = (\widehat{kQ}, d)$ *to* A*, where* Q *is a graded quiver concentrated in non-positive degrees and* $d : \widehat{kQ} \to \widehat{kQ}$ *is a continuous* k*-linear differential satisfying the graded Leibniz rule and* $d(\widehat{\mathfrak{m}}) \subseteq \widehat{\mathfrak{m}}^2$*, such that* e *is the image of a sum* $\tilde{e}$ *of some trivial paths of* Q*, then* B *is quasi-isomorphic to* $\tilde{A}/\overline{\tilde{A}\tilde{e}\tilde{A}}$*. Here* $\widehat{kQ}$ *is the completion of the path algebra* kQ *with respect to the* $\mathfrak{m}$*-adic topology in the category of graded algebras for the ideal* $\mathfrak{m}$ *of* kQ *generated by all arrows, and* $\overline{\tilde{A}\tilde{e}\tilde{A}}$ *is the closure of* $\tilde{A}\tilde{e}\tilde{A}$ *under the* $\widehat{\mathfrak{m}}$*-adic topology for the ideal* $\widehat{\mathfrak{m}}$ *of* $\widehat{kQ}$ *generated by all arrows.*

Thanks to the following lemma due to Keller, Theorem 3 (g) becomes practical when the global dimension of A is 2.

Lemma 4. *Let* $A = \widehat{kQ'}/\overline{(R)}$ *be of global dimension* 2*, where* Q' *is a finite (ordinary) quiver and* R *is a finite set of minimal relations. Let* Q *be the graded quiver obtained from* Q' *by adding an arrow* ρ_r *of degree* -1 *from the source of* r *to the target of* r *for* *each relation $r \in R$. Let d be the unique continuous k-linear automorphism of $\widehat{kQ}$ which satisfies the graded Leibniz rule and which takes ρ_r to r for each relation $r \in R$. Then there is a quasi-isomorphism from $(\widehat{kQ}, d)$ to A.*

Example 5. Let A be as in Example 1. Let Q be the graded quiver

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] & 2 \arrow[l, shift left, "\beta"] \arrow[loop right, dotted, "\rho"] \end{tikzcd}$$

where α and β are in degree 0 and ρ is in degree -1. Let d be the unique continuous k-linear automorphism of $\widehat{kQ}$ which satisfies the graded Leibniz rule and which takes ρ to $\alpha\beta$. Then the obvious map from $(\widehat{kQ}, d)$ to A is a quasi-isomorphism.

Let $e = e_1$. The associated dg algebra B as in Theorem 3 is (quasi-isomorphic to) the dg algebra $k[\rho]$ with ρ in degree -1 and with vanishing differential.

2. Application to singularity categories

Let k be a field, and let R be a *Iwanaga–Gorenstein k-algebra*, *i.e.* R is left and right noetherian as a ring and R has finite injective dimension both as left R-module and as right R-module. Let $\mathsf{mod}\, R$ denote the category of finitely generated right R-modules. On the one hand, one defines the *singularity category*

$$\mathcal{D}_{sg}(R) := \mathcal{D}^b(\mathsf{mod}\, R)/K^b(\mathsf{proj}\, R),$$

which measures the complexity of the singularity of R. ($K^b(\mathsf{proj}\, R)$ is considered as the smooth part.) On the other hand, one defines the category $\mathsf{MCM}(R)$ of maximal Cohen–Macaulay R-modules

$$\mathsf{MCM}(R) := \{M \in \mathsf{mod}\, R \mid \mathsf{Ext}^i_R(M, R) = 0 \text{ for any } i > 0\}.$$

The following nice result of Buchweitz relates these categories.

Theorem 6. ([4]) $\mathsf{MCM}(R)$ *is a Frobenius category whose full subcategory of projective-injective objects is precisely* $\mathsf{proj}\, R$. *Moreover, the embedding* $\mathsf{MCM}(R) \to \mathsf{mod}\, R$ *induces a triangle equivalence from the stable category* $\underline{\mathsf{MCM}}(R)$ *to the singularity category* $\mathcal{D}_{sg}(R)$.

Let $M_1, \ldots, M_r \in \mathsf{MCM}(R)$ be pairwise non-isomorphic non-projective R-modules and let $M = R \oplus M_1 \oplus \ldots \oplus M_r$. Let $A = \mathsf{End}_R(M)$ and $e = id_R$ considered as an element of A. Then $R = eAe$ and $A/AeA = \mathsf{End}_{\underline{\mathsf{MCM}}(R)}(M)$. For example, the ring $R = k[x]/x^2$ has a unique simple module S, and letting $M = R \oplus S$ we obtain that $A = \mathsf{End}_R(M)$ is the algebra given in Example 1.

There is always an embedding of $K^b(\mathsf{proj}\, R)$ into $K^b(\mathsf{proj}\, A)$ with essential image being $\mathsf{thick}(eA)$. If the following condition is satisfied

(c1) A has finite global dimension,

then A becomes a *non-commutative/categorical resolution* of R. The condition (c1) has an interesting consequence: the object M generates $\underline{\mathsf{MCM}}(R)$ as a triangulated category.

Cluster-tilting theory comes into the story because cluster-tilting objects are closely related to Van den Bergh's non-commutative crepant resolutions [16], see [10].

The triangle quotient $K^b(\mathsf{proj}\, A)/\mathsf{thick}(eA)$ measures the difference between the resolution and the smooth part of the singularity, see [5]. So $K^b(\mathsf{proj}\, A)/\mathsf{thick}(eA)$ is in some sense a 'categorical exceptional locus'. A natural question is: how is $K^b(\mathsf{proj}\, A)/\mathsf{thick}(eA)$ related to $\mathcal{D}_{sg}(R)$?

Consider the following condition

(c2) $\underline{\mathsf{MCM}}(R)$ is Hom-finite.

Theorem 7. ([12]) *Keep the above notations and assume that (c1) and (c2) hold. There is a dg algebra B with a morphism $f : A \to B$ such that f induces a triangle equivalence*

$$\mathsf{per}(B) \cong (K^b(\mathsf{proj}\, A)/\mathsf{thick}(eA))^{\omega}.$$

Moreover, B satisfies the following properties:

(a) $B^i = 0$ *for any* $i > 0$,

(b) $H^0(B) \cong A/AeA$,

(c) $\mathcal{D}_{fd}(B) \subseteq \mathsf{per}(B)$,

(d) $\mathsf{per}(B)$ *is Hom-finite,*

(e) *there is a triangle equivalence*

$$\mathcal{D}_{sg}(R)^{\omega} \cong (\mathsf{per}(B)/\mathcal{D}_{fd}(B))^{\omega}.$$

Theorem 7 (a–d) are obtained by applying Theorem 3, and part (e) needs more work. This theorem was proved by Thanhoffer de Völcsey and Van den Bergh in [15] for R being a local complete commutative Gorenstein k-algebra with isolated singularity. As an application, they proved the following result, which was independently proved by Amiot, Iyama and Reiten.

Theorem 8. ([2, 15]) *Let $d \in \mathbb{N}$. Let $G \subset SL_d(k)$ be a finite subgroup, acting naturally on $S = k[\![x_1, \ldots, x_d]\!]$ and let $R = S^G$ be the ring of invariants. Then $\underline{\mathsf{MCM}}(R)$ is a generalized $(d-1)$-cluster category in the sense of Amiot* [1] *and Guo* [9].

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INTRODUCTION TO REPRESENTATION THEORY OF COHEN-MACAULAY MODULES AND THEIR DEGENERATIONS

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Abstract. This is a quick introduction to the theory of representation theory of Cohen-Macaulay modules and their degenerations.

1. Representation theory of Cohen-Macaulay modules.

Let k be a field and let R be commutative noetherian complete local k-algebra with unique maximal ideal $\mathfrak{m}$. We assume $k \cong R/\mathfrak{m}$ naturally. Then it is known that there is a regular local k-subalgebra T of R such that R is a module-finite T-algebra. (Cohen's structure theorem for complete local rings.) Note that T is isomorphic to a formal power series ring over k.

Definition 1. (1) R is called a **Cohen-Macaulay ring** (a CM ring for short) if R is free as a T-module.

(2) A finitely generated R-module M is called a **Cohen-Macaulay module** over R, or a maximal Cohen-Macaulay module (a CM module or an MCM module for short) if M is free as a T-module.

Given a CM module M, since $M \cong T^n$ for some $n \geq 0$, we have a k-algebra homomorphism $R \to \mathrm{End}_T(M) \cong T^{n\times n}$, which is a matrix-representation of R over T.

In the following we always assume that R **is a CM complete local k-algebra**. We denote by $\mathrm{mod}(R)$ (res. $\mathrm{CM}(R)$) the category of finitely generated R-modules (resp. CM modules over R) and R-homomorphisms.

$$\mathrm{CM}(R) := \{\text{ CM modules over } R\} \subseteq \mathrm{mod}(R) := \{\text{finitely generated } R\text{-modules }\}$$

Since R is complete, $\mathrm{mod}(R)$ and $\mathrm{CM}(R)$ are Krull-Schmidt categories. Note that $\mathrm{CM}(R)$ is a resolving subcategory of $\mathrm{mod}(R)$ in the following sense: Suppose there is an exact sequence $0 \to L \to M \to N \to 0$ in $\mathrm{mod}(R)$.

(i) If $L, N \in \mathrm{CM}(R)$ then $M \in \mathrm{CM}(R)$.

(ii) If $M, N \in \mathrm{CM}(R)$ then $L \in \mathrm{CM}(R)$.

Let d be the Krull-dimension of the ring R (so that we can take $T = k[[t_1, \dots, t_d]]$ on which R is finite). If $d = 1$ and if R is reduced, then CM modules are just torsion-free modules. If $d = 2$ and if R is normal, then CM modules are nothing but reflexive modules. In general, if $d \geq 3$ and if R is normal, then $\mathrm{CM}(R) \subseteq \{\text{reflexive modules}\}$ but this is not necessarily an equality. If R is regular (i.e. gl-dim$R < \infty$) then all CM modules over R are free.

Let $K_R := \mathrm{Hom}_T(R, T)$ and call it the canonical module of R. Since R is a CM ring, $K_R \in \mathrm{CM}(R)$. For any $X \in \mathrm{mod}(R)$, we have a natural isomorphism $\mathrm{Hom}_R(X, K_R) \cong$

$\mathrm{Hom}_T(X,T)$. It follows that $\mathrm{Hom}_R(-,K_R)$ gives duality $\mathrm{CM}(R) \to \mathrm{CM}(R)^{op}$. Grothendieck's local duality theorem claims the existence of natural isomorphisms

$$\mathrm{Ext}^i_R(M,K_R) \cong \mathrm{Hom}_R(H^{d-i}_{\mathfrak{m}}(M), E_R(k)) \quad (\forall i \in \mathbb{N})$$

whenever R is a CM complete ring and $M \in \mathrm{mod}(R)$. Thus it is easy to see the following

Lemma 2. *The following are equivalent for $M \in \mathrm{mod}(R)$:*

(1) $M \in \mathrm{CM}(R)$,
(2) $\mathrm{Ext}^i_R(M,K_R) = 0 \quad (\forall i > 0)$,
(3) $H^j_{\mathfrak{m}}(M) = 0 \quad (\forall j < d)$,
(4) $\mathrm{Ext}^i_R(k,M) = 0 \quad (\forall i < d)$.

Now recall that R is called an **isolated singularity** if $R_{\mathfrak{p}}$ is a regular local ring for each prime $\mathfrak{p} \neq \mathfrak{m}$. It is not hard to prove the following

Lemma 3. *Let R be a CM local ring as above. The R is an isolated singularity if and only if $\mathrm{Ext}^1_R(M,N)$ is of finite length for each $M, N \in \mathrm{CM}(R)$.*

Definition 4. A CM local ring R is said to be **of finite CM representation type** if $\mathrm{CM}(R)$ has only a finite number of isomorphism classes of indecomposable modules.

The first celebrated result about finiteness of CM representation type was due to M. Auslander.

Theorem 5. [Auslander, 1986] *Let R be a CM complete local ring. If R is of finite CM representation type, then R is an isolated singularity.*

We prove this theorem by using an idea of Huneke and Leuschke [6]. By virtue of Lemma 3 it is enough to prove the following:

(*) Let $a_1, a_2, a_3, \ldots$ be any countable sequence of elements in $\mathfrak{m}$ and let $M, N \in \mathrm{CM}(R)$ be any indecomposable CM modules. Then there is an integer n such that
$a_1 a_2 \cdot \cdots a_n \mathrm{Ext}^1_R(M,N) = 0$.

Actually this will imply that a power of $\mathfrak{m}$ annihilates $\mathrm{Ext}^1_R(M,N)$, hence the length of $\mathrm{Ext}^1_R(M,N)$ is finite. To prove $(*)$, take a $\sigma \in \mathrm{Ext}^1_R(M,N)$ that corresponds to a short exact sequence $\sigma : 0 \to N \to E_0 \to M \to 0$. Now assume the corresponding sequence to $a_1 a_2 \cdots a_n \sigma \in \mathrm{Ext}^1_R(M,N)$ is $0 \to N \to E_n \to M \to 0$ for any integer n. Note that each E_n is a direct sum of indecomposable CM modules and the multiplicity (or the rank if it is defined) $e(E_n)$ is constantly equal to $e(M) + e(N)$. Therefore the possibilities of such E_n are finite, and hence there are integers n and $r > 0$ such that $E_n \cong E_{n+r}$. By definition, there is a commutative diagram with exact rows:

$$\begin{tikzcd}
a_1 \cdots a_n \sigma : & 0 \arrow[r] & N \arrow[r, "j"] \arrow[d, "b := a_{n+1} \cdots a_{n+r}"'] & E_n \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "="] & 0 \\
a_1 \cdots a_{n+r} \sigma \ : & 0 \arrow[r] & N \arrow[r] & E_{n+r} \arrow[r] & M \arrow[r] & 0,
\end{tikzcd}$$

where the first square is a push-out. Hence,

$$0 \longrightarrow N \xrightarrow{\binom{j}{b}} E_n \oplus N \longrightarrow E_{n+r} \longrightarrow 0$$

is exact. Since $E_n \cong E_{n+r}$, Miyata's theorem forces that $\binom{j}{b}$ is a split monomorphism. Then one can see that j is also a split monomorphism. ($pj + qb = 1_N$ in the local ring $\mathrm{End}_R(N)$.) Hence $a_1 \cdots a_n \sigma = 0$ as an element of $\mathrm{Ext}^1_R(M, N)$. $\square$

By a similar idea to the proof above, Huneke and Leuschke [7] was able to prove the following theorem which had been conjectured by F.-O.Schreyer in 1987.

Theorem 6. [Huneke-Leuschke 2003] *Let R be a CM complete local ring and assume that R is of countable CM representation type (i.e.* $\mathrm{CM}(R)$ *has only a countable number of isomorphism classes of indecomposable modules). Then the singular locus of R has at most one-dimension, i.e. $R_\mathfrak{p}$ is regular for each prime $\mathfrak{p}$ with* $\dim R/\mathfrak{p} > 1$.

(Proof) Let $\{M_i \mid i = 1, 2, \ldots\}$ be a complete list of isomorphism classes of indecomposable CM modules, and set

$$\Lambda = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} = \mathrm{Ann}_R\ \mathrm{Ext}^1_R(M_i, M_j) \ \text{ for some } \ i, j \ \text{ and } \ \dim R/\mathfrak{p} = 1\},$$

which is a countable set of prime ideals. Let J be an ideal defining the singular locus of $\mathrm{Spec}(R)$ and we want to show $\dim R/J \leq 1$. Assume contrarily $\dim R/J \geq 2$. If $\mathfrak{p} \in \Lambda$ then, since $(M_i)_\mathfrak{p}$ is not free, we have $J \subseteq \mathfrak{p}$. Thus $J \subseteq \bigcap_{\mathfrak{p}\in\Lambda} \mathfrak{p}$. By countable prime avoidance, there is an $f \in \mathfrak{m} \backslash \bigcup_{\mathfrak{p}\in\Lambda} \mathfrak{p}$, and we can find a prime $\mathfrak{q}$ so that $\mathfrak{q} \supseteq J + fR$ and $\dim R/\mathfrak{q} = 1$. Set $X_i = \Omega^i_R(R/\mathfrak{q})$ the ith syzygy for $i \geq 0$. Then $X_i \in \mathrm{CM}(R)$ if $i \geq d$ and one can show that $\mathrm{Ann}_R\ \mathrm{Ext}^1_R(X_d, X_{d+1}) = \mathfrak{q}$. The CM modules X_d and X_{d+1} is a direct sum of indecomposables as $X_d \cong \bigoplus_{u=1}^r M_{i_u}$ and $X_{d+1} \cong \bigoplus_{v=1}^s M_{j_u}$. Thus since $\mathfrak{q} = \bigcap_{u,v} \mathrm{Ann}_R\ \mathrm{Ext}^1_R(M_{i_u}, M_{j_v})$, we have $\mathfrak{q} = \mathrm{Ann}_R\ \mathrm{Ext}^1_R(M_{i_u}, M_{j_v})$ for some u, v. Thus $\mathfrak{q} \in \Lambda$, but this is a contradiction for $f \in \mathfrak{q}$. $\square$

Auslander's original proof of Theorem 5 uses AR-sequences.

Definition 7. A non-split short exact sequence $0 \to N \to E \xrightarrow{p} M \to 0$ in $\mathrm{CM}(R)$ is called an **AR-sequence** (ending in M) if

(1) M and N are indecomposable,
(2) if $f : X \to M$ is any morphism in $\mathrm{CM}(R)$ that is not a splitting epimorphism, then f factors through p.

We say that the category $\mathrm{CM}(R)$ admits AR-sequences if, for any indecomposable $M \in \mathrm{CM}(R)$, there is an AR-sequence ending in M.

M.Auslander proved the following theorems.

Theorem 8. *Let R be a CM complete local ring and assume that R is of finite CM representation type. Then* $\mathrm{CM}(R)$ *admits AR-sequences.*

Theorem 9. *Let R be a CM complete local ring. Then* $\mathrm{CM}(R)$ *admits AR-sequences if and only if R is an isolated singularity.*

The most difficult part of the proofs of Theorems 8 and 9 is to show the implication "being isolated singularity $\Rightarrow$ admitting AR-sequences". This implication follows from the following isomorphism which is called the Auslander-Reiten duality :

Theorem 10. *Assume that a CM complete local ring R is an isolated singularity of dimension d. Then, for any $M, N \in \mathrm{CM}(R)$, there is a natural isomorphism*

$$\mathrm{Ext}_R^d(\underline{\mathrm{Hom}}_R(N, M), K_R) \cong \mathrm{Ext}_R^1(M, \mathrm{Hom}_R(\Omega_R^d \mathrm{tr}(N), K_R)).$$

Now we discuss some generalities about stable categories. For this let R be a CM complete local ring of dimension d. We denote by $\underline{\mathrm{CM}}(R)$ the stable category of $\mathrm{CM}(R)$. By definition, $\underline{\mathrm{CM}}(R)$ is the factor category $\mathrm{CM}(R)/[R]$. Recall that the objects of $\underline{\mathrm{CM}}(R)$ is CM modules over R, and the morphisms of $\underline{\mathrm{CM}}(R)$ are elements of $\underline{\mathrm{Hom}}_R(M, N) := \mathrm{Hom}_R(M, N)/P(M, N)$ for $M, N \in \underline{\mathrm{CM}}(R)$, where $P(M, N)$ denotes the set of morphisms from M to N factoring through projective R-modules. For a CM module M we denote it by $\underline{M}$ to indicate that it is an object of $\underline{\mathrm{CM}}(R)$.

Since R is a complete local ring, note that $\underline{M}$ is isomorphic to $\underline{N}$ in $\underline{\mathrm{CM}}(R)$ if and only if $M \oplus P \cong N \oplus Q$ in $\mathrm{CM}(R)$ for some projective (hence free) R-modules P and Q.

For any R-module M, we denote the first syzygy module of M by $\Omega_R M$. We should note that $\Omega_R M$ is uniquely determined up to isomorphism as an object in the stable category. The nth syzygy module $\Omega_R^n M$ is defined inductively by $\Omega_R^n M = \Omega_R(\Omega_R^{n-1} M)$, for any nonnegative integer n.

We say that R is a Gorenstein ring if $K_R \cong R$. If R is Gorenstein, then it is easy to see that the syzygy functor $\Omega_A : \underline{\mathrm{CM}}(R) \to \underline{\mathrm{CM}}(R)$ is an autoequivalence. Hence, in particular, one can define the cosyzygy functor Ω_R^{-1} on $\underline{\mathrm{CM}}(R)$ which is the inverse of Ω_R. We note from [3, 2.6] that $\underline{\mathrm{CM}}(R)$ is a triangulated category with shifting functor $[1] = \Omega_R^{-1}$. In fact, if there is an exact sequence $0 \to L \to M \to N \to 0$ in $\mathrm{CM}(R)$, then we have the following commutative diagram by taking the pushout:

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r] & P \arrow[r] & \Omega^{-1} L \arrow[r] & 0,
\end{tikzcd}$$

where P is projective (hence free). We define the triangles in $\underline{\mathrm{CM}}(R)$ are the sequences

$$\underline{L} \longrightarrow \underline{M} \longrightarrow \underline{N} \longrightarrow \underline{L}[1]$$

obtained in such a way.

Now we remark one of the fundamental dualities called the *Auslander-Reiten-Serre duality*, which essentially follows from Theorem 10.

Theorem 11. *Let R be a Gorenstein complete local ring of dimension d. Suppose that R is an isolated singularity. Then, for any $\underline{X}, \underline{Y} \in \underline{\mathrm{CM}}(R)$, we have a functorial isomorphism*

$$\mathrm{Ext}_R^d(\underline{\mathrm{Hom}}_R(X, Y), R) \cong \underline{\mathrm{Hom}}_R(Y, X[d-1]).$$

Therefore the triangulated category $\underline{\mathrm{CM}}(R)$ is a $(d-1)$-Calabi-Yau category.

2. Degenerations of modules

Let us recall the definition of degeneration of finitely generated modules over a noetherian algebra, which is given in [12].

Let R be an associative k-algebra where k is any field. We take a discrete valuation ring (V, tV, k) which is a k-algebra and t is a prime element. We denote by K the quotient field of V. We denote by $\mathrm{mod}(R)$ the category of all finitely generated left R-modules and R-homomorphisms as before. Then we have the natural functors

$$\mathrm{mod}(R) \xleftarrow{\ r\ } \mathrm{mod}(R \otimes_k V) \xrightarrow{\ \ell\ } \mathrm{mod}(R \otimes_k K),$$

where $r = - \otimes_V V/tV$ and $\ell = - \otimes_V K$. ("r" for residue, and "ℓ" for localization.)

Definition 12. For modules $M, N \in \mathrm{mod}(R)$, we say that M **degenerates to** N if there exist a discrete valuation ring (V, tV, k) which is a k-algebra and a module $Q \in \mathrm{mod}(R \otimes_k V)$ that is V-flat such that $\ell(Q) \cong M \otimes_k K$ and $r(Q) \cong N$.

The module Q, regarded as a bimodule ${}_RQ_V$, is a flat family of R-modules with parameter in V. At the closed point in the parameter space $\mathrm{Spec}V$, the fiber of Q is N, which is a meaning of the isomorphism $r(Q) \cong N$. On the other hand, the isomorphism $\ell(Q) \cong M \otimes_k K$ means that the generic fiber of Q is essentially given by M.

Example 13. Let $R = k[[x, y]]/(x^2)$, where k is a field. In this case, a pair of matrices

$$(\varphi, \psi) = \left(\begin{pmatrix} x & y^2 \\ 0 & x \end{pmatrix}, \begin{pmatrix} x & -y^2 \\ 0 & x \end{pmatrix} \right)$$

over $k[[x, y]]$ is a matrix factorization of x^2, giving a CM R-module N that is isomorphic to an ideal $I = (x, y^2)R$. Thus there is a periodic free resolution of N;

$$\cdots \longrightarrow R^2 \xrightarrow{\ \psi\ } R^2 \xrightarrow{\ \varphi\ } R^2 \xrightarrow{\ \psi\ } R^2 \xrightarrow{\ \varphi\ } R^2 \longrightarrow N \longrightarrow 0.$$

Now we deform the matices to

$$(\Phi, \Psi) = \left(\begin{pmatrix} x + ty & y^2 \\ -t^2 & x - ty \end{pmatrix}, \begin{pmatrix} x - ty & -y^2 \\ t^2 & x + ty \end{pmatrix} \right)$$

over $R \otimes_k V$. Since this is a matrix factorization of x^2 again, we have a free resolution

$$\cdots \xrightarrow{\ \Phi\ } (R \otimes_k V)^2 \xrightarrow{\ \Psi\ } (R \otimes_k V)^2 \xrightarrow{\ \Phi\ } (R \otimes_k V)^2 \longrightarrow Q \longrightarrow 0.$$

It is obvious to see that $r(Q) = Q/tQ \cong N$, since $\Phi \otimes_V V/tV = \varphi$. On the other hand, since t^2 is a unit in $R \otimes_k K$, we have $\Phi \otimes_V K \cong \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ after elementary transformations of matrices. Hence, $\ell(Q) = Q_t \cong R \otimes_k K$. As a conclusion, we see that R degenerates to $I = (x, y^2)R$!

Theorem 14 ([12])**.** *The following conditions are equivalent for finitely generated left R-modules M and N.*

(1) *M degenerates to N.*
(2) *There is a short exact sequence of finitely generated left R-modules*

$$0 \to Z \xrightarrow{\binom{\phi}{\psi}} M \oplus Z \to N \to 0,$$

such that the endomorphism ψ of Z is nilpotent, i.e. $\psi^n = 0$ for $n \gg 1$.

Example 15. In Example 13, we have an exact sequence

$$0 \longrightarrow \mathfrak{m} \xrightarrow{(-1, \frac{x}{y})} R \oplus \mathfrak{m} \xrightarrow{\binom{x}{y}} I \longrightarrow 0,$$

such that $\frac{x}{y} : \mathfrak{m} \to \mathfrak{m}$ is nilpotent, where $\mathfrak{m} = (x, y)R$.

By virtue of this theorem together with a theorem of Zwara [17, Theorem 1], we see that if R is a finite-dimensional algebra over k, then our definition of degeneration agrees with the classical (geometric) definition of degenerations using module varieties of R-module structures.

We prove here the implication (2) $\Rightarrow$ (1).

Suppose that there is an exact sequence of finitely generated left R-modules

$$0 \to Z \xrightarrow{f=\binom{\phi}{\psi}} M \oplus Z \to N \to 0,$$

such that ψ is nilpotent. Considering a trivial exact sequence

$$0 \to Z \xrightarrow{j=\binom{0}{1}} M \oplus Z \to M \to 0,$$

we shall combine these two exact sequences along a $[0,1]$-interval. More precisely, let V be the discrete valuation ring $k[t]_{(t)}$, where t in an indeterminate over k, and consider a left $R \otimes_k V$-homomorphism

$$g = j \otimes t + f \otimes (1-t) = \begin{pmatrix} \phi \otimes (1-t) \\ 1 \otimes t + \psi \otimes (1-t) \end{pmatrix} : Z \otimes_k V \to (M \oplus Z) \otimes_k V.$$

We can easily show that g is a monomorphism.

Setting the cokernel of the monomorphism g as Q, we have an exact sequence in $\mathrm{mod} R \otimes_k V$:

$$0 \to Z \otimes_k V \xrightarrow{g} (Z \otimes_k V) \oplus (M \otimes_k V) \to Q \to 0.$$

Since $g \otimes_k V/tV = f$ is an injection and since one can easily show $\mathrm{Tor}_1^V(Q, V/tV) = 0$, we conclude that Q is flat over V and $Q/tQ \cong N$.

Finally note that the morphism $g \otimes_k V[\frac{1}{t}]$ is essentially the same as the morphism

$$Z \otimes_k V[\tfrac{1}{t}] \xrightarrow{\begin{pmatrix} s\phi \\ 1+s\psi \end{pmatrix}} M \otimes_k V[\tfrac{1}{t}] \oplus Z \otimes_k V[\tfrac{1}{t}],$$

where $s = \frac{1-t}{t} \in V[\frac{1}{t}]$. Note that $s\psi : Z \otimes_k V[\frac{1}{t}] \to Z \otimes_k V[\frac{1}{t}]$ is nilpotent as well as ψ, hence $1 + s\psi$ is an automorphism on $Z \otimes_k V[\frac{1}{t}]$. Therefore we have an isomorphism $Q[\frac{1}{t}] \cong M \otimes_k V[\frac{1}{t}]$. This completes the proof of the theorem. $\square$

We remark from this proof that we can always take $k[t]_{(t)}$ as V in Definition 12.

We give an outline of the proof of (1) $\Rightarrow$ (2). (See [12] for the detail.)

We can take Q in Definition 12 so that $M \otimes_k V \subseteq Q$. Then we have an exact sequence

$$0 \to Q/(M \otimes_k V) \xrightarrow{t} Q/(M \otimes_k tV) \longrightarrow Q/tQ \to 0$$

Setting $Z = Q/(M \otimes_k V)$, we can see that the middle term will be $M \oplus Z$ and the right term is N. $\square$

Lemma 16. *If there is an exact sequence* $0 \to L \xrightarrow{i} M \xrightarrow{p} N \to 0$ *in* $\mathrm{mod}(R)$*, then* M *degenerates to* $L \oplus N$.

(Proof)

$$0 \longrightarrow L \xrightarrow{\binom{i}{0}} M \oplus L \xrightarrow{\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}} N \oplus L \longrightarrow 0$$

is exact where $0 : L \to L$ is of course nilpotent. □

Such a degeneration given as in the lemma will be called a degeneration by an extension. There is a degeneration which is not a degeneration by an extension. See the degeneration of Example 13.

In the rest we mainly treat the case when R is a commutative ring.

Remark 17. Let R be a commutative noetherian algebra over k, and suppose that a finitely generated R-module M degenerates to a finitely generated R-module N. Then:

(1) The modules M and N give the same class in the Grothendieck group, i.e. $[M] = [N]$ as elements of $K_0(\mathrm{mod}(R))$. This is actually a direct consequence of $0 \to Z \to M \oplus Z \to N \to 0$. In particular, rank $M =$ rank N if the ranks are defined for R-modules. Furthermore, if $(R, \mathfrak{m})$ is a local ring, then $e(I, M) = e(I, N)$ for any $\mathfrak{m}$-primary ideal I, where $e(I, M)$ denotes the multiplicity of M along I.

(2) If L is an R-module of finite length, then we have the following inequalities of lengths for any integer i:

$$\begin{cases} \mathrm{length}_R(\mathrm{Ext}^i_R(L, M)) \leqq \mathrm{length}_R(\mathrm{Ext}^i_R(L, N)), \\ \mathrm{length}_R(\mathrm{Ext}^i_R(M, L)) \leqq \mathrm{length}_R(\mathrm{Ext}^i_R(N, L)). \end{cases}$$

In particular, when R is a local ring, then

$$\nu(M) \leqq \nu(N), \quad \beta_i(M) \leqq \beta_i(N) \text{ and } \mu^i(M) \leqq \mu^i(N) \ \ (i \geqq 0),$$

where ν, β_i and μ^i denote the minimal number of generators, the ith Betti number and the ith Bass number respectively.

(3) We also have $\mathrm{pd}_R M \leq \mathrm{pd}_R N$, $\mathrm{depth}_R M \geq \mathrm{depth}_R N$ and similar inequalities like $\text{G-dim}_R M \leq \text{G-dim}_R N$. Roughly speaking, when there is a degeneration from M to N, then M is a better module than N.

Recall that a finitely generated R-module is called rigid if it satisfies $\mathrm{Ext}^1_R(N, N) = 0$.

Lemma 18. *Let R be a complete local k-algebra and let $M, N \in \mathrm{mod}(R)$. Assume that N is rigid. If M degenerates to N, then $M \cong N$.*

(Proof) From the sequence $0 \to Z \xrightarrow{\binom{\phi}{\psi}} M \oplus Z \to N \to 0$, we have an exact sequence

$$\mathrm{Ext}^1_R(N, Z) \xrightarrow{\binom{\phi}{\psi}} \mathrm{Ext}^1_R(N, M) \oplus \mathrm{Ext}^1_R(N, Z) \to \mathrm{Ext}^1_R(N, N),$$

where ψ is nilpotent and $\mathrm{Ext}^1_R(N, N) = 0$. Thus we have $\mathrm{Ext}^1_R(N, Z) = 0$. It follows the first sequence splits, and thus $M \oplus Z \cong N \oplus Z$. Since R is complete, it forces $M \cong N$. □

We recall the definition of the Fitting ideal of a finitely presented module. Suppose that a module M over a commutative ring R is given by a finitely free presentation

$$R^m \xrightarrow{C} R^n \longrightarrow M \longrightarrow 0,$$

where C is an $n \times m$-matrix with entries in R. Then recall that the ith Fitting ideal $\mathcal{F}_i^R(M)$ of M is defined to be the ideal $I_{n-i}(C)$ of R generated by all the $(n-i)$-minors of the matrix C. (We use the convention that $I_r(C) = R$ for $r \leqq 0$ and $I_r(C) = 0$ for $r > \min\{m, n\}$.) It is known that $\mathcal{F}_i^R(M)$ depends only on M and i, and independent of the choice of free presentation, and $\mathcal{F}_0^R(M) \subseteq \mathcal{F}_1^R(M) \subseteq \cdots \subseteq \mathcal{F}_n^R(M) = R$. The following lemma will be used to prove the theorem.

Lemma 19. *Let $f : A \to B$ be a ring homomorphism and let M be an A-module which possesses a finitely free presentation. Then $\mathcal{F}_i^B(M \otimes_A B) = f(\mathcal{F}_i^A(M))B$ for all $i \geqq 0$.*

(Proof) If M has a presentation $A^m \xrightarrow{C} A^n \to M \to 0$, then $M \otimes_A B$ has a presentation $B^m \xrightarrow{f(C)} B^n \to M \otimes_A B \to 0$. Thus $\mathcal{F}_i^B(M \otimes_A B) = I_{n-i}(f(C)) = f(I_{n-i}(C))B = f(\mathcal{F}_i^A(M))B$. $\square$

Theorem 20. [Y, 2011] *Let R be a noetherian commutative algebra over k, and M and N finitely generated R-modules. Suppose M degenerates to N. Then we have $\mathcal{F}_i^R(M) \supseteqq \mathcal{F}_i^R(N)$ for all $i \geqq 0$.*

(Proof) By the assumption there is a finitely generated $R \otimes_k V$-module Q such that $Q_t \cong M \otimes_k K$ and $Q/tQ \cong N$, where $V = k[t]_{(t)}$ and $K = k(t)$. Note that $R \otimes_k V \cong S^{-1}R[t]$ where $S = k[t]\backslash(t)$. Since Q is finitely generated, we can find a finitely generated $R[t]$-module Q' such that $Q' \otimes_{R[t]} (R \otimes_k V) \cong Q$. For a fixed integer i we now consider the Fitting ideal $J := \mathcal{F}_i^{R[t]}(Q') \subseteqq R[t]$. Apply Lemma 19 to the ring homomorphism $R[t] \to R = R[t]/tR[t]$, and noting that $Q' \otimes_{R[t]} R \cong N$, we have

$$\mathcal{F}_i^R(N) = J + tR[t]/tR[t] \tag{2.1}$$

as an ideal of $R = R[t]/tR[t]$. On the other hand, applying Lemma 19 to $R[t] \to R \otimes_k K = T^{-1}R[t]$ where $T = k[t]\backslash\{0\}$, we have $\mathcal{F}_i^R(M)T^{-1}R[t] = JT^{-1}R[t]$. Therefore there is an element $f(t) \in T$ such that $f(t)J \subseteq \mathcal{F}_i^R(M)R[t]$.

Now to prove the inclusion $\mathcal{F}_i^R(N) \subseteqq \mathcal{F}_i^R(M)$, take an arbitrary element $a \in \mathcal{F}_i^R(N)$. It follows from (2.1) that there is a polynomial of the form $a + b_1 t + b_2 t^2 + \cdots + b_r t^r$ ($b_i \in R$) that belongs to J. Then, we have $f(t)(a + b_1 t + b_2 t^2 + \cdots + b_r t^r) \in \mathcal{F}_i^R(M)R[t]$. Since $f(t)$ is a non-zero polynomial whose coefficients are all in k, looking at the coefficient of the non-zero term of the least degree in the polynomial $f(t)(a + b_1 t + \cdots + b_r t^r)$, we have that $a \in \mathcal{F}_i^R(M)$. $\square$

Example 21. Let $R = k[[x, y]]/(x^2, y^2)$. Note that R is an artinian Gorenstein local ring. Now consider the modules $M_\lambda = R/(x - \lambda y)R$ for all $\lambda \in k$. We denote by k the unique simple module $R/(x, y)R$ over R.

(1) R degenerates to $M_\lambda \oplus M_{-\lambda}$ for $\forall \lambda \in k$, since there is an exact sequence $0 \to M_{-\lambda} \to R \to M_\lambda \to 0$.

(2) There is a sequence of degenerations from $R \oplus k^2$ to $M_\lambda \oplus M_\mu \oplus k^2$ for any choice of $\lambda, \mu \in k$. ([9, Example 3.1])

(Proof) There are exact sequences; $0 \to \mathfrak{m} \to R \oplus \mathfrak{m}/(xy) \to R/(xy) \to 0$, $0 \to M_\lambda \to \mathfrak{m} \to k \to 0$ and $0 \to k \xrightarrow{x - \mu y} R/(xy) \to M_\mu \to 0$ for any $\lambda, \mu \in k$. Noting $\mathfrak{m}/(xy) \cong k^2$, we have a sequence of degenerations $R \oplus k^2 \Rightarrow \mathfrak{m} \oplus R/(xy) \Rightarrow (M_\mu \oplus k) \oplus (M_\lambda \oplus k) = M_\lambda \oplus M_\mu \oplus k^2$. □

(3) There is no sequence of degenerations from R to $M_\lambda \oplus M_\mu$ if $\lambda + \mu \neq 0$.

(Proof) If there are such degenerations, then we have an inclusion of Fitting ideals; $\mathcal{F}_n^R(M_\lambda \oplus M_\mu) \subseteq \mathcal{F}_n^R(R)$ for all n. Note that $\mathcal{F}_0^R(R) = 0$, and

$$\mathcal{F}_0^R(M_\lambda \oplus M_\mu) = \mathcal{F}_0^R(M_\lambda)\mathcal{F}_0^R(M_\mu) = (x - \lambda y)(x - \mu y)R = (\lambda + \mu)xyR.$$

Hence we must have $\lambda + \mu = 0$. □

This example shows the cancellation law does not hold for degeneration.

Example 22. Let $R = k[[t]]$ be a formal power series ring over a field k with one variable t and let M be an R-module of length n. It is easy to see that there is an isomorphism

$$M \cong R/(t^{p_1}) \oplus \cdots \oplus R/(t^{p_n}), \tag{2.2}$$

where

$$p_1 \geq p_2 \geq \cdots \geq p_n \geq 0 \quad \text{and} \quad \sum_{i=1}^{n} p_i = n. \tag{2.3}$$

In this case the ith Fitting ideal of M is given as

$$\mathcal{F}_i^R(M) = (t^{p_{i+1}+\cdots+p_n}) \quad (i \geq 0).$$

We denote by p_M the sequence $(p_1, p_2, \cdots, p_n)$ of non-negative integers. Recall that such a sequence satisfying (2.3) is called a partition of n.

Conversely, given a partition $p = (p_1, p_2, \cdots, p_n)$ of n, we can associate an R-module of length n by (2.2), which we denote by $M(p)$. In such a way there is a one-one correspondence between the set of partitions of n and the set of isomorphism classes of R-modules of length n.

Let $p = (p_1, p_2, \cdots, p_n)$ and $q = (q_1, q_2, \cdots, q_n)$ be partitions of n. Then we denote $p \succeq q$ if it satisfies $\sum_{i=1}^{j} p_i \geq \sum_{i=1}^{j} q_i$ for all $1 \leq j \leq n$. This $\succeq$ is known to be a partial order on the set of partitions of n and called the dominance order.

Then we can show that there is a degeneration from M to N if and only if $p_M \succeq p_N$.

3. Stable degenerations of CM modules

In this section we are interested in the stable analogue of degenerations of Cohen-Macaulay modules over a commutative Gorenstein local ring. For this purpose, $(R, \mathfrak{m}, k)$ always denotes a Gorenstein local ring which is a k-algebra, and $V = k[t]_{(t)}$ and $K = k(t)$ where t is a variable. We note that $R \otimes_k V$ and $R \otimes_k K$ are Gorenstein as well as R and we have the equality of Krull dimension;

$$\dim R \otimes_k V = \dim R + 1, \quad \dim R \otimes_k K = \dim R.$$

If $\dim R = 0$ (i.e. R is artinian), then the rings $R \otimes_k V$ and $R \otimes_k K$ are local. However we should note that $R \otimes_k V$ and $R \otimes_k K$ will never be local rings if $\dim R > 0$. Since $R \otimes_k K$ is non-local, there may be a lot of projective modules which are not free.

Example 23. Let $R = k[[x,y]]/(x^3 - y^2)$. It is known that the maximal ideal $\mathfrak{m} = (x,y)$ is a unique non-free indecomposable Cohen-Macaulay module over R. See [10, Proposition 5.11]. In fact it is given by a matrix factorization of the polynomial $x^3 - y^2$;

$$(\varphi, \psi) = \left(\begin{pmatrix} y & x \\ x^2 & y \end{pmatrix}, \quad \begin{pmatrix} y & -x \\ -x^2 & y \end{pmatrix} \right).$$

Therefore there is an exact sequence

$$\cdots \longrightarrow R^2 \xrightarrow{\varphi} R^2 \xrightarrow{\psi} R^2 \xrightarrow{\varphi} R^2 \longrightarrow \mathfrak{m} \longrightarrow 0.$$

Now we deform these matrices and consider the pair of matrices over $R \otimes_k K$;

$$(\Phi, \Psi) = \left(\begin{pmatrix} y - xt & x - t^2 \\ x^2 & y + xt \end{pmatrix}, \quad \begin{pmatrix} y + xt & -x + t^2 \\ -x^2 & y - xt \end{pmatrix} \right).$$

Define the $R \otimes_k K$-module P by the following exact sequence;

$$\cdots \longrightarrow (R \otimes_k K)^2 \xrightarrow{\Psi} (R \otimes_k K)^2 \xrightarrow{\Phi} (R \otimes_k K)^2 \longrightarrow P \longrightarrow 0.$$

In this case we can prove that $\underline{P \text{ is a projective module of rank one over } R \otimes_k K \text{ but non-free.}}$ (Hence the Picard group of $R \otimes_k K$ is non-trivial.)

Let A be a commutative Gorenstein ring which is not necessarily local. We say that a finitely generated A-module M is CM if $\mathrm{Ext}^i_A(M, A) = 0$ for all $i > 0$. We consider the category of all CM modules over A with all A-module homomorphisms:

$$\mathrm{CM}(A) := \{ M \in \mathrm{mod}(A) \mid M \text{ is a Cohen-Macaulay module over } A \}.$$

We can then consider the stable category of $\mathrm{CM}(A)$, which we denote by $\underline{\mathrm{CM}}(A)$. This is similarly defined as in local cases, but the morphisms of $\underline{\mathrm{CM}}(A)$ are elements of $\underline{\mathrm{Hom}}_A(M,N) := \mathrm{Hom}_A(M,N)/P(M,N)$ for $M, N \in \underline{\mathrm{CM}}(A)$, where $P(M,N)$ denotes the set of morphisms from M to N factoring through projective A-modules (not necessarily free).

Note that $\underline{M} \cong \underline{N}$ in $\underline{\mathrm{CM}}(A)$ if and only if there are projective A-modules P_1 and P_2 such that $M \oplus P_1 \cong N \oplus P_2$ in $\mathrm{CM}(A)$.

Under such circumstances it is known that $\underline{\mathrm{CM}}(A)$ has a structure of triangulated category as well as in local cases.

Let $x \in A$ be a non-zero divisor on A. Note that x is a non-zero divisor on every CM module over A. Thus the functor $- \otimes_A A/xA$ sends a CM module over A to that over A/xA. Therefore it yields a functor $\mathrm{CM}(A) \to \mathrm{CM}(A/xA)$. Since this functor maps projective A-modules to projective A/xA-modules, it induces the functor $\mathcal{R} : \underline{\mathrm{CM}}(A) \to \underline{\mathrm{CM}}(A/xA)$. It is easy to verify that $\mathcal{R}$ is a triangle functor.

Now let $S \subset A$ be a multiplicative subset of A. Then, by a similar reason to the above, we have a triangle functor $\mathcal{L} : \underline{\mathrm{CM}}(A) \to \underline{\mathrm{CM}}(S^{-1}A)$ which maps $\underline{M}$ to $\underline{S^{-1}M}$.

As before, let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring that is a k-algebra and let $V = k[t]_{(t)}$ and $K = k(t)$. Since $R \otimes_k V$ and $R \otimes_k K$ are Gorenstein rings, we can apply the observation above. Actually, $t \in R \otimes_k V$ is a non-zero divisor on $R \otimes_k V$ and there are isomorphisms of k-algebras; $(R \otimes_k V)/t(R \otimes_k V) \cong R$ and $(R \otimes_k V)_t \cong R \otimes_k K$. Thus there are triangle functors $\mathcal{L} : \underline{\mathrm{CM}}(R \otimes_k V) \to \underline{\mathrm{CM}}(R \otimes_k K)$ defined by the localization by t, and

$\mathcal{R} : \underline{\mathrm{CM}}(R \otimes_k V) \to \underline{\mathrm{CM}}(R)$ defined by taking $- \otimes_{R \otimes_k V} (R \otimes_k V)/t(R \otimes_k V) = - \otimes_V V/tV$. Now we define the stable degeneration of CM modules.

Definition 24. Let $\underline{M},\ \underline{N} \in \underline{\mathrm{CM}}(R)$. We say that $\underline{M}$ **stably degenerates to** $\underline{N}$ if there is a Cohen-Macaulay module $\underline{Q} \in \underline{\mathrm{CM}}(R \otimes_k V)$ such that $\mathcal{L}(\underline{Q}) \cong \underline{M \otimes_k K}$ in $\underline{\mathrm{CM}}(R \otimes_k K)$ and $\mathcal{R}(\underline{Q}) \cong \underline{N}$ in $\underline{\mathrm{CM}}(R)$.

Lemma 25. [15, Lemma 4.2, Proposition 4.3]

(1) *Let $M, N \in \mathrm{CM}(R)$. If M degenerates to N, then $\underline{M}$ stably degenerates to $\underline{N}$.*

(2) *Suppose that there is a triangle in $\underline{\mathrm{CM}}(R)$;*

$$\underline{L} \xrightarrow{\ \alpha\ } \underline{M} \xrightarrow{\ \beta\ } \underline{N} \xrightarrow{\ \gamma\ } \underline{L}[1].$$

Then $\underline{M}$ stably degenerates to $\underline{L} \oplus \underline{N}$.

Lemma 26. [15, Proposition 4.4] *Let $\underline{M},\ \underline{N} \in \underline{\mathrm{CM}}(R)$ and suppose that $\underline{M}$ stably degenerates to $\underline{N}$. Then the following hold.*

(1) *$\underline{M}[1]$ (resp. $\underline{M}[-1]$) stably degenerates to $\underline{N}[1]$ (resp. $\underline{N}[-1]$).*

(2) *$\underline{M^*}$ stably degenerates to $\underline{N^*}$, where M^* denotes the R-dual $\mathrm{Hom}_R(M, R)$.*

Lemma 27. [15, Proposition 4.5] *Let $\underline{M},\ \underline{N},\ \underline{X} \in \underline{\mathrm{CM}}(R)$. If $\underline{M} \oplus \underline{X}$ stably degenerates to $\underline{N}$, then $\underline{M}$ stably degenerates to $\underline{N} \oplus \underline{X}[1]$.*

Remark 28. The zero object in $\underline{\mathrm{CM}}(R)$ can stably degenerate to a non-zero object. In fact, in Example 13 the free module R degenerates to an ideal N. Hence it follows from Proposition 25(1) that $\underline{0} = \underline{R}$ stably degenerates to $\underline{N}$.

For another example, note that there is a triangle

$$\underline{X} \longrightarrow \underline{0} \longrightarrow \underline{X}[1] \xrightarrow{\ 1\ } \underline{X}[1],$$

for any $\underline{X} \in \underline{\mathrm{CM}}(R)$. Hence $\underline{0}$ stably degenerates to $\underline{X} \oplus \underline{X}[1]$ by Proposition 25(2).

Let $(R, \mathfrak{m}, k)$ be a Gorenstein complete local k-algebra and assume for simplicity that k is an infinite field. For Cohen-Macaulay R-modules M and N we consider the following four conditions:

(1) $R^m \oplus M$ degenerates to $R^n \oplus N$ for some $m, n \in \mathbb{N}$.

(2) There is a triangle $\underline{Z} \xrightarrow{\binom{\phi}{\psi}} \underline{M} \oplus \underline{Z} \to \underline{N} \to \underline{Z}[1]$ in $\underline{\mathrm{CM}}(R)$, where $\underline{\psi}$ is a nilpotent element of $\underline{\mathrm{End}}_R(Z)$.

(3) $\underline{M}$ stably degenerates to $\underline{N}$.

(4) There exists an $X \in \mathrm{CM}(R)$ such that $M \oplus R^m \oplus X$ degenerates to $N \oplus R^n \oplus X$ for some $m, n \in \mathbb{N}$.

In [15] we proved the following implications and equivalences of these conditions:

Theorem 29. (*i*) *In general,* $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ *holds.*

(*ii*) *If* $\dim R = 0$*, then* $(1) \Leftrightarrow (2) \Leftrightarrow (3)$ *holds.*

(*iii*) *If R is an isolated singularity of any dimension, then* $(2) \Leftrightarrow (3)$ *holds.*

(*iv*) *There is an example of isolated singularity of* $\dim R = 1$ *for which* $(2) \Rightarrow (1)$ *fails.*

(*v*) *There is an example of* $\dim R = 0$ *for which* $(4) \Rightarrow (3)$ *fails.*

We give here an outline of some of the proofs.

Proof of (1) $\Rightarrow$ (2) : By Theorem 14, there exists an exact sequence

$$0 \to Z \xrightarrow{\binom{\phi}{\psi}} (R^m \oplus M) \oplus Z \to (R^n \oplus N) \to 0,$$

where ψ is nilpotent. In such a case Z ia a Cohen-Macaulay module as well. Then converting this into a triangle in $\underline{\mathrm{CM}}(R)$, and noting that the nilpotency of $\psi \in \mathrm{End}_R(Z)$ forces the nilpotency of $\underline{\psi} \in \underline{\mathrm{End}}_R(Z)$, we can see that (2) holds. □

Proof of (2) $\Rightarrow$ (3): Suppose that there exists a triangle $\underline{Z} \xrightarrow{\binom{\phi}{\psi}} \underline{M} \oplus \underline{Z} \to \underline{N} \to \underline{Z}[1]$, where $\underline{\psi}$ is nilpotent. Then we have a triangle of the form;

$$\underline{Z \otimes_k V} \xrightarrow{\binom{\phi}{t+\psi}} \underline{M \otimes_k V} \oplus \underline{Z \otimes_k V} \longrightarrow \underline{Q} \longrightarrow \underline{Z \otimes_k V}[1],$$

for a $\underline{Q} \in \underline{\mathrm{CM}}(R \otimes_k V)$. Note $\mathcal{L}(t + \underline{\psi})$ is an isomorphism in $\underline{\mathrm{CM}}(R \otimes_k K)$. Thus $\mathcal{L}(\underline{Q}) \cong \mathcal{L}(\underline{M \otimes_k V}) = \underline{M \otimes_k K}$. On the other hand, since $\mathcal{R}(t + \underline{\psi}) = \underline{\psi}$, $\mathcal{R}(\underline{Q}) \cong N$. Thus $\underline{M}$ stably degenerates to $\underline{N}$. □

Proof of (3) $\Rightarrow$ (1) when $\dim R = 0$: In this proof we assume $\dim R = 0$. Suppose that $\underline{M}$ stably degenerates to $\underline{N}$. Then there is a $\underline{Q} \in \underline{\mathrm{CM}}(R \otimes_k V)$ with $\mathcal{L}(\underline{Q}) \cong \underline{M \otimes_k K}$ and $\mathcal{R}(\underline{Q}) \cong \underline{N}$. By definition, we have isomorphisms $Q_t \oplus P_1 \cong (M \otimes_k K) \oplus P_2$ in $\mathrm{CM}(R \otimes_k K)$ for some projective $R \otimes_k K$-modules P_1, P_2, and $Q/tQ \oplus R^a \cong N \oplus R^b$ in $\mathrm{CM}(R)$ for some $a, b \in \mathbb{N}$. Since $R \otimes_k K$ is a local ring, P_1 and P_2 are free. Thus $Q_t \oplus (R \otimes_k K)^c \cong (M \otimes_k K) \oplus (R \otimes_k K)^d$ for some $c, d \in \mathbb{N}$. Setting $\tilde{Q} = Q \oplus (R \otimes_k V)^{a+c}$, we have isomorphisms

$$\tilde{Q}_t \cong (M \oplus R^{a+d}) \otimes_k K, \quad \tilde{Q}/t\tilde{Q} \cong N \oplus R^{b+c}.$$

Since $\tilde{Q}$ is V-flat, $M \oplus R^{a+d}$ degenerates to $N \oplus R^{b+c}$. □

The difficult part of the proof is to show the implications (3) $\Rightarrow$ (4) and (3) $\Rightarrow$ (2). Actually it is technically difficult to show the existence of a Cohen-Macaulay module Z and X in each case. To get over this difficulty, we use the following lemma called Swan's Lemma in Algebraic K-Theory.

Lemma 30. [8, Lemma 5.1] *Let R be a noetherian ring and t a variable. Assume that an $R[t]$-module L is a submodule of $W \otimes_R R[t]$ with W being a finitely generated R-module. Then there is an exact sequence of $R[t]$-modules;*

$$0 \longrightarrow X \otimes_R R[t] \longrightarrow Y \otimes_R R[t] \longrightarrow L \longrightarrow 0,$$

where X and Y are finitely generated R-modules.

By virtue of Swan's lemma we can prove the following proposition that will play an essential role in the proof of Theorem 29.

Proposition 31. *Let R be a Gorenstein local k-algebra, where k is an infinite field. Suppose we are given a Cohen-Macaulay $R \otimes_k V$-module P' satisfying that the localization*

$P = P'_t$ *by* t *is a projective* $R \otimes_k K$*-module. Then there is a Cohen-Macaulay* R*-module* X *with a triangle in* $\underline{\mathrm{CM}}(R \otimes_k V)$ *of the following form:*

$$\underline{X \otimes_k V} \longrightarrow \underline{X \otimes_k V} \longrightarrow \underline{P'} \longrightarrow \underline{X \otimes_k V}[1]. \tag{3.1}$$

As a direct consequence of Theorem 29, we have the following corollary.

Corollary 32. *Let* $(R_1, \mathfrak{m}_1, k)$ *and* $(R_2, \mathfrak{m}_2, k)$ *be Gorenstein complete local* k*-algebras. Assume that the both* R_1 *and* R_2 *are isolated singularities, and that* k *is an infinite field. Suppose there is a* k*-linear equivalence* $F : \underline{\mathrm{CM}}(R_1) \to \underline{\mathrm{CM}}(R_2)$ *of triangulated categories. Then, for* $\underline{M}$, $\underline{N} \in \underline{\mathrm{CM}}(R_1)$*,* $\underline{M}$ *stably degenerates to* $\underline{N}$ *if and only if* $F(\underline{M})$ *stably degenerates to* $F(\underline{N})$*.*

Remark 33. Let $(R_1, \mathfrak{m}_1, k)$ and $(R_2, \mathfrak{m}_2, k)$ be Gorenstein complete local k-algebras as above. Then it hardly occurs that there is a k-linear equivalence of categories between $\mathrm{CM}(R_1)$ and $\mathrm{CM}(R_2)$. In fact, if it occurs, then R_1 is isomorphic to R_2 as a k-algebra. (See [4, Proposition 5.1].)

On the other hand, an equivalence between $\underline{\mathrm{CM}}(R_1)$ and $\underline{\mathrm{CM}}(R_2)$ may happen for non-isomorphic k-algebras. For example, let $R_1 = k[[x, y, z]]/(x^n+y^2+z^2)$ and $R_2 = k[[x]]/(x^n)$ with characteristic of k not being 2 and $n \in \mathbb{N}$. Then, by Knoerrer's periodicity ([10, Theorem 12.10]), we have an equivalence $\underline{\mathrm{CM}}(k[[x, y, z]]/(x^n + y^2 + z^2)) \cong \underline{\mathrm{CM}}(k[[x]]/(x^n))$. Since $k[[x]]/(x^n)$ is an artinian Gorenstein ring, the stable degeneration of modules over $k[[x]]/(x^n)$ is equivalent to a degeneration up to free summands by Theorem 29(ii). Moreover the degeneration problem for modules over $k[[x]]/(x^n)$ is known to be equivalent to the degeneration problem for Jordan canonical forms of square matrices of size n. (See Example 22.) Thus by virtue of Corollary 32, it is easy to describe the stable degenerations of Cohen-Macaulay modules over $k[[x, y, z]]/(x^n + y^2 + z^2)$.

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SUBCATEGORIES OF EXTENSION MODULES RELATED TO SERRE SUBCATEGORIES

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Abstract. We consider subcategories consisting of the extensions of modules in two given Serre subcategories to find a method of constructing Serre subcategories of the module category. We shall give a criterion for this subcategory to be a Serre subcategory.

1. Introduction

Let R be a commutative Noetherian ring. We denote by R-Mod the category of R-modules and by R-mod the full subcategory consisting of finitely generated R-modules.

In [2], P. Gabriel showed that one has lattice isomorphisms between the set of Serre subcategories of R-mod, the set of Serre subcategories of R-Mod which are closed under arbitrary direct sums and the set of specialization closed subsets of $\mathrm{Spec}\,(R)$. By this result, Serre subcategories of R-mod are classified. However, it has not yet classified Serre subcategories of R-Mod. In this paper, we shall give a way of constructing Serre subcategories of R-Mod by considering subcategories of extension modules related to Serre subcategories.

2. The definition of a subcategory of extension modules by Serre subcategories

We assume that all full subcategories of R-Mod are closed under isomorphisms. We recall that a subcategory $\mathcal{S}$ of R-Mod is said to be Serre subcategory if the following condition is satisfied: For any short exact sequence

$$0 \to L \to M \to N \to 0$$

of R-modules, it holds that M is in $\mathcal{S}$ if and only if L and N are in $\mathcal{S}$. In other words, $\mathcal{S}$ is called a Serre subcategory if it is closed under submodules, quotient modules and extensions.

We give the definition of a subcategory of extension modules by Serre subcategories.

Definition 1. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be Serre subcategories of R-Mod. We denote by $(\mathcal{S}_1, \mathcal{S}_2)$ a subcategory consisting of R-modules M with a short exact sequence

$$0 \to X \to M \to Y \to 0$$

of R-modules where X is in $\mathcal{S}_1$ and Y is in $\mathcal{S}_2$, that is

$$(\mathcal{S}_1, \mathcal{S}_2) = \left\{ M \in R\text{-Mod} \;\middle|\; \begin{array}{c} \text{there are } X \in \mathcal{S}_1 \text{ and } Y \in \mathcal{S}_2 \text{ such that} \\ 0 \to X \to M \to Y \to 0 \\ \text{is a short exact sequence.} \end{array} \right\}.$$

The detailed version of this paper has been submitted for publication elsewhere.

Remark 2. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be Serre subcategories of R-Mod.

(1) Since the zero module belongs to any Serre subcategory, one has $\mathcal{S}_1 \subseteq (\mathcal{S}_1, \mathcal{S}_2)$ and $\mathcal{S}_2 \subseteq (\mathcal{S}_1, \mathcal{S}_2)$.

(2) It holds $\mathcal{S}_1 \supseteq \mathcal{S}_2$ if and only if $(\mathcal{S}_1, \mathcal{S}_2) = \mathcal{S}_1$.

(3) It holds $\mathcal{S}_1 \subseteq \mathcal{S}_2$ if and only if $(\mathcal{S}_1, \mathcal{S}_2) = \mathcal{S}_2$.

(4) A subcategory $(\mathcal{S}_1, \mathcal{S}_2)$ is closed under finite direct sums.

Example 3. We denote by $\mathcal{S}_{f.g.}$ the subcategory consisting of finitely generated R-modules and by $\mathcal{S}_{Artin}$ the subcategory consisting of Artinian R-modules. If R is a complete local ring, then a subcategory $(\mathcal{S}_{f.g.}, \mathcal{S}_{Artin})$ is known as the subcategory consisting of Matlis reflexive R-modules. Therefore, $(\mathcal{S}_{f.g.}, \mathcal{S}_{Artin})$ is a Serre subcategory of R-Mod.

The following example shows that a subcategory $(\mathcal{S}_1, \mathcal{S}_2)$ needs not be a Serre subcategory for Serre subcategories $\mathcal{S}_1$ and $\mathcal{S}_2$.

Example 4. We shall see that the subcategory $(\mathcal{S}_{Artin}, \mathcal{S}_{f.g.})$ needs not be closed under extensions.

Let R be a one dimensional Gorenstein local ring with a maximal ideal $\mathfrak{m}$. Then one has a minimal injective resolution

$$0 \to R \to \bigoplus_{\substack{\mathfrak{p} \in \operatorname{Spec}(R) \\ \operatorname{ht}\mathfrak{p} = 0}} E_R(R/\mathfrak{p}) \to E_R(R/\mathfrak{m}) \to 0$$

of R. ($E_R(M)$ denotes the injective hull of an R-module M.) We note that R and $E_R(R/\mathfrak{m})$ are in $(\mathcal{S}_{Artin}, \mathcal{S}_{f.g.})$.

Now, we assume that a subcategory $(\mathcal{S}_{Artin}, \mathcal{S}_{f.g.})$ is closed under extensions. Then $E_R(R) = \oplus_{\operatorname{ht}\mathfrak{p}=0} E_R(R/\mathfrak{p})$ is in $(\mathcal{S}_{Artin}, \mathcal{S}_{f.g.})$. It follows from the definition of $(\mathcal{S}_{Artin}, \mathcal{S}_{f.g.})$ that there exists an Artinian R-submodule X of $E_R(R)$ such that $E_R(R)/X$ is a finitely generated R-module.

If $X = 0$, then $E_R(R)$ is a finitely generated injective R-module. It follows from the Bass formula that one has $\dim R = \operatorname{depth} R = \operatorname{inj\,dim} E_R(R) = 0$. However, this equality contradicts $\dim R = 1$. On the other hand, if $X \neq 0$, then X is a non-zero Artinian R-module. Therefore, one has $\operatorname{Ass}_R(X) = \{\mathfrak{m}\}$. Since X is an R-submodule of $E_R(R)$, one has

$$\operatorname{Ass}_R(X) \subseteq \operatorname{Ass}_R(E_R(R)) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \operatorname{ht}\mathfrak{p} = 0\}.$$

This is contradiction as well.

3. The main result

In this section, we shall give a criterion for a subcategory $(\mathcal{S}_1, \mathcal{S}_2)$ to be a Serre subcategory for Serre subcategories $\mathcal{S}_1$ and $\mathcal{S}_2$.

First of all, it is easy to see that the following assertion holds.

Proposition 5. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be Serre subcategories of R-*Mod*. Then a subcategory $(\mathcal{S}_1, \mathcal{S}_2)$ is closed under submodules and quotient modules.*

Lemma 6. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be Serre subcategories of R-Mod. We suppose that a sequence $0 \to L \to M \to N \to 0$ of R-modules is exact. Then the following assertions hold.*

(1) *If $L \in \mathcal{S}_1$ and $N \in (\mathcal{S}_1, \mathcal{S}_2)$, then $M \in (\mathcal{S}_1, \mathcal{S}_2)$.*

(2) *If $L \in (\mathcal{S}_1, \mathcal{S}_2)$ and $N \in \mathcal{S}_2$, then $M \in (\mathcal{S}_1, \mathcal{S}_2)$.*

Proof. (1) We assume that L is in $\mathcal{S}_1$ and N is in $(\mathcal{S}_1, \mathcal{S}_2)$. Since N belongs to $(\mathcal{S}_1, \mathcal{S}_2)$, there exists a short exact sequence

$$0 \to X \to N \to Y \to 0$$

of R-modules where X is in $\mathcal{S}_1$ and Y is in $\mathcal{S}_2$. Then we consider the following pull buck diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & Y & = & Y & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

of R-modules with exact rows and columns. Since $\mathcal{S}_1$ is a Serre subcategory, it follows from the first row in the diagram that X' belongs to $\mathcal{S}_1$. Consequently, we see that M is in $(\mathcal{S}_1, \mathcal{S}_2)$ by the middle column in the diagram.

(2) We can show that the assertion holds by the similar argument in the proof of (1). $\square$

Now, we can show the main purpose of this paper.

Theorem 7. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be Serre subcategories of R-Mod. Then the following conditions are equivalent:*

(1) *A subcategory $(\mathcal{S}_1, \mathcal{S}_2)$ is a Serre subcategory;*

(2) *One has $(\mathcal{S}_2, \mathcal{S}_1) \subseteq (\mathcal{S}_1, \mathcal{S}_2)$.*

Proof. (1) $\Rightarrow$ (2) We assume that M is in $(\mathcal{S}_2, \mathcal{S}_1)$. By the definition of a subcategory $(\mathcal{S}_2, \mathcal{S}_1)$, there exists a short exact sequence

$$0 \to Y \to M \to X \to 0$$

of R-modules where X is in $\mathcal{S}_1$ and Y is in $\mathcal{S}_2$. We note that X and Y are also in $(\mathcal{S}_1, \mathcal{S}_2)$. Since a subcategory $(\mathcal{S}_1, \mathcal{S}_2)$ is closed under extensions by the assumption (1), we see that M is in $(\mathcal{S}_1, \mathcal{S}_2)$.

(2) $\Rightarrow$ (1) We only have to prove that a subcategory $(\mathcal{S}_1, \mathcal{S}_2)$ is closed under extensions by Proposition 5. Let $0 \to L \to M \to N \to 0$ be a short exact sequence of R-modules such that L and N are in $(\mathcal{S}_1, \mathcal{S}_2)$. We shall show that M is also in $(\mathcal{S}_1, \mathcal{S}_2)$.

Since L is in $(\mathcal{S}_1, \mathcal{S}_2)$, there exists a short exact sequence

$$0 \to S \to L \to L/S \to 0$$

of R-modules where S is in $\mathcal{S}_1$ such that L/S is in $\mathcal{S}_2$. We consider the following push out diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & S \arrow[r, equal] \arrow[d] & S \arrow[d] & & \\
0 \arrow[r] & L \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L/S \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & N \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

of R-modules with exact rows and columns. Next, since N is in $(\mathcal{S}_1, \mathcal{S}_2)$, we have a short exact sequence

$$0 \to T \to N \to N/T \to 0$$

of R-modules where T is in S_1 such that N/T is in $\mathcal{S}_2$. We consider the following pull back diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & L/S \arrow[r] \arrow[d, equal] & P' \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L/S \arrow[r] & P \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & & N/T \arrow[r, equal] \arrow[d] & N/T \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

of R-modules with exact rows and columns.

In the first row of the second diagram, since L/S is in $\mathcal{S}_2$ and T is in $\mathcal{S}_1$, P' is in $(\mathcal{S}_2, \mathcal{S}_1)$. Now here, it follows from the assumption (2) that P' is in $(\mathcal{S}_1, \mathcal{S}_2)$. Next, in the middle column of the second diagram, we have the short exact sequence such that P' is in $(\mathcal{S}_1, \mathcal{S}_2)$ and N/T is in $\mathcal{S}_2$. Therefore, it follows from Lemma 6 that P is in $(\mathcal{S}_1, \mathcal{S}_2)$. Finally, in the middle column of the first diagram, there exists the short exact sequence such that S is in $\mathcal{S}_1$ and P is in $(\mathcal{S}_1, \mathcal{S}_2)$. Consequently, we see that M is in $(\mathcal{S}_1, \mathcal{S}_2)$ by Lemma 6.

The proof is completed. □

Corollary 8. *A subcategory $(\mathcal{S}_{f.g.}, \mathcal{S})$ is a Serre subcategory for a Serre subcategory $\mathcal{S}$ of* R-Mod.

Proof. Let $\mathcal{S}$ be a Serre subcategory of R-Mod. To prove our assertion, it is enough to show that one has $(\mathcal{S}, \mathcal{S}_{f.g.}) \subseteq (\mathcal{S}_{f.g.}, \mathcal{S})$ by Theorem 7. Let M be in $(\mathcal{S}, \mathcal{S}_{f.g.})$. Then there exists a short exact sequence $0 \to Y \to M \to M/Y \to 0$ of R-modules where Y is in $\mathcal{S}$ such that M/Y is in $\mathcal{S}_{f.g.}$. It is easy to see that there exists a finitely generated R-submodule X of M such that $M = X + Y$. Since $X \oplus Y$ is in $(\mathcal{S}_{f.g.}, \mathcal{S})$ and M is a homomorphic image of $X \oplus Y$, M is in $(\mathcal{S}_{f.g.}, \mathcal{S})$ by Proposition 5. □

We note that a subcategory $\mathcal{S}_{Artin}$ consisting of Artinian R-modules is a Serre subcategory which is closed under injective hulls. (Also see [1, Example 2.4].) Therefore we can see that a subcategory $(\mathcal{S}, \mathcal{S}_{Artin})$ is also Serre subcategory for a Serre subcategory of R-Mod by the following assertion.

Corollary 9. *Let $\mathcal{S}_2$ be a Serre subcategory of* R-Mod *which is closed under injective hulls. Then a subcategory $(\mathcal{S}_1, \mathcal{S}_2)$ is a Serre subcategory for a Serre subcategory $\mathcal{S}_1$ of* R-Mod.

Proof. By Theorem 7, it is enough to show that one has $(\mathcal{S}_2, \mathcal{S}_1) \subseteq (\mathcal{S}_1, \mathcal{S}_2)$.

We assume that M is in $(\mathcal{S}_2, \mathcal{S}_1)$ and shall show that M is in $(\mathcal{S}_1, \mathcal{S}_2)$. Then there exists a short exact sequence

$$0 \to Y \to M \to X \to 0$$

of R-modules where X is in $\mathcal{S}_1$ and Y is in $\mathcal{S}_2$. Since $\mathcal{S}_2$ is closed under injective hulls, we note that the injective hull $E_R(Y)$ of Y is also in $\mathcal{S}_2$. We consider a push out diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & M & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & E_R(Y) & \longrightarrow & T & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

of R-modules with exact rows and injective vertical maps. The second exact sequence splits, and we have an injective homomorphism $M \to X \oplus E_R(Y)$. Since there is a short exact sequence

$$0 \to X \to X \oplus E_R(Y) \to E_R(Y) \to 0$$

of R-modules, the R-module $X \oplus E_R(Y)$ is in $(\mathcal{S}_1, \mathcal{S}_2)$. Consequently, we see that M is also in $(\mathcal{S}_1, \mathcal{S}_2)$ by Proposition 5.

The proof is completed. □

Example 10. Let R be a domain but not a field and let Q be a field of fractions of R. We denote by $\mathcal{S}_{Tor}$ a subcategory consisting of torsion R-modules, that is

$$\mathcal{S}_{Tor} = \{M \in R\text{-Mod} \mid M \otimes_R Q = 0\}.$$

Then we shall see that one has

$$(\mathcal{S}_{Tor}, \mathcal{S}_{f.g.}) \subsetneqq (\mathcal{S}_{f.g.}, \mathcal{S}_{Tor}) = \{M \in R\text{-Mod} \mid \dim_Q M \otimes_R Q < \infty\}.$$

Therefore, a subcategory $(\mathcal{S}_{f.g.}, \mathcal{S}_{Tor})$ is a Serre subcategory by Corollary 8, but a subcategory $(\mathcal{S}_{Tor}, \mathcal{S}_{f.g.})$ is not closed under extensions by Theorem 7.

First of all, we shall show that the above equality holds. We suppose that M is in $(\mathcal{S}_{f.g.}, \mathcal{S}_{Tor})$. Then there exists a short exact sequence

$$0 \to X \to M \to Y \to 0$$

of R-modules where X is in $\mathcal{S}_{f.g.}$ and Y is in $\mathcal{S}_{Tor}$. We apply an exact functor $- \otimes_R Q$ to this sequence. Then we see that one has $M \otimes_R Q \cong X \otimes_R Q$ and this module is a finite dimensional Q-vector space.

Conversely, let M be an R-module with $\dim_Q M \otimes_R Q < \infty$. Then we can denote $M \otimes_R Q = \sum_{i=1}^{n} Q(m_i \otimes 1_Q)$ with $m_i \in M$ and the unit element 1_Q of Q. We consider a short exact sequence

$$0 \to \sum_{i=1}^{n} Rm_i \to M \to M/\sum_{i=1}^{n} Rm_i \to 0$$

of R-modules. It is clear that $\sum_{i=1}^{n} Rm_i$ is in $\mathcal{S}_{f.g.}$ and $M/\sum_{i=1}^{n} Rm_i$ is in $\mathcal{S}_{Tor}$. So M is in $(\mathcal{S}_{f.g.}, \mathcal{S}_{Tor})$. Consequently, the above equality holds.

Next, it is clear that $M \otimes_R Q$ has finite dimension as Q-vector space for an R-module M of $(\mathcal{S}_{Tor}, \mathcal{S}_{f.g.})$. Thus, one has $(\mathcal{S}_{Tor}, \mathcal{S}_{f.g.}) \subseteq (\mathcal{S}_{f.g.}, \mathcal{S}_{Tor})$.

Finally, we shall see that a field of fractions Q of R is in $(\mathcal{S}_{f.g.}, \mathcal{S}_{Tor})$ but not in $(\mathcal{S}_{Tor}, \mathcal{S}_{f.g.})$, so one has $(\mathcal{S}_{Tor}, \mathcal{S}_{f.g.}) \subsetneqq (\mathcal{S}_{f.g.}, \mathcal{S}_{Tor})$. Indeed, it follows from $\dim_Q Q \otimes_R Q = 1$ that Q is in $(\mathcal{S}_{f.g.}, \mathcal{S}_{Tor})$. On the other hand, we assume that Q is in $(\mathcal{S}_{Tor}, \mathcal{S}_{f.g.})$. Since R is a domain, a torsion R-submodule of Q is only the zero module. It means that Q must be a finitely generated R-module. But, this is a contradiction.

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