QUANTUM UNIPOTENT SUBGROUP AND DUAL CANONICAL BASIS

YOSHIYUKI KIMURA

Abstract. In a series of works [13, 16, 14, 15, 18, 19], Geiß-Leclerc-Schröer defined the cluster algebra structure on the coordinate ring $\mathbb{C}[N(w)]$ of the unipotent subgroup, associated with a Weyl group element w. And they proved cluster monomials are contained in Lusztig's *dual semicanonical basis* $\mathcal{S}^*$. We give a set up for the quantization of their results and propose a conjecture which relates the quantum cluster algebras in [3] to the *dual canonical basis* $\mathbf{B}^{\mathrm{up}}$. In particular, we prove that the quantum analogue $\mathcal{O}_q[N(w)]$ of $\mathbb{C}[N(w)]$ has the induced basis from $\mathbf{B}^{\mathrm{up}}$, which contains quantum flag minors and satisfies a factorization property with respect to the 'q-center' of $\mathcal{O}_q[N(w)]$. This generalizes Caldero's results [4, 5, 6] from finite type to an arbitrary symmetrizable Kac-Moody Lie algebra.

1. Introduction

1.1. **The canonical basis $\mathbf{B}$ and the dual canonical basis $\mathbf{B}^{\mathrm{up}}$.** Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra, $\mathbf{U}_q(\mathfrak{g})$ its associated quantized enveloping algebra, and $\mathbf{U}_q^-(\mathfrak{g})$ its negative part. In [24], Lusztig constructed the canonical basis $\mathbf{B}$ of $\mathbf{U}_q^-(\mathfrak{g})$ by a geometric method when $\mathfrak{g}$ is symmetric. In [21], Kashiwara constructed the (lower) global basis $G^{\mathrm{low}}(\mathcal{B}(\infty))$ by a purely algebraic method. Grojnowski-Lusztig [20] showed that the two bases coincide when $\mathfrak{g}$ is symmetric. We call the basis the *canonical basis*. There are two remarkable properties of the canonical basis, one is the positivity of structure constants of multiplication and comultiplication, and another is Kashiwara's crystal structure $\mathcal{B}(\infty)$, which is a combinatorial machinery useful for applications to representation theory, such as tensor product decomposition.

Since $\mathbf{U}_q^-(\mathfrak{g})$ has a natural pairing which makes it into a (twisted) self-dual bialgebra, we consider the dual basis $\mathbf{B}^{\mathrm{up}}$ of the canonical basis in $\mathbf{U}_q^-(\mathfrak{g})$. We call it the *dual canonical basis.*

1.2. **Cluster algebras.** Cluster algebras were introduced by Fomin and Zelevinsky [10] and intensively studied also with Berenstein [11, 1, 12] with an aim of providing a concrete and combinatorial setting for the study of Lusztig's (dual) canonical basis and total positivity. Quantum cluster algebras were also introduced by Berenstein and Zelevinsky [3], Fock and Goncharov [8, 9, 7] independently. The definition of (quantum) cluster algebra was motivated by Berenstein and Zelevinsky's earlier work [2] where combinatorial and multiplicative structures of the dual canonical basis were studied for $\mathfrak{g} = \mathfrak{sl}_n$ $(2 \leq n \leq 4)$. In [1], it was shown that the coordinate ring of the double Bruhat cell contains a cluster algebra as a subalgebra, which is conjecturally equal to the whole algebra.

The detailed version of this paper [22] will be published from Kyoto Journal of Mathematics.

A cluster algebra $\mathcal{A}$ is a subalgebra of rational function field $\mathbb{Q}(x_1, x_2, \cdots, x_r)$ of r indeterminates which is equipped with a distinguished set of generators (*cluster variables*) which is grouped into overlapping subsets (*clusters*) consisting of precisely r elements. Each subset is defined inductively by a sequence of certain combinatorial operation (*seed mutations*) from the initial seed. The monomials in the variables of a given single cluster are called *cluster monomials*. However, it is not known whether a cluster algebra have a basis, related to the dual canonical basis, which includes all cluster monomials in general.

1.3. **Cluster algebra and the semicanonical basis.** In a series of works [13, 16, 14, 15, 18, 19], Geiß, Leclerc and Schröer introduced a cluster algebra structure on the coordinate ring $\mathbb{C}[N(w)]$ of the unipotent subgroup associated with a Weyl group element w. Furthermore they show that the *dual semicanonical basis* $\mathcal{S}^*$ is compatible with the inclusion $\mathbb{C}[N(w)] \subset U(\mathfrak{n})^*_{\mathrm{gr}}$ and contains all cluster monomials. Here the dual semicanonical basis is the dual basis of the semicanonical basis of $U(\mathfrak{n})$, introduced by Lusztig [25, 28], and "compatible" means that $\mathcal{S}^* \cap \mathbb{C}[N(w)]$ forms a $\mathbb{C}$-basis of $\mathbb{C}[N(w)]$. It is known that canonical and semicanonical bases share similar combinatorial properties (crystal structure), but they are different. Geiß, Leclerc and Schröer conjecture that certain dual semicanonical basis elements are specialization of the corresponding dual canonical basis elements. This is called the *open orbit conjecture*.

Acknowledgement. The author is grateful to Professor Osamu Iyama for giving opportunity to talk in Okayama University.

2. Quantum unipotent subgroup and the dual canonical basis

2.1. **Notations.** Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra and $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_- = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ be its triangular decomposition and its root decomposition. Let W be a Weyl group which is associated with $\mathfrak{g}$. Let $\Delta_\pm$ be the set of positive (resp. negative) roots. For a Weyl group element $w \in W$, we set $\Delta(w) := \Delta_+ \cap w\Delta_- = \{\alpha \in \Delta_+ \mid w^{-1}\alpha < 0\} \subset \Delta_+$. For a Weyl group element w, let $\overrightarrow{w} = (i_1, i_2, \ldots, i_\ell)$ be a reduced expression of w. We set $\beta_k := s_{i_1} \ldots s_{i_{k-1}}(\alpha_{i_k})$ for each $1 \leq k \leq \ell$. Then it is known that $\Delta(w) = \{\beta_k \mid 1 \leq k \leq \ell\}$. Let $\mathfrak{n}(w)$ be the nilpotent Lie subalgebra which is associated with $\Delta(w)$, that is

$$\mathfrak{n}(w) = \bigoplus_{1 \leq k \leq \ell} \mathfrak{g}_{\beta_k}.$$

For $i \in I$, we have Lusztig's braid symmetry T_i on $\mathbf{U}_q(\mathfrak{g})$, see [26, Chapter 32] for more details. It is known that $\{T_i\}_{i \in I}$ satisfies braid relations. Hence the composite $T_w := T_{i_1} \cdots T_{i_\ell}$does not depend on a choice of reduced word $\overrightarrow{w} = (i_1, i_2, \ldots, i_\ell)$ of w. In this article, we set $T_i = T'_{i,-1}$.

2.2. **Poincaré-Birkhoff-Witt basis.** Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra and $\mathbf{U}_q(\mathfrak{g})$ be the corresponding quantized enveloping algebra. We have a standard generators $\{E_i\}_{i \in I} \cup \{q^h\} \cup \{F_i\}_{i \in I}$ Let $\mathbf{U}_q^-(\mathfrak{g})$ be the $\mathbb{Q}(q)$-subalgebra which is generated by $\{F_i\}_{i \in I}$. It is known that $\mathbf{U}_q^-(\mathfrak{g})$ is isomophic to the $\mathbb{Q}(q)$-algebra which is defined by

$\{F_i\}_{i\in I}$ and q-Serre relations

$$\sum_{k=0}^{1-a_{ij}}(-1)^k F_i^{(k)}F_jF_i^{(1-a_{ij}-k)},$$

where $\{a_{ij}\}$ is the generalized Cartamn matrix which defines $\mathfrak{g}$ and $F_i^{(k)}$ is the divided power which is defined by $F_i^{(k)} := F_i^k/[k]_i!$. Let $\mathbf{U}_q^-(\mathfrak{g})_{\mathbb{Q}}$ be the $\mathbb{Q}[q^{\pm 1}]$-subalgebra which is generated by $\{F_i^{(n)}\}_{i\in I, n\in\mathbb{Z}_{\geq 0}}$. This $\mathbb{Q}[q^{\pm 1}]$-algebra is called Lusztig's $\mathbb{Q}[q^{\pm 1}]$-form.

We define root vectors associated with a reduced word$\overrightarrow{w} = (i_1, i_2, \ldots, i_\ell)$ for a Weyl group element $w \in W$. See [26, Proposition 40.1.3, Proposition 41.1.4] for more detail. For a Weyl group element$w \in W$ and a reduced word $\overrightarrow{w} = (i_1, i_2, \ldots, i_\ell)$, we define β_k as above. We define the *root vectors* $F(\beta_k)$ associated with $\beta_k \in \Delta(w)$

$$F(\beta_k) = T_{i_1}\ldots T_{i_{k-1}}(F_{i_k}).$$

It is known that $F(\beta_k) \in \mathbf{U}_q^-(\mathfrak{g})$ for all $1 \leq k \leq \ell$. We also define its divided power by $F(c\beta_k) = T_{i_1}\ldots T_{i_{k-1}}(F_{i_k}^{(c)})$. For an ℓ tuple of non-negative integers $\mathbf{c} = (c_1, c_2, \ldots, c_\ell)$, we set

$$F(\mathbf{c}, \overrightarrow{w}) := F(c_\ell\beta_\ell)\cdots F(c_1\beta_1).$$

It is known that $F(\mathbf{c}, \overrightarrow{w}) \in \mathbf{U}_q^-(\mathfrak{g})_{\mathbb{Q}}$.

Theorem 1 ([26, Proposition 40.2.1, Proposition 41.1.3])**.**

Theorem 2. (1) *Then* $\{F(\mathbf{c}, \overrightarrow{w})\}_{\mathbf{c}\in\mathbb{Z}_{\geq 0}^\ell}$ *forms a* $\mathbb{Q}(q)$*-basis of a subspace defined to be* $\mathbf{U}_q^-(w)$ *of* $\mathbf{U}_q^-(\mathfrak{g})$ *which does not depend on* $\overrightarrow{w}$*.*

(2) *We have* $F(\mathbf{c}, \overrightarrow{w}) \in \mathbf{U}_q^-(\mathfrak{g})_{\mathbb{Q}}$ *for all* $\mathbf{c} \in \mathbb{Z}_{\geq 0}^\ell$*.*

We consider the total order on $\Delta(w)$ as follows:

$$\beta_1 < \beta_2 < \cdots < \beta_\ell.$$

We have the following convex properties on $\{F(\beta_k)\}_{1\leq k\leq\ell}$.

Theorem 3 ([29, Proposition 3.6], [23, 5.5.2 Proposition])**.** *For* $j < k$*, let us write*

$$F(c_j\beta_j)F(c_k\beta_k) - q^{-(c_j\beta_j, c_k\beta_k)}F(c_k\beta_k)F(c_j\beta_j) = \sum_{\mathbf{c}'\in\mathbb{Z}_{\geq 0}^\ell} f_{\mathbf{c}'}F(\mathbf{c}', \overrightarrow{w})$$

$f_{\mathbf{c}'} \in \mathbb{Q}(q)$*. If* $f_{\mathbf{c}'} \neq 0$*, then* $c'_j < c_j$ *and* $c'_k < c_k$ *with* $\sum_{j\leq m\leq k} c'_m\beta_m = c_j\beta_j + c_k\beta_k$*.*

By the above formula, it is shown that $\mathbf{U}_q^-(w)$ is a $\mathbb{Q}(q)$-algebra which is generated by $\{F(\beta_k)\}_{1\leq k\leq\ell}$.

2.3. **PBW basis and crystal basis.** Let $\mathcal{L}(\infty)$ be the crystal lattice of $\mathbf{U}_q^-(\mathfrak{g})$ and $\mathcal{B}(\infty)$ be the crystal basis and $\mathbf{B}$ the canoncial basis.

The following result is due to Saito and Lusztig.

Theorem 4 ([30, Theorem 4.1.2], [27, Proposition 8.2]). (1) *We have* $F(\mathbf{c}, \overrightarrow{w}) \in \mathcal{L}(\infty)$ *and*

$$b(\mathbf{c}, \overrightarrow{w}) := F(\mathbf{c}, \overrightarrow{w}) \mod q\mathcal{L}(\infty) \in \mathcal{B}(\infty).$$

(2) *The map* $\mathbb{Z}_{\geq 0}^{\ell} \to \mathcal{B}(\infty)$ *which is defined by* $\mathbf{c} \mapsto b(\mathbf{c}, \overrightarrow{w})$ *is injective and the image* $\mathcal{B}(w)$ *does not depend on the choice of* $\overrightarrow{w}$.

2.4. **Dual canonical basis.** Let $(\ , \ )_K$ be the inner product on $\mathbf{U}_q^-(\mathfrak{g})$ defined by Kashiwara and $\mathbf{U}_q^-(\mathfrak{g})_{\mathbb{Q}}^{\mathrm{up}}$ be the dual $\mathbb{Q}[q^{\pm 1}]$-lattice of $\mathbf{U}_q^-(\mathfrak{g})_{\mathbb{Q}}$. Let $\mathbf{B}^{\mathrm{up}}$be the dual basis of $\mathbf{B}$ with respect to $(\ , \ )_K$ and this is called *dual canonical basis.* We set

$$F^{\mathrm{up}}(\mathbf{c}, \overrightarrow{w}) := \frac{1}{(F(\mathbf{c}, \overrightarrow{w}), F(\mathbf{c}, \overrightarrow{w}))_K} F(\mathbf{c}, \overrightarrow{w}).$$

Proposition 5. (1) *We have* $F^{\mathrm{up}}(\beta_k) \in \mathbf{B}^{\mathrm{up}}$.

(2) *Let* $\mathbf{U}_q^-(w)_{\mathbb{Q}}^{\mathrm{up}}$ *be the* $\mathbb{Q}[q^{\pm 1}]$*-span of* $\{F^{\mathrm{up}}(\mathbf{c}, \overrightarrow{w})\}_{\mathbf{c} \in \mathbb{Z}_{\geq 0}^{\ell}}$. *Then* $\mathbf{U}_q^-(w)_{\mathbb{Q}}^{\mathrm{up}}$ *is the* $\mathbb{Q}[q^{\pm 1}]$*-algebra generated by* $\{F^{\mathrm{up}}(\beta_k)\}_{1 \leq k \leq \ell}$.

Using the above proposition we obtain the following compabitility. This is a quantum analogue of the Geiß-Leclerc-Schroër's result.

Theorem 6. *Let* $\mathbf{B}^{\mathrm{up}}(w) := \mathbf{B}^{\mathrm{up}} \cap \mathbf{U}_q^-(w)_{\mathbb{Q}}^{\mathrm{up}}$. *Then* $\mathbf{B}^{\mathrm{up}}(w)$ *is a* $\mathbb{Q}[q^{\pm 1}]$*-basis of* $\mathbf{B}^{\mathrm{up}}(w)$.

2.5. **Specialization at** $q = 1$**.** For the Lusztig form, we have the specilization isomorphism $\mathbb{C} \otimes_{\mathbb{Q}[q^{\pm 1}]} \mathbf{U}_q^-(\mathfrak{g})_{\mathbb{Q}} \simeq U(\mathfrak{n})$. Dually, we have the $\mathbb{C}$-algebra isomorphism $\Phi^{\mathrm{up}} \colon \mathbb{C} \otimes_{\mathbb{Q}[q^{\pm 1}]} \mathbf{U}_q^-(\mathfrak{g})_{\mathbb{Q}}^{\mathrm{up}} \simeq \mathbb{C}[N]$.

Under the isomorphism $\mathbb{C} \otimes_{\mathbb{Q}[q^{\pm 1}]} \mathbf{U}_q^-(\mathfrak{g})_{\mathbb{Q}}^{\mathrm{up}} \simeq \mathbb{C}[N]$, as a corollay of the above theorem, we obtain the following result for $\mathbf{U}_q^-(w)$ which concerns the specialization at $q = 1$.

Corollary 7. *Under the* $\mathbb{C}$*-algebra isomorphism* Φ^{up}, *we have*

$$\mathbb{C} \otimes_{\mathbb{Q}[q^{\pm 1}]} \mathbf{U}_q^-(w)_{\mathbb{Q}}^{\mathrm{up}} \simeq \mathbb{C}[N(w)],$$

where $N(w)$ *is the unipotent subgroup associated with the nilpotent Lie algebra* $\mathfrak{n}(w)$.

3. Quantum closed unipotent subgroup and Dual canonical basis

For a Weyl group element $w \in W$ and a reduced word $\overrightarrow{w} = (i_1, \ldots, i_\ell)$, we set

$$\mathbf{U}_w^- := \sum_{\mathbf{a} = (a_1, \ldots, a_\ell) \in \mathbb{Z}_{\geq 0}^{\ell}} \mathbb{Q}(q) F_{i_1}^{(a_1)} \ldots F_{i_\ell}^{(a_\ell)}.$$

This is called Demazure-Schubert filtration. It is known that $\mathbf{U}_w^-$ is compatible with the canonical basis $\mathbf{B}$, that is $\mathbf{B} \cap \mathbf{U}_w^-$ is a $\mathbb{Q}[q^{\pm 1}]$-basis of $\mathbf{U}_w^-$. We denote the correponding subset by $\mathcal{B}(w, \infty)$. Hence we set

$$\mathcal{O}_q[\overline{N_w}] := \mathbf{U}_q^-(\mathfrak{g}) / (\mathbf{U}_w^-)^{\perp},$$

where $(\mathbf{U}_w^-)^{\perp}$ is the annhilator of $\mathbf{U}_w^-$ with respect to Kashiwara's bilinear form $(\ , \ )_K$. Since $(\mathbf{U}_w^-)^{\perp}$ is compatible with $\mathbf{B}^{\mathrm{up}}$, the canonical projection induces the dual canonical basis on $\mathcal{O}_q[\overline{N}_w]$.

Theorem 8. (1)*Let* $\mathbf{U}_q^-(w) \to \mathbf{U}_q^-(\mathfrak{g}) \to \mathcal{O}_q[\overline{N_w}]$ *be the inclusion and the canonical projectinon. Then the composite is monomorphism of algebra.*

(2) *We have* $\mathcal{B}(w) \subset \mathcal{B}(w, \infty)$.

4. Quantum flag minor and Its multiplicative proprerties

For a dominant integral weight $\lambda \in P_+$, let $V(\lambda)$ be the corresponding integrable highest weight module with highest weight vector u_λ. We have symmetric bilinear form $(\ ,\ )_\lambda$on $V(\lambda)$. Let $\pi_\lambda\colon \mathbf{U}_q^-(\mathfrak{g}) \twoheadrightarrow V(\lambda)$ be the projection defined by $x \mapsto xu_\lambda$. Let j_λ be dual of π_λ, that is $j_\lambda\colon V(\lambda) \hookrightarrow \mathbf{U}_q^-(\mathfrak{g})$. For a Weyl group element $w \in W$, we have the extremal vector $u_{w\lambda}$ of weight λ. It is known that $u_{w\lambda}$ is contained in the canoical basis and the dual canoncial basis. We set quantum unipotent minro $D_{w\lambda,\lambda}$by

$$D_{w\lambda,\lambda} := j_\lambda(u_{w\lambda}).$$

It is known that $D_{w\lambda,\lambda} \in \mathbf{B}^{\mathrm{up}}$. The following is main result in our study.

Theorem 9. (1) *For* $w \in W$ *and* $\lambda \in P_+$*, we have* $D_{w\lambda,\lambda} \in \mathbf{U}_q^-(w)$.

(2) *For arbitrary* $b \in \mathcal{B}(w)$*, there exists* $N \in \mathbb{Z}$ *such that* $q^N G^{\mathrm{up}}(b) D_{w\lambda,\lambda} \in \mathbf{B}^{\mathrm{up}}(w)$*, there* $G^{\mathrm{up}}(b)$ *is the dual canonical basis element which is associated with* $b \in \mathcal{B}(w)$.

Using the above theorem, we obtain the following quantum seed.

For a Weyl group element w, a reduced word $\overrightarrow{w} = (i_1, i_2, \ldots, i_\ell)$ and $\mathbf{c} = (c_1, \ldots, c_\ell) \in \mathbb{Z}_{\geq 0}^\ell$, we set

$$D^{\overrightarrow{w}}(\mathbf{c}) := \prod_{1 \leq k \leq \ell} D_{s_{i_1} \cdots s_{i_k} c_k \varpi_{i_k}, c_k \varpi_{i_k}}.$$

Then $\{D^{\overrightarrow{w}}(\mathbf{c})\}_{\mathbf{c} \in \mathbb{Z}_{\geq 0}^\ell}$ forms a mutually commmuting familty and $\{D^{\overrightarrow{w}}(\mathbf{c})\}_{\mathbf{c} \in \mathbb{Z}_{\geq 0}^\ell}$ is linear independent over $\mathbb{Z}[q^{\pm 1}]$. $\{D^{\overrightarrow{w}}(\mathbf{c})\}_{\mathbf{c} \in \mathbb{Z}_{\geq 0}^\ell}$ can be considered as a quantum analogue of the initial seed in [18] and we can form the corresponding quantum cluster algebra by it. Our conjecture is an $\mathbb{Q}[q^{\pm 1}]$-algebra isomorphism between the quantum cluster algebra and the quantum unipotent subgroup $\mathcal{O}_q[N(w)]$ and the set of quantum cluster monomials is contained by the dual canonical basis $\mathbf{B}^{\mathrm{up}}(w)$. This is just a quantum analogue of [18] and this is compatible with their open orbit conjecture for symmetric $\mathfrak{g}$. Recently the $\mathbb{Q}(q)$-algebra isomorphism is obtained by [17].

References

[1] A. Berenstein, S. Fomin, and A. Zelevinsky. Cluster algebras. III. Upper bounds and double Bruhat cells. *Duke Math. J.*, 126(1):1–52, 2005.

[2] A. Berenstein and A. Zelevinsky. String bases for quantum groups of type A_r. In *I. M. Gelfand Seminar*, volume 16 of *Adv. Soviet Math.*, pages 51–89. Amer. Math. Soc., Providence, RI, 1993.

[3] A. Berenstein and A. Zelevinsky. Quantum cluster algebras. *Adv. Math.*, 195(2):405–455, 2005.

[4] P. Caldero. On the q-commutations in $U_q(\mathfrak{n})$ at roots of one. *J. Algebra*, 210(2):557–576, 1998.

[5] P. Caldero. Adapted algebras for the Berenstein-Zelevinsky conjecture. *Transform. Groups*, 8(1):37–50, 2003.

[6] P. Caldero. A multiplicative property of quantum flag minors. *Represent. Theory*, 7:164–176 (electronic), 2003.

[7] V. V. Fock and A. B. Goncharov. The quantum dilogarithm and representations of quantum cluster varieties. *Invent. Math.*, 175(2):223–286, 2009.

[8] V.V. Fock and A.B. Goncharov. Cluster ensembles, quantization and the dilogarithm. *Ann. Sci. Éc. Norm. Supér. (4)*, 42(6):865–930, 2009.

[9] V.V. Fock and A.B. Goncharov. Cluster ensembles, quantization and the dilogarithm. II. the intertwiner. In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. I*, volume 269 of *Progr. Math.*, pages 655–673. Birkhäuser Boston Inc., Boston, MA, 2009.

[10] S. Fomin and A. Zelevinsky. Cluster algebras. I. Foundations. *J. Amer. Math. Soc.*, 15(2):497–529 (electronic), 2002.

[11] S. Fomin and A. Zelevinsky. Cluster algebras. II. Finite type classification. *Invent. Math.*, 154(1):63–121, 2003.

[12] S. Fomin and A. Zelevinsky. Cluster algebras. IV. Coefficients. *Compos. Math.*, 143(1):112–164, 2007.

[13] C. Geiß, B. Leclerc, and J. Schröer. Semicanonical bases and preprojective algebras. *Ann. Sci. École Norm. Sup. (4)*, 38(2):193–253, 2005.

[14] C. Geiß, B. Leclerc, and J. Schröer. Verma modules and preprojective algebras. *Nagoya Math. J.*, 182:241–258, 2006.

[15] C. Geiß, B. Leclerc, and J. Schröer. Cluster algebra structures and semicanonical bases for unipotent groups. E-print arXiv http://arxiv.org/abs/math/0703039, 2007.

[16] C. Geiß, B. Leclerc, and J. Schröer. Semicanonical bases and preprojective algebras. II. A multiplication formula. *Compos. Math.*, 143(5):1313–1334, 2007.

[17] C. Geiss, B. Leclerc, and J. Schröer. Cluster structures on quantum coordinate rings. E-print arXiv http://arxiv.org/abs/1104.0531v2, 04 2011.

[18] C. Geiß, B. Leclerc, and J. Schröer. Kac–moody groups and cluster algebras. *Advances in Mathematics*, 228(1):329–433, Sep 2011.

[19] Christof Geiß, Bernard Leclerc, and Jan Schröer. Generic bases for cluster algebras and the chamber ansatz. *J. Amer. Math. Soc. 25 (2012), 21-76*, 04 2010.

[20] I. Grojnowski and G. Lusztig. A comparison of bases of quantized enveloping algebras. In *Linear algebraic groups and their representations (Los Angeles, CA, 1992)*, volume 153 of *Contemp. Math.*, pages 11–19. Amer. Math. Soc., Providence, RI, 1993.

[21] M. Kashiwara. On crystal bases of the Q-analogue of universal enveloping algebras. *Duke Math. J.*, 63(2):465–516, 1991.

[22] Y. Kimura. Quantum unipotent subgroup and dual canonical basis. E-print arXiv http://arxiv.org/abs/1010.4242, to appear in Kyoto J. of Math., 2010.

[23] S. Levendorskiĭ and Y. Soĭbelman. Algebras of functions on compact quantum groups, Schubert cells and quantum tori. *Comm. Math. Phys.*, 139(1):141–170, 1991.

[24] G. Lusztig. Quivers, perverse sheaves, and quantized enveloping algebras. *J. Amer. Math. Soc.*, 4(2):365–421, 1991.

[25] G. Lusztig. Affine quivers and canonical bases. *Inst. Hautes Études Sci. Publ. Math.*, (76):111–163, 1992.

[26] G. Lusztig. *Introduction to quantum groups*, volume 110 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1993.

[27] G. Lusztig. Braid group action and canonical bases. *Adv. Math.*, 122(2):237–261, 1996.

[28] G. Lusztig. Semicanonical bases arising from enveloping algebras. *Adv. Math.*, 151(2):129–139, 2000.

[29] H. Nakajima. Quiver varieties and canonical bases of quantum affine algebras. Available at http://www4.ncsu.edu/ jing/conf/CBMS/cbms10.html, 2010.

[30] Y. Saito. PBW basis of quantized universal enveloping algebras. *Publ. Res. Inst. Math. Sci.*, 30(2):209–232, 1994.

Research Insitute for Mathematical Sciences
Graduate School of Science
Kyoto University
Kitashiwakawa Oiwake-cho, Sakyo-ku, Kyoto, Yamanashi 606-8502 JAPAN

E-mail address: ykimura@kurims.kyoto-u.ac.jp