REALIZING STABLE CATEGORIES AS DERIVE CATEGORIES

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Abstract. In this paper, we compare two different kinds of triangulated categories. First one is the stable category $\underline{\mathrm{mod}}A$ of the category of $\mathbb{Z}$-graded modules over a positively grade self-injective algebra A. Second one is the derived category $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Lambda)$ of the category of modules over an algebra Λ. Our aim is give the complete answer to the following question. For a positively graded self-injective algebra A, when is $\underline{\mathrm{mod}}A$ triangle-equivalent to $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Lambda)$ for some algebra Λ ? The main result of this paper gives the following very simple answer. $\underline{\mathrm{mod}}A$ is triangle-equivalent to $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Lambda)$ for some algebra Λ if and only if the 0-th subring A_0 of A has finite global dimension.

1. Main Result

There are two kinds of triangulated categories which are important for representation theory for algebras. First one is the derived category $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Lambda)$ of the category $\mathrm{mod}A$ of modules over an algebra Λ. Second one is algebraic triangulated categories, that is the stable categories of Frobenius categories (cf. [5]). A typical example is the stable category $\underline{\mathrm{mod}}A$ of the category $\mathrm{mod}A$ of modules over a self-injective algebra A.

In this paper, our aim is to compare derived categories of algebras and the stable categories of self-injective algebras, and find a "nice" relationship between them. If we find it, then those triangulated categories can be investigated from mutual viewpoints.

There several method to compare derived categories of algebras and the stable categories of self-injective algebras. We focus on the following Happel's result. For any algebra Λ, one can associate a self-injective algebra A which is called the trivial extension of Λ. A admits a natural positively grading such that $A_0 = \Lambda$ where A_0 is the 0-th subring. Therefore A is a positively graded self-injective algebra. So the stable category $\underline{\mathrm{mod}}^{\mathbb{Z}}A$ of the category $\mathrm{mod}^{\mathbb{Z}}A$ of $\mathbb{Z}$-graded A-modules has the structure of triangulated category. In this setting, D. Happel [6] showed that Λ has finite global dimension if and only if there exists a triangle-eqiuvalence

$$\underline{\mathrm{mod}}^{\mathbb{Z}}A \simeq \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Lambda). \tag{1.1}$$

This equivalence gives a "nice" relationship between derived category $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Lambda)$ and the stable categories $\underline{\mathrm{mod}}^{\mathbb{Z}}A$. The above result asserts that sometimes representation theory of Λ and that of A are deeply related.

We consider the drastic generalization of the above Happel's result. Happel started from an algebra Λ, and constructed the special positively graded self-injective algebra of A. In contrast, we start from a positively graded self-injective algebra $A = \bigoplus_{i\geq 0} A_i$, and suggest the following question.

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Question. When is $\underline{\mathrm{mod}}^{\mathbb{Z}} A$ triangle-equivalent to the derived category $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Lambda)$ for some algebra Λ ?

The following result is main theorem of this paper which gives the complete answer to our question.

Theorem 1. *Let A be a positively graded self-injective algebra. Then the following are equivalent.*

(1) *The global dimension of A_0 is finite.*
(2) *There exists an algebra Λ, and a triangle-equivalence*

$$\underline{\mathrm{mod}}^{\mathbb{Z}} A \simeq \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Lambda). \tag{1.2}$$

The aim of the rest of this paper is to give an explanation of the proof of Theorem 1, and some examples. Our plan is as follows.

In Section 2, we give two preliminaries. First we recall that $\mathrm{mod}^{\mathbb{Z}} A$ for a positively graded algebra A is a Frobenius category, and so its stable category $\underline{\mathrm{mod}}^{\mathbb{Z}} A$ is an algebraic triangulated category. Secondly we give an explanation of Keller's tilting theorem. Our approach to the question is using Keller's titling theorem for algebraic triangulated categories. B. Keller [7] introduced and investigated differential graded categories and its derived categories. In his work, it was determine when is an algebraic triangulated category triangle-equivalent to the derived category of some algebra by the existence of tilting objects (tilting theorem). In Section 3, we apply Keller's tilting theorem to our study.

In Section 3, we give an outline of the proof of Theorem 1. We omit the proof of (2) $\Rightarrow$ (1). We give proofs (1) $\Rightarrow$ (2). We start from finding a concrete tilting object in $\underline{\mathrm{mod}}^{\mathbb{Z}} A$ which has "good" properties. After finding it, we show two ways to prove (1) $\Rightarrow$ (2). The first proof is based on Keller's tilting theorem, namely we entrust with constructing the triangle-equivalence (1.2). The second proof is direct more than the first one, namely we construct the triangle-equivalence (1.2) explicitly.

In Section 4, we give some examples of Theorem 1. In particular as an application of our main theorem, we show Happel's result, and its generalization shown by X-W Chen [2].

Throughout this paper, let K be an algebraically closed field. An algebra means a finite dimensional associative algebra over K. We always deal with finitely generated right modules over algebras. For an algebra Λ, we denote by $\mathrm{mod}\Lambda$ the category of Λ-modules, $\mathrm{proj}\Lambda$ the category of projective Λ-modules. The same notations is used for graded case. For an additive category $\mathcal{A}$, we denote by $\mathsf{K}^{\mathrm{b}}(\mathcal{A})$ the homotopy category of bounded complexes of $\mathcal{A}$. For an abelian category $\mathcal{A}$, we denote by $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$ the bounded derived category of $\mathcal{A}$.

2. Preliminaries

In this section, we recall basic facts about representation theory of a positively graded algebras, and tilting theorem for algebraic triangulated categories for the readers convenient.

2.1. **Positively graded self-injective algebras.** In this subsection, our aim is to recall that the stable category of $\mathbb{Z}$-graded modules over positively graded self-injective algebras are algebraic triangulated categories. Most of results stated here are due to Gordon-Green [3, 4]. In details, readers should refer to [3, 4].

We start with setting notations. Let $A = \bigoplus_{i \geq 0} A_i$ be a positively graded self-injective algebra. We say that an A-module is $\mathbb{Z}$-gradable if it can be regarded as a $\mathbb{Z}$-graded A-module. For a $\mathbb{Z}$-graded A-module X, we write X_i the i-degree part of X. We denote by $\mathrm{mod}^{\mathbb{Z}}A$ the category of $\mathbb{Z}$-graded A-modules. For $\mathbb{Z}$-graded A-modules X and Y, we write $\mathrm{Hom}_A(X,Y)_0$ the morphism space in $\mathrm{mod}^{\mathbb{Z}}A$ from X to Y.

We recall that $\mathrm{mod}^{\mathbb{Z}}A$ has two important functors. The first one is the grading shift functor. For $i \in \mathbb{Z}$, we denote by

$$(i) : \mathrm{mod}^{\mathbb{Z}}A \to \mathrm{mod}^{\mathbb{Z}}A$$

the grading shift functor, that is defined as follows. For a $\mathbb{Z}$-graded A-module X,

- $X(i) := X$ as an A-module,
- $\mathbb{Z}$-grading on $X(i)$ is defined by $X(i)_j := X_{j+i}$ for any $j \in \mathbb{Z}$.

This is an autofunctor on $\mathrm{mod}^{\mathbb{Z}}A$ whose inverse is $(-i)$.

The second one is the K-dual. It is already known that there is the standard duality

$$D := \mathrm{Hom}_K(-, K) : \mathrm{mod}A \to \mathrm{mod}A^{\mathrm{op}}.$$

This functor induces the following duality. For a $\mathbb{Z}$-graded A-module X, we regard DX as a $\mathbb{Z}$-graded A^{op}-module by defining $(DX)_i := D(X_{-i})$ for any $i \in \mathbb{Z}$. By this observation, we have the duality

$$D : \mathrm{mod}^{\mathbb{Z}}A \to \mathrm{mod}^{\mathbb{Z}}A^{\mathrm{op}}.$$

Next we recall a few important facts about objects and morphism spaces in $\mathrm{mod}^{\mathbb{Z}}A$. The following results are two of the most basic categorical properties of $\mathrm{mod}^{\mathbb{Z}}A$.

Proposition 2. $\mathrm{mod}^{\mathbb{Z}}A$ *is a Hom-finite Krull-Schmidt category*

Proposition 3. [3, Theorem 3.2. Theorem 3.3.] *The following assertions hold.*

(1) *A $\mathbb{Z}$-graded A-module is indecomposable in* $\mathrm{mod}^{\mathbb{Z}}A$ *if and only if it is an indecompsable A-module.*
(2) *Any direct summand of a $\mathbb{Z}$-gradable A-module is also $\mathbb{Z}$-gradable.*
(3) *Let X and Y be indecomposable $\mathbb{Z}$-graded A-modules. If X and Y are isomorphic to each other in* $\mathrm{mod}A$*, then there exists $i \in \mathbb{Z}$ such that X and $Y(i)$ are isomorphic to each other in* $\mathrm{mod}^{\mathbb{Z}}A$.

Next we recall what are projective objects and injective objects in $\mathrm{mod}^{\mathbb{Z}}A$. A is naturally regarded as a $\mathbb{Z}$-graded A-module. By Proposition 3 (2), any projective A-modules are $\mathbb{Z}$-gradable. Moreover it is easy to check that all projective object in $\mathrm{mod}^{\mathbb{Z}}A$ is given by projective A-modules. By the standard duality, the same argument hold for injective objects in $\mathrm{mod}^{\mathbb{Z}}A$.

Proposition 4. *A complete list of indecomposable projective objects in* $\mathrm{mod}^{\mathbb{Z}}A$ *is given by*

$$\{P(i) \mid i \in \mathbb{Z}, P \text{ is an indecomposable projective } A\text{-module}\}.$$

Dually a complete list of indecomposable injective objects in $\mathrm{mod}^{\mathbb{Z}}A$ *is given by*

$$\{I(i) \mid i \in \mathbb{Z}, I \textit{ is an indecomposable injective } A\textit{-module}\}.$$

If A is self-injective, then $\mathrm{mod}^{\mathbb{Z}}A$ is a Frobenius category by Proposition 3 and Proposition 4. So in this case, the stable category $\underline{\mathrm{mod}}^{\mathbb{Z}}A$ has a structure of triangulated category by [5].

Lemma 5. *If A is self-injective, the following assertions hold.*

(1) $\mathrm{mod}^{\mathbb{Z}}A$ *is a Frobenius category.*

(2) $\underline{\mathrm{mod}}^{\mathbb{Z}}A$ *has a structure of triangulated category whose shift functor* [1] *is given by the graded cosyzygy functor* $\Omega^{-1} : \underline{\mathrm{mod}}^{\mathbb{Z}}A \to \underline{\mathrm{mod}}^{\mathbb{Z}}A$.

2.2. **Tilting theorem for algebraic triangulated categories.** In this subsection, we recall tilting theorem for algebraic triangulated categories which is due to Keller [7]. It is a theorem which provides a method for comparison of given triangulated category and homotopy category of bounded complexes of projective modules over some algebra.

First let us recall the definition of algebraic triangulated categories again.

Definition 6. A triangulated category $\mathcal{T}$ is *algebraic* if it is triangle-equivalent to the stable category of some Frobenius category.

A class of algebraic triangulated categories contains the following important examples.

Example 7. (1) Let $\mathbb{Z}$ be an abelian group, and A a $\mathbb{Z}$-graded self-injective algebra. Then $\mathrm{mod}^{\mathbb{Z}}A$ is a Frobenius category, and the stable category $\underline{\mathrm{mod}}^{\mathbb{Z}}A$ is an algebraic triangulated category (Lemma 5).

(2) Let Λ be an algebra. The category $\mathsf{C}^{\mathrm{b}}(\mathrm{proj}\Lambda)$ of bounded complexes of projective Λ-modules can be regarded as a Frobenius category whose stable category is the homotopy category $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)$ of bounded complexes of projective Λ-modules (cf. [5]).

In tilting theory, tilting objects which is defined as follows play an important role.

Definition 8. Let $\mathcal{T}$ be a triangulated category. An object $T \in \mathcal{T}$ is called a *tilting object* in $\mathcal{T}$ if it satisfies the following conditions.

(1) $\mathrm{Hom}_{\mathcal{T}}(T, T[i]) = 0$ for $i \neq 0$.

(2) $\mathcal{T} = \mathrm{thick}T$.

Here $\mathrm{thick}T$ is the smallest triangulated full subcategory of $\mathcal{T}$ which contains T, and is closed under direct summands.

The following is a typical example of tilting objects.

Example 9. Let Λ be a ring. Λ can be regarded as a complex which concentrates in degree 0. So Λ is contained in a triangulated category $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)$. It is a tilting object in $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)$.

The following result is Keller's tilting theorem which determine when is an algebraic triangulated category triangle-equivalent to $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)$ for some algebra Λ,

Theorem 10. [7, Theorem 4.3.] *Let $\mathcal{T}$ be an algebraic triangulated category. If $\mathcal{T}$ has a tilting object T, then there exists a triangle-equivalence up to direct summands*

$$\mathcal{T} \simeq \mathsf{K}^{\mathrm{b}}(\mathrm{projEnd}_{\mathcal{T}}(T)).$$

By the above result, finding tilting objects is a basic problem for the study of a given algebraic triangulated category. We will consider this problem for Example 7 (1) in the next section (Theorem 11).

3. Triangle-equivalences between stable categories and derived categories

Throughout this section, let A be a positively graded self-injective algebra. In this section, we discuss triangle-equivalences between the stable category $\underline{\mathrm{mod}}^{\mathbb{Z}}A$ and derived categories of algebras.

First we prove Theorem 1 in the half of this section. We omit the proof of (2) $\Rightarrow$ (1). We prove (1) $\Rightarrow$ (2). We begin the proof from giving the necessary and sufficient condition for existence of tilting objects in the stable category $\underline{\mathrm{mod}}^{\mathbb{Z}}A$. The necessary and sufficient condition is described by important homological property of the subring A_0 of A which is stated as follows.

Theorem 11. $\underline{\mathrm{mod}}^{\mathbb{Z}}A$ *has a tilting object if and only if* A_0 *has finite global dimension.*

We omit the proof of only if part of Theorem 11. In the following, we show the proof of if part of Theorem 11 which is given by constructing a tilting object in $\underline{\mathrm{mod}}^{\mathbb{Z}}A$. To construct it, we consider truncation functors

$$(-)_{\geq i} : \mathrm{mod}A \to \mathrm{mod}A$$

and

$$(-)_{\leq i} : \mathrm{mod}A \to \mathrm{mod}A$$

which are defined as follows. For a $\mathbb{Z}$-graded A-module X, $X_{\geq i}$ is a $\mathbb{Z}$-graded sub A-module of X defined by

$$(X_{\geq i})_j := \begin{cases} 0 & (j < i) \\ X_j & (j \geq i), \end{cases}$$

and $X_{\leq i}$ is a $\mathbb{Z}$-graded factor A-module $X/X_{\geq i+1}$ of X.

Now we define

$$T := \bigoplus_{i \geq 0} A(i)_{\leq 0}. \tag{3.1}$$

which is an object in $\mathrm{Mod}^{\mathbb{Z}}A$ but not an object in $\mathrm{mod}^{\mathbb{Z}}A$. However since $A(i)_{\leq 0} = A(i)$ for enough large i, T can be regarded as an object in $\underline{\mathrm{mod}}^{\mathbb{Z}}A$.

Then we have the following result.

Theorem 12. *Under the above setting, the following assertions hold.*

(1) T *is a tilting object in* $\mathrm{thick}T$.
(2) *If* A_0 *has finite global dimension, then* T *is a tilting object in* $\underline{\mathrm{mod}}^{\mathbb{Z}}A$.

It is proved that T satisfies the first condition in Definition 8 with no assumptions for A, and T satisfies the second condition in Definition 8 if A_0 has finite global dimension. Then we finish the proof of if part of Theorem 11. $\square$

Now we keep the notation as above and put

$$\Gamma := \underline{\mathrm{End}}_A(T)_0.$$

the endomorphism algebra of T in $\underline{\mathrm{mod}}^{\mathbb{Z}}A$. This endomorphism algebra Γ has a nice homological property if so does A_0.

Theorem 13. *If A_0 has finite global dimension, then so does Γ.*

Now we ready to prove Theorem 1 (1) $\Rightarrow$ (2).

Theorem 14. *Under the above setting, the following assertions hold.*

(1) *There exists a triangle-equivalence*

$$\mathrm{thick}T \longrightarrow \mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Gamma).$$

(2) *If A_0 has finite global dimension, then there exists a triangle-equivalence*

$$\underline{\mathrm{mod}}^{\mathbb{Z}}A \longrightarrow \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Gamma).$$

Proof. (1) By Theorem 10 and Theorem 12 (1), we have the triangle-equivalence $\mathrm{thick}T \longrightarrow \mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Gamma)$.

(2) We assume that A_0 has finite global dimension. First by Theorem 10 and Theorem 12 (2), we have the triangle-equivalence $\underline{\mathrm{mod}}^{\mathbb{Z}}A \longrightarrow \mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Gamma)$. Next by Theorem 13, the natural triangle-functor $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda) \longrightarrow \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Gamma)$ is an equivalence. Finally by composing these equivalences, we have a triangle-equivalence

$$\underline{\mathrm{mod}}^{\mathbb{Z}}A \longrightarrow \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Gamma).$$

$\square$

In the above proof, the triangle-equivalence $\underline{\mathrm{mod}}^{\mathbb{Z}}A \to \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Gamma)$ was given by the existence of tilting object T in $\underline{\mathrm{mod}}^{\mathbb{Z}}A$ and Keller's Theorem 10 automatically. In the rest of this section, we construct a triangle-equivalence $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Gamma) \to \underline{\mathrm{mod}}^{\mathbb{Z}}A$ by derived tensor functor directly.

To construct the triangle-equivalence, first we want to consider the derived tensor functor $- \overset{\mathbb{L}}{\otimes}_\Gamma T : \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Gamma) \to \mathsf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}}A)$. However Γ does not act on T naturally since Γ is defined by the morphism space in the stable category $\underline{\mathrm{mod}}^{\mathbb{Z}}A$. To solve this problem, we give the description of T in $\underline{\mathrm{mod}}^{\mathbb{Z}}A$ below. The description allow us to realize Γ as the morphism space in the category $\mathrm{mod}^{\mathbb{Z}}A$.

Proposition 15. *T is decomposed as $T = \underline{T} \oplus P$ where $\underline{T}$ is a direct sum of all indecomposable non-projective direct summand of T. Then the following assertions hold.*

(1) *$\underline{T}$ is in $\mathrm{mod}^{\mathbb{Z}}A$.*
(2) *T and $\underline{T}$ are isomorphic to each other in $\underline{\mathrm{mod}}^{\mathbb{Z}}A$.*
(3) *There exists an algebra isomorphism $\Gamma \simeq \mathrm{End}_A(\underline{T})_0$.*

Let $T = \underline{T} \oplus P$ be the decomposition which was given in Proposition 15. By Proposition 15 (3), $\underline{T}$ is regarded as a $\mathbb{Z}$-graded $\Gamma^{\mathrm{op}} \otimes_K A$-module naturally. So we have the left derived tensor functor

$$- \overset{\mathbb{L}}{\otimes}_\Gamma \underline{T} : \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Gamma) \to \mathsf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}}A).$$

Next we consider the quotient category $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}}A)/\mathsf{K}^{\mathrm{b}}(\mathrm{proj}^{\mathbb{Z}}A)$ of $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}}A)$, and the quotient functor

$$\mathsf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}}A) \longrightarrow \mathsf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}}A)/\mathsf{K}^{\mathrm{b}}(\mathrm{proj}^{\mathbb{Z}}A).$$

The following triangle-equivalence is the realization of $\underline{\mathrm{mod}}^{\mathbb{Z}}A$ as the quotient category $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}}A)/\mathsf{K}^{\mathrm{b}}(\mathrm{proj}^{\mathbb{Z}}A)$. The ungraded version of this realization was studied by several authors [1], [8] and [9].

Theorem 16. [9, Theorem 2.1.] *The natural embedding* $\mathrm{mod}^{\mathbb{Z}}A \to \mathsf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}}A)$ *induces a triangle-equivalence*

$$\underline{\mathrm{mod}}^{\mathbb{Z}}A \longrightarrow \mathsf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}}A)/\mathsf{K}^{\mathrm{b}}(\mathrm{proj}^{\mathbb{Z}}A)$$

Now we consider the following composition of the above three functors

$$G : \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Gamma) \xrightarrow{-\overset{\mathbb{L}}{\otimes}_{\Gamma}\underline{T}} \mathsf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}}A) \longrightarrow \mathsf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}}A)/\mathsf{K}^{\mathrm{b}}(\mathrm{proj}^{\mathbb{Z}}A) \longrightarrow \underline{\mathrm{mod}}^{\mathbb{Z}}A.$$

where the second one is the quotient functor, and the third one is a quasi-inverse of Theorem 16. This is the triangle-functor which we want.

Theorem 17. *Under the above setting, the following assertions hold.*

(1) *G is fully faithful on $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Gamma)$.*
(2) *A_0 has finite global dimension if and only if G is a triangle-equivalence.*

Proof. (1) It is easy to check that $G(\Gamma)$ is isomorphic to $\underline{T}$, so is isomorphic to T. Moreover by Theorem 12 (1), G induces an isomorphism

$$\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Gamma)}(\Gamma, \Gamma[i]) \simeq \underline{\mathrm{Hom}}_A(G(\Gamma), G(\Gamma)[i])_0$$

for any $i \in \mathbb{Z}$. By this and $\mathrm{thick}\Gamma = \mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Gamma)$, G is fully faithful on $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Gamma)$. Thus G induces a triangle-equivalence $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Gamma) \to \mathrm{thick}T$.

(2) We assume that A_0 has finite global dimension. Then Γ has finite global dimension by Theorem 13. Thus we have $\mathrm{thick}\Gamma = \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Gamma)$, and so G is fully faithful. Again since A_0 has finite global dimension, we have $\mathrm{thick}T = \underline{\mathrm{mod}}^{\mathbb{Z}}A$ by Theorem 12 (2). Thus G is dense.

We omit the proof of converse. □

4. Examples

In this section, we show some examples and applications of results which was shown in previous section.

First example is famous Happel's result [6], which gives a relationship between representation theory of algebras and that of the trivial extensions. We show it as an application of Theorem 1.

Example 18. If an algebra is given, then we can always construct a positively graded self-injective algebra called trivial extension, which contains original algebra as a subalgebra. Let us recall the definition of trivial extensions. Let Λ be an algebra. The trivial extension A of Λ is defined as follows.

- $A := \Lambda \oplus D\Lambda$ as an abelian group.

- The multiplication on A is defined by

$$(x, f) \cdot (y, g) := (xy, xg + fy).$$

for any $x, y \in \Lambda$ and $f, g \in D\Lambda$. Here xg and fy is defined by (Λ, Λ)-bimodule structure on $D\Lambda$.

This A becomes an algebra with respect to the above operations. Moreover it is known that A is self-injective.

Now we introduce a positively grading on A by

$$A_i := \begin{cases} \Lambda & (i = 0), \\ D\Lambda & (i = 1), \\ 0 & (i \geq 2). \end{cases}$$

Then obviously $A = \bigoplus_{i \geq 0} A_i$ becomes a positively graded self-injective algebra.

Under the above setting, we apply Theorem 17 to the trivial extension A of an algebra Λ. Then we have the following Happel's triangle-equivalence.

Theorem 19. [6, Theorem 2. 3.] *Under the above setting, the following are equivalent.*

(1) Λ *has finite global dimension.*

(2) *There exists an triangle-equivalence*

$$\underline{\mathrm{mod}}^{\mathbb{Z}} A \simeq \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Lambda).$$

Proof. We calculate T constructed in (3.1) for our setting. Then one can check that $T = \Lambda$, and $\underline{\mathrm{End}}_A(T)_0 = \mathrm{End}_A(T)_0 \simeq \Lambda$. Thus the assertion follows from this and Theorem 17. □

Next example is X-W Chen's result [2] which gives a generalization of Happel's result.

Example 20. Chen [2] studied relationship between the stable category $\underline{\mathrm{mod}}^{\mathbb{Z}} A$ of a positively graded self-injective algebra A which has Gorenstein parameter and the derived category $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Gamma)$ of the Beilinson algebra Γ of A. The notion of Gorenstein parameter is defined as follows.

Definition 21. Let A be a positively graded self-injective algebra. We say that A has *Gorenstein parameter* ℓ if $\mathrm{Soc}A$ is contained in A_ℓ.

Let A be a positively graded self-injective algebra of Gorenstein parameter ℓ. The Beilinson algebra Γ of A is defined by

$$\Gamma := \begin{pmatrix} A_0 & A_1 & \cdots & A_{\ell-2} & A_{\ell-1} \\ & A_0 & \cdots & A_{\ell-3} & A_{\ell-2} \\ & & \ddots & \vdots & \vdots \\ & & & A_0 & A_1 \\ 0 & & & & A_0 \end{pmatrix}.$$

Then Chen showed the following result.

Theorem 22. [2, Corollary 1.2.] *Under the above setting, the following are equivalent.*

(1) A_0 *has finite global dimension.*

(2) *There exists a triangle-equivalence*

$$\underline{\mathrm{mod}}^{\mathbb{Z}} A \simeq \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Gamma).$$

As an application of Theorem 12, we give a proof of the above result. Let T be the object defined in (3.1), and $\underline{T}$ the direct summand of T defined in Proposition 15. We calculate $\underline{T}$ and the endomorphism algebra $\underline{\mathrm{End}}_A(T)_0$. Then since A has Gorenstein parameter ℓ, those can be represented as the following explicit form.

Proposition 23. *Under the above setting, the following assertions hold*

(1) $\underline{T} = \bigoplus_{i=0}^{\ell-1} A(i)_{\leq 0}$.
(2) *There exists an algebra isomorphism* $\underline{\mathrm{End}}_A(T)_0 \simeq \Gamma$.

Proof. Since A has Gorenstein parameter ℓ, we have $\underline{T} = \bigoplus_{i=0}^{\ell-1} A(i)_{\leq 0}$ by the definition of $\underline{T}$. Moreover it is easy to calculate that there is an algebra isomorphism $\underline{\mathrm{End}}_A(T)_0 = \underline{\mathrm{End}}_A\left(\bigoplus_{i=0}^{\ell-1} A(i)_{\leq 0}\right)_0 \simeq \Gamma$. □

Proof of Theorem 22. The assertion follows from Theorem 17 and Proposition 23. □

Remark 24. The trivial extensions of algebras are positively graded self-injective algebras of Gorenstein parameter 1. Thus Theorem 22 contains Theorem 19.

Next we show a concrete examples.

Example 25. We consider $A := K[x]/(x^{n+1})$, and define a grading on A by $\deg x := 1$. Then A is a positively graded self-injective algebra of Gorenstein parameter n.

Since the global dimension of $A_0 = K$ is equal to zero, $\underline{\mathrm{mod}}^{\mathbb{Z}} A$ has a tilting object by Theorem 11. Let T be the object in $\mathrm{mod}^{\mathbb{Z}} A$ which was defined in (3.1). Since A has a unique chain

$$A \supset (x)/(x^{n+1}) \supset (x^2)/(x^{n+1}) \supset \cdots \supset (x^n)/(x^{n+1})$$

of $\mathbb{Z}$-graded A-submodules of A, it is easy to calculate that the endomorphism algebra $\Gamma := \underline{\mathrm{End}}_A(T)_0$ of T is isomorphic to the $n \times n$ upper triangular matrix algebra over K. By Theorem 12, there exists a triangle-equivalence

$$\underline{\mathrm{mod}}^{\mathbb{Z}} A \simeq \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Gamma).$$

We observe the above triangle-equivalences by considering the case that $n = 2$, namely the case that $A = K[x]/(x^3)$. For $i = 1, 2$, we put $X^i := (x^i)/(x^3)$ the $\mathbb{Z}$-graded A-submodule of A. Then we have a chain $A \supset X^1 \supset X^2$ of $\mathbb{Z}$-graded A-submodules of A. It is known that $\{X^i(j) \mid i = 1, 2,\ j \in \mathbb{Z}\}$ is a complete set of indecomposable non-projective $\mathbb{Z}$-graded A-modules.

The Auslander-Reiten quiver of $\underline{\mathrm{mod}}^{\mathbb{Z}} A$ is as follows.

$$\begin{tikzcd}[column sep=tiny]
 & & X^1(-2) \arrow[dr] & & X^1(-1) \arrow[ll, dotted] \arrow[dr] & & X^1 \arrow[ll, dotted] \arrow[dr] & & X^1(1) \arrow[ll, dotted] \arrow[dr] & & X^1(2) \arrow[ll, dotted] & \\
\cdots\cdots & X^2(-2) \arrow[ur] & & X^2(-1) \arrow[ll, dotted] \arrow[ur] & & X^2 \arrow[ll, dotted] \arrow[ur] & & X^2(1) \arrow[ll, dotted] \arrow[ur] & & X^2(2) \arrow[ll, dotted] \arrow[ur] & & \cdots\cdots
\end{tikzcd}$$

Here dotted arrows mean the Auslander-Reiten translation in $\underline{\mathrm{mod}}^{\mathbb{Z}} A$. We can observe that the Auslander-Reiten translation coincides with the graded shift functor (-1).

Next we write the Auslander-Reiten quiver of $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Gamma)$. In this case, $\Gamma = \underline{\mathrm{End}}_A(T)_0$ is isomorphic to 2×2 upper triangular matrix algebra over K. We put $P^1 := (KK)$, $P^2 := (0K)$ and $I^1 := (K0)$. It is known that the set $\{P^1, P^2, I^1\}$ is a complete set of indecomposable Γ-modules, and the Auslander-Reiten quiver of $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Gamma)$ is as follows.

$$\begin{tikzcd}[column sep=small]
 & & I^1[-1] \arrow[dr] & & P^1 \arrow[ll, dotted] \arrow[dr] & & P^2[1] \arrow[ll, dotted] \arrow[dr] & & I^1[1] \arrow[ll, dotted] \arrow[dr] & & P^1[2] \arrow[ll, dotted] & \\
\cdots\cdots & P^1[-1] \arrow[ur] & & P^2 \arrow[ll, dotted] \arrow[ur] & & I^1 \arrow[ll, dotted] \arrow[ur] & & P^1[1] \arrow[ll, dotted] \arrow[ur] & & P^2[2] \arrow[ll, dotted] \arrow[ur] & & \cdots\cdots
\end{tikzcd}$$

Here dotted arrows mean the Auslander-Reiten translation in $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Gamma)$.

From shape of the above Auslander-Reiten quivers, one can see that $\underline{\mathrm{mod}}^{\mathbb{Z}} A$ and $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Gamma)$ should be equivalent to each other. In fact, we gave a triangle-equivalence between those.

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