ON A DEGENERATION PROBLEM FOR COHEN-MACAULAY MODULES

NAOYA HIRAMATSU

1. Introduction

The aim of this article is to give an outline of the paper [3], which is a joint work with Yuji Yoshino.

In this note, we would like to give several examples of degenerations of maximal Cohen-Macaulay modules and to show how we can describe them (Theorem 12). This result depends heavily on the recent work by Yoshino about the stable analogue of degenerations for Cohen-Macaulay modules over a Gorenstein local algebra [9]. In Section 3 we also investigate the relation among the extended versions of the degeneration order, the extension order and the AR order (Theorem 22).

2. Examples of degenerations

In this section, we recall the definition of degeneration and state several known results on degenerations.

Definition 1. Let R be a noetherian algebra over a field k, and let M and N be finitely generated left R-modules. We say that M degenerates to N, or N is a degeneration of M, if there is a discrete valuation ring (V, tV, k) that is a k-algebra (where t is a prime element) and a finitely generated left $R \otimes_k V$-module Q which satisfies the following conditions:

(1) Q is flat as a V-module.
(2) $Q/tQ \cong N$ as a left R-module.
(3) $Q[1/t] \cong M \otimes_k V[1/t]$ as a left $R \otimes_k V[1/t]$-module.

The following characterization of degenerations has been proved by Yoshino [7].

Theorem 2. [7, Theorem 2.2] *The following conditions are equivalent for finitely generated left R-modules M and N.*

(1) *M degenerates to N.*
(2) *There is a short exact sequence of finitely generated left R-modules*

$$0 \longrightarrow Z \xrightarrow{\binom{\varphi}{\psi}} M \oplus Z \longrightarrow N \longrightarrow 0,$$

such that the endomorphism ψ of Z is nilpotent, i.e. $\psi^n = 0$ for $n \gg 1$.

Remark 3. Let R be a noetherian k-algebra.

(1) Suppose that a finitely generated R-module M degenerates to a finitely generated module N. Then as a discrete valuation ring V in Definition 1 we can always take the ring $k[t]_{(t)}$. See [7, Corollary 2.4.]. Thus we always take $k[t]_{(t)}$ as V.

(2) Assume that there is an exact sequence of finitely generated left R-modules
$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0.$$
Then M degenerates to $L \oplus N$. See [7, Remark 2.5] for the detail.

(3) Let M and N be finitely generated R-modules and suppose that M degenerates to N. Then the modules M and N give the same class in the Grothendieck group, *i.e.* $[M] = [N]$ as an element of $K_0(\mathrm{mod}(R))$, where $\mathrm{mod}(R)$ denotes the category of finitely generated R-modules and R-homomorphisms.

We are mainly interested in degenerations of modules over commutative rings. Henceforth, in the rest of the paper, all the rings are assumed to be commutative.

Definition 4. Let M and N be finitely generated modules over a commutative noetherian k-algebra R.

(1) We denote by $M \leq_{deg} N$ if N is obtained from M by iterative degenerations, i.e. there is a sequence of finitely generated R-modules $L_0, L_1, \dots, L_r$ such that $M \cong L_0$, $N \cong L_r$ and each L_i degenerates to L_{i+1} for $0 \leq i < r$.

(2) We say that M degenerates by an extension to N if there is a short exact sequence $0 \to U \to M \to V \to 0$ of finitely generated R-modules such that $N \cong U \oplus N$.

We denote by $M \leq_{ext} N$ if N is obtained from M by iterative degenerations by extensions, i.e. there is a sequence of finitely generated R-modules $L_0, L_1, \dots, L_r$ such that $M \cong L_0$, $N \cong L_r$ and each L_i degenerates by an extension to L_{i+1} for $0 \leq i < r$.

If R is a local ring, then $\leq_{deg}$ and $\leq_{ext}$ are known to be partial orders on the set of isomorphism classes of finitely generated R-modules, which are called the degeneration order and the extension order respectively. See [6] for the detail.

Remark 5. By virtue of Remark 3, if $M \leq_{ext} N$ then $M \leq_{deg} N$. However the converse is not necessarily true.

For example, consider a ring $R = k[[x, y]]/(x^2)$. A pair of matrices over $k[[x, y]]$;
$$(\varphi, \psi) = \left(\begin{pmatrix} x & y^2 \\ 0 & x \end{pmatrix}, \begin{pmatrix} x & -y^2 \\ 0 & x \end{pmatrix} \right)$$
is a matrix factorization of the equation x^2, hence it gives a maximal Cohen-Macaulay R-module N that is isomorphic to the ideal $(x, y^2)R$. It is known that N is indecomposable. Then we can show that R degenerates to $(x, y^2)R$ in this case, and hence $R \leq_{deg} (x, y^2)R$. See [3, Remark 2.5.].

In general if $M \leq_{ext} N$ and if $M \ncong N$, then N is a non-trivial direct sum of modules. Since $N \cong (x, y^2)R$ is indecomposable, we see that $R \leq_{ext} (x, y^2)R$ can never happen.

Remark 6. We remark that if finitely generated R-modules M and N satisfy the relation $M \leq_{ext} N$, then M degenerates to N.

Now we note that the following lemma holds.

Lemma 7. *Let I be a two-sided ideal of a noetherian k-algebra R, and let M and N be finitely generated left R/I-modules. Then $M \leq_{deg} N$ (resp. $M \leq_{ext} N$) as R-modules if and only if so does as R/I-modules.* $\square$

We make several other remarks on degenerations for the later use.

Remark 8. Let R be a noetherian k-algebra, and let M and N be finitely generated R-modules. Suppose that M degenerates to N. The ith Fitting ideal of M contains that of N for all $i \geq 0$. Namely, denoting the ith Fitting ideal of an R-module M by $\mathcal{F}_i^R(M)$, we have $\mathcal{F}_i^R(M) \supseteq \mathcal{F}_i^R(N)$ for all $i \geqq 0$. (See [9, Theorem 2.5]).

Let $R = k[[x]]$ be a formal power series ring over a field k with one variable x and let M be an R-module of length n. It is easy to see that there is an isomorphism

$$M \cong R/(x^{p_1}) \oplus \cdots \oplus R/(x^{p_n}), \tag{2.1}$$

where

$$p_1 \geq p_2 \geq \cdots \geq p_n \geq 0 \quad \text{and} \quad \sum_{i=1}^{n} p_i = n. \tag{2.2}$$

In this case the finite presentation of M is given as follows:

$$0 \longrightarrow R^n \xrightarrow{\begin{pmatrix} x^{p_1} & & \\ & \ddots & \\ & & x^{p_n} \end{pmatrix}} R^n \longrightarrow M \longrightarrow 0.$$

Note that we can easily compute the ith Fitting ideal of M from this presentation;

$$\mathcal{F}_i^R(M) = (x^{p_{i+1}+\cdots+p_n}) \quad (i \geq 0).$$

We denote by p_M the sequence $(p_1, p_2, \cdots, p_n)$ of non-negative integers. Recall that such a sequence satisfying (2.2) is called a partition of n.

Conversely, given a partition $p = (p_1, p_2, \cdots, p_n)$ of n, we can associate an R-module of length n by (2.1), which we denote by $M(p)$. In such a way we see that there is a one-one correspondence between the set of partitions of n and the set of isomorphism classes of R-modules of length n.

Definition 9. Let n be a positive integer and let $p = (p_1, p_2, \cdots, p_n)$ and $q = (q_1, q_2, \cdots, q_n)$ be partitions of n. Then we denote $p \succeq q$ if it satisfies $\sum_{i=1}^{j} p_i \geq \sum_{i=1}^{j} q_i$ for all $1 \leq j \leq n$.

We note that $\succeq$ is known to be a partial order on the set of partitions of n and called the dominance order (see for example [4, page 7]).

In the following proposition we show the degeneration order for R-modules of length n coincides with the opposite of the dominance order of corresponding partitions.

Proposition 10. *Let $R = k[[x]]$ as above, and let M, N be R-modules of length n. Then the following conditions are equivalent:*

(1) $M \leq_{deg} N$,
(2) $M \leq_{ext} N$,
(3) $p_M \succeq p_N$.

Proof. First of all, we assume M degenerates to N, and let $p_M = (p_1, p_2, \cdots, p_n)$ and $p_N = (q_1, q_2, \cdots, q_n)$. Then, by definition, we have the equalities of the Fitting ideals; $\mathcal{F}_i^R(M) = (x^{p_{i+1}+\cdots+p_n})$ and $\mathcal{F}_i^R(M) = (x^{q_{i+1}+\cdots+q_n})$ for all $i \geq 0$. Since M degenerates to N, it follows from Remark 8 that $\mathcal{F}_i^R(M) \supseteq \mathcal{F}_i^R(N)$ for all i. Thus $p_{i+1} + \cdots + p_n \leq q_{i+1} + \cdots + q_n$. Since $\sum_{i=1}^n p_i = n = \sum_{i=1}^n q_i$, it follows that $p_1 + \cdots + p_i \geq q_1 + \cdots + q_i$ for all $i \geq 0$. Therefore $p_M \succeq p_N$, so that we have proved the implication (1) $\Rightarrow$ (3).

Finally we shall prove (3) $\Rightarrow$ (2). To this end let $p = (p_1, p_2, \cdots, p_n)$ and $q = (q_1, q_2, \cdots, q_n)$ be partitions of n. Note that it is enough to prove that the corresponding R-module $M(p)$ degenerates by an extension to $M(q)$ whenever q is a predecessor of p under the dominance order. (Recall that q is called a predecessor of p if $p \succeq q$ and there are no partitions r with $p \succeq r \succeq q$ other than p and q.)

Assume that q is a predecessor of p under the dominance order. Then it is easy to see that there are numbers $1 \leq i < j \leq n$ with $p_i - p_j \geq 2$, $p_i > p_{i+1}$, $p_{j-1} > p_j$ such that the equality $q = (p_1, \cdots, p_i - 1, p_{i+1}, \cdots, p_j + 1, \cdots, p_n)$ holds. In this case, setting $L = M((p_1, \cdots, p_{i-1}, p_{i+1}, \cdots, p_{j-1}, p_j, \cdots, p_n))$, we have $M(p) = L \oplus M((p_i, p_j))$ and $M(q) = L \oplus M((p_i - 1, p_j + 1))$. Note that, in general, if M degenerates by an extension to N, then $M \oplus L$ degenerates by an extension to $N \oplus L$, for any R-modules L. Hence it is enough to show that $M((a, b))$ degenerates by an extension to $M((a-1, b+1))$ if $a \geq b + 2$. However there is a short exact sequence of the form:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R/(x^{a-1}) & \longrightarrow & R/(x^a) \oplus R/(x^b) & \longrightarrow & R/(x^{b+1}) & \longrightarrow & 0 \\ & & 1 & \longrightarrow & (x, 1) & & & & \end{array}$$

Thus $M((a, b)) = R/(x^a) \oplus R/(x^b)$ degenerates by an extension to $M((a-1, b+1)) = R/(x^{a-1}) \oplus R/(x^{b+1})$. $\square$

Combining Proposition 10 with Lemma 7, we have the following corollary which will be used latter.

Corollary 11. *Let $R = k[[x]]/(x^m)$, where k is a field and m is a positive integer, and let M, N be finitely generated R-modules. Then $M \leq_{deg} N$ holds if and only if $M \leq_{ext} N$ holds.* $\square$

Next we describe another example.

Let k be a field of characteristic 0 and $R = k[[x_0, x_1, x_2, \cdots, x_d]]/(f)$, where f is a polynomial of the form

$$f = x_0^{n+1} + x_1^2 + x_2^2 + \cdots + x_d^2 \qquad (n \geq 1).$$

Recall that such a ring R is call the ring of simple singularity of type (A_n). Note that R is a Gorenstein complete local ring and has finite Cohen-Macaulay representation type. (Recall that a Cohen-Macaulay k-algebra R is said to be of finite Cohen-Macaulay representation type if there are only a finite number of isomorphism classes of objects in $\mathrm{CM}(R)$. See [5].) We shall show the following whose proof will be given in the last part of this section.

Theorem 12. *Let k be an algebraically closed field of characteristic 0 and let $R = k[[x_0, x_1, x_2, \cdots, x_d]]/(x_0^{n+1} + x_1^2 + x_2^2 + \cdots + x_d^2)$ as above, where we assume that d is even. For maximal Cohen-Macaulay R-modules M and N, if $M \leq_{deg} N$, then $M \leq_{ext} N$.*

To prove the theorem, we need several results concerning the stable degeneration which was introduced by Yoshino in [9].

Let A be a commutative Gorenstein ring. We denote by $\mathrm{CM}(A)$ the category of all maximal Cohen-Macaulay A-module with all A-homomorphisms. And we also denote by $\underline{\mathrm{CM}}(A)$ the stable category of $\mathrm{CM}(A)$. For a maximal Cohen-Macaulay module M we denote it by $\underline{M}$ to indicate that it is an object of $\underline{\mathrm{CM}}(A)$. Since A is Gorenstein, it is known that $\underline{\mathrm{CM}}(A)$ has a structure of triangulated category.

The following theorem proved by Yoshino [9] shows the relation between stable degenerations and ordinary degenerations.

Theorem 13. [9, Theorem 5.1, 6.1, 7.1] *Let $(R, \mathfrak{m}, k)$ be a Gorenstein complete local k-algebra, where k is an infinite field. Consider the following four conditions for maximal Cohen-Macaulay R-modules M and N:*

(1) *$R^m \oplus M$ degenerates to $R^n \oplus N$ for some $m, n \in \mathbb{N}$.*
(2) *There is a triangle*

$$\underline{Z} \xrightarrow{\binom{\underline{\varphi}}{\underline{\psi}}} \underline{M} \oplus \underline{Z} \longrightarrow \underline{N} \longrightarrow \underline{Z}[1]$$

in $\underline{\mathrm{CM}}(R)$, where $\underline{\psi}$ is a nilpotent element of $\underline{\mathrm{End}}_R(Z)$.
(3) *$\underline{M}$ stably degenerates to $\underline{N}$.*
(4) *There exists an $X \in \mathrm{CM}(R)$ such that $M \oplus R^m \oplus X$ degenerates to $N \oplus R^n \oplus X$ for some $m, n \in \mathbb{N}$.*

Then, in general, the implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ hold. If R is an isolated singularity, then (2) and (3) are equivalent. Furthermore, if R is an artinian ring, then the conditions (1), (2) and (3) are equivalent.

Corollary 14. [9, Corollary 6.6] *Let $(R_1, \mathfrak{m}_1, k)$ and $(R_2, \mathfrak{m}_2, k)$ be Gorenstein complete local k-algebras. Assume that the both R_1 and R_2 are isolated singularities, and that k is an infinite field. Suppose there is a k-linear equivalence $F : \underline{\mathrm{CM}}(R_1) \to \underline{\mathrm{CM}}(R_2)$ of triangulated categories. Then, for $\underline{M}$, $\underline{N} \in \underline{\mathrm{CM}}(R_1)$, $\underline{M}$ stably degenerates to $\underline{N}$ if and only if $F(\underline{M})$ stably degenerates to $F(\underline{N})$.* □

We now consider the stable analogue of the degeneration by an extension.

Definition 15.

(1) We denote by $\underline{M} \leq_{st} \underline{N}$ if $\underline{N}$ is obtained from $\underline{M}$ by iterative stable degenerations, i.e. there is a sequence of objects $\underline{L_0}, \underline{L_1}, \ldots, \underline{L_r}$ in $\underline{\mathrm{CM}}(R)$ such that $\underline{M} \cong \underline{L_0}$, $\underline{N} \cong \underline{L_r}$ and each $\underline{L_i}$ stably degenerates to $\underline{L_{i+1}}$ for $0 \leq i < r$.
(2) We say that $\underline{M}$ stably degenerates by a triangle to $\underline{N}$, if there is a triangle of the form $\underline{U} \to \underline{M} \to \underline{V} \to \underline{U}[1]$ in $\underline{\mathrm{CM}}(R)$ such that $\underline{U} \oplus \underline{V} \cong \underline{N}$. We denote by $\underline{M} \leq_{tri} \underline{N}$ if there is a finite sequence of modules $\underline{L_0}, \underline{L_1}, \cdots, \underline{L_r}$ in $\underline{\mathrm{CM}}(R)$ such that $\underline{M} \cong \underline{L_0}$, $\underline{N} \cong \underline{L_r}$ and each $\underline{L_i}$ stably degenerates by a triangle to $\underline{L_{i+1}}$ for $0 \leq i < r$.

Remark 16. Let R be a Gorenstein local ring that is a k-algebra.

(1) Let $M, N \in \mathrm{CM}(R)$. If M degenerates to N, then $\underline{M}$ stably degenerates to $\underline{N}$. Therefore that $M \leq_{deg} N$ forces that $\underline{M} \leq_{st} \underline{N}$. (See [9, Lemma 4.2].)

(2) Suppose that there is a triangle

$$\underline{L} \longrightarrow \underline{M} \longrightarrow \underline{N} \longrightarrow \underline{L}[1],$$

in $\underline{\mathrm{CM}}(R)$. Then $\underline{M}$ stably degenerates to $\underline{L} \oplus \underline{N}$, thus $\underline{M} \leq_{st} \underline{L} \oplus \underline{N}$. (See [9, Proposition 4.3].)

We need the following proposition to prove Theorem 12.

Proposition 17. *Let $(R, \mathfrak{m}, k)$ be a Gorenstein complete local ring and let $M, N \in \mathrm{CM}(R)$. Assume $[M] = [N]$ in $K_0(\mathrm{mod}(R))$. Then $\underline{M} \leq_{tri} \underline{N}$ if and only if $M \leq_{ext} N$.* $\square$

Now we proceed to the proof of Theorem 12.

Let k be an algebraically closed field of characteristic 0 and let

$$R = k[[x_0, x_1, x_2, \cdots, x_d]]/(x_0^{n+1} + x_1^2 + x_2^2 + \cdots + x_d^2)$$

as in the theorem, where we assume that d is even. Suppose that $M \leq_{deg} N$ for maximal Cohen-Macaulay R-modules M and N. We want to show $M \leq_{ext} N$.

Since $M \leq_{deg} N$, we have $\underline{M} \leq_{st} \underline{N}$ in $\underline{\mathrm{CM}}(R)$ and $[M] = [N]$ in $K_0(\mathrm{mod}(R))$, by Remarks 16(1) and 3(3). Now let us denote $R' = k[[x_0]]/(x_0^{n+1})$, and we note that $\underline{\mathrm{CM}}(R)$ and $\underline{\mathrm{CM}}(R')$ are equivalent to each other as triangulated categories. In fact this equivalence is given by using the lemma of the Knörrer's periodicity (cf. [5]), since d is even. Let $\Omega : \underline{\mathrm{CM}}(R) \to \underline{\mathrm{CM}}(R')$ be a triangle functor which gives the equivalence. Then, by virtue of Corollary 14, we have $\Omega(\underline{M}) \leq_{st} \Omega(\underline{N})$ in $\underline{\mathrm{CM}}(R')$. Since R' is an artinian algebra, the equivalence (1) $\Leftrightarrow$ (3) holds in Theorem 13, and thus we have $\tilde{M} \oplus R'^m \leq_{deg} \tilde{N} \oplus R'^n$, where $\tilde{M}$ (resp. $\tilde{N}$) is a module in $\mathrm{CM}(R')$ with $\underline{\tilde{M}} \cong \Omega(\underline{M})$ (resp. $\underline{\tilde{N}} \cong \Omega(\underline{N})$) and m, n are non-negative integers. It then follows from Corollary 11 that $\tilde{M} \oplus R'^m \leq_{ext} \tilde{N} \oplus R'^n$. Hence, by Proposition 17, we have that $\Omega(\underline{M}) \leq_{tri} \Omega(\underline{N})$ in $\underline{\mathrm{CM}}(R')$. Noting that the partial order $\leq_{tri}$ is preserved under a triangle functor, we see that $\underline{M} \leq_{tri} \underline{N}$ in $\underline{\mathrm{CM}}(R)$. Since $[M] = [N]$ in $K_0(\mathrm{mod}(R))$, applying Proposition 17, we finally obtain that $M \leq_{ext} N$. $\square$

Example 18. Let $R = k[[x_0, x_1, x_2]]/(x_0^3 + x_1^2 + x_2^2)$, where k is an algebraically closed field of characteristic 0. Let $\mathfrak{p}$ and $\mathfrak{q}$ be the ideals generated by $(x_0, x_1 - \sqrt{-1}\ x_2)$ and $(x_0^2, x_1 + \sqrt{-1}\ x_2)$ respectively. It is known that the set $\{R, \mathfrak{p}, \mathfrak{q}\}$ is a complete list of the isomorphism classes of indecomposable maximal Cohen-Macaulay modules over R. The Hasse diagram of degenerations of maximal Cohen-Macaulay R-modules of rank 3 is a disjoint union of the following diagrams:

$$\begin{tikzcd}[arrows=dash]
\mathfrak{p}\oplus\mathfrak{p}\oplus\mathfrak{p} \arrow[dr] & & \mathfrak{q}\oplus\mathfrak{q}\oplus\mathfrak{q} \arrow[dl] & \mathfrak{p}\oplus\mathfrak{p}\oplus\mathfrak{q} \arrow[d] & \mathfrak{p}\oplus\mathfrak{q}\oplus\mathfrak{q} \arrow[d] \\
& R\oplus\mathfrak{p}\oplus\mathfrak{q} \arrow[d] & & R\oplus\mathfrak{q}\oplus\mathfrak{q} \arrow[d] & R\oplus\mathfrak{p}\oplus\mathfrak{p} \arrow[d] \\
& R^3, & & R^2\oplus\mathfrak{p}, & R^2\oplus\mathfrak{q}.
\end{tikzcd}$$

3. Extended orders

In the rest of this paper R denotes a (commutative) Cohen-Macaulay complete local k-algebra, where k is any field.

We shall show that any extended degenerations of maximal Cohen-Macaulay R-modules are generated by extended degenerations of Auslander-Reiten (abbr.AR) sequences if R is of finite Cohen-Macaulay representation type. For the theory of AR sequences of maximal Cohen-Macaulay modules, we refer to [5]. First of all we recall the definitions of the extended orders generated respectively by degenerations, extensions and AR sequences.

Definition 19. [6, Definition 4.11, 4.13] The relation $\leq_{DEG}$ on $\mathrm{CM}(R)$, which is called the extended degeneration order, is a partial order generated by the following rules:

(1) If $M \leq_{deg} N$ then $M \leq_{DEG} N$.
(2) If $M \leq_{DEG} N$ and if $M' \leq_{DEG} N'$ then $M \oplus M' \leq_{DEG} N \oplus N'$
(3) If $M \oplus L \leq_{DEG} N \oplus L$ for some $L \in \mathrm{CM}(R)$ then $M \leq_{DEG} N$.
(4) If $M^n \leq_{DEG} N^n$ for some natural number n then $M \leq_{DEG} N$.

Definition 20. [6, Definition 3.6] The relation $\leq_{EXT}$ on $\mathrm{CM}(R)$, which is called the extended extension order, is a partial order generated by the following rules:

(1) If $M \leq_{ext} N$ then $M \leq_{EXT} N$.
(2) If $M \leq_{EXT} N$ and if $M' \leq_{EXT} N'$ then $M \oplus M' \leq_{EXT} N \oplus N'$
(3) If $M \oplus L \leq_{EXT} N \oplus L$ for some $L \in \mathrm{CM}(R)$ then $M \leq_{EXT} N$.
(4) If $M^n \leq_{EXT} N^n$ for some natural number n then $M \leq_{EXT} N$.

Definition 21. [6, Definition 5.1] The relation $\leq_{AR}$ on $\mathrm{CM}(R)$, which is called the extended AR order, is a partial order generated by the following rules:

(1) If $0 \to X \to E \to Y \to 0$ is an AR sequence in $\mathrm{CM}(R)$, then $E \leq_{AR} X \oplus Y$.
(2) If $M \leq_{AR} N$ and if $M' \leq_{AR} N'$ then $M \oplus M' \leq_{AR} N \oplus N'$
(3) If $M \oplus L \leq_{AR} N \oplus L$ for some $L \in \mathrm{CM}(R)$ then $M \leq_{AR} N$.
(4) If $M^n \leq_{AR} N^n$ for some natural number n then $M \leq_{AR} N$.

The following is the main theorem of this section.

Theorem 22. *Let R be a Cohen-Macaulay complete local k-algebra as above. Adding to this, we assume that R is of finite Cohen-Macaulay representation type.Then the following conditions are equivalent for $M, N \in \mathrm{CM}(R)$:*

(1) $M \leq_{DEG} N$,
(2) $M \leq_{EXT} N$,
(3) $M \leq_{AR} N$.

Proof. The implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are clear from the definitions.

To prove (1) $\Rightarrow$ (2), it suffices to show that $M \leq_{EXT} N$ whenever M degenerates to N. If M degenerates to N, then, by virtue of Theorem 2, we have a short exact sequence $0 \to Z \to M \oplus Z \to N \to 0$ with $Z \in \mathrm{CM}(R)$. Thus $M \oplus Z \leq_{ext} N \oplus Z$, hence $M \leq_{EXT} N$.

It remains to prove that (2) $\Rightarrow$ (3), for which we need the following lemma which is essentially due to Auslander and Reiten [1].

Lemma 23. *Under the same assumptions on R as in Theorem 22, let $0 \to L \to M \to N \to 0$ be a short exact sequence in* $\mathrm{CM}(R)$*. Then there are a finite number of AR sequences in* $\mathrm{CM}(R)$*;*

$$0 \to X_i \to E_i \to Y_i \to 0 \quad (1 \leq i \leq n),$$

such that there is an equality in $\mathrm{G}(\mathrm{CM}(R))$*;*

$$L - M + N = \sum_{i=1}^{n} (X_i - E_i + Y_i).$$

Here, $\mathrm{G}(\mathrm{CM}(R)) = \bigoplus \mathbb{Z} \cdot X$ *where X runs through all isomorphism classes of indecomposable objects in* $\mathrm{CM}(R)$*.*

To prove this lemma, we consider the functor category $\mathrm{Mod}(\mathrm{CM}(R))$ and the Auslander category $\mathrm{mod}(\mathrm{CM}(R))$ of $\mathrm{CM}(R)$. $\square$

Remark 24. In the paper [6], Yoshino introduced the order relation $\leq_{hom}$ as well. Adding to the assumption that R is of finite Cohen-Macaulay representation type, if we assume further conditions on R, such as R is an integral domain of dimension 1 or R is of dimension 2, then he showed that $\leq_{hom}$ is also equal to any of $\leq_{AR}$, $\leq_{EXT}$ and $\leq_{DEG}$.

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Department of general education,
Kure National College of Technology,
2-2-11 Agaminami, Kure, Hiroshima, 737-8506 JAPAN
E-mail address: hiramatsu@kure-nct.ac.jp