THE LOEWY LENGTH OF TENSOR PRODUCTS FOR DIHEDRAL TWO-GROUPS

ERIK DARPÖ AND CHRISTOPHER C. GILL

Abstract. The indecomposable modules of a dihedral 2-group over a field of characteristic 2 were classified by Ringel over 30 years ago. However, relatively little is known about the tensor products of such modules, except in certain special cases. We describe here the main result of our recent work determining the Loewy length of a tensor product of modules for a dihedral 2-group. As a consequence of this result, we can determine precisely when a tensor product has a projective direct summand.

1. Introduction

Let k be a field of positive characteristic p and let G be a finite group. The group algebra kG is a Hopf algebra with coproduct and co-unit given by $\Delta(\sum_{g\in G} r_g g) = \sum_{g\in G} r_g g \otimes g$ and $\epsilon(\sum_{g\in G} r_g g) = \sum_{g\in G} r_g$ for $r_g \in k$. Thus, there is a tensor product operation on the category of kG-modules. If M and N are kG-modules, then the tensor product of M and N is the module with underlying vector space $M \otimes_k N$ and module structure given by $g(m \otimes n) = \Delta(g)(m \otimes n) = gm \otimes gn$ for $g \in G, m \in M, n \in N$. The tensor product is a frequently used tool in the representation theory of finite groups. However, the problem of determining the decomposition of a tensor product of two modules of a finite group G – the Clebsch-Gordan problem – can be extremely difficult.

One approach to understanding tensor products of kG-modules goes via the representation ring, or Green ring, of kG. The isomorphism classes of finite-dimensional kG-modules form a semiring, with addition given by the direct sum, and multiplication by the tensor product of kG-modules. The Green ring, $A(kG)$, is the Groethendieck ring of this semiring, i.e., the ring obtained by formally adjoining additive inverses to all elements in the semiring. Research in this direction was pioneered by J. A. Green [6], who proved the Green ring of a cyclic p-group is semisimple. The question of semisimplicity of the Green ring for other finite groups has been studied by several authors since. Benson and Carlson [2] gave a method to produce nilpotent elements in a Green ring, and determined a quotient of the Green ring which has no nilpotent elements.

This so-called Benson–Carlson quotient of the Green ring was studied by Archer [1] in the case of the dihedral 2-groups, who realised it as an integral group ring of an abelian, infinitely generated, torsion-free group. Archer gave a precise statement relating the multiplication of two elements in this infinite group to the Auslander–Reiten quiver of kD_{4q}. The Green ring of the Klein four group V_4 was completely determined by Conlon [4]; a summary of this result can be found in [1].

For the dihedral 2-groups D_{4q}, the indecomposable modules, over fields of characteristic 2, were classified by Ringel [7] over 30 years ago. However, very little progress has been made towards understanding the behaviour of the tensor product of the kD_{4q}-modules. In particular, the decomposition of a tensor product of two indecomposable kD_{4q}-modules is not known, other than in some very special cases. One example is the work of Bessenrodt [3], classifying the endotrivial kD_{4q}-modules, thus determining the kD_{4q}-modules M for which the tensor product of M with its dual M^* is the direct sum of a trivial and a projective module.

In recent work [5], we have continued the study of tensor products of kD_{4q}-modules, determining completely the Loewy length of the tensor product of any two indecomposable kD_{4q}-modules. This provides an additional piece of information towards the understanding of the Green rings of the dihedral 2-groups, and gives certain bounds on which modules can occur as direct summands of a tensor product. In particular, it determines precisely when a tensor product of two modules has a projective direct summand.

The *Loewy length* $\ell(M)$ of a module M is, by definition, the common length of the radical series and the socle series of M, that is, $\ell(M) = \min\{t \in \mathbb{N} \mid \mathrm{rad}^t(kD_{4q})M = 0\}$.

In the next section, we recall Ringel's classification of the indecomposable kD_{4q}-modules. Section 3 gives a summary of the results in [5], and in Section 4, we give examples illuminating our results and showing how they can be used to determine the direct sum decomposition of a tensor product in certain cases.

2. The indecomposable modules of dihedral 2-groups

Let q be a 2-power, and write $D_{4q} = \langle x, y \mid x^2 = y^2 = 1, (xy)^q = (yx)^q \rangle$ for the dihedral group of order $4q$. There is an isomorphism of algebras

$$kD_{4q} \xrightarrow{\sim} \Lambda_q := \frac{k\langle X, Y\rangle}{(X^2, Y^2, (XY)^q - (YX)^q)},$$

given by $x \mapsto 1 + X$ and $y \mapsto 1 + Y$. Setting $\Delta(X) = 1 \otimes X + X \otimes 1 + X \otimes X$ and $\Delta(Y) = 1 \otimes Y + Y \otimes 1 + Y \otimes Y$ defines a coproduct on Λ_q corresponding under this isomorphism to the Hopf algebra structure of kD_{4q}. Owing to the fact that Λ_q is a special biserial algebra, its non-projective modules split into two classes, known as string modules and band modules. We describe both classes of modules below.

Let $\bar{\mathcal{W}}$ be the set of words in letters a, b and inverse letters a^{-1}, b^{-1} such that a or a^{-1} are always followed by b or b^{-1} and b or b^{-1} are always followed by a or a^{-1}. A *directed subword* of a word $w \in \bar{\mathcal{W}}$ is a word w' in either the letters $\{a, b\}$ or $\{a^{-1}, b^{-1}\}$ such that $w = w_1 w' w_2$ for some words $w_1, w_2 \in \bar{\mathcal{W}}$. Let $\mathcal{W}$ be the subset of $\bar{\mathcal{W}}$ consisting of words in which all directed subwords are of length strictly less than $2q$. Define an equivalence relation $\sim_1$ on $\mathcal{W}$ by $w \sim_1 w'$ if, and only if, $w' = w$ or $w' = w^{-1}$. Given $w = l_1 \dots l_n \in \mathcal{W}$, the *string module determined by* w, denoted by $M(w)$, is the $n+1$-dimensional module with basis $e_0, \dots, e_n$ and Λ_q-action given by the following schema:

$$ke_0 \xleftarrow{l_1} ke_1 \xleftarrow{l_2} ke_2 \longleftarrow \dots \xleftarrow{l_{n-1}} ke_{n-1} \xleftarrow{l_n} ke_n \ .$$

If $l_i \in \{a^{-1}, b^{-1}\}$, the corresponding arrow should be interpreted as going in the opposite direction, from ke_{i-1} to ke_i, and having the label l_i^{-1}. Now X maps e_i to e_j ($j \in \{i-1, i+1\}$) if there is an arrow $ke_i \xrightarrow{a} ke_j$, and as zero if no arrow labelled with a starting in ke_i exists. Similarly, the action of Y is given by arrows labelled with b. Two modules $M(w)$ and $M(w')$, $w, w' \in \mathcal{W}$, are isomorphic if, and only if, $w \sim_1 w'$.

Next, let $\mathcal{W}'$ be the subset of $\mathcal{W}$ consisting of words w of even positive length containing letters from both $\{a, b\}$ and $\{a^{-1}, b^{-1}\}$, and such that w is not a power of a word of smaller length. Given $w = l_1 \dots l_m \in \mathcal{W}'$, and φ an indecomposable linear automorphism of k^n, the *band module determined by w and φ*, $M(w, \varphi)$, is the Λ_q-module with underlying vector space $\bigoplus_{i=0}^{m-1} V_i$ where $V_i = k^n$, and Λ_q-action specified by the following schema:

$$\begin{tikzcd} V_0 \arrow[rrrrr, bend right, "l_m"'] & V_1 \arrow[l, "l_1=\varphi"'] & V_2 \arrow[l, "l_2"'] & \cdots \arrow[l] & V_{m-2} \arrow[l, "l_{m-2}"'] & V_{m-1} \arrow[l, "l_{m-1}"'] \end{tikzcd} .$$

The interpretation of the schema is similar to that for string modules. The elements $l_2, \dots, l_m$ act as the identity map on k^n, l_1 acts as the linear automorphism φ (this means that if $l_1 \in \{a^{-1}, b^{-1}\}$ then V_0 is mapped onto V_1 by φ^{-1} under either X or Y).

Define $\sim_2$ to be the equivalence relation on $\mathcal{W}'$ defined by $w \sim_2 w'$ if, and only if, w of w^{-1} is a cyclic permutation of w'. Two band modules $M(w, \varphi)$ and $M(w', \psi)$ are isomorphic if, and only if, $w \sim_2 w'$ and $\varphi = \phi\psi\phi^{-1}$ for some linear automorphism ϕ. It may be noted that for every $w \in \mathcal{W}'$ there exists a $w' \in \mathcal{W}$ with an even number of maximal directed subwords such that $w \sim_2 w'$. While there are several different choices for w', its maximal directed subwords are uniquely determined, as elements in $\mathcal{W}/\sim_1$, by w.

Every indecomposable non-projective kD_{4q}-module is isomorphic either to a string or a band module, but not both. There is a single, indecomposable projective module, ${}_{kD_{4q}}kD_{4q}$. The Loewy length of any non-projective module is at most $2q$, while $\ell({}_{kD_{4q}}kD_{4q}) = 2q + 1$.

3. Loewy length of tensor products

Here, we give an overview of the results in [5]. We fix the the following conventions and notation. The least natural number is 0. For $l < 2q$, $A_l \in \mathcal{W}$ is the (unique) word of length l in the letters a, b ending in a, and similarly $B_l \in \mathcal{W}$ is the word of length l in the letters a, b ending in b.

If M is a module and $X \subset M$ any set of generators, then $\ell(M) = \max\{\ell(\langle x \rangle) \mid x \in X\}$, and if N is another module then, in a similar fashion, $\ell(M \otimes N) = \max\{\ell(\langle x \rangle \otimes N) \mid x \in X\}$. Any kD_{4q}-module that is generated by a single element is isomorphic to either $M(A_l B_m^{-1})$ or $M(A_l B_m^{-1}, \rho)$ for some $l, m < 2q$ and $\rho \in k \setminus \{0\}$, hence it is sufficient to determine Loewy lengths of tensor products of these types of modules, to solve the problem for arbitrary modules M and N. Refining these ideas a little, one can prove the following results.

Proposition 1. *Let M and N be kD_{4q}-modules. If M is a string module corresponding to a word $w \in \mathcal{W}$ with maximal directed subwords w_i, $i \in \{1, \dots, m\}$, then*

$$\ell(M \otimes N) = \max\{\ell(M(w_i) \otimes N) \mid i \in \{1, \dots, m\}\}.$$

Proposition 2. *Let $M = M(w, \varphi)$, where $w \in \mathcal{W}'$ and φ is an indecomposable automorphism of k^n, $n \geqslant 1$. Let $w' \in \mathcal{W}'$ be a word with an even number of maximal directed subwords w_i, $i \in \{1, \dots, 2m\}$, such that $w \sim_2 w'$. If m and n are not both equal to 1, then*

$$\ell(M(w, \varphi) \otimes N) = \max\{\ell(M(w_i) \otimes N) \mid i \in \{1, \dots, 2m\}\}.$$

Proposition 1 and 2 leave us with determining the Loewy lengths of tensor products of modules of the types $M(A_l)$, $M(B_m)$ and $M(A_lB_m^{-1},\rho)$ for $l,m<2q$, $\rho\in k\setminus\{0\}$. One can note that these are precisely the non-projective kD_{4q}-modules whose top and socle are simple modules.

Given $x\in\mathbb{N}$, denote by $[x]_i$ the ith term of its binary expansion, i.e., $[x]_i\in\{0,1\}$ such that $x=\sum_{i\in\mathbb{N}}[x]_i2^i$. Let $l,m\in\mathbb{N}$, and take $a\in\mathbb{N}$ to be the smallest number such that $[l]_i+[m]_i\leqslant 1$ for all $i\geqslant a$. Set $\lambda=\sum_{i\geqslant a}[l]_i2^i$ and $\mu=\sum_{j\geqslant a}[m]_j2^j$. Now define a binary operation $\#$ on $\mathbb{N}$ by setting

$$l\#m=\lambda+\mu+2^a-1\,.$$

If the binary expansions of l and m are disjoint, that is, if $[l]_i+[m]_i\leqslant 1$ for all $i\in\mathbb{N}$, we write $l\perp m$. Now observe that if $l\perp m$ then $a=0$ and $l\#m=l+m$, while $l\#m<l+m$ otherwise.

Example 3. We have $85\#38=119$. Namely, $85=2^0+2^2+2^4+2^6$ and $38=2^1+2^2+2^5$, hence $a=3$ for these two numbers, and therefore $85\#38=(2^4+2^6)+2^5+2^3-1=119$. Clearly, $85\#38=119<123=85+38$, which was to be expected, since $85\not\perp 38$.

The relevance of the operation $\#$ is that it neatly describes the Loewy length of a tensor product of uniserial modules, that is, modules of the type $M(A_l)$ and $M(B_l)$. If u is a generating element in the module $M(A_l)$, and v a generating element in $M(A_m)$, then $\ell(\langle u\otimes v\rangle)=l\#m+1$ (observe that $u\otimes v$ does not generate $M(A_l)\otimes M(A_m)$, unless l or m equals zero). Showing this is the most important step in the proof of our principal theorem, which gives the Loewy lengths of tensor products of kD_{4q}-modules with simple top and simple socle.

Theorem 4. *Let $l,m\in\mathbb{N}$, $l_1,l_2,m_1,m_2\in\mathbb{N}\setminus\{0\}$, $\rho,\sigma\in k\setminus\{0\}$.*

1. String with string:

$$\ell(M(A_l)\otimes M(B_m))=\begin{cases}1+l\#m=1+l+m & \text{if}\quad l\perp m,\\ 2+l\#m & \text{if}\quad l\not\perp m,\end{cases}$$

$$\ell(M(A_l)\otimes M(A_m))=\begin{cases}1+l\#m & \text{if } [l]_t=[m]_t=0 \text{ for all } 0\leqslant t<a-1,\\ 2+l\#m & \text{otherwise.}\end{cases}$$

where $a=\min\{r\in\mathbb{N}\mid [l]_t+[m]_t\leqslant 1,\ \forall t\geqslant r\}$.

2. Band with string:

$$\ell\left(M(A_{l_1}B_{l_2}^{-1},\rho)\otimes M(A_m)\right)=\begin{cases}2+(l_1-1)\#m & \text{if } \rho=1,\ l_1=l_2 \text{ and}\\ & \quad l_1\perp m,\ l_1\perp(m-1),\\ \ell\left(M\left(A_{l_1}B_{l_2}^{-1}\right)\otimes M(A_m)\right) & \text{otherwise.}\end{cases}$$

3. Band with band: *Let $M=M\left(A_{l_1}B_{l_2}^{-1},\rho\right)$, $N=M\left(A_{m_1}B_{m_2}^{-1},\sigma\right)$.*
   (a) *If $l_1\neq l_2$, then*

$$\ell(M\otimes N)=\ell(M\left(A_{l_1}B_{l_2}^{-1}\right)\otimes N).$$

Assume $l_1 = l_2$, $m_1 = m_2$.

(b) *If $l_1 \not\perp m_1$, $l_1 \not\perp (m_1 - 1)$, $(l_1 - 1) \not\perp m_1$ then*

$$\ell(M \otimes N) = 2 + (l_1 - 1)\#(m_1 - 1).$$

(c) *If $l_1 \perp m_1$, $(l_1 - 1) \perp m_1$, then*

$$\ell(M \otimes N) = \begin{cases} 2 + (l_1 - 1)\#(m_1 - 1) & \text{if} \quad \sigma = 1, \\ l_1 + m_1 + 1 & \text{otherwise.} \end{cases}$$

(d) *If $l_1 \perp m_1$, $l_1 \perp (m_1 - 1)$, then*

$$\ell(M \otimes N) = \begin{cases} 2 + (l_1 - 1)\#(m_1 - 1) & \text{if} \quad \rho = 1, \\ l_1 + m_1 + 1 & \text{otherwise.} \end{cases}$$

(e) *If $(l_1 - 1) \perp m_1$, $l_1 \perp (m_1 - 1)$, then*

$$\ell(M \otimes N) = \begin{cases} 2 + (l_1 - 1)\#(m_1 - 1) & \text{if} \quad \rho = \sigma = 1, \\ l_1 + m_1 & \text{if} \quad \rho = \sigma \neq 1, \\ l_1 + m_1 + 1 & \text{otherwise.} \end{cases}$$

We remark that if any one of the statements $l \perp m$, $(l-1) \perp m$ and $l \perp (m-1)$ holds true, then so does precisely one of the remaining ones. Hence 3(b)–3(e) in the theorem give a complete list of cases. As a consequence of Theorem 4:1, it is not difficult to prove the following sequence of inequalities:

$$\ell(M(A_l) \otimes M(A_m)) \leqslant \ell(M(A_l) \otimes M(B_m)) \leqslant \ell(M(A_l) \otimes M(A_{m+1}))\,. \tag{3.1}$$

It is entirely possible that each of these inequalities are identities. This is the case for example if $l = 8$, $m = 9$: $\ell(M(A_8) \otimes M(A_9)) = 2 + 8\#9 = 2 + 15 = 2 + 8\#10 = \ell(M(A_8) \otimes M(A_{10}))$.

Corollary 5. *Let $l, m < 2q$, $0 < l_1, l_2, m_1, m_2 < 2q$, and $\rho, \sigma \in k \setminus \{0\}$.*

1. *$M(A_l) \otimes M(B_m)$ has a projective direct summand if, and only if, $l + m \geqslant 2q$,*
2. *$M(A_l) \otimes M(A_m)$ has a projective direct summand if, and only if, $l + m \geqslant 2q + 1$.*
3. *$M(A_{l_1} B_{l_2}^{-1}, \rho) \otimes M(A_m)$ has a projective direct summand precisely when*

$$\max\{l_1 + m - 1, l_2 + m\} \geqslant 2q.$$

4. *If $l_1 \neq l_2$ or $m_1 \neq m_2$, then $M\left(A_{l_1} B_{l_2}^{-1}, \rho\right) \otimes M\left(A_{m_1} B_{m_2}^{-1}, \sigma\right)$ has a projective direct summand if, and only if,*

$$\max\{l_1 + m_1 - 1, l_1 + m_2, l_2 + m_1, l_2 + m_2 - 1\} \geqslant 2q\,.$$

5. *If $l_1 = l_2$, $m_1 = m_2$ then $M\left(A_{l_1} B_{l_2}^{-1}, \rho\right) \otimes M\left(A_{m_1} B_{m_2}^{-1}, \sigma\right)$ has projective direct summands if, and only if,*
   (a) *$l_1 \perp (m_1 - 1)$, $\rho \neq \sigma$ and $l_1 + m_1 = 2q$, or*
   (b) *$l_1 \not\perp (m_1 - 1)$, and $l_1 + m_1 \geqslant 2q$.*

We remark that, for $l, m < 2q$, the condition $l + m \geqslant 2q$ implies $l \not\perp m$. Thus, in particular, in 5(a) above, the condition $l_1 \perp (m_1 - 1)$ is equivalent to $(l_1 - 1) \perp m_1$, and similarly, in 5(b), $l_1 \not\perp (m_1 - 1)$ could be replaced by $(l_1 - 1) \not\perp m_1$.

4. Examples

Example 6. Let $M = M(A_5 B_7^{-1} B_4)$, $N = M(A_6 B_4^{-1})$. By Proposition 1,

$$\begin{aligned}\ell(M \otimes N) &= \max\left\{\ell(M(w) \otimes M(w')) \mid w \in \{A_5, B_7^{-1}, B_4\}, w' \in \{A_6, B_4^{-1}\}\right\} \\ &= \ell(M(B_7^{-1}) \otimes M(A_6)) = \ell(M(B_7) \otimes M(A_6))\,.\end{aligned}$$

Since $7 \not\preceq 6$, by Theorem 4:1 we have, $\ell(M(B_7) \otimes M(A_6)) = 2 + 7\#6 = 9$. Hence, the Loewy length of $M \otimes N$ is 9 and, seen as a kD_{16}-module, $M \otimes N$ has a projective direct summand.

Example 7. By Proposition 1 and the inequality (3.1), we have

$$\begin{aligned}\ell(M(A_l) \otimes M(A_{m+1} B_m^{-1})) &= \max\{\ell(M(A_l) \otimes M(A_{m+1})), \ell(M(A_l) \otimes M(B_m))\} \\ &= \ell(M(A_l) \otimes M(A_{m+1}))\,.\end{aligned}$$

for all $l, m \in \mathbb{N}$.

Example 8. While it is clear that $\ell(M(A_l B_l^{-1}, 1) \otimes N) \leqslant \ell(M(A_l B_l^{-1}) \otimes N)$, the difference between the lengths of the two tensor products may be zero, or arbitrarily large. For example, if $N = M(A_m)$ and $l \not\preceq m$, then $\ell(M(A_l B_l^{-1}, 1) \otimes N) = \ell(M(A_l B_l^{-1}) \otimes N)$ by Theorem 4:2. If, on the other hand, $l = 2^r$ and $m = 2^s$ with $r > s$ then

$$\ell(M(A_l B_l^{-1}, 1) \otimes M(A_m)) = 2 + (l-1)\#m = 2 + 2^r - 1 = 1 + 2^r,$$

while

$$\ell(M(A_l B_l^{-1}) \otimes M(A_m)) = \ell(M(B_l) \otimes M(A_m)) = 1 + l + m = 1 + 2^r + 2^s\,.$$

Example 9. Let $M = M(a) = M(A_1)$ and $N = M(b(ab)^l) = M(B_{2l+1})$ for some $l \in \mathbb{N}$. Now $1 \not\preceq (2l+1)$, and $1\#(2l+1) = 2l+1$, so by Theorem 4:1, $\ell(M \otimes N) = 2l+3$. In this case, the Loewy length actually provides the missing piece of information to compute the isomorphism type of $M \otimes N$.

Namely, since k is the unique simple module, we have

$$\begin{aligned}\dim \operatorname{soc}(M \otimes N) &= \dim \operatorname{Hom}_{kD_{4q}}(k, M \otimes N) = \dim \operatorname{Hom}_{kD_{4q}}(N^*, M) \\ &= \dim \operatorname{Hom}_{kD_{4q}}(N, M) = 1\end{aligned}$$

and similarly,

$$\dim \operatorname{top}(M \otimes N) = \dim \operatorname{Hom}_{kD_{4q}}(M \otimes N, k) = 1\,.$$

Hence $M \otimes N$ is a module with simple top and simple socle, of dimension $4(l+1)$, and Loewy length $2l+3$. A module satisfying these conditions is indecomposable, and must be isomorphic to $M(A_{2l+2} B_{2l+2}^{-1}, \rho)$ for some $\rho \in k \smallsetminus \{0\}$. Now if k is the prime field, that is the Galois field with two elements, this means that $\rho = 1$. From this follows that $\rho = 1$ also in the general case, since extension of scalars commutes with taking tensor products. Hence, we have $M \otimes N \simeq M(A_{2l+2} B_{2l+2}^{-1}, 1)$.

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Darpö: Graduate School of Mathematics, Nagoya University, Furocho, Chikusaku, Nagoya, Japan

E-mail address: darpo@math.nagoya-u.ac.jp

Gill: Department of Algebra, Charles University, Sokolovska 83, Praha 8, 186 75, Czech Republic

E-mail address: gill@karlin.mff.cuni.cz