Proceedings of the 42nd Symposium on Ring Theory and Representation Theory

October 10 (Sat.) - 12 (Mon.), 2009
Osaka Kyoiku University, Japan

Edited by
Shigeto Kawata
Osaka City University

JSPS Grants-in Aid for 2009 Scientific Research (B)
(Representative Researcher: Kunio Yamagata)

January, 2010
Osaka, JAPAN

第 42 回 環論および表現論シンポジウム報告集

２００９年１０月１０日（土）－１２日（月）
大阪教育大学

編集:　河田成人 (大阪市立大学)

平成 21 年度科学研究費補助金 基盤研究（B）
(代表：山形邦夫)

２０１０年1月
大阪市立大学

Organizing Committee of The Symposium on Ring Theory and Representation Theory

The Symposium on Ring Theory and Representation Theory has been held annually in Japan and the Proceedings have been published by the organizing committee. The first Symposium was organized in 1968 by H. Tominaga, H. Tachikawa, M. Harada and S. Endo. After their retirement, a new committee was organized in 1997 for managing the Symposium. The present members of the committee are H. Asashiba (Shizuoka Univ.), S. Ikehata (Okayama Univ.), S. Koshitani (Chiba Univ.), M. Sato (Yamanashi Univ.) and K. Yamagata (Tokyo Noukou Univ.).

The Proceedings of each Symposium is edited by program organizer. Anyone who wants these Proceedings should ask to the program organizer of each Symposium or one of the committee members.

The Symposium in 2010 will be held at Naruto Kyoiku University for Sep. 10(Fri.)-12(Sun.), and the program will be arranged by H. Komatsu (Okayama Pref. Univ.).

Concerning several information on ring theory group in Japan containing schedules of meetings and symposiums as well as the addresses of members in the group, you should refer the following homepage, which is arranged by M. Sato (Yamanashi Univ.):

http://fuji.cec.yamanashi.ac.jp/˜ring/ (in Japanese)

fuji.cec.yamanashi.ac.jp/˜ring/japan/ (in English)

Masahisa Sato
Yamanashi Japan
December, 2009

CONTENTS

PREFACE

The 42nd Symposium on Ring Theory and Representation Theory was held at Osaka Kyoiku University on October 10th - 12th, 2009. The symposium and the proceedings are financially supported by Kunio Yamagata (Tokyo University of Agriculture and Technology) JSPS Grant-in-Aid for 2009 Scientific Research (B), No. 21340003.

This volume consisits of the articles presented at the symposium. We would like to thank all speakers and coauthors for their contributions.

We would also like to express our thanks to all the members of the orgnizing commitee (Professors Hideto Asashiba, Shûichi Ikehata, Shigeo Koshitani, Masahisa Sato and Kunio Yamagata) for their helpful suggestions concerning the symposium. Finally we would like to express our gratitude to Professor Yoshitomo Baba and students of Osaka Kyoiku University who contributed in the organization of the symposium.

Shigeto Kawata
Osaka, Japan
January, 2010

第42回 環論および表現論シンポジウム
プログラム

１０月１０日（土曜日）

09：00 ー 09：45　池畑 秀一（岡山大学）
George Szeto (Bradley University)
Lianyong Xue (Bradley University)

On Galois Extensions with an Inner Galois Group and a Galois Commutator Subring

10：00 ー 10：45　Edward Poon（Embry-Riddle University）
Hisaya Tsutsui (Embry-Riddle University)
平野 康之（鳴門教育大学）

Fully Weakly Prime Rings

11：00 ー 12：00　Fred Van Oystaeyen (Antwerp University)

Crystalline graded rings

13：30 ー 14：15　大関 一秀（明治大学）
後藤 四郎（明治大学）
西田 康二（千葉大学）

On the structure of Sally modules of rank one

14：30 ー 15：15　早坂 太（明治大学）

The Buchsbaum-Rim function of a parameter module

15：30 ー 16：15　亀山 統胤（信州大学）

Extension of the Matlis duality to a filtered Noetherian ring

16：30 ー 17：15　柳川 浩二（関西大学）

Dualizing complex of the Stanley ring associated with a simplicial poset

１０月１１日（日曜日）

09：00 ー 09：45　高橋 亮（信州大学）

Thick subcategories of the stable category of Cohen-Macaulay modules

10：00 ー 10：45　源 泰幸（京都大学）

Ampleness of two-sided tilting complexes and Fano algebras

11：00 ー 12：00　Fred Van Oystaeyen (Antwerp University)

The projective scheme of the blow-up ring

13：30 − 14：15　阿部 弘樹（筑波大学）
星野 光男（筑波大学）

Derived equivalences for endomorphism rings

14：30 − 15：15　山浦 浩太（名古屋大学）

The classification of tilting modules over Harada algebras

15：30 − 16：15　本瀬 香（弘前市）

The Stickelberger relation and Loewy series of group algebras Map($\mathbb{F}_q, \mathbb{F}_q$)

16：30 − 17：30　Changchang Xi (Beijing Normal University)

Homological conjectures and radical-full extensions, I

１０月１２日（月曜日）

09：00 − 09：45　久田見 守（山口大学）

Almost comparability and related comparabilities in von Neumann regular rings

10：00 − 10：45　Martin Herschend（名古屋大学）

The Clebsch-Gordan problem for quiver representations

11：00 − 12：00　Changchang Xi (Beijing Normal University)

Homological conjectures and radical-full extensions, II

The 42nd Symposium on Ring Theory and Representation Theory

Program

October 10 (Saturday)

09:00 – 09:45 Shuichi Ikehata (Okayama University)
George Szeto (Bradley University)
Lianyong Xue (Bradley University)
On Galois Extensions with an Inner Galois Group and a Galois Commutator Subring

10:00 – 10:45 Edward Poon (Embry-Riddle University)
Hisaya Tsutsui (Embry-Riddle University)
Yasuyuki Hirano (Naruto Kyoiku University)
Fully Weakly Prime Rings

11:00 – 12:00 Fred Van Oystaeyen (Antwerp University)
Crystalline graded rings

13:30 – 14:15 Kazuho Ozeki (Meiji Institute for Advanced Study of Mathematical Sciences)
Shiro Goto (Meiji University)
Koji Nishida (Chiba University)
On the structure of Sally modules of rank one

14:30 – 15:15 Futoshi Hayasaka (Meiji University)
The Buchsbaum-Rim function of a parameter module

15:30 – 16:15 Noritsugu Kameyama (Shinshu University)
Extension of the Matlis duality to a filtered Noetherian ring

16:30 – 17:15 Kohji Yanagawa (Kansai University)
Dualizing complex of the Stanley ring associated with a simplicial poset

October 11 (Sunday)

09:00 – 09:45 Ryo Takahashi (Shinshu University)
Thick subcategories of the stable category of Cohen-Macaulay modules

10:00 – 10:45 Hiroyuki Minamoto (Kyoto University)
Ampleness of two-sided tilting complexes and Fano algebras

11:00 – 12:00 Fred Van Oystaeyen (Antwerp University)
The projective scheme of the blow-up ring

13:30 – 14:15 Hiroki Abe (University of Tsukuba)
Mitsuo Hoshino (University of Tsukuba)
Derived equivalences for endomorphism rings

14:30 – 15:15 Kota Yamaura (Nagoya University)
The classification of tilting modules over Harada algebras

15:30 – 16:15 Kaoru Motose (Hirosaki)
The Stickelberger relation and Loewy series of group algebras Map($\mathbb{F}_q, \mathbb{F}_q$)

16:30 – 17:30 Changchang Xi (Beijing Normal University)
Homological conjectures and radical-full extensions, I

October 12 (Monday)

09:00 – 09:45 Mamoru Kutami (Yamaguchi University)
Almost comparability and related comparabilities in von Neumann regular rings

10:00 – 10:45 Martin Herschend (Nagoya University)
The Clebsch-Gordan problem for quiver representations

11:00 – 12:00 Changchang Xi (Beijing Normal University)
Homological conjectures and radical-full extensions, II

DERIVED EQUIVALENCES FOR ENDOMORPHISM RINGS

HIROKI ABE AND MITSUO HOSHINO

Abstract. We provide derived equivalences for endomorphism rings associated with a certain exact sequences.

1. Notation

For a ring A, we denote by Mod-A the category of right A-modules, by mod-A the full subcategory of Mod-A consisting of finitely presented modules and by $\mathcal{P}_A$ the full subcategory of Mod-A consisting of finitely generated projective modules. For $M \in$ Mod-A, we denote by proj dim M_A (resp., inj dim M_A) the projective (resp., injective) dimension of M, where we use the notation M_A to stress that the module M considered is a right A-module, and by $\Omega^n M$ the nth syzygy of M. For a ring A, we denote by gl dim A the global dimension of A. For an object X in an additive category $\mathcal{B}$, we denote by $\mathrm{add}(X)$ the full subcategory of $\mathcal{B}$ whose objects are direct summands of finite direct sums of copies of X.

2. Main result

In [1], we have shown the following.

Theorem 1 ([1, Lemma 1.1])**.** *Let $0 \to Y \xrightarrow{\mu} E \xrightarrow{\varepsilon} X \to 0$ be an exact sequence in an abelian category $\mathcal{A}$ and P an object of $\mathcal{A}$. Assume that $E \in \mathrm{add}(P)$ and that both $\mathrm{Hom}_{\mathcal{A}}(P, \varepsilon)$ and $\mathrm{Hom}_{\mathcal{A}}(\mu, P)$ are epic. Then $\mathrm{End}_{\mathcal{A}}(X \oplus P)$ and $\mathrm{End}_{\mathcal{A}}(Y \oplus P)$ are derived equivalent to each other.*

The next two propositions are direct consequences of Theorem 1.

Proposition 2. *Let A be a right noetherian ring and $M \in$ mod-A. If $\mathrm{Ext}^i_A(M, A) = 0$ for $1 \le i \le n$, then $\mathrm{End}_A(M \oplus A)$ and $\mathrm{End}_A(\Omega^n M \oplus A)$ are derived equivalent to each other.*

Proposition 3. *Let A be an Artin algebra, $P \in$ mod-A and $0 \to Y \to E \to X \to 0$ an almost split sequence in mod-A. If $E \in \mathrm{add}(P)$ and $X, Y \notin \mathrm{add}(P)$, then $\mathrm{End}_A(X \oplus P)$ and $\mathrm{End}_A(Y \oplus P)$ are derived equivalent to each other.*

The propositions above enable us to construct many derived equivalences between endomorphism rings. For example, we obtain the following.

Example 4. Let k be a field, $R = k[X_1, \cdots, X_n]/\langle X_i^2 - X_j^2, X_iX_j \mid 1 \le i \ne j \le n\rangle$ with $n \ge 2$ and S the simple R-module. Then the following hold.

The detailed version of this paper will be submitted for publication elsewhere.

(1) $\mathrm{End}_R(\Omega^{-i-1}S \oplus \Omega^{-i}S)$ and $\mathrm{End}_R(\Omega^{-1}S \oplus S)$ are derived equivalent to each other for all $i \geq 1$, where the first algebra has global dimension three and the last algebra has global dimension two.
(2) $\mathrm{End}_R(\Omega^{i}S \oplus \Omega^{i+1}S)$ and $\mathrm{End}_R(S \oplus \Omega S)$ are derived equivalent to each other for all $i \geq 1$, where the first algebra has global dimension three and the last algebra has global dimension two.
(3) $\mathrm{End}_R(\Omega^{i}S \oplus \Omega^{i+1}S \oplus R)$ and $\mathrm{End}_R(S \oplus \Omega S \oplus R)$ are derived equivalent to each other for all $i \in \mathbb{Z}$, where these algebras have global dimension three.
(4) $\mathrm{End}_R(\Omega^{i}S \oplus \Omega^{i+1}S)$ is isomorphic to a trivial extension of $\begin{pmatrix} k & k^n \\ 0 & k \end{pmatrix}$ for all $i \in \mathbb{Z}$.

3. Auslander algebra

In this section, we apply the results of the previous section to Auslander algebras. We start by recalling the definition of Auslander algebras (see e.g. [3] for details).

Definition 5. Let Λ be an Artin algebra and $0 \to \Lambda \to I^0 \to I^1 \to \cdots$ a minimal injective resolution in mod-Λ. Set $\mathrm{dom\,dim}\,\Lambda = \sup\{k \in \mathbb{Z} \mid I^i \in \mathcal{P}_\Lambda \text{ for } 0 \leq i \leq k-1\}$, which is called the dominant dimension of Λ. Then Λ is said to be an Auslander algebra provided $\mathrm{gl\,dim}\,\Lambda \leq 2$ and $\mathrm{dom\,dim}\,\Lambda \geq 2$.

Let A be a representation-finite Artin algerba and assume that A is basic and connected. Let $M_1, \cdots, M_m$ be a complete set of nonisomorphic indecomposable modules in mod-A and set $I = \{1, \cdots, m\}$. We assume that $m \geq 2$, i.e., A is not simple. Then, setting $M = \bigoplus_{i\in I} M_i$, we have an Auslander algebra $\Lambda = \mathrm{End}_A(M)$, which will be called the Auslander algebra of A. For each indecomposable module $X \in$ mod-A, since there exists a unique $i_X \in I$ such that $X \cong M_{i_X}$, we set $I(X) = I \setminus \{i_X\}$, $M_X = \bigoplus_{i\in I(X)} M_i$ and $\Lambda_X = \mathrm{End}_A(M_X)$. Then by Proposition 3 we have the following.

Proposition 6. *The following hold.*
(1) *If X is not projective then Λ_X is derived equivalent to $\Lambda_{\tau X}$, where τ denotes the Auslander-Reiten translation.*
(2) *If X is not injective then Λ_X is derived equivalent to $\Lambda_{\tau^{-1}X}$.*

We can calculate the global dimension and the dominant dimension of Λ_X.

Lemma 7. *Assume that X is not projective, not injective and $\tau X \cong X$. Then A is a local Nakayama algebra and the following hold.*
(1) *If $m = 2$, then $\Lambda_X \cong A$ as algebras.*
(2) *If $m > 2$, then* $\mathrm{inj\,dim}\,\Lambda_X = 2$.

Proposition 8. *The following hold.*
(1) *If X is projective (resp., injective), then* $\mathrm{gl\,dim}\,\Lambda_X \leq 2$.
(2) *If X is not projective, not injective and $\tau X \ncong X$, then* $\mathrm{gl\,dim}\,\Lambda_X = 3$.
(3) *If X is not projective, not injective and $\tau X \cong X$, then* $\mathrm{gl\,dim}\,\Lambda_X = \infty$.

Proposition 9. *The following hold.*
(1) *If X is projective (resp., injective), not injective (resp., not projective) and not simple, then* $\mathrm{dom\,dim}\,\Lambda_X = 0$.

(2) *If X is projective (resp., injective), not injective (resp., not projective) and simple, then* dom dim $\Lambda_X = 1$.
(3) *If X is projective and injective, then* dom dim $\Lambda_X \geq 2$.
(4) *If X is not projective and not injective, then* dom dim $\Lambda_X \geq 2$.

It follows by the propositions above that Λ_X is an Auslander algebra if and only if X is projective and injective.

Consider next the case where X is a simple projective module with inj dim $X_A = 1$. Let $P_1, \cdots, P_n = X$ be a complete set of nonisomorphic indecomposable modules in $\mathcal{P}_A$ and set $T = (\bigoplus_{i=1}^{n-1} P_i) \oplus \tau^{-1}X$. Then T is a classical tilting module, i.e., a tilting module of projective dimension ≤ 1 (cf. [2]). Set $B = \mathrm{End}_A(T)$ and $Y = \mathrm{Ext}^1_A(T, X) \in$ mod-B. Then Y is a simple injective module with proj dim $Y_B = 1$. We set $N_Y = \mathrm{Hom}_A(T, M_X)$, $N = N_Y \oplus Y$, $\Gamma = \mathrm{End}_B(N)$ and $\Gamma_Y = \mathrm{End}_B(N_Y)$. Note that Γ is the Auslander algebra of B.

Proposition 10. *We have $\Gamma_Y \cong \Lambda_X$ as algebras and hence for any $i, j \geq 0$, if $\tau^i Y, \tau^{-j} X$ are nonzero, $\Gamma_{\tau^i Y}$ and $\Lambda_{\tau^{-j} X}$ are derived equivalent to Λ_X.*

Remark 11. Set $\tilde{T} = \mathrm{Hom}_A(M, M_X) \oplus \mathrm{Ext}^1_A(M, X) \in$ mod-Λ. Then the following hold.

(1) $\mathrm{End}_\Lambda(\tilde{T}) \cong \Gamma$ as algebras.
(2) proj dim $\tilde{T}_\Lambda = 2$.
(3) there exists an exact sequence $0 \to \Lambda \to T^0 \to T^1 \to T^2 \to 0$ in mod-Λ with the $T^i \in \mathrm{add}(\tilde{T})$.
(4) $\mathrm{Ext}^1_\Lambda(\tilde{T}, \tilde{T}) = 0$.
(5) $\mathrm{Ext}^2_\Lambda(\tilde{T}, \tilde{T}) = 0$ if and only if $A \cong \begin{pmatrix} D & D \\ 0 & D \end{pmatrix}$ with $D = \mathrm{End}_A(X)$.

4. Tilting module

Finally, we point out that the exact sequence in Theorem 1 enables us to construct another tilting module from a given tilting module by exchanging direct summands.

Proposition 12. *Let A be a ring, $P \in$ Mod-A and $0 \to Y \xrightarrow{\mu} E \xrightarrow{\varepsilon} X \to 0$ an exact sequence in* Mod-A*. Assume that $E \in \mathrm{add}(P)$ and that both $\mathrm{Hom}_A(P, \varepsilon)$ and $\mathrm{Hom}_A(\mu, P)$ are epic. Then $X \oplus P$ is a tilting module if and only if so is $Y \oplus P$. In particular, if $X \oplus P$ is a classical tilting module, then so is $Y \oplus P$.*

Corollary 13. *Let A be a Noether algebra and $X \in$* mod-A*. Assume that there exists $T \in$* mod-A *such that $X \oplus T$ is a tilting module. Then the following hold.*

(1) *If there exists an epimorphism of the form $f : T^{(l)} \to X$, then there exists an epimorphism $\varepsilon : T^{(r)} \to X$ such that* Ker $\varepsilon \oplus T$ *is a tilting module. In particular, if $X \oplus T$ is a classical tilting module, then so is* Ker $\varepsilon \oplus T$.
(2) *If there exists a monomorphism of the form $g : X \to T^{(l)}$, then there exists a monomorphism $\mu : X \to T^{(r)}$ such that* Cok $\mu \oplus T$ *is a tilting module.*

References

[1] H. Abe and M. Hoshino, Gorenstein orders associated with modules, Comm. Algebra, to appear.
[2] M. Auslander, M. I. Platzeck and I. Reiten, Coxeter functors without diagrams, Trans. Amer. Math. Soc., **250** (1979), 1–12.
[3] M. Auslander, I. Reiten and S. O. Smalø, *Representation theory of artin algebras*, Cambridge studies in advanced mathematics., 36, Cambridge University Press, 1995.

Institute of Mathematics
University of Tsukuba
Ibaraki 305-8571 JAPAN
E-mail address: abeh@math.tsukuba.ac.jp

Institute of Mathematics
University of Tsukuba
Ibaraki 305-8571 JAPAN
E-mail address: hoshino@math.tsukuba.ac.jp

THE BUCHSBAUM-RIM FUNCTION OF A PARAMETER MODULE

FUTOSHI HAYASAKA

Abstract. This note is basically a summary of a part of the paper [11] with Eero Hyry (University of Tampere). In this note we prove that the Buchsbaum-Rim function $\ell_A(\mathcal{S}_{\nu+1}(F)/N^{\nu+1})$ of a parameter module N in F is bounded above by $e(F/N)\binom{\nu+d+r-1}{d+r-1}$ for every integer $\nu \geq 0$. Moreover, it turns out that the base ring A is Cohen-Macaulay once the equality holds for some integer ν. As a direct consequence, we observe that the first Buchsbaum-Rim coefficient $e_1(F/N)$ of a parameter module N is always non-positive.

1. Introduction

Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d. Let $F = A^r$ be a free module of rank $r > 0$, and let $S = \mathcal{S}_A(F)$ be the symmetric algebra of F, which is a polynomial ring over A. For a submodule M of F, let $\mathcal{R}(M)$ denote the image of the natural homomorphism $\mathcal{S}_A(M) \to \mathcal{S}_A(F)$, which is a standard graded subalgebra of S. Assume that the quotient F/M has finite length and $M \subseteq \mathfrak{m}F$. Then we can consider the function

$$\lambda : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0} \ ; \quad \nu \mapsto \ell_A(S_\nu/M^\nu)$$

where S_ν and M^ν denote the homogeneous components of degree ν of S and $\mathcal{R}(M)$, respectively. Buchsbaum and Rim studied this function in [4] in order to generalize the notion of the usual Hilbert-Samuel multiplicity of an $\mathfrak{m}$-primary ideal. They proved that $\lambda(\nu)$ eventually coincides with a polynomial $P(\nu)$ of degree $d + r - 1$. This polynomial can then be written in the form

$$P(\nu) = \sum_{i=0}^{d+r-1} (-1)^i e_i(F/M)\binom{\nu+d+r-2-i}{d+r-1-i}$$

with integer coefficients $e_i(F/M)$. The coefficients $e_i(F/M)$ are called the *Buchsbaum-Rim coefficients* of F/M. The *Buchsbaum-Rim multiplicity of* F/M, denoted by $e(F/M)$, is now defined to be the leading coefficient $e_0(F/M)$.

In their article Buchsbaum and Rim also introduced the notion of a parameter module (matrix), which generalizes the notion of a parameter ideal (system of parameters). The module N in F is said to be *a parameter module in* F, if the following three conditions are satisfied: (i) F/N has finite length, (ii) $N \subseteq \mathfrak{m}F$, and (iii) $\mu_A(N) = d + r - 1$, where $\mu_A(N)$ is the minimal number of generators of N.

A starting point of this note is the characterization of the Cohen-Macaulay property of A given in [4, Corollary 4.5] by means of the equality $\ell_A(F/N) = e(F/N)$ for every

The detailed version of this paper has been submitted for publication elsewhere.

parameter module N of rank r in $F = A^r$. Brennan, Ulrich and Vasconcelos observed in [1, Theorem 3.4] that if A is Cohen-Macaulay, then in fact

$$\ell_A(S_{\nu+1}/N^{\nu+1}) = e(F/N)\binom{\nu+d+r-1}{d+r-1}$$

for all integers $\nu \geq 0$. Our main result is now as follows:

Theorem 1. *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$.*

(1) *For any rank $r > 0$, the inequality*

$$\ell_A(S_{\nu+1}/N^{\nu+1}) \geq e(F/N)\binom{\nu+d+r-1}{d+r-1}$$

always holds true for every parameter module N in $F = A^r$ and for every integer $\nu \geq 0$.

(2) *The following statements are equivalent:*
   - (*i*) *A is a Cohen-Macaulay local ring;*
   - (*ii*) *There exists an integer $r > 0$ and a parameter module N of rank r in $F = A^r$ such that the equality*

$$\ell_A(S_{\nu+1}/N^{\nu+1}) = e(F/N)\binom{\nu+d+r-1}{d+r-1}$$

holds true for some integer $\nu \geq 0$.

This generalizes our previous result [10, Theorem 1.3] where we assumed that $\nu = 0$. The equivalence of (i) and (ii) in (2) seems to contain some new information even in the ideal case. Indeed, it improves a recent observation that the ring A is Cohen-Macaulay if there exists a parameter ideal Q in A such that $\ell_A(A/Q^{\nu+1}) = e(A/Q)\binom{\nu+d}{d}$ for all $\nu \gg 0$ (see [8, 12]). Moreover, as a direct consequence of (1), we have the non-positivity of the first Buchsbaum-Rim coefficient of a parameter module.

Corollary 2. *For any rank $r > 0$, the inequality*

$$e_1(F/N) \leq 0$$

always holds true for every parameter module N in $F = A^r$.

Mandal and Verma have recently proved that $e_1(A/Q) \leq 0$ for any parameter ideal Q in A (see [15], and also [8]). Corollary 2 can be viewed as the module version of this fact. However, our proof based on the inequality in Theorem 1 (1) is completely different from theirs and is considerably more simpler.

2. Preliminaries

Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d. Let $F = A^r$ be a free module of rank $r > 0$. Let $S = \mathcal{S}_A(F)$ be the symmetric algebra of F. Let N be a parameter module in F, that is, N is a submodule of F satisfying the conditions: (i) $\ell_A(F/N) < \infty$, (ii) $N \subseteq \mathfrak{m}F$, and (iii) $\mu_A(N) = d+r-1$. We put $n = d+r-1$. Let N^ν be the homogeneous component of degree ν of the standard graded subalgebra $\mathcal{R}(N) = \mathrm{Im}(\mathcal{S}_A(N) \to S)$ of S. Let $\tilde{N} = (c_{ij})$ be the matrix associated to a minimal free presentation

$$A^n \xrightarrow{\tilde{N}} F \to F/N \to 0$$

of F/N. Let $X = (X_{ij})$ be a generic matrix of the same size $r \times n$. We denote by $I_s(X)$ the ideal in the polynomial ring $A[X] = A[X_{ij} \mid 1 \le i \le r, 1 \le j \le n]$ generated by the s-minors of X. Let $B = A[X]_{(\mathfrak{m},X)}$ be the ring localized at the graded maximal ideal $(\mathfrak{m}, X)$ of $A[X]$. The substitution map $A[X] \to A$ where $X_{ij} \mapsto c_{ij}$ now induces a map $\varphi : B \to A$. We consider the ring A as a B-algebra via the map φ. Let

$$\mathfrak{b} = \mathrm{Ker}\,\varphi = (X_{ij} - c_{ij} \mid 1 \le i \le r, 1 \le j \le n)B.$$

Set $G = B^r$, and let L denote the submodule $\mathrm{Im}(B^n \xrightarrow{X} G)$ of G. Let G_ν and L^ν be the homogeneous components of degree ν of the graded algebras $\mathcal{S}_B(G)$ and $\mathcal{R}(L)$, respectively. Then one can check the following.

Lemma 3. *For any integers $\nu \ge 0$, we have the following:*

(1) $(G_{\nu+1}/L^{\nu+1}) \otimes_B (B/\mathfrak{b}) \cong S_{\nu+1}/N^{\nu+1}$;
(2) $\mathrm{Supp}_B(G_{\nu+1}/L^{\nu+1}) = \mathrm{Supp}_B(B/I_r(X)B)$;
(3) *The ideal $\mathfrak{b}$ is generated by a system of parameters of the module $G_{\nu+1}/L^{\nu+1}$.*

The following fact concerning $G_{\nu+1}/L^{\nu+1}$ is known by [3, Corollary 3.2] (see also [13, Proposition 3.3]).

Lemma 4. *For any integer $\nu \ge 0$, we have $G_{\nu+1}/L^{\nu+1}$ is a perfect B-module of grade d.*

The following plays a key role in the proof of Theorem 1. See [11, Proposition 2.4] for the proof.

Proposition 5. *For any $\mathfrak{p} \in \mathrm{Min}_B(B/I_r(X)B)$, the equality*

$$\ell_{B_\mathfrak{p}}\left((G_{\nu+1}/L^{\nu+1})_\mathfrak{p}\right) = \ell_{B_\mathfrak{p}}\left((B/I_r(X)B)_\mathfrak{p}\right)\binom{\nu+d+r-1}{d+r-1}$$

holds true for all integers $\nu \ge 0$.

3. Proof of Theorem 1

In order to prove Theorem 1, we need to introduce more notation. For any matrix $\mathfrak{a}$ of size $r \times n$ over an arbitrary ring, we denote by $K_\bullet(\mathfrak{a})$ its Eagon-Northcott complex [6]. When $r = 1$, the complex $K_\bullet(\mathfrak{a})$ is just the ordinary Koszul complex of the sequence $\mathfrak{a}$. See [7, Appendix A2] for the definition and more details of complexes of this type. Recall in particular that if N is a parameter module in a free module F as in section 2, then

$$e(F/N) = \chi(K_\bullet(\tilde{N})),$$

where $\chi(K_\bullet(\tilde{N}))$ denotes the Euler-Poincaré characteristic of the complex $K_\bullet(\tilde{N})$ (see [4] and [14]). Moreover, one can check the following by computing $\mathrm{Tor}_p^B(B/IB, A)$ for any $p \ge 0$ (see [5]).

Lemma 6. *Using the setting and notation of section 2, we have*

$$\chi(K_\bullet(\mathfrak{b}) \otimes_B (B/I_r(X)B)) = \chi(K_\bullet(\tilde{N})).$$

Now we can give the proof of Theorem 1.

Proof of Theorem 1. We use the same notation as in section 2. Put $I = I_r(X)$.

(1): Fix integers $\nu \geq 0$. The ideal $\mathfrak{b}$ being generated by a system of parameters of the module $G_{\nu+1}/L^{\nu+1}$, we get

$$\begin{aligned} & \ell_A(S_{\nu+1}/N^{\nu+1}) \\ = \ & \ell_B((G_{\nu+1}/L^{\nu+1}) \otimes_B (B/\mathfrak{b})) \\ \geq \ & e(\mathfrak{b}; G_{\nu+1}/L^{\nu+1}) \\ = \ & \sum_{\mathfrak{p} \in \operatorname{Assh}_B(G_{\nu+1}/L^{\nu+1})} e(\mathfrak{b}; B/\mathfrak{p}) \cdot \ell_{B_\mathfrak{p}}((G_{\nu+1}/L^{\nu+1})_\mathfrak{p}) \\ = \ & \sum_{\mathfrak{p} \in \operatorname{Assh}_B(B/IB)} e(\mathfrak{b}; B/\mathfrak{p}) \cdot \ell_{B_\mathfrak{p}}((B/IB)_\mathfrak{p}) \binom{\nu+d+r-1}{d+r-1} \\ = \ & e(\mathfrak{b}; B/IB) \binom{\nu+d+r-1}{d+r-1} \\ = \ & \chi(K_\bullet(\mathfrak{b}) \otimes_B (B/IB)) \binom{\nu+d+r-1}{d+r-1} \\ = \ & \chi(K_\bullet(\tilde{N})) \binom{\nu+d+r-1}{d+r-1} \\ = \ & e(F/N) \binom{\nu+d+r-1}{d+r-1} \end{aligned}$$

as desired, where $e(\mathfrak{b}; *)$ denotes the multiplicity of $*$ with respect to $\mathfrak{b}$.

(2): The other implication being clear, by the ideal case, for example, it is enough to show that (ii) implies (i). Assume thus that

$$\ell_A(S_{\nu+1}/N^{\nu+1}) = e(F/N) \binom{\nu+d+r-1}{d+r-1}$$

for some $\nu \geq 0$. The above argument then gives

$$\ell_B((G_{\nu+1}/L^{\nu+1}) \otimes_B (B/\mathfrak{b})) = e(\mathfrak{b}; G_{\nu+1}/L^{\nu+1}).$$

It follows that $G_{\nu+1}/L^{\nu+1}$ is a Cohen-Macaulay B-module of dimension rn ([2, (5.12) Corollary]). By Lemma 4, $G_{\nu+1}/L^{\nu+1}$ is a perfect B-module of grade d. Thus, by the Auslander-Buchsbaum formula,

$$\begin{aligned} \operatorname{depth} B &= \operatorname{depth}_B(G_{\nu+1}/L^{\nu+1}) + \operatorname{pd}_B(G_{\nu+1}/L^{\nu+1}) \\ &= \dim_B(G_{\nu+1}/L^{\nu+1}) + \operatorname{grade}_B(G_{\nu+1}/L^{\nu+1}) \\ &= rn + d \\ &= \dim B. \end{aligned}$$

Therefore B is Cohen-Macaulay so that A is Cohen-Macaulay, too. □

References

[1] J. Brennan, B. Ulrich and W. V. Vasconcelos, *The Buchsbaum-Rim polynomial of a module*, J. Algebra **241** (2001), 379–392

[2] W. Bruns and U. Vetter, *Determinantal Rings*, Lecture Notes in Math. **1327**, Springer-Verlag Berlin Heidelberg, 1988

[3] D. A. Buchsbaum and D. Eisenbud, *Generic free resolutions and a family of generically perfect ideals*, Adv. in Math. **18** (1975), 245–301

[4] D. A. Buchsbaum and D. S. Rim, *A generalized Koszul complex. II. Depth and multiplicity*, Trans. Amer. Math. Soc. **111** (1964), 197–224

[5] D. A. Buchsbaum and D. S. Rim, *A generalized Koszul complex. III. A Remark on Generic Acyclicity*, Proc. Amer. Math. Soc. **16** (1965), 555–558

[6] J. A. Eagon and D. G. Northcott, *Ideals defined by matrices and a certain complex associated with them*, Proc. Roy. Soc. Ser. A **269** (1962), 188–204

[7] D. Eisenbud, *Commutative algebra. With a view toward algebraic geometry*, Graduate Texts in Mathematics, **150**, Springer-Verlag, New York, 1995

[8] L. Ghezzi, S. Goto, J. Hong, K. Ozeki, T. T. Phuong, and W. V. Vasconcelos, *Cohen-Macaulayness versus the vanishing of the first Hilbert coefficient of parameters*, to appear in J. London Math. Soc.

[9] F. Hayasaka and E. Hyry, *A family of graded modules associated to a module*, Communications in Algebra, Volume **36** (2008), Issue 11, 4201–4217

[10] F. Hayasaka and E. Hyry, *A note on the Buchsbaum-Rim multiplicity of a parameter module*, Proc. Amer. Math. Soc. **138** (2010), 545–551

[11] F. Hayasaka and E. Hyry, *A note on the Buchsbaum-Rim function of a parameter module*, Preprint 2009 (submitted)

[12] Y. Kamoi, *Remark on the polynomial type Poincaré series*, Proceedings of The Second Japan-Vietnam Joint Seminar on Commutative Algebra (2006), 162–168

[13] D. Katz and C. Naudé, *Prime ideals associated to symmetric powers of a module*, Comm. Algebra **23** (1995), no. 12, 4549–4555

[14] D. Kirby, *On the Buchsbaum-Rim multiplicity associated with a matrix*, J. London Math. Soc. (2) **32** (1985), no. 1, 57–61

[15] M. Mandal and J. K. Verma, *On the Chern number of an ideal*, Preprint 2008

Department of Mathematics
School of Science and Technology
Meiji University
1-1-1 Higashimita, Tama-ku, Kawasaki 214–8571, JAPAN
E-mail address: hayasaka@isc.meiji.ac.jp

REPRESENTATION RINGS OF STRING ALGEBRAS

MARTIN HERSCHEND

Abstract. String algebras are a class of algebras given by certain quivers with monomial relations. Thus the category of finite dimensional left modules over a string algebra is equipped with a tensor product defined point-wise and arrow-wise on the level of quiver representations. We describe the corresponding representation ring for any string algebra.

1. Introduction

The category of finite dimensional representations of a group G is equipped with a tensor product defined by diagonal action. Thus the set of isoclasses of such representations has the structure of a semi-ring, where addition is given by the direct sum and multiplication by the tensor product. From this semi-ring one constructs the representation ring $R(G)$ by including formal additive inverses.

It would be interesting to generalise this procedure to the category of left modules over an associative algebra A instead of group representations. However, in general there is no know way of defining a tensor product on this category. Now assume that A is given as the path algebra of a quiver Q with monomial relations, i.e. $A = kQ/\langle X\rangle$ for some set X of paths in Q and a field k. Then finite dimensional left A-modules are given by finite dimensional representations of Q satisfying the relations X (we call such representations (Q, X)-representations). Thus we can define a tensor product point-wise and arrow-wise. Moreover, as in the case of group representations we obtain a representation ring $R(Q, X)$, which we denote simply by $R(Q)$ in case X is empty. Our aim is to describe this ring.

By the Krull-Schmidt Theorem $R(Q, X)$ has a $\mathbb{Z}$-basis consisting of the isoclasses of indecomposable (Q, X)-representations. Thus, describing the multiplicative structure of $R(Q, X)$ amounts to solving the following problem: given to indecomposable (Q, X)-representations V, W decompose $V \otimes W$ into indecomposables. This problem is called the Clebsch-Gordan problem and has its origin in the study of binary algebraic forms by Clebsch and Gordan [2].

The most classical case is when Q is the loop quiver. For k algebraically closed of characteristic zero, the solution to the Clebsch-Gordan problem for the loop was found by Aitken [1]. The case when k is algebraically closed of positive characteristic was solved by Iima-Iwamatsu [11] and the case when k is perfect was treated in [3].

For Q a Dynkin quiver, $R(Q)$ was described for type $\mathbb{A}$ and $\mathbb{D}$ in [10] and for type $\mathbb{E}_6$ in [9]. The remaining cases $\mathbb{E}_7$ and $\mathbb{E}_8$ are still unsolved to my knowledge.

The detailed version of this paper has been submitted for publication elsewhere.

For extended Dynkin quivers of type $\tilde{\mathbb{A}}$ the solution to the Clebsch-Gordan problem was found in [7]. There is also a solution in case Q is the double loop quiver

$$\alpha \circlearrowright \bullet \circlearrowleft \beta$$

with relations $\alpha^n = \beta^n = \alpha\beta = \beta\alpha = 0$, found in [8]. These two cases are instances of string algebras. In the present article we shall describe the representation ring for each quiver with relations corresponding to a string algebra.

Gelfand and Ponomarev classified the indecomposable representations of the double loop quiver appearing above in [6], as part of their classification of Harish-Chandra modules over the Lorentz group. The indecomposables in this case fall into two classes called strings and bands. This type of classification was later used in other settings by Ringel [13] and Donovan-Freislich [4]. A well-rounded setting to which it applies is that of string algebras.

2. Preliminaries

Let us recall some definitions and set notation. More detail can be found in [5]. Throughout fix a perfect field k. A quiver Q consists of a set of vertices Q_0 and a set of arrows Q_1. Moreover, it is equipped with two maps $t, h : Q_1 \to Q_0$ mapping each arrow α to its tail $t\alpha$ and head $h\alpha$ respectively. We depict this by $t\alpha \xrightarrow{\alpha} h\alpha$.

A representation V of Q consists of a collection of finite dimensional k-vector spaces V_x, where $x \in Q_0$ and linear maps $V(\alpha) : V_x \to V_y$ where $x \xrightarrow{\alpha} y \in Q_1$. Let X be a set of paths in Q. We call V a (Q, X)-representation if for every path $\alpha_1 \cdots \alpha_n \in X$ the equality $V(\alpha_1) \cdots V(\alpha_n) = 0$ holds. The category of (Q, X)-representations is denoted $\mathrm{rep}_k(Q, X)$.

Given two (Q, X)-representation their tensor product $V \otimes W$ is defined as follows. For each $x \in Q_0$, $\alpha \in Q_1$ set

$$(V \otimes W)_x = V_x \otimes W_x \text{ and } (V \otimes W)(\alpha) = V(\alpha) \otimes W(\alpha).$$

It is routine to check that $V \otimes W$ is a (Q, X)-representation.

Let $S(Q, X)$ be the set of isoclasses of (Q, X)-representations. For all $[V], [W] \in S(Q, X)$ set

$$[V] + [W] = [V \oplus W] \text{ and } [V][W] = [V \otimes W].$$

This endows $S(Q, X)$ with the structure of a semi-ring. Let $R(Q, X)$ be the corresponding Grothendieck ring [12].

Our aim is to describe $R(Q, X)$ in case (Q, X) corresponds to a string algebra. Of particular importance is the case when Q is the loop quiver:

$$\bullet \circlearrowleft \alpha$$

In this case there is an equivalence of categories

$$\mathrm{rep}_k Q \xrightarrow{\sim} \mathrm{mod}\, k[x]$$

defined for each representation V by letting x act on $V_\bullet$ by $V(\alpha)$. The tensor product induced by this equivalence on $\mathrm{mod}\, k[x]$ comes from the coproduct $k[x] \to k[x] \otimes k[x]$, $x \mapsto x \otimes x$.

Let V_n correspond to the indecomposable $k[x]/x^n$ under the above equivalence for every $n > 0$. We have the following result from [3].

Proposition 1. *Let Q be the loop quiver and V a Q-representation such that $V(\alpha)$ is an invertible linear operator. Then the following statements hold.*

(1) $V_n \otimes V_m \xrightarrow{\sim} (m-n+1)V_n \oplus \bigoplus_{i=1}^{n-1} 2V_i$ *for* $n \leq m$.
(2) $V \otimes V_n = (\dim V)V_n$ *for all* n.

Let $I_s \subset R(Q)$ be the $\mathbb{Z}$-span of $\{[V_n] \mid n > 0\}$. By Proposition 1, I_s is an ideal in $R(Q)$ and $R(Q)/I_s \xrightarrow{\sim} R(k[x,x^{-1}])$. The structure of $R(k[x,x^{-1}])$ depends heavily on the field k. Let us recall its description from [3]. We need to construct another ring which we denote by R'.

If $\operatorname{char} k = 0$, then set $R' = \mathbb{Z}[T]$.

If $\operatorname{char} k = p > 0$, then R' is constructed as follows. For each $i \in \mathbb{N}$ let C_{p^i} be the cyclic group of order p^i and set $R_i = R(kC_{p^i})$. There are canonical inclusions $R_i \subset R_{i+1}$, and we set $R' = \bigcup_{i\in\mathbb{N}} R_i$.

Let $\overline{k}^\iota$ be the group of invertible elements in the algebraic closure of k and $\mathbb{Z}\overline{k}^\iota$ the corresponding group ring. The absolute Galois group $G = \operatorname{Gal}(\overline{k}/k)$ acts on $\overline{k}^\iota$ and consequently on $\mathbb{Z}\overline{k}^\iota$. Denote by $(\mathbb{Z}\overline{k}^\iota)^G$, the ring of invariants. The following Theorem is from [3].

Theorem 2. *There is an isomorphim*

$$R(k[x,x^{-1}]) \xrightarrow{\sim} (\mathbb{Z}\overline{k}^\iota)^G \otimes_{\mathbb{Z}} R'.$$

3. String algebras

As before fix a quiver Q and a set X of paths in Q. Set $I = \langle X \rangle$ and $A = kQ/I$.

Definition 3. The algebra A is called a string algebra if it is finite dimensional and satisfies the following conditions.

(1) Each $x \in Q_0$ is the tail, respectively head, of at most two arrows.
(2) For each $\alpha \in Q_1$ there is at most one $\beta \in Q_1$ and at most one $\gamma \in Q_1$ such that $\beta\alpha \notin I$ and $\alpha\gamma \notin I$.

Example 4. The following quiver with relations defines a string algebra for every $n > 0$.

$$\begin{tikzcd} \bullet \arrow[r, shift left, "\alpha"] & \bullet \arrow[l, shift left, "\beta"] \arrow[loop right, "\gamma"] \end{tikzcd} \qquad \beta\alpha = \alpha\beta = (\beta\alpha\gamma)^n = 0$$

We proceed to describe the indecomposable modules over string algebras.

Definition 5. A *quiver morphism* $F : P \to Q$ consists of two maps $F : P_0 \to Q_0$, $F : P_1 \to Q_1$ such that for any arrow $x \xrightarrow{\alpha} y$ we get $Fx \xrightarrow{F\alpha} Fy$.

We call $\mathbf{F} = (F, P)$ a *shape* over Q if for any two distinct arrows $x_1 \xrightarrow{\alpha_1} y_1, x_2 \xrightarrow{\alpha_2} y_2 \in Q_1$ we have that $F\alpha_1 = F\alpha_2$ implies $x_1 \neq y_1$ and $x_2 \neq y_2$.

A *morphism of shapes* $(F, P) \to (F', P')$ is a quiver morphism $G : P \to P'$ such that $F = F'G$. Denote by $|\mathbf{F}' : \mathbf{F}|$, the number of morphisms $\mathbf{F} \to \mathbf{F}'$.

We only consider shapes (F, P) and such that for any path $\alpha_1 \cdots \alpha_n$ in P we have that $F\alpha_1 \cdots F\alpha_n \notin I$.

With each shape (F, P) we associate two functors

$$\begin{tikzcd} \operatorname{rep}_k P \arrow[d, shift left, "F_*"] \\ \operatorname{rep}_k(Q, X) \arrow[u, shift left, "F^*"] \end{tikzcd}$$

defined as follows.

For each $V \in \operatorname{rep}_k(Q, X)$, $x \in P_0$ and $\alpha \in P_1$ set $(F^*V)_x = V_{Fx}$ and $(F^*V)(\alpha) = V(Fx)$. For each $W \in \operatorname{rep}_k P$, and $x' \in Q_0$ set

$$(F_*W)_x = \bigoplus_{Fx=x'} W_x.$$

Let $x' \xrightarrow{\alpha'} y' \in Q_1$. Write the linear map

$$(F_*W)(\alpha') : \bigoplus_{Fx=x'} W_x \to \bigoplus_{Fy=y'} W_y$$

as a matrix A with elements

$$A_{yx} = \begin{cases} W(\alpha) & \text{if there is } x \xrightarrow{\alpha} y \text{ such that } F\alpha = \alpha', \\ 0 & \text{else.} \end{cases}$$

Definition 6. A shape $\mathbf{F} = (F, L)$ is called *linear* if L is Dynkin of type $\mathbb{A}$, i.e. if its underlying graph is

$$\bullet \text{ --- } \cdots \text{ --- } \bullet.$$

We define the L-representation V by

$$k \overset{1}{\text{ --- }} \cdots \overset{1}{\text{ --- }} k.$$

The *string* associated to $\mathbf{F}$ is the (Q, X)-module $S_{\mathbf{F}} := F_*V$. It is always indecomposable.

A shape $\mathbf{G} = (G, Z)$ is called *cyclic* if it has trivial automorphism group and Z is extended Dynkin of type $\tilde{\mathbb{A}}$, i.e. if its underlying graph is

$$\begin{tikzcd}[column sep=small] & \bullet \arrow[dl, dash] \arrow[dr, dash] & \\ \bullet \arrow[r, dash] & \cdots \arrow[r, dash] & \bullet. \end{tikzcd}$$

Now let M be a $k[x, x^{-1}]$-module and $\gamma \in Z_1$. We define the Z-representation W by

$$\begin{tikzcd}[column sep=small] & M \arrow[dl, dash, "1"'] \arrow[dr, dash, "x"] & \\ M \arrow[r, dash, "1"'] & \cdots \arrow[r, dash, "1"'] & M \end{tikzcd}$$

where the arrow acting as x is γ. The *band* associated with $(\mathbf{G}, M, \gamma)$ is the (Q, X)-module $B_{\mathbf{G}}(M, \gamma) := G_*W$. It is indecomposable if and only if M is indecomposable. For $\gamma' \in Z_1$ we say that γ and γ' are oriented equally if when cycling through the vertices of Z we encounter $t\gamma$ and $h\gamma$ in the same order as we encounter $t\gamma'$ and $h\gamma'$. In that case

$B_{\mathbf{G}}(M,\gamma') \tilde{\to} B_{\mathbf{G}}(M,\gamma)$. Otherwise $B_{\mathbf{G}}(M,\gamma') \tilde{\to} B_{\mathbf{G}}(M^{-1},\gamma)$, where M^{-1} is obtained from M by inverting the action of x.

The following Theorem follows from [14].

Theorem 7. *Assume that A is a string algebra. Then strings and (indecomposable) bands classify all indecomposables, i.e.*

1. *Each indecomposable A-module is isomorphic to either a string or band.*
2. *No strings are isomorphic to bands.*
3. *Two strings $S_{\mathbf{F}}$ and $S_{\mathbf{F'}}$ are isomorphic if and only if they have isomorphic shapes.*
4. *Two bands $B_{\mathbf{G}}(M,\gamma)$ and $B_{\mathbf{G'}}(M',\gamma')$ are isomorphic if and only if their shapes are isomorphic via some H such that M' is isomorphic to M if $H(\gamma)$ and γ' are equally oriented and M' is isomorphic to M^{-1} otherwise.*

Let $\mathcal{L}$ be the set of isoclasses of linear shapes and $\mathcal{Z}$ be the set of isoclasses of cyclic shapes.

We need the following preliminary result.

Proposition 8. *Let (F,P) be a shape over Q, $V \in \operatorname{rep}_k(Q,X)$ and $W \in \operatorname{rep}_k P$. Then*

$$F_*W \otimes V \tilde{\to} F_*(W \otimes F^*V)$$

Let $I_s \subset R(Q,X)$ be the $\mathbb{Z}$-span of $\{[S_{\mathbf{F}}] \mid \mathbf{F} \in \mathcal{L}\}$. By Proposition 8, it is an ideal, since $[S_{\mathbf{F}}][V] = [F_*(F^*V)] \in I_s$.

The following Theorem completely describes the structure of $R(Q,X)$ in the case A is a string algebra.

Theorem 9. *Assume that A is a string algebra. Then the ideal I_s has a unique $\mathbb{Z}$-basis of pair-wise orthogonal idempotents $\{e_{\mathbf{F}} = e_{\overline{\mathbf{F}}}\}_{\overline{\mathbf{F}} \in \mathcal{L}}$, such that the following statements hold:*

1. *For each linear shape $\mathbf{F}$*
$$[S_{\mathbf{F}}] = \sum_{\overline{\mathbf{F'}} \in \mathcal{L}} |\mathbf{F} : \mathbf{F'}| e_{\mathbf{F'}}.$$
2. *For each cyclic shape $\mathbf{G} = (G,Z)$, $\gamma \in Z_1$ and $k[x,x^{-1}]$-module M*
$$[B_{\mathbf{G}}(M,\gamma)]e_{\mathbf{F'}} = \dim M |\mathbf{G} : \mathbf{F'}| e_{\mathbf{F'}}.$$
3. *For each pair of non-isomorphic cyclic shapes $\mathbf{G_1} = (G_1, Z^1)$, $\mathbf{G_2} = (G_2, Z^2)$, $\gamma_1 \in Z^1_1$, $\gamma_2 \in Z^2_1$ and $k[x,x^{-1}]$-modules M, N*
$$[B_{\mathbf{G_1}}(M,\gamma_1)][B_{\mathbf{G_2}}(N,\gamma_2)] = \sum_{\overline{\mathbf{F'}} \in \mathcal{L}} \dim M \dim N |\mathbf{G_1} : \mathbf{F'}||\mathbf{G_2} : \mathbf{F'}| e_{\mathbf{F'}}.$$
Moreover,
$$[B_{\mathbf{G_1}}(M,\gamma_1)][B_{\mathbf{G_1}}(N,\gamma_1)] = [B_{\mathbf{G_1}}(M \otimes N,\gamma_1)] + \sum_{\overline{\mathbf{F'}} \in \mathcal{L}} \dim M \dim N |\mathbf{G_1} : \mathbf{F'}|(|\mathbf{G_1} : \mathbf{F'}| - 1) e_{\mathbf{F'}}.$$

We end with the following observations. As a (non-unital) subring $I_s \tilde{\to} \bigoplus_{\overline{\mathbf{F}} \in \mathcal{L}} \mathbb{Z}$. On the other hand, $R(Q,X)/I_s \tilde{\to} \bigoplus_{\overline{\mathbf{G}} \in \mathcal{Z}} R(k[x,x^{-1}])$.

References

[1] A. C. Aitken, *The normal form of compound and induced matrices*, Proc. London Math. Soc. **38** (1935), 354–376.

[2] A. Clebsch and P. Gordan, *Theorie der abelschen functionen*, B. G. Teubner, 1866.

[3] E. Darpö and M. Herschend, *On the representation ring of the polynomial algebra over a perfect field*, Math. Z. (2009), to appear.

[4] P. W. Donovan and M.-R. Freislich, *The indecomposable modular representations of certain groups with dihedral Sylow subgroup*, Math. Ann. **238** (1978), no. 3, 207–216.

[5] P. Gabriel and A. V. Roiter, *Representations of finite-dimensional algebras*, Algebra, VIII, Encyclopaedia of Mathematical Sciences, vol. 73, Springer, 1992, With a chapter by B. Keller, pp. 1–177.

[6] I. M. Gelfand and V. A. Ponomarev, *Indecomposable representations of the Lorentz group*, Usp. Mat. Nauk **23** (1968), no. 2 (140), 3–60.

[7] M. Herschend, *Solution to the Clebsch-Gordan problem for representations of quivers of type* $\tilde{\mathbb{A}}_n$, J. Algebra Appl. **4** (2005), no. 5, 481–488.

[8] ———, *Galois coverings and the Clebsch-Gordan problem for quiver representations*, Colloq. Math. **109** (2007), no. 2, 193–215.

[9] ———, *On the representation rings of quivers of exceptional Dynkin type*, Bull. Sci. Math. **132** (2008), no. 5, 395–418.

[10] ———, *On the representation ring of a quiver*, Algebr. Represent. Theory **12** (2009), no. 6, 513–541.

[11] K. Iima and R. Iwamatsu, *On the Jordan decomposition of tensored matrices of Jordan canonical forms*, 2006, http://arxiv.org/abs/math/0612437.

[12] S. Mac Lane, *Categories for the working mathematician*, Graduate Texts in Mathematics, Springer-Verlag, 1998.

[13] C. M. Ringel, *The indecomposable representations of the dihedral* 2*-groups*, Math. Ann. **214** (1975), 19–34.

[14] B. Wald and J. Waschbüsch, *Tame biserial algebras*, J. Algebra **95** (1985), no. 2, 480–500.

Graduate School of Mathematics
Nagoya University
Chikusa-ku, Nagoya, 464-8602 JAPAN
E-mail address: martinh@math.nagoya-u.ac.jp

FULLY WEAKLY PRIME RINGS

YASUYUKI HIRANO, EDWARD POON AND HISAYA TSUTSUI

Abstract. Anderson and Smith studied weakly prime ideals for a commutative ring with identity. Blair and Tsutsui studied the structure of a ring in which every ideal is prime. In this paper we investigate the structure of rings, not necessarily commutative, in which all ideals are weakly prime.

1. Introduction

Anderson-Smith [1] defined a proper ideal P of a commutative ring R with identity to be weakly prime if $0 \neq ab \in P$ implies $a \in P$ or $b \in P$. They proved that every proper ideal in a commutative ring R with identity is weakly prime if and only if either R is a quasilocal ring (possibly a field) whose maximal ideal is square zero, or R is a direct sum of two fields [1, Theorem 8]. On the other hand, *Blair-Tsutsui* [2] studied the structure of a ring in which every ideal is prime. In this paper we first consider the structure of rings, not necessarily commutative nor with identity, in which all ideals are weakly prime. A necessary and sufficient condition for a ring to have such property is given and several examples to support given propositions are constructed. We then further investigate commutative rings in which every ideal is weakly prime and the structure of such rings under assumptions that generalize commutativity of rings. At the end, we consider the structure of rings in which every right ideal is weakly prime.

2. General results

We generalize the definition of a weakly prime ideal to arbitrary (not necessarily commutative) rings as follows.

Definition. A proper ideal I of a ring R is *weakly prime* if $0 \neq JK \subseteq I$ implies either $J \subseteq I$ or $K \subseteq I$ for any ideals J, K of R.

Our first proposition is Theorem 1 of *Anderson-Smith* [1] in a more general setting.

Proposition 1. *If P is weakly prime but not prime, then $P^2 = 0$.*

Proof. Since P is weakly prime but not prime, there exist ideals $I \not\subseteq P$ and $J \not\subseteq P$ but $0 = IJ \subseteq P$. But if $P^2 \neq 0$, then $0 \neq P^2 \subseteq (I + P)(J + P) \subseteq P$, which implies $I \subseteq P$ or $J \subseteq P$, a contradiction.

2000 *Mathematics Subject Classification.* 16N60, 16W99.

Key words and phrases. Weakly prime.

The detailed version of this paper has been submitted for publication elsewhere.

Proposition 2. *Let P be an ideal in a ring R with identity. The following statements are equivalent:*
(1) *P is a weakly prime ideal.*
(2) *If J, K are right (left) ideals of R such that $0 \neq JK \subseteq P$, then $J \subseteq P$ or $K \subseteq P$.*
(3) *If $a, b \in R$ such that $0 \neq aRb \subseteq P$, then $a \in P$ or $b \in P$.*

Proof. The implications (1) => (2) and (2) => (3) are easy. The implication (3) => (1) can be verified by checking a number of cases.

Since weakly prime ideals are defined to be proper ideals, we shall say that every ideal of a ring R is weakly prime when every proper ideal of R is weakly prime. In this case we say that R is fully weakly prime.

If $R^2 = 0$, then clearly every ideal of R is weakly prime. In particular, if an ideal I of a ring R is weakly prime but not a prime ideal, then every ideal of I as a ring is weakly prime by Proposition 1.

Proposition 3. *Every ideal of a ring R is weakly prime if and only if for any ideals I and J of R, $IJ = I$,$IJ = J$, or $IJ = 0$.*

Corollary 1. *Let R be a ring in which every ideal of R is weakly prime. Then for any ideal I of R, either $I^2 = I$ or $I^2 = 0$.*

Example 1. Let F be a field and $R = F \oplus F \oplus F$. Then every ideal of R is idempotent but the ideal $I = F \oplus 0 \oplus 0$ is evidently not weakly prime, showing that the converse of Corollary 1 is false.

Suppose that a ring R with identity has a maximal ideal M and $M^2 = 0$. One can readily check that R is fully weakly prime, and M is the only prime ideal of R.

Corollary 1 in particular yields that if a ring R has the property that every ideal is weakly prime, then either $R^2 = R$, or $R^2 = 0$. Notice that R^2 is neither 0 nor R in the example given below.

Example 2. Let S be a ring such that $S^2 = 0$, and let F be a field. Then the ring $R = F \oplus S \oplus S$ with component-wise addition and multiplication has a maximal ideal $M = 0 \oplus S \oplus S$ and $M^2 = 0$. However, $I = F \oplus 0 \oplus S$ is not weakly prime since $0 \neq (F \oplus S \oplus 0)^2 \subseteq I$.

If a ring R satisfying $R^2 = R$ has a maximal ideal M and $M^2 = 0$, then every proper ideal of R is contained in M. However, it is possible that $MR \neq M$. Thus, such a ring does not necessarily have the property that every ideal is weakly prime as the following example shows.

Example 3. Let F be a field and $S = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & c & d \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a, b, c, d \in F \right\}$. Then S has a unique maximal ideal $L = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & d \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a, b, d \in F \right\}$.

Let $N = \left\{ \begin{bmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \, b \in F \right\}$. Consider the factor ring $R = S/N$. While $R^2 = R$ and $M = L/N$ is a maximal ideal whose square is zero, the proper ideals RM and MR are not weakly prime.

Proposition 4. *If every ideal of a ring R is weakly prime and $R^2 = R$, then R has at most two maximal ideals.*

Proof. By contradiction.

The following example shows that the condition $R^2 = R$ in Proposition 4 cannot be dropped.

Example 4. Let R be the unique maximal ideal of $\mathbb{Z}_4$.Then $S = R \oplus R \oplus R$ is an example of a ring all of whose ideals are weakly prime and having more than 2 maximal ideals.

Proposition 5. *Suppose that every ideal of a ring R is weakly prime. If R has two maximal ideals M_1 and M_2, then their product is zero. Furthermore, if R has an identity element, then R is a direct sum of two simple rings.*

Proof. Note $M_1 M_2 \subseteq M_1 \cap M_2$. If R has an identity, then $M_1 \cap M_2 = (M_1 \cap M_2)(M_1 + M_2) = 0$.

We denote the prime radical of R by $P(R)$, and the sum of all ideals whose square is zero by $N(R)$.

Theorem 1. *Suppose that every ideal of a ring R is weakly prime and $R^2 = R$. Then $P(R) = N(R)$ and $(P(R))^2 = (N(R))^2 = 0$.*

Proof. Any finite sum of square-zero ideals is nilpotent, and hence square-zero, so $N(R)^2 = 0$. Thus $N(R) \subseteq P(R)$.

Either $P(R)$ is not prime (in which case $P(R)^2 = 0$, so $P(R) \subseteq N(R)$), or $P(R)$ is prime, in which case $N(R)$ can be shown to be prime (apply Theorem 1.2 of Blair-Tsutsui [2] to $R/P(R)$).

Corollary 2. *Suppose that every ideal of a right Noetherian ring R with identity is weakly prime and $R^2 = R$. Then $P(R) = N(R) = J(R)$ and $(J(R))^2 = 0$, where $J(R)$ is the Jacobson radical of R.*

Proof. If $(J(R))^2 = J(R)$, then $J(R) = 0 \subseteq P(R)$ by Nakayama's lemma. If $(J(R))^2 = 0$, then $J(R) \subseteq P(R)$.

Note that for a ring R in which every ideal is weakly prime, in general it is possible that $P(R) = N(R) \neq J(R)$ [2, §5 *An Example*].

Corollary 3. *Suppose that every ideal of a ring R is weakly prime. Then every nonzero ideal of $R/N(R)$ is prime.*

Corollary 4. *Suppose every ideal of a ring R is weakly prime. Then $(N(R))^2 = 0$ and every prime ideal contains $N(R)$. There are three possibilities:*

(a) $N(R) = R$.

(b) $N(R) = P(R)$ is the smallest prime ideal and all other prime ideals are idempotent and prime ideals are linearly ordered. If $N(R) \neq 0$, then it is the only non-idempotent prime ideal.
(c) $N(R) = P(R)$ is not a prime ideal. In this case, there exist two nonzero minimal prime ideals J_1 and J_2 with $N(R) = J_1 \cap J_2$ and $J_1J_2 = J_2J_1 = 0$. All other ideals containing $N(R)$ also contain $J_1 + J_2$ and they are linearly ordered.

Proof. Use Theorem 1. If we are not in case (a) or (b), apply [2, Theorem 1.2] and [4, Theorem 2.1] to $R/N(R)$.

Example 5. Let R be a ring and M an R-bimodule. Define

$$R * M = \{(r,\, m) | r \in R,\, m \in M\}$$

with component-wise addition and multiplication

$$(r,\, m)(s, n) = (rs,\, rn + ms).$$

Then $R * M$ is a ring whose ideals are precisely of the form $I * N$ where I is an ideal of R and N is a submodule (a bimodule) of M containing IM and MI.

(a) Let R be a prime ring with exactly one nonzero proper ideal P. For example, the ring of linear transformations of a vector space V over a field F where $\dim_F V = \aleph_0$ has such a property. Then every ideal of $S_1 = R * P$ is weakly prime: the maximum ideal $P_1 = P * P$ is idempotent and the nonzero minimal ideal $P_2 = 0 * P$ is nilpotent, both of which are prime.

(b) Every ideal of $S_2 = S_1 * P_2$ is weakly prime: The maximum ideal $Q_1 = P_1 * P_2$ is idempotent and the three nonzero nilpotent ideals are $Q_2 = P_2 * P_2$, $Q_3 = 0 * P_2$, and $Q_4 = P_2 * 0$.

(c) If we redefine the multiplication above as

$$(r,\, m)(s, n) = (rs,\, rn + ms + mn),$$

then S_1 in (a) has an additional minimal ideal $P_3 = \{(p,\, -p) |\, p \in P\}$. In this case, $N(S_1) = P_3 \cap P_2 = 0$.

We don't know of an example of Corollary 4, case (c) where $N(R) \neq 0$.

3. Commutative Rings and Generalizations thereof

We now consider the structure of rings in which every ideal is weakly prime under the assumption of the ring being commutative or with commutative-like conditions.

Proposition 6. *Let R be a commutative ring in which every ideal is weakly prime. If $R^2 = R$, then R has a maximal ideal.*

Proof. If $N(R)$ is not maximal there exists a prime ideal I. Apply [2, Theorem 1.3] to R/I.

We note that a commutative ring R with the property $R^2 = R$ does not necessarily have a maximal ideal. For example, if a commutative ring S has a unique nonzero maximal ideal M and $M^2 = M$, then M as a ring cannot have a maximal ideal.

The next corollary follows from Propositions 4 and 6.

Corollary 5. *Let R be a commutative ring all of whose ideals are weakly prime. Suppose that $R^2 = R$. Then R has either a unique maximal ideal or exactly two maximal ideals.*

Theorem 2. *Let R be a commutative ring all of whose ideals are weakly prime. Suppose that $R^2 = R$.*

(1) If R has a unique maximal ideal M, then $M^2 = 0$.

(2) If R has two maximal ideals M and N, then $MN = 0$.

Proof. By contradiction.

Proposition 7. *Let R be a commutative ring all of whose ideals are weakly prime. Suppose that $R^2 = R$. Then every proper ideal is contained in a maximal ideal.*

Proof. Use the preceding theorem.

Corollary 6. *Let R be a commutative ring and suppose that every ideal of R is weakly prime. If $R^2 = R$, then R has an identity element.*

Proof. We show that if a commutative ring R satisfies the following conditions, then R has an identity element:
(a) $R^2 = R$,
(b) every proper ideal is contained in a maximal ideal, and
(c) R has a finite number of maximal ideals $M_1, M_2, \ldots, M_n$.

Choose $x \in R$ such that $x \notin M_j$ for any j. Let $(x) = \{xR + nx | n \in \mathbb{Z}\}$. If $xR \subseteq M_j$, then $R = R^2 = (M_j + (x))^2 \subseteq M_j$, a contradiction. Hence $xR = R$ and consequently, R has an identity element.

Corollary 7. *Let R be a commutative ring all of whose ideals are weakly prime. Then one of the following holds:*
(a) $R^2 = 0$,
(b) R is a ring with identity and a square zero maximal ideal M, or
(c) R is a direct sum of two fields.

For the case (b) in Corollary 7, the following theorem further determines the structure of R.

Theorem 3. *Let R be a commutative ring with a square-zero maximal ideal M and $R^2 = R$. If $(ch(R/M) = 0$, then R is isomorphic to $(R/M) * M$ (as defined in Example 5).*

Proof. Note that if E is a subfield of R/M and $\psi : E \to R$ is a homomorphism satisfying $\pi \circ \psi = id|_E$, then the map $\varphi : E * M \to R$ given by $\varphi((\bar{x}, m)) = \psi(\bar{x}) + m$ is a monomorphism. So, it suffices to show such a map ψ exists for $E = R/M$ (in this case φ is also onto); the proof proceeds by defining ψ on successively larger subfields $E \subseteq R/M$.

Using the same idea, the result also holds for $ch(R/M) = p$ if $pR = 0$ and R/M is separable over F. In general, however, the theorem is false if $ch(R/M) = p \neq 0$.

Example 6. **(a)** Let $R = Z_{p^2}$ where p is prime. Then R has maximal ideal $M = pR \neq 0$ but $p(R/M * M) = 0$.

(b) Let $R = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$ where F is a field. Then $R^2 = R \neq 0$, and every ideal of R is weakly prime but R does not contain an identity element.

As a natural generalization of commutative rings, we next consider polynomial identity (PI) rings.

Theorem 4. *Let R be a PI-ring with identity. If every ideal of R is weakly prime, then one of the following folds:*
(a) $R/P(R)$ is a finite dimensional central simple algebra.
(b) R is a direct sum of two finite dimensional central simple algebras.

Proof. Use Corollary 4 and [2, Theorem 3.3].

More general than the class of PI-rings is the class of fully bounded rings. Using [2, Theorem 3.4] yields the following theorem.

Theorem 5. *Let R be a ring with identity in which every ideal is weakly prime. If R is a right fully bounded, right Noetherian ring, then one of the following holds:*

(a) $R/P(R)$ is a simple Artinian ring.
(b) R is a direct sum of two simple Artinian rings.

4. Rings in which every right ideal is weakly prime

Definition. We define a proper right ideal I of a ring R to be weakly prime if $0 \neq JK \subseteq I$ implies either $J \subseteq I$ or $K \subseteq I$ for any right ideals J, K of R.

For a ring R that is not square zero, Koh[3] showed that R is simple and $a \in aR$ for all $a \in R$ if and only if every right ideal of R is prime. Now consider the structure of rings in which every right ideal is weakly prime. For the commutative case, it is evident that such rings need not be simple. Example 6 (b) gives an example of a ring $R = R^2$ in which every right and left ideal is weakly prime.

Unlike the case of weakly prime two sided ideals, there exists a nonzero idempotent weakly prime right ideal that is not prime. For example, if $R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \middle| \, a, b, c \in F \right\}$, then $K = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & t \end{bmatrix} \middle| \, t \in F \right\}$ is a weakly prime right ideal and $K^2 = K \neq 0$. But K is not a prime right ideal.

We conclude with the following generalization of Corollary 7.

Theorem 6. *Suppose that every right ideal of a ring R is weakly prime. Then one of the following holds:*
(a) $R^2 = 0$.
(b) R has a square zero maximal ideal.
(c) R is a direct sum of two division rings.

Under an additional condition we can say more about case (b).

Proposition 8. *Let R be a ring all of whose right ideals are weakly prime. Suppose R has a square zero maximal ideal $N \neq 0$. If $NR = 0$, then $RN = N$ and either:*

(a) R/N is a simple dense ring of endomorphisms over the infinite-dimensional vector space N (and every nonzero endomorphism is surjective), or

(b) R/N is a division ring and R is isomorphic to

$$\left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in R/N \right\}.$$

References

[1] D. D. Anderson and E. Smith, *Weakly Prime Ideals*, Houston J. Math **29** (2003), no. 4, 831-840.

[2] W. D. Blair and H. Tsutsui, *Fully Prime Rings*, Comm. Algebra **22** (1994), no. 13, 5389-5400.

[3] K. Koh, *On one-sided ideals of a prime type*, Proc. Amer. Math. Soc. **28** (1971), 321-329.

[4] Tsutsui, *Fully Prime Rings II*, Comm. Algebra **24** (1996), no. 9, 2981-2989.

Y. Hirano
Department of Mathematics
Naruto University of Education
Naruto, Japan

E-mail address: yahirano@naruto-u.ac.jp

E. Poon & H. Tsutsui
Department of Mathematics
Embry-Riddle Aeronautical University
Prescott, AZ USA

E-mail address: edward.poon@erau.edu, hisaya.tsutsui@erau.edu

ON GALOIS EXTENSIONS WITH AN INNER GALOIS GROUP AND A GALOIS COMMUTATOR SUBRING

SHÛICHI IKEHATA, GEORGE SZETO AND LIANYONG XUE

Abstract. Properties of a Galois ring extension with an inner Galois group are given, and equivalent conditions for a Galois extension with a Galois commutator subring are shown.

1. Introduction

In 1960's, Galois theory was developed for rings by M. Auslander-O.Goldman ([2]), S.U. Chase-D.K. Harrison-A. Rosenberg ([3]), F.R. DeMeyer ([4], [5]), M. Harada ([7]), Y. Miyashita ([13]), T. Nagahara ([14]), T. Kanzaki ([12]), K. Sugano ([15], [16]), and others. It was shown ([4], Theorem 6, [5], Theorem 3) that B is a central Galois algebra over its center C with an inner Galois group G if and only if it is an Azumaya projective group algebra CG_f where $f : G \times G \longrightarrow$ units of C is a factor set. In section 3, we shall generalize the above theorem to any Galois extension B with an inner Galois group G where $G = \{g \in G \,|\, g(x) = U_g x U_g^{-1}$ for some $U_g \in B$ and for all $x \in B\}$. It is shown that B contains a projective group algebra CG_f. An equivalent condition for a central Galois algebra CG_f with Galois group induced by G is given, and characterizations for a Galois extension B with an inner Galois group G generated by $\{U_g \,|\, g \in G\}$ over B^G are obtained. When B is also an Azumaya algebra, in section 4, some properties are given for a Galois extension B with an inner Galois group G. We note that any Galois extension with an inner Galois group G is a Hirata separable extension of B^G ([17], Corollary 3). For a Hirata separable Galois extension B with Galois group G (not necessarily inner), in [17], Sugano investigated the Galois commutator subring $V_B(B^G)$ of B^G in B. We shall study when $V_B(B^G)$ is a Galois extension with Galois group induced by G for any Galois extension B with Galois group G in section 5. Equivalent conditions are given in terms of a composition Galois extensions: $B \supset B^G \cdot V_B(B^G) \supset B^G$ and crossed products respectively. Some examples are also given to demonstrate the results.

2. Basic Definitions and Notations

Let B be a ring with identity 1, C the center of B, G a finite automorphism group of B, B^G the set of elements in B fixed under each element in G. Following the definitions as given in the references, we call B a Galois extension of B^G with Galois group G if there exist elements $\{a_i, b_i$ in B, $i = 1, 2, ..., m$ for some integer $m\}$ such that $\sum_{i=1}^m a_i g(b_i) = \delta_{1,g}$ for each $g \in G$ ([4]). Such a set $\{a_i, b_i\}$ is called a G-Galois system for B. A Galois extension B of B^G is called a Galois algebra if B^G is contained in C ([21]), and a central Galois algebra if $B^G = C$ ([20]). We call B a center Galois extension with Galois group

The detailed version of this paper will be submitted for publication elsewhere.

G if C is a Galois algebra over C^G with Galois group $G|_C \cong G$, and a commutator Galois extension of B^G with Galois group G if $V_B(B^G)$ is a Galois extension of $(V_B(B^G))^G$ with Galois group $G|_{V_B(B^G)} \cong G$. Let A be a subring of B with the same identity 1. We denote $V_B(A)$ the commutator (also called centralizer) subring of A in B, that is, $V_B(A) = \{b \in B | bx = xb$ for all $x \in A\}$. We call B a separable extension of A if there exist $\{a_i, b_i$ in B, $i = 1, 2, ..., m$ for some integer $m\}$ such that $\sum a_i b_i = 1$, and $\sum b a_i \otimes b_i = \sum a_i \otimes b_i b$ for all b in B where $\otimes$ is over A. An Azumaya algebra is a separable extension of its center. A Galois extension B of B^G with Galois group G is called an Azumaya Galois extension if B^G is an Azumaya C^G-algebra ([1]). A Galois extension B of B^G with Galois group G is called a DeMeyer-Kanzaki Galois extension if B is an Azumaya algebra over C which is a Galois algebra over C^G with Galois group $G|_C \cong G$. A ring B is called a Hirata separable extension of A if $B \otimes_A B$ is isomorphic to a direct summand of a finite direct sum of B as a B-bimodule, and B is called a Hirata separable Galois extension of B^G if it is a Galois and a Hirata separable extension of B^G. Let R be a commutative ring with 1 and $U(R)$ the set of units of R. As given in [4], for a factor set f: $G \times G \longrightarrow U(R)$ (that is, $f(g,h)f(gh,k) = f(h,k)f(g,hk)$ for all g, h, and k in G), $RG_f = \sum_{g \in G} RU_g$ is called a projective group algebra over R if RG_f is an algebra with a free basis $\{U_g \bigm| g \in G\}$ over R where U_g is an invertible element for each $g \in G$, the multiplications are given by $(r_g U_g)(r_h U_h) = r_g r_h U_g U_h$ and $U_g U_h = f(g,h)U_{gh}$ for $r_g, r_h \in R$ and $g, h \in G$; that is, $f(g,h) = U_g U_h U_{gh}^{-1}$.

3. Galois Extensions with an Inner Galois Group

Let B be a Galois extension of B^G with an inner Galois group G whose order $|G|$ is invertible in B where $G = \{g \in G \,|\, g(x) = U_g x U_g^{-1}$ for some $U_g \in B$ and for all $x \in B\}$. We shall show that B contains a projective group algebra CG_f where C is the center of B. An equivalent condition is given for a central Galois algebra CG_f. Thus several characterizations are obtained for B generated by $\{U_g \,|\, g \in G\}$ over B^G. These characterizations generalize the results for a central Galois algebra with an inner Galois group ([4], Theorem 6).

Theorem 3.1. ([23], Theorem 2.1) *Let B be a Galois extension of B^G with an inner Galois group G, $G = \{g \,|\, g(x) = U_g x U_g^{-1}$ for some $U_g \in B$ and for all $x \in B\}$, and C the center of B. Then B contains a projective group algebra CG_f of G over C with a factor set $f : G \times G \longrightarrow$ units of C.*

Proof. We first claim that $\{U_g \,|\, g \in G\}$ are linearly independent over C. Let $\{x_i, y_i \in B \,|\, i = 1, 2, ..., m$ for some integer $m\}$ be a G-Galois system such that $\sum_{i=1}^m x_i g(y_i) = \delta_{1,g}$ for each $g \in G$. Let $\sum_{g \in G} a_g U_g = 0$ for some $a_g \in C$. Then

$$\sum_{i=1}^{m} x_i \sum_{g \in G} a_g U_g h^{-1}(y_i) = 0 \text{ for each } h \in G \text{ and}$$

$$\sum_{g \in G} a_g \sum_{i=1}^{m} x_i g h^{-1}(y_i) U_g = \sum_{g \in G} a_g \delta_{1,gh^{-1}} U_g = a_h U_h.$$

Noting that $a_g \in C$ and $U_g h^{-1}(y_i) = gh^{-1}(y_i)U_g$, we have that

$$\sum_{i=1}^{m} x_i \sum_{g\in G} a_g U_g h^{-1}(y_i) = \sum_{g\in G} a_g \sum_{i=1}^{m} x_i g h^{-1}(y_i) U_g;$$

and so $a_h U_h = 0$. But U_h is invertible in B, so $a_h = 0$ for each $h \in G$. Also, noting that $U_{gh}^{-1} U_g U_h$ is a unit in C, we have a factor set $f : G \times G \longrightarrow$ units of C by $f(g,h) = U_{gh}^{-1} U_g U_h$. Thus $\sum_{g\in G} CU_g = CG_f \subset B$.

Let Z be the center of G and $\overline{G}$ the restriction of G to CG_f. Then $\overline{G} \cong G/K$ where $K = \{g \in Z \,|\, f(g,h) = f(h,g)$ for all $h \in G\}$. Next is necessary and sufficient condition for a central Galois algebra CG_f with an inner Galois group $\overline{G}$.

Theorem 3.2. ([23], Theorem 2.2) *Let B be a Galois extension of B^G with an inner Galois group G of order n invertible in B and CG_f as given in Theorem 3.1. Then CG_f is a central Galois algebra over its center S with an inner Galois group $\overline{G}$ if and only if $\{U_{\overline{g}} \,|\, \overline{g} \in \overline{G}\}$ are linearly independent over S where $U_{\overline{g}} = U_g$ for each $g \in G$.*

Proof. ($\Longrightarrow$) Since CG_f is a central Galois algebra with an inner Galois group $\overline{G}$, $CG_f = S\overline{G}_{\overline{f}}$ ([4], Theorem 6). Thus $\{U_{\overline{g}} \,|\, \overline{g} \in \overline{G}\}$ are linearly independent over S.

($\Longleftarrow$) Since $\{U_{\overline{g}} \,|\, \overline{g} \in \overline{G}\}$ are linearly independent over S, $S\overline{G}_{\overline{f}} = \oplus_{\overline{g}\in\overline{G}} SU_{\overline{g}}$ is a projective group algebra of $\overline{G}$ over S with factor set $f : \overline{G} \times \overline{G} \longrightarrow$ units of S induced by $f : G \times G \longrightarrow$ units of C. Noting that $\{U_g \,|\, g \in K\} \subset S$, we have that $CG_f = \oplus_{\overline{g}\in\overline{G}} SU_{\overline{g}} = S\overline{G}_{\overline{f}}$. But CG_f is an Azumaya S-algebra (for n is a unit in C), so $S\overline{G}_f$ is an Azumaya S-algebra. Thus $S\overline{G}_f$ is a central Galois S-algebra with an inner Galois group $\overline{G}$ ([5], Theorem 3). Therefore CG_f is a central Galois algebra over S with an inner Galois group $\overline{G}$.

Theorem 3.2 can be generalized to a projective group ring RG_f of a group G over a ring R (not necessarily commutative) with a factor set $f : G \times G \longrightarrow$ units of the center of R.

Theorem 3.3. ([22], Theorem 3.2) *Let RG_f be a Galois projective group ring of G over a ring R, C the center of RG_f, and R_0 the center of R. Then the following are equivalent: (1) RG_f is a Galois extension of $(RG_f)^{\overline{G}}$ with an inner Galois group $\overline{G}$ induced by $\{U_g \,|\, g \in G\}$. (2) $C\overline{G}_{\overline{f}}$ is a central Galois projective group algebra of $\overline{G}$ over C with factor set $\overline{f} : \overline{G} \times \overline{G} \longrightarrow$ units of C induced by $f : G \times G \longrightarrow$ units of R_0. (3) $\{U_{\overline{g}} \,|\, \overline{g} \in \overline{G}\}$ are free over RC and $RC = \oplus \sum_{g\in K} RU_g$ where $U_{\overline{g}} = U_g$ for each $g \in G$ and $K = \{g \in$ the center of $G \,|\, f(g,g') = f(g',g)$ for all $g' \in G\}$.*

Proof. Let Z be the center of G. We first note that $\overline{G} \cong G/K$ where $K = \{g \in Z \,|\, f(g,g') = f(g',g)$ for all $g' \in G\}$ and that $\{U_{\overline{g}} \,|\, \overline{g} \in \overline{G}\}$ are free over C where $U_{\overline{g}} = U_g$ for each $g \in G$ by the argument used in the proof of Theorem 3.1. Next we prove (1) $\Longrightarrow$ (2) and leave other implications (2) $\Longrightarrow$ (1) and (2) $\Longrightarrow$ (3) $\Longrightarrow$ (2) to readers.

Since RG_f is a Galois extension of $(RG_f)^{\overline{G}}$ with an inner Galois group $\overline{G}$, $\{U_{\overline{g}} \,|\, \overline{g} \in \overline{G}\}$ are free over RC. Noting that $\overline{f} : \overline{G} \times \overline{G} \longrightarrow$ units of R_0 contained in C, we have that $C\overline{G}_{\overline{f}}$ is a projective group algebra of $\overline{G}$ over C with factor set $\overline{f} : \overline{G} \times \overline{G} \longrightarrow$ units of C where $\overline{f}$ is induced by $f : G \times G \longrightarrow$ units of R_0. Moreover, since $R_0 K_f \subset C$, $\sum_{\overline{g} \in \overline{G}} (R_0 K_f) U_{\overline{g}} \subset C\overline{G}_{\overline{f}}$. But $\overline{G} = G/K$, so

$$RG_f = \sum_{g \in G} RU_g = R(R_0 G_f) \subset R(\sum_{\overline{g} \in \overline{G}} CU_{\overline{g}}) = R(C\overline{G}_{\overline{f}}) \subset RG_f.$$

Hence $RG_f = R(C\overline{G}_{\overline{f}})$. Thus $\overline{G}|_{C\overline{G}_{\overline{f}}} \cong \overline{G}$. Next we claim that C is also the center of $\sum_{\overline{g} \in \overline{G}} CU_{\overline{g}}$ $(= C\overline{G}_{\overline{f}})$. In fact, clearly, C is contained in the center of $C\overline{G}_{\overline{f}}$. Conversely, for any $x \in$ the center of $C\overline{G}_{\overline{f}}$, x is in the center of $\sum_{\overline{g} \in \overline{G}} CU_{\overline{g}}$. Also, for any $r \in R$, $rx = xr$, so x is in the center of $R(\sum_{\overline{g} \in \overline{G}} CU_{\overline{g}})$ which is RG_f. Thus $x \in C$. Therefore $C\overline{G}_f$ is an Azumaya C-algebra; and so $C\overline{G}_f$ is a central Galois C-algebra with an inner Galois group $\overline{G}|_{C\overline{G}_{\overline{f}}} \cong \overline{G}$ ([4], Theorem 6).

We give two examples of Galois extensions with an inner Galois group G.

Example 1. Let $R[i,j,k]$ be the real quaternion algebra over real field R with inner automorphism group $G = \{1, \bar{i}, \bar{j}, \bar{k}\}$ where $\bar{i}(x) = ixi^{-1}$, $\bar{j}(x) = jxj^{-1}$, and $\bar{k}(x) = kxk^{-1}$ for $x \in R[i,j,k]$. Then $R[i,j,k] = R \oplus Ri \oplus Rj \oplus Rk$, a projective group algebra RG_f with center R; and so it is a central Galois algebra over R with an inner Galois group G.

Example 2. Let $T = R[i] \subset R[i,j,k]$ as given in Example 1 and $H_i = \{1, \bar{i}\} \subset G$. Then $(R[i,j,k])^{H_i} = R[i]$ and $R[i,j,k]$ is a noncommutative Galois extension of $R[i]$ with a cyclic Galois group H_i. We note that any Galois algebra with a cyclic Galois group is commutative ([4], Theorem 11).

By using Theorem 3.2, we derive some characterizations for a Galois extension B as given in Theorem 3.2 which is generated by $\{U_g \,|\, g \in G\}$ over B^G. We recall that C is the center of B, S the center of CG_f, Z the center of G, and $K = \{g \in Z \,|\, f(g,h) = f(h,g)$ for all $h \in G\}$.

Theorem 3.4. ([23], Theorem 2.3) *Let B be a Galois extension of B^G with an inner Galois group G of order n invertible in B. Then the following are equivalent:*

(1) $B = \sum_{g \in G} B^G U_g$, i.e., B is generated by $\{U_g \,|\, g \in G\}$ over B^G;

(2) $B = B^G G_f$, a projective group ring of G over B^G with factor set $f : G \times G \longrightarrow$ units of C;

(3) $C = S$;

(4) $\sum_{g \in G} CU_g$, the subring of B generated by $\{U_g \,|\, g \in G\}$ over C, is a central Galois C-algebra with Galois group $\overline{G} \cong G$;

(5) $\sum_{g \in G} CU_g$ is an Azumaya C-algebra;

(6) $K = \langle 1 \rangle$ and $\{U_{\overline{g}} \,|\, \overline{g} \in \overline{G}\}$ are linearly independent over S.

4. The Azumaya Algebra

Let B be a Galois extension of B^G with an inner Galois group G whose order n is invertible in B as given in Theorem 3.2, $G = \{g \in G \,|\, g(x) = U_g x U_g^{-1}$ for some $U_g \in B$ and for all $x \in B\}$, C the center of B, Z the center of G, and $K = \{g \in Z \,|\, f(g,h) = f(h,g)$ for all $h \in G\}$. Assume that B is an Azumaya C-algebra. We shall show an equivalent condition for a central Galois algebra CG_f in terms of the Galois extension B^K of B^G with Galois group G/K.

Theorem 4.1. ([23], Theorem 3.1) *Let B be given in Theorem 3.2. If B is an Azumaya C-algebra, then $V_B(B^G) = CG_f$.*

Proof. Since n is invertible in B, CG_f is a separable subalgebra of the Azumaya C-algebra B. Hence $V_B(V_B(CG_f)) = CG_f$. Noting that $V_B(CG_f) = B^G$, we have that $V_B(B^G) = CG_f$.

Theorem 4.2. ([23], Theorem 3.2) *Let B be given in Theorem 3.2. Assume B is an Azumaya C-algebra. Then CG_f is a central Galois algebra over its center S with Galois group $\overline{G}$ $(= G/K)$ if and only if $B^K = B^G \cdot (CG_f)$.*

Proof. ($\Longrightarrow$) Since CG_f is a central Galois algebra with Galois group $\overline{G}$ $(= G/K)$, CG_f has a $\overline{G}$-Galois system. Clearly, $CG_f \subset B^G \cdot (CG_f) \subset B^K$ and $(B^G \cdot (CG_f))^G = (B^K)^G = B^G$, so $B^G \cdot (CG_f)$ and B^K are also Galois extensions with the same Galois system as CG_f by noting that the restrictions of G to $B^G \cdot (CG_f)$ and B^K are isomorphic with $\overline{G}$ $(= G/K)$. Thus $B^K = B^G \cdot (CG_f)$.

($\Longleftarrow$) By hypothesis, B is a Galois extension of B^G with an inner Galois group G of order n invertible in B, so B^K is a Galois extension of B^G with an inner Galois group G/K. Let S be the center of CG_f. Since CG_f is a separable C-subalgebra of the Azumaya C-algebra B, $V_B(V_B(CG_f)) = CG_f$. Hence CG_f, B^G $(= V_B(CG_f))$, and $B^G \cdot (CG_f)$ have the same center S. By hypothesis, $B^K = B^G \cdot (CG_f)$. Thus S is the center of B^K. But B^K is a Galois extension of B^G with an inner Galois group $\overline{G}$ $(= G/K)$, so B^K contains the separable projective group algebra $S\overline{G}_{\overline{f}}$ where $f : \overline{G} \times \overline{G} \longrightarrow$ units of S induced by $f : G \times G \longrightarrow$ units of C by Theorem 3.1. Thus $\{U_{\overline{g}} \,|\, \overline{g} \in \overline{G}\}$ are linearly independent over S. Therefore CG_f is a central Galois algebra with Galois group $\overline{G}$ by Theorem 3.2.

Corollary 4.3. ([23], Corollary 3.1) *Let B be given in Theorem 4.2. Then B^K is a Galois projective group ring of $\overline{G}$ over $B^G S$ with factor set $\overline{f} : \overline{G} \times \overline{G} \longrightarrow$ units of C.*

Proof. By Theorem 4.2, $B^K = B^G \cdot (CG_f)$ and $CG_f = S\overline{G}_{\overline{f}}$, so $B^K = B^G \cdot (CG_f) = B^G(S\overline{G}_{\overline{f}}) = (B^G S)\overline{G}_{\overline{f}}$ which is a Galois projective group ring of $\overline{G}$ over $B^G S$ with factor set $\overline{f} : \overline{G} \times \overline{G} \longrightarrow$ units of C.

5. The Galois Commutator Subring

We note that a Galois extension with an inner Galois group G is a Hirata separable extension of B^G ([17], Corollary 3). In [17], let B be a Hirata separable Galois extension of B^G with Galois group G and $\Delta = V_B(B^G) = \{b \in G \mid ba = ab$ for each element $a \in B^G\}$, the commutator subring of B^G in B. A sufficient condition was given for Δ being a Galois algebra with Galois group G/N where $N = \{g \in G \mid g(x) = x$ for all $x \in \Delta\}$. We shall study the problem for a Galois extension B of B^G with Galois group G such that Δ is a Galois extension with Galois group G/N. Such a Galois extension B with Galois group G will be characterized in terms of a composition of two Galois extensions: $B \supset B^G \cdot V_B(B^G) \supset B^G$ and in terms of crossed products respectively.

We begin with two lemmas whose proofs are straightforward.

Lemma 5.1. ([24], Lemma 3.1) *Let T be a ring and G an automorphism group of T. Then (1) $V_T(T^G)$ is a G-invariant subring of T and (2) $(V_T(T^G))^G$ is contained in the center of $V_T(T^G)$ (hence $V_T(T^G)$ is an algebra over $(V_T(T^G))^G$).*

Lemma 5.2. ([24], Lemma 3.2) *Let B be a Galois extension of B^G with Galois group G and A a G-invariant subring of B under the action of G. If A is a Galois extension of B^G with Galois group induced by and isomorphic with G, then $A = B$.*

Theorem 5.3. ([24], Theorem 3.3) *Let B be a Galois extension of B^G with Galois group G, $\Delta = V_B(B^G)$, and $D = \Delta^G$. Then the following statements are equivalent: (1) Δ is a Galois algebra over D with Galois group induced by and isomorphic with G/N where $N = \{g \in G \mid g(x) = x$ for all $x \in \Delta\}$. (2) $B^G\Delta$ is a Galois extension of B^G with Galois group induced by and isomorphic with G/N and Δ is a finitely generated and projective module over D. (3) B is a composition of two Galois extensions: $B \supset B^G\Delta$ with Galois group N and $B^G\Delta \supset B^G$ with Galois group induced by and isomorphic with G/N such that $J_{\overline{g}}^{(\Delta)}$ is a finitely generated projective module over D for each $\overline{g} \in G/N$ where $J_{\overline{g}}^{(\Delta)} = \{b \in \Delta \mid bx = g(x)b$ for all $x \in \Delta\}$.*

Proof. (1) $\Longrightarrow$ (2) Since the automorphism groups induced by G/N on $B^G\Delta$ and Δ are isomorphic and Δ is a Galois algebra over D where $D = \Delta^G$, $B^G\Delta$ is a Galois extension of $(B^G\Delta)^G$ $(= B^G)$ with Galois group induced by and isomorphic with G/N.

(2) $\Longrightarrow$ (1) Since $B^G\Delta \supset B^G$ is a Galois extension with Galois group induced by and isomorphic with G/N, the crossed product

$$(B^G\Delta) * (G/N) \cong \mathrm{Hom}_{B^G}(B^G\Delta, B^G\Delta).$$

Denoting G/N by $\overline{G}$, we have that

$$\alpha : (B^G\Delta) * \overline{G} \cong \mathrm{Hom}_{B^G}(B^G\Delta, B^G\Delta)$$

by $(\alpha(\sum_{\overline{g}\in\overline{G}} a_{\overline{g}}\overline{g}))(x) = \sum_{\overline{g}\in\overline{G}} a_{\overline{g}}\overline{g}(x)$ for each $x \in B^G\Delta$. Then

$$\Delta * \overline{G} = V_{B^G\Delta*\overline{G}}(B^G) \cong V_{\mathrm{Hom}_{B^G}(B^G\Delta, B^G\Delta)}(\alpha(B^G)).$$

It can be verified that $V_{\mathrm{Hom}_{B^G}(B^G\Delta, B^G\Delta)}(\alpha(B^G)) = \mathrm{Hom}_D(\Delta, \Delta)$ where $D = \Delta^{\overline{G}} = \Delta^G$. But Δ is a finitely generated and projective module over D, so Δ is a Galois algebra over D with Galois group isomorphic with $\overline{G}$.

(2) $\Longrightarrow$ (3) Since $B^G\Delta \subset B^N$ such that $(B^G\Delta)^G = B^G = (B^N)^G$ and $B^G\Delta$ is a Galois extension of B^G with Galois group induced by and isomorphic with $\overline{G}$ $(= G/N)$, $B^N = B^G\Delta$ by Lemma 5.2. Moreover, noting that $V_{B^G\Delta}(B^G) = \Delta = \oplus \sum_{\overline{g}\in\overline{G}} J_{\overline{g}}^{(\Delta)}$ ([12], Proposition 1 and Theorem 1), we conclude that $J_{\overline{g}}^{(\Delta)}$ is a finitely generated projective module over D for each $\overline{g} \in G/N$.

(3) $\Longrightarrow$ (2) is clear.

By Theorem 5.3, we shall derive some consequences for several well known classes of Galois extensions. We recall that B is a center Galois extension with Galois group G if its center C is a Galois algebra over C^G with Galois group $G|_C \cong G$, and B is a commutator Galois extension of B^G with Galois group G if $V_B(B^G)$ is a Galois extension of $(V_B(B^G))^G$ with Galois group $G|_{V_B(B^G)} \cong G$.

Corollary 5.4. *Let B be a Galois extension of B^G with Galois group G. If $B = B^GC$ such that C is finitely generated and projective over C^G, then B a center Galois extension with Galois group G.*

Corollary 5.5. *Let B be a Galois extension of B^G with Galois group G. If $B = B^G\Delta$ such that Δ is finitely generated and projective over Δ^G, then B a commutator Galois extension with Galois group G.*

Remark. Since a DeMeyer-Kanzaki Galois extension is also a center Galois extension ([4], Lemma 2) and an Azumaya Galois extension is a commutator Galois extension ([1], Theorem 2), Corollary 5.4 and Corollary 5.5 hold for the classes of DeMeyer-Kanzaki Galois extensions and Azumaya Galois extensions.

Corollary 5.6. *Let B be a Hirata separable Galois extension of B^G with Galois group G. If $B = B^G\Delta$, then Δ is a Galois algebra with Galois group induced by and isomorphic with G/N.*

Proof. Since B is a Hirata separable Galois extension of B^G with Galois group G, J_g is a finitely generated and projective rank one module over C^G for each $g \in G$ ([17], Theorem 2). The corollary holds by Theorem 5.3.

We continue to characterize a Galois commutator subring Δ in terms of crossed products.

Theorem 5.7. *Keeping the notations of Theorem 5.3, the following statements are equivalent: (1) Δ is a Galois algebra over Δ^G with Galois group induced by and isomorphic with G/N where $N = \{g \in G \,|\, g(x) = x$ for all $x \in \Delta\}$. (2) Let $\Delta * (G/N)$ be the crossed* *product of G/N over Δ with trivial factor set. Then $\Delta * (G/N)$ is an Azumaya algebra over Δ^G. (3) Let $(B^G\Delta) * (G/N)$ be the crossed product of G/N over $B^G\Delta$ with trivial factor set. Then $(B^G\Delta) * (G/N)$ is a Hirata separable extension of B^G such that B^G is a direct summand of $(B^G\Delta) * (G/N)$ as a B^G-bimodule.*

Proof. (1) $\Longrightarrow$ (2) Since Δ is a Galois algebra over Δ^G with Galois group $\overline{G}$ induced by and isomorphic with G/N, $\Delta * \overline{G} \cong \mathrm{Hom}_{\Delta^G}(\Delta, \Delta)$ where Δ is a finitely generated and projective module over Δ^G. Noting that Δ is an algebra with 1 over Δ^G, we have that $\mathrm{Hom}_{\Delta^G}(\Delta, \Delta)$ is an Azumaya algebra over Δ^G. Hence $\Delta * \overline{G}$ is an Azumaya algebra over Δ^G.

(2) $\Longrightarrow$ (1) By hypothesis, $\Delta * \overline{G}$ is an Azumaya algebra over Δ^G, so $\Delta * \overline{G}$ is a Hirata separable extension of Δ ([8], Theorem 1). Since Δ is a progenerator of Δ, Δ is a progenerator of $\Delta * \overline{G}$. Thus Δ is a Galois algebra over Δ^G with Galois group isomorphic with $\overline{G}$.

(2) $\Longrightarrow$ (3) Since $\Delta * \overline{G}$ is an Azumaya algebra over Δ^G, $B^G \otimes_{\Delta^G} (\Delta * \overline{G})$ is a Hirata separable extension of B^G; and so, as a homomorphism image of $(B^G \otimes_{\Delta^G} \Delta) * \overline{G}$, $(B^G\Delta) * \overline{G}$ is also a Hirata separable extension of B^G. Since $\Delta * \overline{G}$ is an Azumaya algebra over Δ^G again, Δ is a Galois algebra over Δ^G with Galois group $\overline{G}$ by (2) $\Longrightarrow$ (1). Hence there exists an element $d \in \Delta$ such that $\mathrm{tr}_{\overline{G}}(d) = 1$ ([12], proof of Proposition 5) where $\mathrm{tr}_{\overline{G}}(\ ) = \sum_{\overline{g} \in \overline{G}} \overline{g}(\ )$. Thus $\mathrm{tr}_{\overline{G}}(\ ) : B^G\Delta \longrightarrow B^G \longrightarrow 0$ is exact as B^G-bimodule homomorphism, and so B^G is a direct summand of $B^G\Delta$ as B^G-bimodule homomorphism. Noting that $B^G\Delta$ a direct summand of $(B^G\Delta) * \overline{G}$ as a B^G-bimodule, we conclude that so is B^G.

(3) $\Longrightarrow$ (2) Since $(B^G\Delta) * \overline{G}$ is a Hirata separable extension of B^G such that B^G is a direct summand of $(B^G\Delta) * \overline{G}$ as a B^G-bimodule, $V_{(B^G\Delta)*\overline{G}}(B^G)$ is a separable algebra over the center of $(B^G\Delta) * \overline{G}$ ([16], Theorem 1). But $V_{(B^G\Delta)*\overline{G}}(B^G) = \Delta * \overline{G}$, so $\Delta * \overline{G}$ is a separable algebra over the center of $(B^G\Delta) * \overline{G}$. We claim that the centers of $\Delta * \overline{G}$ and $(B^G\Delta) * \overline{G}$ are Δ^G. In fact, by hypothesis, $(B^G\Delta) * \overline{G}$ is a Hirata separable extension of B^G such that B^G is a direct summand of $(B^G\Delta) * \overline{G}$ as a B^G-bimodule again, $V_{(B^G\Delta)*\overline{G}}(V_{(B^G\Delta)*\overline{G}}(B^G)) = B^G$ ([16], Theorem 1). Hence the center of $(B^G\Delta) * \overline{G}$ is contained in B^G; and so it is contained in the center of B^G. Conversely, the center of B^G is clearly contained in the center of $(B^G\Delta) * \overline{G}$. Thus, the center of $(B^G\Delta) * \overline{G}$ is equal to the center of B^G. Moreover, since the center of B^G is Δ^G, the center of $(B^G\Delta) * \overline{G}$ is Δ^G. But the centers of $\Delta * \overline{G}$ and $(B^G\Delta) * \overline{G}$ are the same, so the center of $(B^G\Delta) * \overline{G}$ is Δ^G. Therefore, $\Delta * \overline{G}$ is an Azumaya algebra over Δ^G.

Corollary 5.8. *Let B satisfy the equivalent conditions of Theorem 5.7. Then $N = \langle 1 \rangle$ if and only if $B = B^G\Delta$ such that $\Delta^G = C^G$ where C is the center of B.*

Corollary 5.9. *Let B satisfy the equivalent conditions of Theorem 5.7. If N is a maximal subgroup of G, then Δ is a commutative Galois algebra over Δ^G with a cyclic Galois group G/N ([4], Theorem 11).*

References

[1] R. Alfaro and G. Szeto, *Skew Group Rings which are Azumaya*, Comm. in Algebra **23**(6) (1995), 2255–2261.

[2] M. Auslander and O.Goldman, *The Brauer Group of a Commutative Ring*, Transactions of the American Math. Soc. **97**(3) (1960), 367-409.

[3] S.U. Chase, D.K. Harrison, A. Rosenberg, *Galois Theory and Galois Cohomology of Commutative Rings*, Memoirs Amer. Math. Soc. , No. 52, 1965.

[4] F.R. DeMeyer, *Some Notes on the General Galois Theory of Rings*, Osaka J. Math. **2** (1965) 117-127.

[5] F.R. DeMeyer, *Galois Theory in Separable Algebras over Commutative Rings*, Illinois J. Math. **10** (1966), 287-295.

[6] F.R. DeMeyer and E. Ingraham, *"Separable algebras over commutative rings"*, Lecture Notes in Mathematics, Volume **181**, Springer Verlag, Berlin, Heidelberg, New York, 1971.

[7] M. Harada, *Supplementary Results on Galois Extension*, Osaka J. Math. **2** (1965), 343-350.

[8] S. Ikehata, *Note on Azumaya Algebras and H-Separable Extensions*, Math. J. Okayama Univ. **23** (1981), 17-18.

[9] S. Ikehata, *On H-separable polynomials of prime degree*, Math. J. Okayama Univ. **33** (1991), 21-26.

[10] S. Ikehata and G. Szeto, *On H-separable polynomials in skew polynomial rings of automorphism type*, Math. J. Okayama Univ. **34** (1992), 49-55.

[11] S. Ikehata and G. Szeto, *On H-skew polynomial rings and Galois extensions*, Rings, Extension and Cohomology (Evanston, IL, 1993), 113-121, Lecture Notes in Pure and Appl. Math., 159, Dekker, New York, 1994.

[12] T. Kanzaki, *On Galois algebra over a commutative ring*, Osaka J. Math. **2** (1965), 309-317.

[13] Y. Miyashita, *On a skew polynomial ring*, J. Math. Soc. Japan **311** (1979), 317-330.

[14] T. Nagahara, *On Galois conditions and Galois groups of simple rings*, Trans. Amer. Math. Soc. **116** (1965), 417-434.

[15] K. Sugano, *Note on semisimple extensions and separable extensions*, Osaka J. Math. **4** (1967), 265-270.

[16] K. Sugano, *On Centralizers in Separable Extensions II*, Osaka J. Math. **8** (1971), 465-469.

[17] K. Sugano, *On a Special Type of Galois Extensions*, Hokkaido J. Math. **9** (1980), 123-128.

[18] G. Szeto and L. Xue, *The general Ikehata theorem for H-separable crossed products*, International Journal of Mathematics and Mathematical Sciences **23**(10) (2000), 657-662.

[19] G. Szeto and L. Xue, *On the Ikehata theorem for H-separable skew polynomial Rings*, Math. J. Okayama Univ. **40** (2000), 27-32.

[20] G. Szeto and L. Xue, *The Structure of Galois Algebras*, Journal of Algebra **237**(1) (2001), 238-246.

[21] G. Szeto and L. Xue, *The Galois Algebra with Galois Group which is the Automorphism Group*, Journal of Algebra **293**(1) (2005), 312-318.

[22] G. Szeto and L. Xue, *On Projective Group Rings with an Inner Automorphism Group*, Far East Journal of Mathematical Sciences **31**(1) (2008), 1-8.

[23] G. Szeto and L. Xue, *On Galois Extensions with an Inner Galois Group*, Recent Developments in Algebra and Related Area, ALM 8, 239-245, Higher Education Press and International Press Beijing-Boston, 2008.

[24] G. Szeto and L. Xue, *Galois extensions with a Galois commutator subring*, (submitted).

Department of Environmental and Mathematical Science
Faculty of Environmental Science and Technology
Okayama University
Tsushima, Okayama 700-8530, Japan

E-mail address: ikehata@ems.okayama-u.ac.jp

Department of Mathematics
Bradley University
Peoria, Illinois 61625, U.S.A.
E-mail address: szeto@bradley.edu

Department of Mathematics
Bradley University
Peoria, Illinois 61625, U.S.A.
E-mail address: lxue@bradley.edu

EXTENSION OF THE MATLIS DUALITY TO A FILTERED NOETHERIAN RING

N.KAMEYAMA*

Abstract. A ring theoretic investigation of the Iwasawa algebra is accomplished. Therefore, we look at a filtered pseudocompact algebra (abbreviation:FPC algebra) which is a reasonable generalization of the Iwasawa algebra (1.1). It is shown that an FPC algebra has the Matlis duality between suitable categories. When an FPC algebra is Auslander regular and with homogeneity condition, we study the local cohomology and local dualty.

Key Words: Iwasawa algebra, pseudocompact algebra, local cohomology, local duality.

2000 *Mathematics Subject Classification*: 13D45, 16E30, 16E65, 16P40

1. Introduction

This paper is a summary of [12]. A class of (non)commutative Iwasawa algebras, studied as main objects in Iwasawa theory, occupies quite interesting position in that of noncommutative Noetherian rings. Moreover, they possess a filtered ring structure which is an algebraic device of topological notion. In the present paper, we study ring theoretic properties of Iwasawa algebras, through a filtered pseudocompact algebra, FPC algebra, for short.

Let us explain essential properties of Iwasawa algebras shortly. Let p be a prime number, and G a compact p-adic analytic group. The Iwasawa algebra is defined by

$$\Lambda(G) := \varprojlim \mathbb{Z}_p[G/U],$$

where U ranges over all open normal subgroups of G. A key fact for us is the following. Assume that G is a uniform pro-p group. Then $\Lambda(G)$ is a right and left Noetherian ring ([6], Corollary 7.25) and local Auslander-regular domain ([2], 4.1, 4.3, 5.1, 5.2). It has a J-adic filtration $F\Lambda(G)$, where $J = \mathrm{rad}\Lambda(G)$, with $F_i\Lambda(G) := J^{-i}$ $(i < 0)$, $= \Lambda(G)$ $(i \geq 0)$. $\Lambda(G)$ is complete with respect to this filtration ([2], 3.5). Suppose that G is a p-valued compact p-adic Lie group or a uniform extra-powerful pro-p group, then the filtration $F\Lambda(G)$ is Zariskian ([5], §7 or [20], Theorem 3.22, see also [9], Chapter II §2, 2.1.2 Theorem (4)). For these cases, $\Lambda(G)$ is a typical example of an FPC algebra.

We can say that a reasonably generalized algebra of the Iwasawa algebra is a pseudocompact algebra due, for example, to [3], [19]. There is a duality between the category of pseudocompact Γ-modules and that of discrete Γ-modules for a pseudocompact algebra Γ. This is a basic result for homological study of such algebras. To begin with, we make this duality over for the suitable categories over an FPC algebra (see 1.4). Then we study the local cohomology and local duality over such algebras. This provides a generalization

The detailed version of this paper will be submitted for publication elsewhere.

of [20], §§5,6. Further, we expect that the homological properties such that Bass number, Gorensteiness etc. are within view as module-finite algebras [8].

2. Duality over a filtered pseudocompact algebra

2.1. **Assumption.** Let Λ be a left and right Noetherian filtered ring with a Zariskian filtration $F\Lambda = \{F_i\Lambda\}_{i\in\mathbb{Z}}$ ([9], Chapter II, §2) such that

(a1) $H_i = F_i\Lambda$ is an ideal of Λ for every $i \in \mathbb{Z}$,
(a2) Λ is complete with respect to $F\Lambda$,
(a3) Λ/H_i is of finite length as a right and left Λ-module for every $i \in \mathbb{Z}$.

For further use, it is desirable that Λ is an algebra over a commutative ring. Let $(R, \mathfrak{m}, k)$ be a commutative local Noetherian ring and Λ an R-algebra. We consider that R is a subring of Λ via a structure map $R \to \Lambda$.

Put $I_i := R \cap H_i (i \in \mathbb{Z})$ and $FR = \{I_i\}(i \in \mathbb{Z})$. Then FR is a filtration of R. We assume that

(b1) R is complete with respect to FR,
(b2) R/I_i is a finite length R-module for every $i \in \mathbb{Z}$,
(b3) $\mathfrak{m}^n$ is open for all $n > 0$, i.e., $\mathfrak{m}^n \supset I_i$ for some $i \in \mathbb{Z}$,
(b4) Λ/H_i is a module-finite R/I_i-algebra for every $i \in \mathbb{Z}$, i.e., Λ/H_i is a finitely generated R/I_i-module.

We call an R-algebra satisfying all above assumptions a *filtered pseudocompact algebra* and FPC algebra for short. Moreover, Λ/H_i is a finite length R-module for every i. Therefore all finite length Λ-modules are finite length R-modules.We sometimes consider a filtered Λ-module (M, FM) as a filtered R-module with the same filtration FM, but regard as an R-module. We assume that all filtrations are separated.

Let $E := E_R(k)$ be an injective hull of k as an R-module. It follows that E is an injective cogenerator of Mod$R$. Put $E_i := \{x \in E | I_{-i}x = 0\}$ an R-submodule of E for every $i \in \mathbb{Z}$. The assumption (b3) and [13], Theorem 18.4 implies $E = \cup E_i$, so E is a filtered R-module with a filtration $FE = \{E_i\}_{i\in\mathbb{Z}}$.

2.2. **Filtration and filtration topology.** Let R be a filtered ring and M, N filtered R-modules. Let $F_p\mathrm{HOM}_R(M, N) = \{f \in \mathrm{Hom}_R(M, N) | f(F_iM) \subset F_{i+p}N$ for all $i \in \mathbb{Z}\}$. Put $\mathrm{HOM}_R(M, N) := \bigcup_{p\in\mathbb{Z}} F_p\mathrm{HOM}_R(M, N)$.

In some cases, all homomorphisms are of finite degree. In particular, the following will be used frequently.

Proposition 1. *Let M be a filtered R-module with a filtration $FM = \{F_iM\}_{i\in\mathbb{Z}}$. Assume that M is of finite length. Then* $\mathrm{Hom}_R(M, E) = \mathrm{HOM}_R(M, E)$.

2.3. **Pseudocompact modules and copseudocompact modules.** We put the category $\mathcal{F}_\Lambda$ as follows,

Objects: all filtered Λ-modules,
Morphisms: all Λ-homomorphisms of finite degree, i.e.,
the elements of $\mathrm{HOM}_\Lambda(M, N)$ for M, $N \in \mathcal{F}_\Lambda$.

We put $M^\vee := \mathrm{HOM}_R(M, E)$ by regarding M as a filtered R-module with a filtration $\{F_iM\}_{i\in\mathbb{Z}}$. Then $(-)^\vee = \mathrm{HOM}_R(-, E)$ turns out to be a contravariant functor between $\mathcal{F}$ and $\mathcal{F}^{op}$. We put $(-)' := \mathrm{Hom}_R(-, E)$, which induces usual Matlis Duality. Let M be a filtered Λ-module with a filtration FM. We call M *pseudocompact*, if $M \cong \varprojlim M/F_iM$, that is, M is complete ([9], Chapter I, §3, 3.5) and $H_iM \subset F_iM$ for every $i \in \mathbb{Z}$ (cf. [3], [19]). Dually, a filtered Λ-module N with a filtration FN is called *copseudocompact*, if $N \cong \varinjlim F_iN$ and $H_{-i}F_iN = 0$ for every $i \in \mathbb{Z}$.

Proposition 2. *Let $M, N \in \mathcal{F}_\Lambda$. Then*
(1) If M is pseudocompact, then $M^\vee \cong \varinjlim(M/F_iM)^\vee$.
(2) If N is copseudocompact, then $N^\vee \cong \varprojlim(F_iN)^\vee$.

2.4. **Duality.** Let $\mathcal{C}$ be a full subcategory of $\mathcal{F}_\Lambda$ consisting of all finitely generated pseudocompact Λ-modules, and $\mathcal{D}$ a full subcategory of $\mathcal{F}_\Lambda$ consisting of all finitely cogenerated copseudocompact Λ-modules. Here, a module is finitely cogenerated if and only if its socle is essential and finitely generated (cf. [1], Proposition 10.7).

Theorem 3. *Let M, $N \in \mathcal{F}_\Lambda$. Then*
(1) If M is pseudocompact, then $M^\vee$ is copseudocompact.
(2) If N is copseudocompact, then $N^\vee$ is pseudocompact.
(3) $\Lambda \cong \Lambda^{\vee\vee}$ and $\Lambda^\vee$ is Artinian.

Theorem 4. *Let M, $N \in \mathcal{F}_\Lambda$. Then*
(1) If $M \in \mathcal{C}$ then $M^\vee \in \mathcal{D}$ and $M^{\vee\vee} \cong M$.
(2) If $N \in \mathcal{D}$ then $N^\vee \in \mathcal{C}$ and $N^{\vee\vee} \cong N$.

Proof. (1): Since M is finitely generated, there is an epimorphism $f : \Lambda^n \to M$. Dualizing it, we have a monomorphism $f^\vee : M^\vee \to \Lambda^{\vee n}$, so $M^\vee$ is Artinian, and $M^\vee \in \mathcal{D}$. We see

$$M^{\vee\vee} \cong \varprojlim(M/F_iM)^{\vee\vee} \cong \varprojlim(M/F_iM)'' \cong \varprojlim M/F_iM \cong M.$$

(2)similarly. □

3. Local cohomology

3.1. **Depth and Auslander-Buchsbaum Formula.** We assume that a FPC algebra Λ is
1) Auslander regularity with gl.dimΛ=d ([9], Chapter Ⅲ, §2, 2.1.7),
2) the homogeneity condition,
where the homogeneity condition (cf. [7], (hc13) and (hc14), p.326) is that every simple left (respectively, right) Λ-module is contained in E^d (respectively, E'^d), where $0 \to \Lambda \to E^0 \to E^1 \to \cdots \to E^d \to 0$ (respectively, $0 \to \Lambda \to E'^0 \to E'^1 \to \cdots \to E'^d \to 0$) is a minimal injective resolution of Λ as a left (respectively, right) Λ-module. Let $J = \mathrm{rad}\Lambda$ be a Jacobson radical of Λ.

Definition 5. The *depth* of a finitely generated Λ-module M is defined by

$$\mathrm{depth}M := \min\{i \geq 0 | \mathrm{Ext}^i_\Lambda(\Lambda/J, M) \neq 0\}.$$

It equals ∞, whenever $\mathrm{Ext}^i_\Lambda(\Lambda/J, M) = 0$ for all $i \geq 0$.

The direct consequence of homogeneity condition is the following determination of depthΛ.

Proposition 6. *It holds that* depth$\Lambda = d$.

Theorem 7. *Let* $M \in \mathrm{mod}\Lambda$. *Then*

$$\mathrm{pd}M + \mathrm{depth}M = \mathrm{depth}\Lambda = d.$$

3.2. **Local cohomology and local duality.** Let M be a Λ-module. Put $\Gamma(M) := \{x \in M | H_{-i}x = 0 \text{ for some } i \geq 0\}$. Then Γ is a left exact additive functor: $\mathrm{Mod}\Lambda \to \mathrm{Mod}\Lambda$ such that $\Gamma(M) \cong \varinjlim \mathrm{Hom}_\Lambda(\Lambda/H_{-i}, M)$.

Definition 8. The local cohomology functors, denoted by $H^i(-)$, are the right derived functors of $\Gamma(-)$.

The following lemma is indispensable for proving the important property of local cohomology modules.

We can determine the structure of the last term E^d of a minimal injective resolution of Λ.

Proposition 9. *Let* $M \in \mathrm{Mod}\Lambda$. *Then, for any* $i \geq 1$,
(1) $H^i(M)$ *is a copseudocompact module for some filtration. Moreover, if* M *is finitely generated, then* $H^i(M) \in \mathcal{D}$,
(2) $H^i(M) \cong \varinjlim \mathrm{Ext}^i_\Lambda(\Lambda/H_{-p}, M)$.

As is usually done, we describe depth using the local cohomology modules.

Theorem 10. *Let* M *be a finitely generated* Λ*-module. Then*

$$\mathrm{depth}\, M = \min\{i \geq 0 : H^i(M) \neq 0\}.$$

Proof. The proof is done by modifying that of [20], Lemma 5.5. Note that $H_{-p} \subset J$ for every $p > 0$. □

We also observe that $\Lambda^\vee$ is copseudocompact. We write $X|Y$, When X is isomorphic to a direct summand of copies of Y. Then by the above corollary, we see $E^d|\Lambda^\vee$ and $\Lambda^\vee|E^d$. As concerns the local cohomology module of Λ, we see

Proposition 11. *There is an isomorphism* $H^d(\Lambda) \cong E^d$.

Proposition 12. *Assume that* Λ *is basic. Then there is the isomorphisms* $\Lambda^\vee \cong E^d \cong H^d(\Lambda)$.

Proposition 13. *Assume that* Λ *is basic. Let a* Λ*-module* M *be in* $\mathcal{C}$ *or* $\mathcal{D}$. *Then* $\mathrm{Hom}_\Lambda(M, H^d(\Lambda)) \cong M^\vee$.

We establish the local duality theorem using the above results. All the assumptions for Λ given before are preserved, that is, 1.1, 2.1 and to be basic.

Theorem 14. *(Local duality) Let* M *be an arbitrary finitely generated* Λ*-module. Then, for all integers* i*, there are natural isomorphisms*

$$H^i(M) \cong \mathrm{Ext}^{d-i}_\Lambda(M, \Lambda)^\vee \quad \text{and}$$
$$\mathrm{Ext}^i_\Lambda(M, \Lambda) \cong H^{d-i}(M)^\vee.$$

Proof. Using the above preparation, we can show the statement by the similar way to the commutative case. □

References

1. F.W. Anderson and K.R. Fuller, *Rings and categories of modules*, GTM13, Springer,1992.
2. K. Ardakov and K.A. Brown, *Ring-theoretic properties of Iwasawa algebras: a survey*, Documenta Math. Extra vol. Coates, 7-33(2006).
3. A. Brumer, *Pseudocompact algebras, profinite groups and class formations*, J. Alg. 4, 442-470(1966).
4. W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge stud. Adv. Math. 39, Cambridge Univ. Press,1993.
5. J. Coates, P. Schneider and R. Sujatha, *Modules over Iwasawa algebras*, J. Inst. Math. Jussieu 2, 73-108(2003).
6. J.D. Dixon, M.P.F. Du Sautoy, A. Mann and D. Segal, *Analytic pro-p groups*, Cambridge stud. Adv. Math. 61, Cambridge Univ. Press,1999.
7. S. Goto and K. Nishida, *Finite modules of finite injective dimension over a Noetherian algebra*, J.London Math. Soc, 63, 319-335(2001)
8. S. Goto and K. Nishida, *Towards a theory of Bass numbers with application to Gorenstein algebras*, Colloq. Math. 91(2), 191-253(2002).
9. L. Huishi and F. Van Oystaeyen, *Zariskian filtrations*, K-Monograph in Mathematics, 2, 1996.
10. Y. Iwanaga, *Duality over Auslander-Gorenstein rings*, Math. Scand. 81(2), 184-190(1997).
11. Y. Iwanaga and H. Sato, *On Auslander's n-Gorenstein rings*, J. Pure Appl. Alegbra, 106, 61-76(1996).
12. N. Kameyama, H. Miyahara and K, Nishida, *Local Duarity for noncommutative Iwasawa algebra*, preprint.
13. H. Matsumura, *Commutative ring theory*, Cambridge stud. Adv. Math. 8, Cambridge Univ. Pres, 1989.
14. C. Năstăsescu, F. Van Oystaeyen, *Graded ring theory*, Math. Library, 28, North Holland, Amsterdam, 1982.
15. J. Neukirch, S. Schmidt, K. Wingberg, *Cohomology of number fields*, Grundlehren math. Wiss., vol. 323, Springer, Berlin, Heidelberg, New York, 2000
16. A. Neumann, *Completed group algebras without zero divisors*, Arch. Math. (Basel) 51, 496-499(1988).
17. J. Rotman, *An introduction to homological algebra*, Academic Press, 1979, *second edition*, Springer, 2009.
18. Bo Stenström, *Rings of quotients*, Springer, 1975.
19. M. Van Gastel and M. Van den Bergh, *Graded modules of Gelfand-Kirillov dimension one over three-dimensional Artin-Schelter regular algebras*, J. Alg. 196, 251-282(1997).
20. O. Venjakob, *On the structure theory of the Iwasawa algebra of a p-adic Lie group*, J. Eur. Math. Soc.4, 271-311(2002).

(*) Graduate School of Science and Technology, Shinshu University, Matsumoto, Japan

ALMOST COMPARABILITY AND RELATED COMPARABILITIES IN VON NEUMANN REGULAR RINGS

MAMORU KUTAMI

Abstract. There are many comparabilities in von Neumann regular rings: general comparability, the comparability axiom, s-comparability, weak comparability, almost comparability etc.. In the article, we mainly investigate von Neumann regular rings satisfying almost comparability, comparing with other comparabilities.

Key Words: Von Neumann regular rings, Comparability.

2000 *Mathematics Subject Classification*: Primary 16E50; Secondary 16D70.

1. Introduction

In the article, we mainly study regular rings satisfying almost comparability, comparing with other related comparabilities: general comparability, the comparability axiom, s-comparability, weak comparability. In section 1, we give definitions and histories of the above related comparabilities. We begin with some notations and elementary definitions which will be needed in the article. For details, we can refee Goodearl's book [5].

Throughout this article, R is a ring with identity and R-modules are unitary right R-modules.

Notation 1. For two R-modules M, N, we use $M \lesssim N$ (resp. $M \lesssim_{\oplus} N$, $M \prec N$, $M \prec_{\oplus} N$) to mean that there exists an isomorphism from M to a submodule of N (resp. a direct summand of N, a proper submodule of N, a proper direct summand of N). For a submodule M of an R-module N, $M \leq_{\oplus} N$ (resp. $M < N, M <_{\oplus} N$) means that M is a direct summand of N (resp. a proper submodule of N, a proper direct summand of N). For a cardinal number k and an R-module M, kM denotes the direct sum of k-copies of M.

Definition 2. A ring R is said to be (von Neumann) *regular* if, for each $x \in R$, there exists an element y of R such that $xyx = x$, and a ring R is said to be *unit-regular* if, for each $x \in R$, there exists a unit element (i.e. an invertible element) u of R such that $xux = x$. It is well-known that a regular ring R is unit-regular if and only if $A \oplus B \cong A \oplus C$ implies $B \cong C$ for any finitely generated projective R-modules A, B, C. An R-module M is *directly finite* provided that M is not isomorphic to a proper direct summand of itself. A ring R is *directly finite* if the R-module R_R is directly finite, and R is said to be *stably finite* if the ring $M_n(R)$ of $n \times n$ matrices over R is directly finite for all positive

This paper is based on the author's talk and the detailed proof of some results in this paper will be submitted for publication elsewhere.

integers n. It is known that a ring R is stably finite if and only if every finitely generated projective R-module is directly finite.

Now, we recall definitions and histories of the related comparabilities.

Definition 3. A regular ring R satisfies *general comparability* if, for each $x, y \in R$, there exists a central idempotent $e \in R$ such that $e(xR) \lesssim e(yR)$ and $(1-e)(yR) \lesssim (1-e)(xR)$.

General comparability is the typical and oldest comparability, which evolved from operator algebras and Baer rings. All right self-injective regular rings are typical examples of regular rings with this comparability, which worked usefully to study these regular rings.

Definition 4. A regular ring R is said to satisfy *the comparability axiom* if, for each $x, y \in R$, either $xR \lesssim yR$ or $yR \lesssim xR$.

The comparability axiom is a special case of general comparability, which means that "each two principal right ideals are comparable". The notion was given by K.R. Goodearl and D. Handelman in 1975. All prime right self-injective regular rings are typical examples of regular rings with this comparability, and they investigated these regular rings using the comparability axiom.

Definition 5. Let s be a positive integer. A regular ring R is said to satisfy *s-comparability* if, for each $x, y \in R$, either $xR \lesssim s(yR)$ or $yR \lesssim s(xR)$. Note that 1-comparability means the comparability axiom above. It is well-known in [4] that s is either 1 or 2 only for any regular rings with s-comparability.

Connecting with the comparability axiom, s-comparability was also given by K.R. Goodearl and D. Handelman in 1976 to characterize uniqueness of rank functions on certain simple regular rings. But, the detailed study of regular rings with s-comparability became after one of regular rings with weak comparability. In the study of regular rings, there is a famous outstanding Open Problem: Is every directly finite simple regular ring always unit-regular? To solve the problem, K.C. O'Meara gave the notion of weak comparability and some interesting result, as follows.

Definition 6 ([11])**.** A regular ring R satisfies *weak comparability* if, for each nonzero $x \in R$, there exists a positive integer n such that $n(yR) \lesssim R_R$ implies $yR \lesssim xR$ for all $y \in R$, where the n depends on x.

Theorem 7 ([11])**.** *Every directly finite simple regular rings with weak comparability are unit-regular.*

After that a criterion of weak comparability for simple regular rings was given, as follows.

Theorem 8 ([3])**.** *For a simple regular ring R, the following are equivalent*:

(a) *R has weak comparability.*

(b) *$nA \prec nB$ implies $A \prec B$ for any finitely generated projective R-modules A, B and any positive integer n.*

2. Almost comparability

In Section 2, we give some fundamental results of almost comparability for finitely generated projective modules over regular rings satisfying almost comparability. We begin with the history for almost comparability. The notion of almost comparability for regular rings was first introduced by Ara and Goodearl [1], for giving an alternative proof of O'Meara's Theorem that every directly finite simple regular rings with weak comparability are unit-regular (Theorem 7). After that the study of almost comparability for simple regular rings was continued by Ara et al. [3], who showed that, for simple regular rings, s-comparability for some positive integer s is equivalent to the ring satisfying almost comparability. Thus directly finite simple regular rings satisfying almost comparability are unit-regular, from a result in O'Meara [11]. Also, Ara et al. [4] studied regular rings with s-comparability, and fixed the relation between s-comparability and almost comparability giving some examples. Here we give the definition of almost comparability, as follows.

Definition 9 ([1]). A regular ring R satisfies *almost comparability* if, for each $x, y \in R$, either $xR \lesssim_a yR$ or $yR \lesssim_a xR$, where $xR \lesssim_a yR$ (called "almost subisomorphic") means that $xR \lesssim yR \oplus C$ for all nonzero principal right ideals C of R.

From the definition of almost comparability, we see that "1-comparability $\Rightarrow$ almost comparability $\Rightarrow$ 2-comparability" obviously. Thus almost comparability is a middle condition between 1-comparability and 2-comparability. But the converse implications do not hold from the following examples.

Example 10 ([4]).
(1) There exists a non-simple stably finite regular ring satisfying almost comparability which is not unit-regular. Hence the ring does not satisfy 1-comparability.
(2) There exists a unit-regular ring with 2-comparability but not almost comparability.

Now we investigate the properties for regular rings satisfying almost comparability, comparing with 1-comparability or 2-comparability. First we ask if almost comparability for regular rings is Morita invariant. To see this, we need the definition of almost comparability for finitely generated projective modules, as follows.

Definition 11. Let R be a regular ring, and P be a finitely generated projective R-module. Then P satisfies *almost comparability* if, for each direct summands A, B of P, either $A \lesssim_a B$ or $B \lesssim_a A$, where $A \lesssim_a B$ means that $A \lesssim B \oplus C$ for all nonzero principal right ideals C of R. Also, P satisfies *strictly almost comparability* if, for each direct summands A, B of P, either $A \prec_a B$ or $B \prec_a A$, where $A \prec_a B$ means that $A \prec B \oplus C$ for all nonzero principal right ideals C of R.

For the above definitions, we can tell that the notion of almost comparability is the just same as one of strictly almost comparability as below, which result can be used as a criterion for almost comparability.

Lemma 12. *Let R be a regular ring, and P be a finitely generated projective R-module. Then the following are equivalent:*
(a) *P satisfies almost comparability.*

(b) *P satisfies strictly almost comparability.*

Lemma 13. *Let R be a regular ring satisfying almost comparability. For every nonzero finitely generated projective R-modules A, B, there exists a nonzero principal right ideal X of R such that both $X \lesssim A$ and $X \lesssim B$. In particular, if S is a simple right ideal of R, then $S \lesssim M$ for all nonzero finitely generated projective R-modules M.*

Here, we recall the definition of separativity for a ring and its criterion, which were born in the study of s-comparability and will be used in proofs of the results after.

Definition 14 ([2]). A ring R is *separative* if $A \oplus A \cong A \oplus B \cong B \oplus B$ implies $A \cong B$ for any finitely generated projective R-modules A, B.

Lemma 15 ([2]). *For a ring R, the following are equivalent*:
(a) *R is separative.*
(b) *For any finitely generated projective R-modules A, B, C, if $A \oplus C \cong B \oplus C$ with $C \lesssim_{\oplus} mA$ and $C \lesssim_{\oplus} nB$ for some positive integers m, n, then $A \cong B$.*

We also recall the definition of exchange rings.

Definition 16. A ring R is said to be *an exchange ring* if the R-module R satisfies the exchange property, where an R-module M *satisfies the exchange property* if for every R-module A and any decompositions $A = M' \oplus N = \oplus_{i \in I} A_i$ with $M' \cong M$, there exist submodules $A'_i \leq A_i$ such that $A = M' \oplus (\oplus_{i \in I} A'_i)$. It is known that regular rings are typical examples of exchange rings.

For exchange rings with s-comparability, we recall an interesting result as follows.

Lemma 17 ([12]). *Any exchange ring with s-comparability is separative. Thus any regular ring satisfying almost comparability is separative.*

Using Lemmas 12,13,15,17 above, we can obtain the following result.

Proposition 18. *Let R be a regular ring, and assume that nR satisfies almost comparability for some positive integer n. Let A, B, C, D be finitely generated projective R-modules, all which are subisomorphic to nR.*
(1) *If $A \prec_a C$ and $B \prec_a D$, then $A \oplus B \prec_a C \oplus D$.*
(2) *If $A \prec_a C$ and $B \prec_a D$, then either $A \oplus D \prec_a B \oplus C$ or $B \oplus C \prec_a A \oplus D$.*

Almost comparability is inherited by direct summands. Hence, using Proposition 18 above and the mathematical induction, we have the following result.

Proposition 19. *Let R be a regular ring. Then the following are equivalent*:
(a) *R satisfies almost comparability.*
(b) *Any finitely generated projective R-module satisfies almost comparability.*
(c) *nR satisfies almost comparability for all positive integers n.*
(d) *There exists a positive integer n such that nR satisfies almost comparability.*

By the way, we have almost subisomorphic relations of the family of all finitely generated submodules between a finitely generated projective R-module over a regular ring R and its endomorphism ring, as Lemma 20 below shows. To see this, for an R-module M_R, we put $add(M_R) = \{$an R-module $N \mid N \lesssim_{\oplus} nM$ for some positive integer $n\}$. Then we see that the lemma follows from equivalences of the Hom and Tensor functors by $Hom_R({}_SM_R, -)$ and $- \otimes_S {}_SM_R$ between the categories $add(M_R)$ and $add(S_S)$, where $S = End_R(M)$.

Lemma 20. *Let R be a regular ring, and P be a finitely generated projective R-module. Set a ring $T = End_R(P)$. Then the following are equivalent:*
(a) *P satisfies almost comparability.*
(b) *T satisfies almost comparability.*

Combining Proposition 19 with Lemma 20, we can answer whether almost comparability for regular rings is Morita invariant, as follows.

Theorem 21. *Let R be a regular ring. Then the following are equivalent*:
(a) *R satisfies almost comparability.*
(b) *For any finitely generated projective R-module P, $End_R(P)$ satisfies almost comparability.*
(c) *Any ring S which is Morita equivalent to R satisfies almost comparability.*
(d) *For all positive integers n, $M_n(R)$ satisfies almost comparability.*
(e) *There exists a positive integer n such that $M_n(R)$ satisfies almost comparability.*

Also we can show Theorem 22 below, from Proposition 19.

Theorem 22. *Let R be a regular ring satisfying almost comparability. Then the family of all finitely generated projective R-modules satisfies almost comparability, which means that either $A \prec_a B$ or $B \prec_a A$ for any finitely generated projective R-modules A, B.*

In addtion, more generally, we can extend almost comparability for the family of all finitely generated projectives to the family of all countably generated projectives, as follows.

Theorem 23. *Let R be a regular ring satisfying almost comparability. Then the family of all countably generated projective R-modules satisfies almost comparability, which means that either $P \prec_a Q$ or $Q \prec_a P$ for any countably generated projective R-modules P, Q.*

The results in Theorems 22,23 above can be used in §3.

3. Cancellation and unperforation

In Section 3, we treat the cancellation and unperforation properties for regular rings satisfying almost comparability. As we mentioned in §2, any directly finite simple regular rings satisfying almost comparability are unit-regular, but there exists a non-simple stably finite regular ring R satisfying almost comparability but not unit-regularity, from which $A \oplus C \lesssim B \oplus C$ does not imply $A \lesssim B$ for some finitely generated projective R-modules A, B, C. Thus, instead of the above property, we consider *the strict cancellation property* for a regular ring R which means that $A \oplus C \prec B \oplus C$ implies $A \prec B$ for any finitely generated projective R-modules A, B, C. Obviously, any unit-regular rings always have the strict cancellation property. First, we ask if any directly finite regular rings satisfying almost comparability have the strict cancellation property. Then we can show the following result.

Theorem 24. *Let R be a regular ring satisfying almost comparability, and A, B, C be directly finite and finitely generated projective R-modules. If $A \oplus C \prec B \oplus C$, then $A \prec B$.*

From the above, we have the following Corollary 25 as desired.

Corollary 25. *Let R be a directly finite regular ring satisfying almost comparability, and A, B, C be finitely generated projective R-modules. If $A \oplus C \prec B \oplus C$, then $A \prec B$.*

We note that Corollary 25 remembers the result in [8] that every directly finite regular ring with weak comparability has the strict cancellation property. By the way, we can give a more general result in Theorem 29 below, by using Theorem 24. To see this, we need the definition and a well-known result for stable range of a ring.

Definition 26. A row $(a_1, \ldots, a_r)$ of elements from a ring R is said to be *right unimodular* if $\sum_{i=1}^{r} a_i R = R$. Given a positive integer n, a ring R is said to *have n in the stable range* provided that for any right unimodular row $(a_1, \ldots, a_r)$ of $r \geq n+1$ elements of R, there exist elements $b_1, \ldots, b_{r-1} \in R$ such that the row $(a_1 + a_r b_1, \ldots, a_{r-1} + a_r b_{r-1})$ is right unimodular. If n is the least positive integer such that R has n in the stable range, then R is said to *have stable range n*. It is well-known that a regular ring R has stable range 1 if and only if R is unit-regular.

We notice that the stable range for a ring nearly relates with the cancellation property, as follows.

Lemma 27 ([13, 14])**.** *Let R be a ring, and A be an R-module such that $End_R(A)$ has n in the stable range for some positive integer n. If B and C are any R-modules such that $A \oplus B \cong A \oplus C$ and $nA \lesssim_{\oplus} B$, then $B \cong C$.*

Here we recall the following interesting result on the stable range for regular rings with 2-comparability.

Lemma 28 ([4])**.** *Let R be a regular ring with 2-comparability, and A be directly finite and finitely generated projective R-module. Then $End_R(A)$ has stable range at most 2.*

Using Theorem 24 and Lemmas 27,28 above, we can show one of main results, as follows.

Theorem 29. *Let R be a regular ring satisfying almost comparability. Let A, B be projective R-modules, and C be directly finite and finitely generated projective R-module. If $A \oplus C \prec_{\oplus} B \oplus C$, then $A \prec_{\oplus} B$.*

Next, we treat the following properties.

Definition 30. A ring R has *the unperforation property* provided that $nA \lesssim nB$ implies $A \lesssim B$ for any finitely generated projective R-modules A, B and any positive integer n. Also, a ring R has *the strict unperforation property* provided that $nA \prec nB$ implies $A \prec B$ for any finitely generated projective R-modules A, B and any positive integer n.

For the above properties, we can recall some interesting results. Goodearl [6] ensured the existence of a simple unit-regular ring R with weak comparability which does not have the unperforation property, and the ring R satisfies 2-comparability too (hence satisfies almost comparability). Thus, simple directly finite regular rings satisfying either almost comparability or weak comparability do not have the unperforation property in general.

On the other hand, it was shown in [9] that every regular ring with weak comparability has the strict unperforation property. But Ara et al. [4] showed that unit-regular rings with 2-comparability do not have the strict unperforation property in general. In spite of the above result, we can show that every regular ring satisfying almost comparability has the strict unperforation property, as follows.

Theorem 31. *Let R be a regular ring satisfying almost comparability, and A, B be finitely generated projective R-modules. If $nA \prec nB$ for some positive integer n, then $A \prec B$.*

Moreover, we can generalize Theorem 31 by using Lemmas below.

Lemma 32. *Let R be a regular ring satisfying almost comparability.*

(1) *Let X_1, X_2, X_3 be finitely generated projective R-modules. If $X_1 \prec_a X_2$, $X_2 \prec_a X_3$, then $X_1 \prec_a X_3$.*

(2) *Let $X_1, \cdots, X_n$ be finitely generated projective R-modules. Then there exists a positive integer k ($1 \leq k \leq n$) such that $X_i \prec_a X_k$ for all $i = 1, \ldots, n$.*

Lemma 33 ([7])**.** *Let R be a regular ring with 2-comparability. Then every directly finite projective R-module is countably generated, and every finite direct sum of directly finite projective R-modules is directly finite.*

Using Lemmas 32,33 and the strict cancellation property (Theorem 29) effectively, we have the following result.

Theorem 34. *Let R be a regular ring satisfying almost comparability, and A, B be projective R-modules such that A is either finitely generated or directly finite. If $nA \prec_\oplus nB$ for some positive integer n, then $A \prec B$.*

We also can show the following result, by using Theorem 34 above.

Theorem 35. *Let R be a regular ring satisfying almost comparability, and A, B be projective R-modules such that A is either finitely generated or directly finite. Then the following are equivalent*:

(a) *$A \prec_a B$.*

(b) *$nA \prec_a nB$ for some positive integer n.*

(c) *$nA \prec_a nB$ for all positive integers n.*

Finally, we inform some interesting problems concerned with the above results. By Corollary 25 and Theorem 31, every directly finite regular ring satisfying almost comparability has the strict cancellation property and every regular ring satisfying almost comparability has the strict unperforation property. Also, any regular rings with weak comparability have similar results, from talks after Corollary 25 and Definition 30. But, there exists a unit-regular ring with 2-comparability which does not have the strict unperforation property, from the talk before Theorem 31. Also we can construct a directly finite regular ring R which does not have the strict cancellation property. For example, we may take $R = S \times T$, where S, T are nonzero stably finite regular rings such that S is not unit-regular (see Example 10(1)). Thus we have the problems:

(A) Which directly finite regular rings have the strict cancellation property?

(B) Which regular rings have the strict unperforation property?

Acknowledgment. This work was supported by JSPS KAKENHI (21540041).

References

[1] P. Ara and K.R. Goodearl, *The almost isomorphism relation for simple regular rings*, Publ. Mat. UAB **36** (1992), 369–388.

[2] P. Ara, K.R. Goodearl, K.C. O'Meara and E. Pardo, *Separative cancellation for projective modules over exchange rings*, Israel J. Math. **105** (1998), 105–137.

[3] P. Ara, K.R. Goodearl, E. Pardo and D.V. Tyukavkin, *K-theoretically simple von Neumann regular rings*, J. Algebra **174** (1995), 659–677.

[4] P. Ara, K.C. O'Meara and D.V. Tyukavkin, *Cancellation of projective modules over regular rings with comparability*, J. Pure Appl. Algebra **107** (1996), 19–38.

[5] K.R. Goodearl, *Von Neumann Regular Rings*, Second Ed. Krieger, Malabar, Florida, 1991.

[6] K.R. Goodearl, *Torsion in* K_0 *of unit-regular rings*, Proc. Edinburgh Math. Soc. **38(2)** (1995), 331–341.

[7] M. Kutami, *On regular rings with* s*-comparability*, Comm. Algebra **27(6)** (1999), 2917–2933.

[8] M. Kutami, *On von Neumann regular rings with weak comparability*, J. Algebra **265** (2003), 285–298.

[9] M. Kutami, *On von Neumann regular rings with weak comparability* II, Comm. Algebra **33(9)** (2005), 3137–3147.

[10] M. Kutami, *On regular rings satisfying almost comparability*, Comm. Algebra **35(7)** (2007), 2171–2182.

[11] K.C. O'Meara, *Simple regular rings satisfying weak comparability*, J. Algebra **141** (1991), 162–186.

[12] E. Pardo, *Comparability, separability, and exchange rings*, Comm. Algebra **24(9)** (1996), 2915–2929.

[13] R.B. Warfield, Jr., *Notes on cancellation, stable range, and related topics*, Univ. of Washington (August 1975).

[14] R.B. Warfield, Jr., *Cancellation of modules and groups and stable range of endomorphism rings*, Pacific J. Math. **91** (1980), 457–485.

Department of Mathematics
Faculty of Science
Yamaguchi University
Yamaguchi 753-8512 JAPAN

E-mail address: kutami@yamaguchi-u.ac.jp

AMPLENESS OF TWO-SIDED TILTING COMPLEXES AND FANO ALGEBRAS

HIROYUKI MINAMOTO

Abstract. From the view point of noncommutative algebraic geometry (NCAG), a two-sided tilting complex is an analog of a line bundle. In this paper we define the notion of ampleness for two-sided tilting complexes over finite dimensional algebras of finite global dimension, and prove its basic properties, which justify the name "ampleness". From the view point of NCAG, Serre functors are considered to be shifted canonical bundles. A finite dimensional algebra A of finite global dimension is called Fano if the shifted Serre functor $A^*[-d]$ is anti-ample. Some classes of algebras studies before are Fano. We show by an example that the property of $A^*[-d]$ from the view point of NCAG captures some representation theoretic property of the algebra A.

From our view point, we give a structure theorem of AS-regular algebras. AS-regular algebras are defined to extract a good homological property of polynomial algebras. Our theorem shows that AS-regular algebra is polynomial algebra in some sense.

1. Introduction

The notion of two-sided tilting complex is introduced independently by Rouquier-Zimmermann [RZ] and Yekutieli [Y] based on the Rickard's derived Morita theory [Ric1]. Let A be a ring. A two-sided tilting complex σ is, by definition, the bounded above complex of A-bimodule such that the derived tensor product $-\otimes^{\mathbf{L}}_{A}\sigma$ gives an autoequivalence of the derived category $D(\text{Mod-}A)$. If algebra A is noetherian and has finite global dimension, then a complex σ of A-bimodules is a two-sided tilting complex if and only if $-\otimes^{\mathbf{L}}_{A}\sigma$ gives an autoequivalence of $D^b(\text{mod-}A)$.

$$-\otimes^{\mathbf{L}}_{A}\sigma : D^b(\text{mod-}A) \xrightarrow{\sim} D^b(\text{mod-}A).$$

From the view point of noncommutative algebraic geometry(NCAG), one thinks of a triangulated category $\mathcal{T}$ as the derived category $D^b(\operatorname{coh} X)$ of coherent sheaves of some "space" X. ¿From this view point, a two-sided tilting complex is an analog of a line bundle. In algebraic geometry, for line bundles ampleness is an important notion. In this paper we define the notion of ampleness of tilting complexes over finite dimensional k-algebras.

We justify this definition by using the theory of noncommutative projective schemes due to Artin-Zhang [AZ] and Polishchuk [Po]. In the theory of noncommutative projective schemes, for a graded coherent ring R over k, we attach an imaginary geometric object $\operatorname{proj} R = \big(\operatorname{cohproj} R, \overline{R}, (1)\big)$. An abelian category $\operatorname{cohproj} R$ is considered as the category of coherent sheaves on $\operatorname{proj} R$.

The final version of this paper will be submitted for publication elsewhere.

In [Le] and [GL], geometric notions are introduced to study certain class of algebras. These works are the inspiration for this work.

Notation and convention. Throughout this paper k denotes a field. For a ring A we denote by Mod-A (resp. mod-A) the abelian category of right A-modules (resp. the abelian category of finite right A-modules). For a k-vector space M, we denote by M^* its k-dual vector space. Let $D^b(\mathcal{A})$ be the derived category of an abelian category $\mathcal{A}$. We denote the standard t-structure in $D^b(\mathcal{A})$ by $(D^{\geq 0}(\mathcal{A}), D^{\leq 0}(\mathcal{A}))$, i.e., $D^{\geq 0}(\mathcal{A})$ (resp. $D^{\leq 0}(\mathcal{A})$) is the full subcategory of $D^b(\mathcal{A})$ with objects $\mathcal{F}^{\cdot}$ such that $\mathrm{H}^i(\mathcal{F}^{\cdot}) = 0$ for $i < 0$ (resp. $i > 0$). If there is no danger of confusion, we identify the two-side tilting complex σ with the autoequivalence $- \otimes^{\mathbf{L}}_A \sigma$. For example $\sigma M := M \otimes^{\mathbf{L}}_A \sigma$ for $M \in D(\text{Mod-}A)$ and $\sigma^n := \sigma \otimes^{\mathbf{L}}_A \cdots \otimes^{\mathbf{L}}_A \sigma$ (n times) for $n \in \mathbb{N}$.

2. Ampleness of two-sided tilting complexes

We start by reformulating the Serre's criteria of ampleness in the theory of derived categories. Let X be a variety over k and let $\mathcal{T} := D^b(\operatorname{coh} X)$ be the bounded derived category of coherent sheaves on X.

Definition 1. Let $\mathcal{L}$ be a line bundle on X. The full subcategory $\mathcal{T}^{\mathcal{L},\geq 0}$ (resp. $\mathcal{T}^{\mathcal{L},\leq 0}$) of $D^b(\operatorname{coh} X)$ consists of objects $\mathcal{F}^{\cdot}$ which satisfy

$$\mathbb{R}\operatorname{Hom}^{\cdot}(\mathcal{O}_X, \mathcal{F}^{\cdot} \otimes^{\mathbf{L}} \mathcal{L}^n) \in D^{\geq 0}(k\text{-}vect) \quad \text{for } n \gg 0$$
$$(\text{resp.} \quad \mathbb{R}\operatorname{Hom}^{\cdot}(\mathcal{O}_X, \mathcal{F}^{\cdot} \otimes^{\mathbf{L}} \mathcal{L}^n) \in D^{\leq 0}(k\text{-}vect). \quad \text{for } n \gg 0)$$

We define $\mathcal{T}^{\mathcal{L}} := (\mathcal{T}^{\mathcal{L},\geq 0}, \mathcal{T}^{\mathcal{L},\leq 0})$.

Theorem 2 (Serre's criteria of ampleness [Har, Propsition III.5.3])**.** *Suppose that X is proper. Then a line bundle $\mathcal{L}$ on X is ample if and only if $\mathcal{T}^{\mathcal{L}}$ is the standard t-structure in $D^b(\operatorname{coh} X)$.*

Reversing this observation, to formulate ampleness in the study of derived categories, we define the following.

Definition 3. Let A be a k-algebra and let σ be a two-sided tilting complex over A. The full subcategory $D^{\sigma,\geq 0}$ (resp. $D^{\sigma,\leq 0}$) of $D^b(\text{mod-}A)$ consists of objects $M^{\cdot}$ which satisfy

$$\sigma^n M \in D^{\geq 0}(\text{Mod-}A) \quad \text{for } n \gg 0$$
$$(\text{resp.} \quad \sigma^n M \in D^{\leq 0}(\text{Mod-}A) \quad \text{for } n \gg 0).$$

We define $D^{\sigma} := (D^{\sigma,\geq 0}, D^{\sigma,\leq 0})$.

Since $\sigma^n M \simeq \mathbb{R}\operatorname{Hom}(A, \sigma^n M)$, we think of A as the "structure sheaf" in Definition 3.

A two-sided tilting complex σ is called *pure* if $\mathrm{H}^i(\sigma) = 0$ for $i \neq 0$. We give the definition of ampleness of two-sided tilting complexes.

Definition 4. Let A be a finite dimensional k-algebra and let σ be a two-sided tilting complex over A.

(1) The two-sided tilting complex σ is called *ample* if σ^n is pure for $n \gg 0$ and D^{σ} is a t-structure in $D^b(\text{mod-}A)$.

(2) The two-sided tilting complex σ is called *very ample* if $\mathrm{H}^i(\sigma)=0$ for $i \geq 1$ and σ is ample.
(3) The two-sided tilting complex σ is called *extremely ample* if σ^n is pure for $n \geq 0$ and D^σ is a t-structure in $D^b(\text{mod-}A)$.

To give a justification of this terminology. we need a bit of notations. Let $S = S_0 \oplus S_1 \oplus S_2 \oplus \cdots$ be a $\mathbb{N}$-graded ring. We denote by $\mathrm{Gr}\, S$ the abelian category of graded right S-modules. An element x of a graded right S-module M is called a *torsion element* if $xS_{\geq n} = 0$ for some $n \in \mathbb{N}$. We define $\mathrm{Tor}\, S$ to be the full subcategory of $\mathrm{Gr}\, S$ consisting of those objects M such that each element $x \in M$ is a torsion element. Note that if S is finitely generated over S_0 then $\mathrm{Tor}\, S$ is a localizing subcategory of $\mathrm{Gr}\, S$. In the case when our graded ring S is finitely generated over S_0 we define $\mathrm{QGr}\, S$ to be the quotient category $\mathrm{Gr}\, S/\,\mathrm{Tor}\, S$.

Definition 5. A right (resp. left) graded S-module M called right (resp. left) coherent if it satisfies the following two conditions:
(a) M is finitely generated;
(b) for every homomorphism $f : \oplus_{i=1}^n S(s_i) \to M$ of right S-modules, the kernel $\ker(f)$ is finitely generated.

A graded ring S is called right (resp. left) coherent if S and $S/S_{\geq 1}$ are right (resp. left) graded coherent as a right (resp. left) graded S-module. A graded ring S is called coherent if S is both right and left coherent.

We denote by $\mathrm{coh}\, S$ the full subcategory of $\mathrm{Gr}\, S$ consisting of right coherent S modules. We define $\mathrm{tor}\, S$ to be the intersection between $\mathrm{Tor}\, S$ and $\mathrm{coh}\, S$. In the case when S is right coherent we define $\mathrm{cohproj}\, S$ to be the quotient category $\mathrm{coh}\, S/\,\mathrm{tor}\, S$.

The following is the one of our main theorem.

Theorem 6. *Let A be a finite dimensional algebra of finite global dimension. Let σ be a two-sided tilting complex over A such that $H^i(\sigma)=0$ for $i \geq 1$ and σ^n is pure for $n \gg 0$. Then the followings holds.*

(1) *There is the following equivalence of k-linear triangulated categories:*

$$D(\mathrm{QGr}\text{-}T) \simeq D(\mathrm{Mod}\text{-}A).$$

(2) *The following conditions are equivalent.*
  (a) *T is a right graded coherent algebra.*
  (b) *$\mathcal{D}^\sigma$ is a t-structure in $D^b(\text{mod-}A)$.*
(3) *If the conditions (a) or (b) holds, then there is the following equivalence of k-linear triangulated categories:*

$$D^b(\mathrm{cohproj}\, T) \simeq D^b(\text{mod-}A).$$

As a corollary we prove the following.

Corollary 7. *Let A be a finite dimensional algebra of finite global dimension and let σ be a very ample two-sided tilting complex. Then there is a natural equivalence of triangulated categories*

$$D^b\,(\text{mod-}A) \xrightarrow{\sim} D^b\,(\mathrm{cohproj}\, T)\,.$$

where $T := T_A(\mathrm{H}^0(\sigma))$ is the tensor algebra of $\mathrm{H}^0(\sigma)$ over A.

In [Be] Beilinson showed that $\mathbb{P}^n$ is derived equivalent to a finite dimensional algebra. This result has been generalized to other varieties. The above corollary gives a partial converse.

3. Fano algebras

Let A be a finite dimensional k-algebra of finite global dimension. The k-dual A^* has the natural A-bimodule structure. It is known that $- \otimes_A^{\mathbf{L}} A^* : D^b(\text{mod-}A) \longrightarrow D^b(\text{mod-}A)$ is the Serre functor ([Hap, I.4.6]). For a nonsingular projective variety X over k, the $[\dim X]$-shifted derived tensor $- \otimes_X^{\mathbf{L}} \omega_X[\dim X]$ of the canonical bundle ω_X is the Serre functor of $D^b(\operatorname{coh} X)$. ¿From a view point of noncommutative algebraic geometry A^* is thought as "shifted canonical bundle". For example, if $(A^*)^m \simeq [n]$ for some positive integers m, n, then A is called *fractional Calabi-Yau of CY dimension* $\dfrac{n}{m}$, which is named after analogy to the property of the derived category of a Calabi-Yau variety.

Definition 8. Let A be a finite dimensional k-algebra of finite global dimension, let d be a integer, and set $\omega := (A^*[-d])$. A is said to be a Fano algebra of Fano dimension d if the two-sided tilting complex ω^{-1} is ample. A is said to be an extremely Fano algebra of Fano dimension d if ω^{-1} is extremely ample.

Let X be a Fano variety or a variety with ample canonical bundle. Then the celebrated Bondal-Orlov's Theorem [BO, Theorem 3.1] state that the k-linear triangulated autoequivalence group (up to natural isomorphisms) is described by the term of algebraic geometry of X. A weaker version holds for Fano algebras and algebras with ample "canonical bundle".

Theorem 9. *Let A be a finite dimensional k-algebra of finite global dimension and let d a natural number. Set $\omega := A^*[-d]$. If the two-sided tilting complex ω or ω^{-1} is ample, then every k-linear triangulated autoequivalence F is standard, i.e. there exist a two-sided tilting complex σ such that there is a natural isomorphism of functors $F \simeq - \otimes_A^{\mathbf{L}} \sigma$.*

Remark 10. It is known that same property holds for hereditary algebras [MY, Theorem 1.8].

The following Theorem gives examples of Fano algebras.

Theorem 11. *Let A be a finite dimensional k-algebra of finite global dimension. Set $\omega := A^*[-1]$. If ω^n (resp. ω^{-n}) is pure for $n \gg 0$. Then D^{ω} (resp. $D^{\omega^{-1}}$) is a t-structure in $D(\text{mod-}A)$.*

4. A noncommutative algebro-geometric characterization of representation type of a quiver

Let Q be a connected finite acyclic quiver, i.e., a connected quiver with finitely many vertexes and finitely many arrows without loops and oriented cycles. Then the path algebra $A = kQ$ of Q is a finite dimensional k-algebra of global dimension 1. Set $\omega_Q := A^*[-1]$. Note that $- \otimes^{\mathbf{L}} \omega_Q^{-1}$ is the inverse of the Auslander-Reiten translation in Happel's derived version of Auslander-Reiten theory [Hap]. By [Hap, II.4.7] if the quiver Q has infinite representation type, then ω_Q^{-n} is pure for any $n \geq 0$. Therefor by theorem 11 we prove the following proposition.

Proposition 12. *Let Q be a connected finite acyclic quiver of infinite representation type. Then the path algebra kQ of Q is a Fano algebra of Fano dimension* 1.

If a finite acyclic quiver Q has finite representation type, then its path algebra kQ is fractional Calabi-Yau. (This fact has been known by specialists. See [MY] for the precise CY dimension of these algebras.)

Now we have the following characterization of representation type of quivers from the view point of noncommutative algebraic geometry.

Theorem 13. *A finite acyclic quiver has finite representation type if and only if its path algebra is fractional Calabi-Yau, and a finite acyclic quiver has infinite representation type if and only if its path algebra is Fano.*

Remark 14. For canonical algebras in the sense of Ringel [Rin] the same type characterization holds.

By Theorem 7 and Theorem 12 we obtain the following corollary.

Corollary 15. *Let Q be a finite acyclic quiver of infinite representation type. Then there is a natural equivalence of triangulated categories*

$$D^b(\text{mod-}kQ) \xrightarrow{\sim} D^b(\operatorname{cohproj}\Pi(Q))$$

where $\Pi(Q)$ is the preprojective algebra of Q.

Remark 16. The similar result is proved in [Le].

5. A structure of AS-regular algebras (joint work with I.Mori.)

Definition 17. A connected graded algebra R is called AS-regular if it has finite global dimension d and satisfies the following Gorenstein property:

$$\underline{\operatorname{Ext}}^q_{\operatorname{Gr} R}(k_R, R) \cong \begin{cases} k(e) \text{ for some } e \in \mathbb{Z} & \text{if } q = d \\ 0 & \text{otherwise} \end{cases}$$

The integer e is called Gorenstein parameter.

Remark 18. In some paper these algebras are called regular algebra. In Artin-Schelter's original definition [AS], (AS-)regular algebras are defined by three conditions: above two conditions and finiteness of Gelfand-Kirillov dimension.

We use the r-th quasi-Veronese algebra introduced by I.Mori.

Definition 19 ([Mo])**.** Let $r \geq 1$ be a natural number. The r-th quasi-Veronese algebra $R^{[r]}$ of R is a graded algebra defined by

$$R^{[r]} := \bigoplus_{n \geq 0} \begin{pmatrix} R_{nr} & R_{nr+1} & \cdots & R_{(n+1)r-1} \\ R_{nr-1} & R_{nr} & \cdots & R_{(n+1)r-2} \\ \cdots & \cdots & \cdots & \cdots \\ R_{(n-1)r+1} & R_{(n-1)r+2} & \cdots & R_{nr} \end{pmatrix}$$

with the multiplication defined as follows: for $(a_{i,j}) \in (R^{[r]})_p$, $(b_{i,j}) \in (R^{[r]})_q$ where $a_{i,j} \in R_{pr+j-i}$, $b_{i,j} \in R_{qr+j-i}$,

$$(a_{i,j})(b_{i,j}) := \left(\sum_{k=0}^{r-1} a_{k,j} b_{i,k} \right) \in (R^{[r]}).$$

We define F to be the degree 0 part $(R^{[e]})$ of e-th quasi Veronese algebra $R^{[e]}$. Note that F is a finite dimensional algebra of finite global dimension. Set $\omega := F^*[-(d-1)]$.

Theorem 20. (1) *ω^{-n} is pure for $n \geq 1$.*

(2) *Let T be the tensor algebra $T_F(\omega^{-1})$ of ω^{-1} over F. There is an automorphism ϕ of T as graded algebras. such that the e-th quasi Veronese algebra $R^{[e]}$ is isomorphic to the twisted algebra T^{ϕ} as graded algebras.*

$$R^{[e]} \cong T_F(\omega^{-1})^{\phi}$$

Artin and Schelter gave the definition of AS-regular algebras to extract good homological property of polynomial algebras. Our structure theorem shows that AS-regular algebras are polynomial algebra in some sense. The point is that we do not consider connected graded algebras over a field k, but consider connected graded algebras over a finite dimensional algebra F.

As an application we reprove the following statement.

Corollary 21 (Piontkovski [Pi]). *An AS-regular algebra R of global dimension 2 is coherent.*

Remark 22. This corollary is already proved by D. Piontkovski [Pi]. He used the description of AS-regular algebras of global dimension 2 obtained by J. J. Zhang [Z2].

In the case when $d = \mathrm{gl.dim}\, R = 0$ the above statement is trivial. Since AS-regular algebra of global dimension 1 is isomorphic to polynomial ring $k[x]$ in one variable, the case $d = 1$ is also trivial.

A.Bondal conjectured that all AS-regular algebras are coherent.

References

[AS] M. Artin, and W.F. Schelter, Graded algebras of global dimension 3, Adv. Math. **66** (1987), pp. 171-216.

[AZ] M. Artin, and J.J. Zhang, Noncommutative projective schemes, Adv. Math. **109** (1994), pp. 228-287.

[Be] A. A. Beilinson, Coherent sheaves on $\mathbb{P}^n$ and problems of linear algebras, Func. Anal. Appl. **12** (1978),pp. 214-216.

[BO] A. Bondal, and D. Orlov, Reconstruction of a variety from the derived category and group of autoequivalence, Compositio Math. **125** (2001), no. 3, pp. 327-344.

[GL] W. Geigle and H. Lenzing. A class of weighted projective curves arising in representation theory of finite dimensional algebras. In: Singularities, representation of algebras, and vector bundles. LNM 1273(1987) pp. 265-297.

[Hap] D. Happel, Triangulated Categories in the Representation Theory of Finite-Dimensional Algebras, London Math. Soc. Lecture Notes **119**, University Press, Cambridge, 1987.

[Har] R. Hartshorne, Algebraic Geometry, Graduated Texts in Mathematics, Springer-Verlag, 1977.

[Le] H. Lenzing, Curve singularities arising from the representation theory of tame hereditary algebras, Representation theory I (Ottawa,Ont.,1984), pp. 199-231, LNM.,1177, Springer,Berlin, 1986

[MM] H. Minamoto, and I. Mori, A structure of AS-regular algebras, preprint
[MY] J. Miyachi, and A. Yekutieli, Derived Picard groups of finite dimensional hereditary algebras, Compositio Math. **129** (2001), no. 3, pp. 341-368.
[Mo] I. Mori, B-construction and C-construction, preprint.
[Pi] D. Piontkovski, coherent algebras and noncommutative projective lines, J. Alg. 319 (2008), pp. 3280-3290.
[Po] A. Polishchuk, Nonncommutative proj and coherent algebras, Math. Res. Lett., **12** (2005),1, pp. 63-74.
[Ric1] J. Rickard, Morita theory for derived categories, J. London Math. Soc. (2) 39 (1989), no. 3, pp. 436-456.
[Ric2] J. Rickard, derived equivalence as derived functors, J. London Math. Soc. (2) 43 (1991), no. 1, pp. 37-48.
[Rin] C. M. Ringel. Tame Algebras and Integral Quadratic Forms. LNM. 1099 (1984).
[RZ] R. Rouquier and A. Zimmermann, Picard groups for derived module categories. Proc. London Math. Soc. (3) 87 (2003) no. 1, pp. 197-225
[Y] A. Yekutieli, Dualizing complexes, Morita equivalence and the derived Picard group of a ring, J. London Math Soc.**60** (1999), pp. 723-746.
[Z1] J. J. Zhang, Twisted graded algebras and equivalences of graded categories, Proc. London math. soc. (3)**72**(1996), no,2. pp. 281-311
[Z2] J. J. Zhang, Non-Noetherian regular rings of dimension 2, Proc. AMS, 126(1998), 6, pp. 1645-1653.

Department of Mathematics,
Kyoto University,
Kyoto 602-8502, JAPAN
E-mail address: minamoto@math.kyoto-u.ac.jp

STICKELBERGER RELATIONS AND LOEWY SERIES OF A GROUP ALGEBRA $\mathbf{Map}(\mathbb{F}_\mathbf{q}, \mathbb{F}_\mathbf{q})$

KAORU MOTOSE

Abstract. In this note, we present a proof of the Stickelberger relation (see [1]) using Loewy series of a group algebra $\mathrm{Map}(\mathbb{F}_q, \mathbb{F}_q)$ of the additive group of a finite field $\mathbb{F}_q$. This relation is essential in a proof of the Eisenstein reciprocity law. We also present partial solutions to the Feit-Thompson conjecture for primes 3 and 5 by a special case of this law.

Key Words : Gaussian sum, Power residue symbol, Feit-Thompson conjecture.

2000 *Mathematics Subject Classification* : Primary 11A15; Secondary 20D05.

§1. Loewy series of group algebras $\mathbf{Map}(\mathbb{F}_q, \mathbb{F}_q)$

Let $\mathbb{F} = \mathbb{F}_q$ be the finite field of order $q = p^f$, where p is a prime, and let $A = \mathrm{Map}(\mathbb{F}, K)$ be the set of mappings from $\mathbb{F}$ to a subring K of a field. We define a convolution product $*$ in A as follows,

$$(f * g)(\alpha) := \sum_{\alpha+\beta=\gamma} f(\alpha)g(\beta) \text{ for } f,\ g \in A \text{ and } \alpha, \beta, \gamma \in \mathbb{F}.$$

We say a character by a group homomorphism from the multiplicative group $\mathbb{F}^*$ to K. Let X be the set of characters. We define the trivial character ϵ by $\epsilon(\alpha) = 1$ for all $\alpha \in \mathbb{F}^*$. It is convenient to set $\epsilon(0) = 1$ and $\chi(0) = 0$ for $\chi \neq \epsilon$. In virtue of this definition, we can see X is contained in A. In case K is a field, X is a group by the usual product, namely, $(\lambda\mu)(\alpha) := \lambda(\alpha)\mu(\alpha)$. This group isomorphic to the group $\mathbb{F}^*$. Let u_α be the characteristic function of $\alpha \in \mathbb{F}_q$, namely,

$$u_\alpha(\beta) := \begin{cases} 1 & \beta = \alpha, \\ 0 & \beta \neq \alpha. \end{cases}$$

This definition shows $u_\alpha * u_\beta = u_{\alpha+\beta}$ and so the set $\{u_\alpha \mid \alpha \in \mathbb{F}\}$ is the additive group of $\mathbb{F}$. Moreover A is a group algebra of the additive group of $\mathbb{F}$ over K. It is easy to see that $\{u_\alpha \mid \alpha \in \mathbb{F}_q\}$ are linearly independent over K and

$$f = \sum_{\alpha \in \mathbb{F}_q} f(\alpha)u_\alpha \text{ for } f \in \mathrm{Map}(\mathbb{F}_q, K).$$

§1, §2 in this note is the detailed proof of Theorem 1 in the published paper [3]. The detailed version of **§3** in this note will be submitted for publication elsewhere.

Thus $\{u_\alpha \mid \alpha \in \mathbb{F}_q\}$ is a basis of A. In case $q-1 \neq 0$ in K, the set $\{u_0\} \cup X$ is also a basis of A because orthogonal relations shows

$$(q-1)u_\alpha = \sum_{\eta \in X} \eta(\alpha^{-1})\eta \text{ for } \alpha \neq 0 \text{ and } \chi = \sum_{\alpha \in \mathbb{F}} \chi(\alpha)u_\alpha.$$

In the remainder of this paper, we assume $K = \mathbb{F}_q$. We define Jacobi sums as follows

$$J_\alpha(\lambda, \mu) = \sum_{\beta+\gamma=\alpha} \lambda(\beta)\mu(\gamma) \text{ for } \lambda,\ \mu \in X \text{ and } \alpha, \beta, \gamma \in \mathbb{F}$$

and we set $J(\lambda, \mu) = J_1(\lambda, \mu)$.

Lemma 1. *We set $\lambda,\ \mu \in X$ and $\alpha \in \mathbb{F}$.*

(1) $J_\alpha(\epsilon, \epsilon) = 0$.
(2) $J_0(\lambda, \mu) = 0$ *for* $\lambda\mu \neq \epsilon$.
(3) $J_\alpha(\lambda, \mu) = \lambda\mu(\alpha)J(\lambda, \mu)$ *for* $\alpha \neq 0$.
(4) $J(\lambda, \lambda^{-1}) = J_0(\lambda, \lambda^{-1}) = -\lambda(-1)$ *for* $\lambda \neq \epsilon$.
(5) $J(\lambda, \mu)$ *is contained in the prime field* $\mathbb{F}_p$.
(6) $\lambda * \mu = J(\lambda, \mu)\lambda\mu$.

Proof. (1) $J_\alpha(\epsilon, \epsilon) = p^f = 0$.

(2) $J_0(\lambda, \mu) = \sum_{\beta \in \mathbb{F}^*} \lambda(\beta)\mu(-\beta) = \mu(-1)\sum_{\beta \in \mathbb{F}^*} \lambda\mu(\beta) = 0$.

(3) $J_\alpha(\lambda, \mu) = \lambda\mu(\alpha)\sum_{\beta+\gamma=\alpha} \lambda(\beta\alpha^{-1})\mu(\gamma\alpha^{-1}) = \lambda\mu(\alpha)J(\lambda, \mu)$.

(4) Using (3), we have

$$\begin{aligned} J_0(\lambda, \lambda^{-1}) - J(\lambda, \lambda^{-1}) &= J_0(\lambda, \lambda^{-1}) + (q-1)J(\lambda, \lambda^{-1}) = \sum_{\alpha \in \mathbb{F}} J_\alpha(\lambda, \lambda^{-1}) \\ &= (\sum_{\beta \in \mathbb{F}} \lambda(\beta))(\sum_{\gamma \in \mathbb{F}} \lambda^{-1}(\gamma)) = 0. \end{aligned}$$

Thus we have

$$\begin{aligned} J(\lambda, \lambda^{-1}) &= J_0(\lambda, \lambda^{-1}) = \sum_{\beta \in \mathbb{F}} \lambda(-\beta)\lambda^{-1}(\beta) = \sum_{\beta \in \mathbb{F}} \lambda(-\beta)\lambda^{-1}(\beta) \\ &= \lambda(-1) \cdot \sum_{\beta \in \mathbb{F}^*} \epsilon(\beta) = \lambda(-1)(q-1) = -\lambda(-1). \end{aligned}$$

(5) The assertion follows from the equation

$$J(\lambda, \mu)^p = \sum_{\beta \in \mathbb{F}} \lambda(\beta)^p \mu(1-\beta)^p = \sum_{\beta \in \mathbb{F}} \lambda(\beta^p)\mu(1-\beta^p) \sum_{\gamma \in \mathbb{F}} \lambda(\gamma)\mu(1-\gamma) = J(\lambda, \mu).$$

(6) We have $J_0(\lambda, \mu)u_0 - J(\lambda, \mu)\lambda\mu(0)u_0 = 0$ from (2) and (4). Thus using (3), we obtain our result.

$$\begin{aligned} \lambda * \mu &= \left(\sum_{\beta \in \mathbb{F}} \lambda(\beta)u_\beta\right)\left(\sum_{\gamma \in \mathbb{F}} \mu(\gamma)u_\gamma\right) = \sum_{\beta,\gamma \in \mathbb{F}} \lambda(\beta)\mu(\gamma)u_{\beta+\gamma} = \sum_{\alpha \in \mathbb{F}} J_\alpha(\lambda, \mu)u_\alpha \\ &= J(\lambda, \mu)\lambda\mu + J_0(\lambda, \mu)u_0 - J(\lambda, \mu)\lambda\mu(0)u_0 = J(\lambda, \mu)\lambda\mu. \end{aligned}$$

Lemma 2. *Let η be a generator of $\mathbb{F}^*$ and $\phi : \eta^k \to \eta^{-k}$ be a generator of X. We set integers $0 < s, t, m < n = q-1$ with $t = p^e$ and $tm \equiv s \bmod n$. Then $J(\phi^s,\ \phi^t) = -m-1$.*

Proof. Let L be a permutation on $B = \{1,\ \ldots,\ n-1\}$ such that $\eta^{L(k)} = 1-\eta^k$ and set $\theta = \eta^t$. Then the order of θ is n. We can easily verify the next equation from the formula of a geometric series.

$$\theta^{-\ell k} \cdot (1-\theta^k)^{-1} = \theta^{-k} + \theta^{-2k} + \cdots + \theta^{-\ell k} + (1-\theta^k)^{-1} \text{ for } k \in B.$$

The next equation follows from the above formula and t is a power of a prime p.

$$\begin{aligned}
J(\phi^s, \phi^t) &= \sum_{k=0}^{n-1} \phi^s(\eta^k)\phi^t(1-\eta^k) = \sum_{k=1}^{n-1} \phi(\eta^{ks} \cdot \eta^{L(k)t}) \\
&= \sum_{k=1}^{n-1} \eta^{-ks}\eta^{-L(k)t} = \sum_{k=1}^{n-1} \eta^{-tmk}(1-\eta^{kt})^{-1} \\
&= \sum_{k=1}^{n-1} \theta^{-mk}(1-\theta^k)^{-1} = \sum_{k=1}^{n-1} \left(\left(\sum_{\ell=1}^{m} \theta^{-\ell k} \right) + \eta^{-L(k)} \right) \\
&= \sum_{\ell=1}^{m} \left(\sum_{k=1}^{n-1} \theta^{-\ell k} \right) + \sum_{k=1}^{n-1} \eta^{-L(k)} = \sum_{\ell=1}^{m} (-1) + \sum_{k=1}^{n-1} \eta^k \\
&= -m-1
\end{aligned}$$

Proposition 3. $\mu_0^{[p-1]} * \mu_1^{[p-1]} * \cdots * \mu_{f-1}^{[p-1]} = \gamma\epsilon \neq 0$ *where* $\mu_k = \phi^{p^k},\ \gamma \in \mathbb{F}$ *and* $\chi^{[\ell]}$ *is the ℓth power by the product $*$.*

Proof. In virtue of Lemma 1 (6), the above product is equal to $\gamma\phi^{q-1} = \gamma\epsilon$ with $\gamma = \prod_{s,t} J(\phi^s, \phi^t)$ where $t = p^k$ for $k = 0, \cdots, f-1$ and $s = (\ell+1)t - 1$ for $\ell = 0, \ldots, p-2$ $((k,\ell) \neq (0,0))$. Thus it remains only to prove $J(\phi^s, \phi^t) \neq 0$. In fact, setting $0 < m = q - q/t + \ell < n = q-1$, It is easily seen that $tm \equiv s \mod n$ and $m \equiv \ell \bmod p$. It follows from Lemma 2 that $J(\phi^s, \phi^t) = -m-1 = -\ell-1 \neq 0$ since $0 < \ell+1 < p$.

§2. Stickelberger relations

Let m be a natural number. let p be a prime do not divide m, and let f be the order of $p \bmod m$. Moreover let D_m be the ring of algebraic integers in $\mathbb{Q}(\zeta_m)$ and let P be a prime ideal containing p, where $\zeta_m = e^{\frac{2\pi i}{m}}$. Then it is well known that q is the order of a finite field $\mathbb{F} = D_m/P$. We consider Gaussian sums $g(\chi^a) = \sum_{a\in\mathbb{F}} \chi^a(\alpha)\zeta_p^{\mathrm{tr}(\alpha)}$ where χ is a generator of X and $\mathrm{tr}(\alpha)$ is the trace of α. Let $\wp$ be the ideal generated by P and $\{1-\zeta_p^k | 0 < k < p\}$ in the ring of algebraic integers D_{mp} of $\mathbb{Q}(\zeta_{mp})$. It is easy to see $\wp$ is a prime ideal generated by P and $1-\zeta_p$. We set $a^* = b_0 + b_1 + \cdots + b_{f-1}$ for a positive integer $a = b_0 + b_1 p + \cdots + b_{f-1}p^{f-1}$.

Theorem 4. $\mathrm{ord}_\wp(g(\chi^a)) = a^*$ *for* $0 < a < q$, *namely,* $\wp^{a^*}$ *divides exactly* $g(\chi^a)$.

Proof. Let ν be a natural homomorphism:

$$\mathrm{Map}(\mathbb{F}, D_m) \to \mathrm{Map}(\mathbb{F}, D_m/P), \text{ where } D_m/P = \mathbb{F},$$

and $\Re$ be the ideal generated by P and $\{u_0 - u_\alpha | \alpha \in \mathbb{F}\}$. Since $\nu(\theta)^{[p]} = 0$ for $\epsilon \neq \theta \in X$, We obtain that $\nu(\theta)$ is contained in $\nu(\Re)$, the radical of the group algebra $\mathrm{Map}(\mathbb{F}, D_m/P)$ and so $\theta \in \Re$. By Proposition 3 together with this implies that $\gamma\chi^a \in \Re^{a^*}$ for the product of Jacobi sums $\gamma \in D_m \setminus P$. The character $u_\beta \to \zeta_p^{\mathrm{tr}(\beta)}$ induces the epimorphism

$$\phi : \mathrm{Map}(\mathbb{F}, D_m) \to D_{mp}$$

with $\phi(\Re) = \wp$ and $\phi(\gamma\chi^a) = \gamma g(\chi^a)$. Thus we have $\mathrm{ord}_\wp(g(\chi^a)) \geqq a^*$. On the other hand, using $\mathrm{ord}_\wp(p) = p - 1$ and $g(\chi^a)g(\chi^{q-1-a}) = g(\chi^a)g(\overline{\chi^a}) = \chi^a(-1)q = \chi^a(-1)p^f$, we have the next

$$\mathrm{ord}_\wp(g(\chi^a)) + \mathrm{ord}_\wp(g(\chi^{q-1-a})) = f(p-1) = a^* + (q-1-a)^*$$

This completes our proof.

From this theorem we have Stickelberger relation and Eisenstein reciprocity law by the same method in [1]. Let σ_t be an automoripism of $\mathbb{Q}(\zeta)$ for $0 < t < m$ and $(m, t) = 1$ such that $\sigma_t(\zeta_m) = \zeta_m^t$.

Theorem 5 (the Stickelberger relation)**.** $g(\chi)^m D_m = \prod_{\sigma_t} \sigma_t(P^t)$ *where* t *runs over* $0 < t < m$ *and* $(t, m) = 1$.

We set $\zeta_\ell = e^{\frac{2\pi i}{\ell}}$ for odd prime ℓ, $\theta_a = \left(\frac{-}{a}\right)_\ell$ is the ℓth power residue symbol and D_ℓ is the ring of algebraic integers in $\mathbb{Q}(\zeta_\ell)$. A non zero and non unit element $\alpha \in D_\ell$ is called primary if α is prime to ℓ and $\alpha \equiv c \bmod (1 - \zeta_\ell)^2$ for some $c \in \mathbb{Z}$.

Theorem 6 (the Eisenstein reciprocity law)**.** *Let* ℓ *be an odd prime,* $a \in \mathbb{Z}$ *and let* $\alpha \in D_\ell$ *be primary. Each pair of* ℓ, a *and* α *is coprime. Then* $\theta_a(\alpha) = \theta_\alpha(a)$.

§3. Partial solutions to the Feit Thompson conjecture for primes 3 and 5

We set $p < q$ are odd primes, and

$$F = \frac{q^p - 1}{q - 1} \text{ and } T = \frac{p^q - 1}{p - 1}.$$

Feit Thompson conjectured that F never divides T. If it would be proved, their odd paper would be greatly simplified (see [4]).

Lemma 7. *We set* $\chi_\eta = \left(\frac{-}{\eta}\right)_p$ *pth power residue symbol,* $\zeta = e^{\frac{2\pi i}{p}}$ *and* $c(q-1) \equiv 1 \bmod p$. *Then* $\eta = \zeta^c(\zeta - q)$ *is primary in the algebraic integer ring of* $\mathbb{Q}(\zeta)$.

(1) $\chi_\eta(1-\zeta)^{2(q-1)} = \chi_{q-1}(\zeta)^{q+1}$. *In particular,* $\chi_\eta(1-\zeta) = 1$ *if* p *divides* $q + 1$.
(2) *if* F *divides* T, *then* $\chi_\eta(p) = 1$ *and* $\chi_\eta(1-\zeta) = \chi_\eta(u)$ *where* $u = \prod_{k=1}^{p-1} \frac{1-\zeta^k}{1-\zeta}$. *In particular, if* p *divides* $q+1$, *then* $\chi_\eta(u) = 1$ *by* (1).

Using this lemma, we obtain

Corollary 8. *F never divides T in either case of the next conditions.*

(1) $p = 3$ *and* $q \not\equiv -1 \bmod 9$.
(2) $p = 5$ *and* $q + 1 = 5\ell$ *with* $(\ell, 5) = 1$.

References

[1] K. Ireland and M. Rosen. A classical introduction to modern number theory. Springer, 2nd ed., 1990.
[2] K. Motose. On Loewy series of group algebras of some solvable groups. J. Alg., 130 (1990), 261-272.
[3] K. Motose. On commutative group algebras. II, Math. J. Okayama Univ., 36 (1994), 23-27.
[4] K. Motose. Notes to the Feit-Thompson conjecture. Proc. Japan, Acad., ser A, 85(2009), 16-17.

Emeritus Professor of Hirosaki University
Toriage 5-13-5, Hirosaki, 036-8171, Japan
E-mail address: moka.mocha_no_kaori@snow.ocn.ne.jp

ON THE STRUCTURE OF SALLY MODULES OF RANK ONE

KAZUHO OZEKI

Abstract. A complete structure theorem of Sally modules of $\mathfrak{m}$-primary ideals I in a Cohen-Macaulay local ring $(A, \mathfrak{m})$ satisfying the equality $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 1$ is given, where $\mathrm{e}_0(I)$ and $\mathrm{e}_1(I)$ denote the first two Hilbert coefficients of I.

Key Words: commutative algebra, Cohen-Macaulay local ring, associated graded ring, Rees algebra, Hilbert coefficient.

2000 *Mathematics Subject Classification*: Primary 13H15, 13D40; Secondary 13H10, 13B22.

1. Introduction

This is based on a joint work with Shiro Goto and Koji Nishida.

Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring with $d = \dim A > 0$ and assume that the residue class field $k = A/\mathfrak{m}$ of A is infinite. Let I be an $\mathfrak{m}$-primary ideal in A and choose a minimal reduction $Q = (a_1, a_2, \cdots, a_d)$ of I. Let

$$\begin{aligned} R &= \mathrm{R}(I) := A[It] \subseteq A[t], \\ T &= \mathrm{R}(Q) := A[Qt], \\ R' &= \mathrm{R}'(I) := A[It, t^{-1}] \subseteq A[t, t^{-1}], \\ G &= \mathrm{G}(I) := R'/t^{-1}R' \cong \bigoplus_{n \geq 0} I^n/I^{n+1} \end{aligned}$$

denote, respectively, the Rees algebras of I and Q, the extended Rees algebras of I and the associated graded ring of I, where t stands for an indeterminate over A.

Let $B = T/\mathfrak{m}T \cong k[X_1, X_2, \cdots, X_d]$, which is the polynomial ring with d indeterminates over the field k. Following W. V. Vasconcelos [10], we then define

$$S = \mathrm{S}_Q(I) = IR/IT$$

and call it the Sally module of I with respect to Q. We notice that the Sally module S is a finitely generated graded T-module, since R is a module-finite extension of the graded ring T.

Let $\ell_A(*)$ stand for the length. Then we have integers $\{\mathrm{e}_i(I)\}_{0 \leq i \leq d}$ such that the equality

$$\ell_A(A/I^{n+1}) = \mathrm{e}_0(I)\binom{n+d}{d} - \mathrm{e}_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d \mathrm{e}_d(I)$$

holds true for all $n \gg 0$. For each integers $0 \leq i \leq d$, we call $e_i = \mathrm{e}_i(I)$ the i-th Hilbert coefficients of I.

The contents of this article are based on [1, 2]. Refer to them for the details.

The Sally module S was introduced by W. V. Vasconcelos [10], where he gave an elegant review, in terms of his *Sally* module, of the works [7, 8, 9] of J. Sally about the structure of $\mathfrak{m}$-primary ideals I with interaction to the structure of the graded ring G and the Hilbert coefficients e_i's of I.

As is well-known, we have the inequality ([5])

$$e_1 \geq e_0 - \ell_A(A/I)$$

and C. Huneke [3] showed that $e_1 = e_0 - \ell_A(A/I)$ if and only if $I^2 = QI$ (cf. Corollary 4). When this is the case, both the graded rings G and $\mathrm{F}(I) = \bigoplus_{n \geq 0} I^n/\mathfrak{m}I^n$ are Cohen-Macaulay, and the Rees algebra R of I is also a Cohen-Macaulay ring, provided $d \geq 2$. Thus, the ideals I with $e_1 = e_0 - \ell_A(A/I)$ enjoy very nice properties.

J. Sally firstly investigated the second border, that is the ideals I satisfying the equality $e_1 = e_0 - \ell_A(A/I) + 1$ but $e_2 \neq 0$ (cf. [9, 10]). The present research is a continuation of [9, 10] and aims to give a complete structure theorem of the Sally module of an $\mathfrak{m}$-primary ideal I satisfying the equality $e_1 = e_0 - \ell_A(A/I) + 1$.

The main result of this paper is the following Theorem 1. Our contribution in Theorem 1 is the implication (1) $\Rightarrow$ (3), the proof of which is based on the new result that the equality $I^3 = QI^2$ holds true if $e_1 = e_0 - \ell_A(A/I) + 1$ (cf. Theorem 7).

Theorem 1. *The following three conditions are equivalent to each other.*

(1) $e_1 = e_0 - \ell_A(A/I) + 1$.

(2) $\mathfrak{m}S = (0)$ *and* $\operatorname{rank}_B S = 1$.

(3) $S \cong (X_1, X_2, \cdots, X_c)B$ *as graded T-modules for some* $0 < c \leq d$, *where* $\{X_i\}_{1 \leq i \leq c}$ *are linearly independent linear forms of the polynomial ring B.*

When this is the case, $c = \ell_A(I^2/QI)$ *and* $I^3 = QI^2$, *and the following assertions hold true.*

(i) $\operatorname{depth} G \geq d - c$ *and* $\operatorname{depth}_T S = d - c + 1$.

(ii) $\operatorname{depth} G = d - c$, *if* $c \geq 2$.

(iii) *Suppose* $c < d$. *Then*

$$\ell_A(A/I^{n+1}) = e_0\binom{n+d}{d} - e_1\binom{n+d-1}{d-1} + \binom{n+d-c-1}{d-c-1}$$

for all $n \geq 0$. *Hence*

$$e_i = \begin{cases} 0 & \text{if } i \neq c+1, \\ (-1)^{c+1} & \text{if } i = c+1 \end{cases}$$

for $2 \leq i \leq d$.

(iv) *Suppose* $c = d$. *Then*

$$\ell_A(A/I^{n+1}) = e_0\binom{n+d}{d} - e_1\binom{n+d-1}{d-1}$$

for all $n \geq 1$. *Hence* $e_i = 0$ *for* $2 \leq i \leq d$.

Thus Theorem 1 settles a long standing problem, although the structure of ideals I with $e_1 = e_0 - \ell_A(A/I) + 2$ or the structure of Sally modules S with $\mathfrak{m}S = (0)$ and $\operatorname{rank}_B S = 2$ remains unknown.

Let us now briefly explain how this paper is organized. We shall prove Theorem 1 in Section 3. In Section 2 we will pick up from the paper [1] some auxiliary results on Sally modules, all of which are known, but let us note them for the sake of the reader's convenience. In Section 4 we will construct one example in order to see the ubiquity of ideals I which satisfy condition (3) in Theorem 1.

In what follows, unless otherwise specified, let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring with $d = \dim A > 0$. We assume that the field $k = A/\mathfrak{m}$ is infinite. Let I be an $\mathfrak{m}$-primary ideal in A and let S be the Sally module of I with respect to a minimal reduction $Q = (a_1, a_2, \cdots, a_d)$ of I. We put $R = A[It], T = A[Qt], R' = A[It, t^{-1}]$, and $G = R'/t^{-1}R'$. Let

$$\tilde{I} = \bigcup_{n \geq 1} [I^{n+1} :_A I^n] = \bigcup_{n \geq 1} [I^{n+1} :_A (a_1^n, a_2^n, \cdots, a_d^n)]$$

denote the Ratliff-Rush closure of I, which is the largest $\mathfrak{m}$-primary ideal in A such that $I \subseteq \tilde{I}$ and $\mathrm{e}_i(\tilde{I}) = e_i$ for all $0 \leq i \leq d$ (cf. [6]). We denote by $\mu_A(*)$ the number of generators.

2. Auxiliary results

In this section let us firstly summarize some known results on Sally modules, which we need throughout this paper. See [1] and [10] for the detailed proofs.

The first two results are basic facts on Sally modules developed by Vasconcelos [10].

Lemma 2. *The following assertions hold true.*

(1) $\mathfrak{m}^\ell S = (0)$ *for integers* $\ell \gg 0$.

(2) *The homogeneous components* $\{S_n\}_{n \in \mathbb{Z}}$ *of the graded* T*-module* S *are given by*

$$S_n \cong \begin{cases} (0) & \text{if } n \leq 0, \\ I^{n+1}/IQ^n & \text{if } n \geq 1. \end{cases}$$

(3) $S = (0)$ *if and only if* $I^2 = QI$.

(4) *Suppose that* $S \neq (0)$ *and put* $V = S/MS$, *where* $M = \mathfrak{m}T + T_+$ *is the graded maximal ideal in* T. *Let* V_n $(n \in \mathbb{Z})$ *denote the homogeneous component of the finite-dimensional graded* T/M*-space* V *with degree* n *and put* $\Lambda = \{n \in \mathbb{Z} \mid V_n \neq (0)\}$. *Let* $q = \max \Lambda$. *Then we have* $\Lambda = \{1, 2, \cdots, q\}$ *and* $\mathrm{r}_Q(I) = q + 1$, *where* $\mathrm{r}_Q(I) = \min\{n \in \mathbb{Z} \mid I^{n+1} = QI^n\}$ *stands for the reduction number of* I *with respect to* Q.

(5) $S = TS_1$ *if and only if* $I^3 = QI^2$.

Proof. See [1, Lemma 2.1]. □

Proposition 3. *Let* $\mathfrak{p} = \mathfrak{m}T$. *Then the following assertions hold true.*

(1) $\mathrm{Ass}_T S \subseteq \{\mathfrak{p}\}$. *Hence* $\dim_T S = d$, *if* $S \neq (0)$.

(2) $\ell_A(A/I^{n+1}) = e_0\binom{n+d}{d} - (e_0 - \ell_A(A/I)) \cdot \binom{n+d-1}{d-1} - \ell_A(S_n)$ *for all* $n \geq 0$.

(3) *We have* $e_1 = e_0 - \ell_A(A/I) + \ell_{T_\mathfrak{p}}(S_\mathfrak{p})$. *Hence* $e_1 = e_0 - \ell_A(A/I) + 1$ *if and only if* $\mathfrak{m}S = (0)$ *and* $\mathrm{rank}_B S = 1$.

(4) *Suppose that* $S \neq (0)$. *Let* $s = \mathrm{depth}_T S$. *Then* $\mathrm{depth}\, G = s - 1$ *if* $s < d$. S *is a Cohen-Macaulay* T*-module if and only if* $\mathrm{depth}\, G \geq d - 1$.

Proof. See [1, Proposition 2.2]. $\square$

Combining Lemma 2 (3) and Proposition 3, we readily get the following results of Northcott [5] and Huneke [3].

Corollary 4 ([3, 5])**.** *We have $e_1 \geq e_0 - \ell_A(A/I)$. The equality $e_1 = e_0 - \ell_A(A/I)$ holds true if and only if $I^2 = QI$. When this is the case, $e_i = 0$ for all $2 \leq i \leq d$.*

The following result is one of the keys for our proof of Theorem 1.

Theorem 5. *The following conditions are equivalent.*

(1) $e_1 = e_0 - \ell_A(A/I) + 1$.

(2) $S \cong \mathfrak{a}$ *as graded T-modules for some graded ideal $\mathfrak{a}$ $(\neq B)$ of B.*

Proof. We have only to show (1) $\Rightarrow$ (2). We have $\mathfrak{m}S = (0)$ and $\mathrm{rank}_B S = 1$ by Proposition 3 (3). Because $S_1 \neq (0)$ and $S = \sum_{n \geq 1} S_n$ by Lemma 2, we have $S \cong B(-1)$ as graded B-modules once S is B-free.

Suppose that S is not B-free. The B-module S is torsionfree, since $\mathrm{Ass}_T S = \{\mathfrak{m}T\}$ by Proposition 3 (1). Therefore, since $\mathrm{rank}_B\, S = 1$, we see $d \geq 2$ and $S \cong \mathfrak{a}(m)$ as graded B-modules for some integer m and some graded ideal $\mathfrak{a}$ $(\neq B)$ in B, so that we get the exact sequence

$$0 \to S(-m) \to B \to B/\mathfrak{a} \to 0$$

of graded B-modules. We may assume that $\mathrm{ht}_B\, \mathfrak{a} \geq 2$, since $B = k[X_1, X_2, \cdots, X_d]$ is the polynomial ring over the field $k = A/\mathfrak{m}$. We then have $m \geq 0$, since $\mathfrak{a}_{m+1} = [\mathfrak{a}(m)]_1 \cong S_1 \neq (0)$ and $\mathfrak{a}_0 = (0)$. We want to show $m = 0$.

Because $\dim B/\mathfrak{a} \leq d - 2$, the Hilbert polynomial of $B/\mathfrak{a}$ has degree at most $d - 3$. Hence

$$\begin{aligned} \ell_A(S_n) &= \ell_A(B_{m+n}) - \ell_A([B/\mathfrak{a}]_{m+n}) \\ &= \binom{m+n+d-1}{d-1} - \ell_A([B/\mathfrak{a}]_{m+n}) \\ &= \binom{n+d-1}{d-1} + m\binom{n+d-2}{d-2} + \text{(lower terms)} \end{aligned}$$

for $n \gg 0$. Consequently

$$\begin{aligned} \ell_A(A/I^{n+1}) &= e_0\binom{n+d}{d} - (e_0 - \ell_A(A/I))\cdot\binom{n+d-1}{d-1} - \ell_A(S_n) \\ &= e_0\binom{n+d}{d} - (e_0 - \ell_A(A/I) + 1)\cdot\binom{n+d-1}{d-1} - m\binom{n+d-2}{d-2} \\ &\quad + \text{(lower terms)} \end{aligned}$$

by Proposition 3 (2), so that we get $e_2 = -m$. Thus $m = 0$, because $e_2 \geq 0$ by Narita's theorem ([4]). $\square$

The following result will enable us to reduce the proof of Theorem 1 to the proof of the fact that $I^3 = QI^2$ if $e_1 = e_0 - \ell_A(A/I) + 1$.

Proposition 6. *Suppose $e_1 = e_0 - \ell_A(A/I) + 1$ and $I^3 = QI^2$. Let $c = \ell_A(I^2/QI)$. Then the following assertions hold true.*

(1) $0 < c \le d$ *and* $\mu_B(S) = c$.
(2) $\operatorname{depth} G \ge d - c$ *and* $\operatorname{depth}_B S = d - c + 1$.
(3) $\operatorname{depth} G = d - c$, *if* $c \ge 2$.
(4) *Suppose $c < d$. Then $\ell_A(A/I^{n+1}) = e_0\binom{n+d}{d} - e_1\binom{n+d-1}{d-1} + \binom{n+d-c-1}{d-c-1}$ for all $n \ge 0$. Hence*
$$e_i = \begin{cases} 0 & \text{if } i \neq c+1 \\ (-1)^{c+1} & \text{if } i = c+1 \end{cases}$$
for $2 \le i \le d$.
(5) *Suppose $c = d$. Then $\ell_A(A/I^{n+1}) = e_0\binom{n+d}{d} - e_1\binom{n+d-1}{d-1}$ for all $n \ge 1$. Hence $e_i = 0$ for $2 \le i \le d$.*

Proof. We have $\mathfrak{m}S = (0)$ and $\operatorname{rank}_B S = 1$ by Proposition 3 (3), while $S = TS_1$ since $I^3 = QI^2$ (cf. Lemma 2 (5)). Therefore by Theorem 5 we have $S \cong \mathfrak{a}$ as graded B-modules where $\mathfrak{a} = (X_1, X_2, \cdots, X_c)B$ is an ideal in B generated by linear forms $\{X_i\}_{1 \le i \le c}$. Hence $0 < c \le d$, $\mu_B(S) = c$, and $\operatorname{depth}_B S = d - c + 1$, so that assertions (1), (2), and (3) follow (cf. Proposition 3 (4)). Considering the exact sequence
$$0 \to S \to B \to B/\mathfrak{a} \to 0$$
of graded B-modules, we get
$$\begin{aligned} \ell_A(S_n) &= \ell_A(B_n) - \ell_A([B/\mathfrak{a}]_n) \\ &= \binom{n+d-1}{d-1} - \binom{n+d-c-1}{d-c-1} \end{aligned}$$
for all $n \ge 0$ (resp. $n \ge 1$), if $c < d$ (resp. $c = d$). Thus assertions (4) and (5) follow (cf. Proposition 3 (2)). $\square$

3. Proof of Theorem 1

The purpose of this section is to prove Theorem 1. See Proposition 3 (3) for the equivalence of conditions (1) and (2) in Theorem 1. The implication (3) ⇒ (2) is clear. So, we must show the implication (1) ⇒ (3) together with the last assertions in Theorem 1. Suppose that $e_1 = e_0 - \ell_A(A/I) + 1$. Then, thanks to Theorem 5, we get an isomorphism
$$\varphi : S \to \mathfrak{a}$$
of graded B-modules, where $\mathfrak{a} \subsetneq B$ is a graded ideal of B. Notice that once we are able to show $I^3 = QI^2$, the last assertions of Theorem 1 readily follow from Proposition 6. On the other hand, since $\mathfrak{a} \cong S = BS_1$ (cf. Lemma 2 (5)), the ideal $\mathfrak{a}$ of B is generated by linearly independent linear forms $\{X_i\}_{1 \le i \le c}$ $(0 < c \le d)$ of B and so, the implication (1) ⇒ (3) in Theorem 1 follows. We have $c = \ell_A(I^2/QI)$, because $\mathfrak{a}_1 \cong S_1 = I^2/QI$ (cf. Lemma 2 (2)). Thus our Theorem 1 has been proven modulo the following theorem.

Theorem 7. *Suppose that $e_1 = e_0 - \ell_A(A/I) + 1$. Then $I^3 = QI^2$.*

Proof. We proceed by induction on d. Suppose that $d = 1$. Then S is B-free of rank one (recall that the B-module S is torsionfree; cf. Proposition 3 (1)) and so, since $S_1 \neq (0)$ (cf. Lemma 2 (3)), $S \cong B(-1)$ as graded B-modules. Thus $I^3 = QI^2$ by Lemma 2 (5).

Let us assume that $d \geq 2$ and that our assertion holds true for $d-1$. Since the field $k = A/\mathfrak{m}$ is infinite, without loss of generality we may assume that a_1 is a superficial element of I. Let

$$\overline{A} = A/(a_1), \quad \overline{I} = I/(a_1), \text{ and } \quad \overline{Q} = Q/(a_1).$$

We then have $\mathrm{e}_i(\overline{I}) = e_i$ for all $0 \leq i \leq d-1$, whence

$$\mathrm{e}_1(\overline{I}) = \mathrm{e}_0(\overline{I}) - \ell_{\overline{A}}(\overline{A}/\overline{I}) + 1.$$

Therefore the hypothesis of induction on d yields $\overline{I}^3 = \overline{Q}\,\overline{I}^2$. Hence, because the element a_1t is a nonzerodivisor on G if $\operatorname{depth} G > 0$, we have $I^3 = QI^2$ in that case.

Assume that $\operatorname{depth} G = 0$. Then, thanks to Sally's technique ([9]), we also have $\operatorname{depth} \mathrm{G}(\overline{I}) = 0$. Hence $\ell_{\overline{A}}(\overline{I}^2/\overline{Q}\,\overline{I}) = d-1$ by Proposition 6 (2), because $\mathrm{e}_1(\overline{I}) = \mathrm{e}_0(\overline{I}) - \ell_{\overline{A}}(\overline{A}/\overline{I}) + 1$. Consequently, $\ell_A(S_1) = \ell_A(I^2/QI) \geq d-1$, because $\overline{I}^2/\overline{Q}\,\overline{I}$ is a homomorphic image of I^2/QI. Let us take an isomorphism

$$\varphi : S \to \mathfrak{a}$$

of graded B-modules, where $\mathfrak{a} \subsetneq B$ is a graded ideal of B. Then, since

$$\ell_A(\mathfrak{a}_1) = \ell_A(S_1) \geq d-1,$$

the ideal $\mathfrak{a}$ contains $d-1$ linearly independent linear forms, say $X_1, X_2, \cdots, X_{d-1}$ of B, which we enlarge to a basis $X_1, \cdots, X_{d-1}, X_d$ of B_1. Hence

$$B = k[X_1, X_2, \cdots, X_d],$$

so that the ideal $\mathfrak{a}/(X_1, X_2, \cdots, X_{d-1})B$ in the polynomial ring

$$B/(X_1, X_2, \cdots, X_{d-1})B = k[X_d]$$

is principal. If $\mathfrak{a} = (X_1, X_2, \cdots, X_{d-1})B$, then $I^3 = QI^2$ by Lemma 2 (5), since $S = BS_1$. However, because $\ell_A(I^2/QI) = \ell_A(\mathfrak{a}_1) = d-1$, we have $\operatorname{depth} G \geq 1$ by Proposition 6 (2), which is impossible. Therefore $\mathfrak{a}/(X_1, X_2, \cdots, X_{d-1})B \neq (0)$, so that we have

$$\mathfrak{a} = (X_1, X_2, \cdots, X_{d-1}, X_d^{\alpha})B$$

for some $\alpha \geq 1$. Notice that $\alpha = 1$ or $\alpha = 2$ by Lemma 2 (4). We must show that $\alpha = 1$.

Assume that $\alpha = 2$. Let us write, for each $1 \leq j \leq d$, $X_j = \overline{a_jt}$ with $a_j \in Q$, where $\overline{a_jt}$ denotes the image of $a_it \in T$ in $B = T/\mathfrak{m}T$. Then $\mathfrak{a} = (\overline{a_1t}, \overline{a_2t}, \cdots, \overline{a_{d-1}t}, \overline{(a_dt)^2})$. We now choose elements $f_i \in S_1$ for $1 \leq i \leq d-1$ and $f_d \in S_2$ so that $\varphi(f_i) = X_i$ for $1 \leq i \leq d-1$ and $\varphi(f_d) = X_d^2$. Let $z_i \in I^2$ for $1 \leq i \leq d-1$ and $z_d \in I^3$ such that $\{f_i\}_{1 \leq i \leq d-1}$ and f_d are, respectively, the images of $\{z_it\}_{1 \leq i \leq d-1}$ and z_dt^2 in S. We now consider the relations $X_if_1 = X_1f_i$ in S for $1 \leq i \leq d-1$ and $X_d^2f_1 = X_1f_d$, that is

$$a_iz_1 - a_1z_i \in Q^2I$$

for $1 \leq i \leq d-1$ and

$$a_d^2z_1 - a_1z_d \in Q^3I.$$

Notice that

$$Q^3 = a_1Q^2 + (a_2, a_3, \cdots, a_{d-1})^2{\cdot}(a_2, a_3, \cdots, a_d) + a_d^2Q$$

and write

$$a_d^2z_1 - a_1z_d = a_1\tau_1 + \tau_2 + a_d^2\tau_3$$

with $\tau_1 \in Q^2I$, $\tau_2 \in (a_2, a_3, \cdots, a_{d-1})^2{\cdot}(a_2, a_3, \cdots, a_d)I$, and $\tau_3 \in QI$. Then

$$a_d^2(z_1 - \tau_3) = a_1(\tau_1 + z_d) + \tau_2 \in (a_1) + (a_2, a_3, \cdots, a_{d-1})^2.$$

Hence $z_1 - \tau_3 \in (a_1) + (a_2, a_3, \cdots, a_{d-1})^2$, because the sequence $a_1, a_2, \cdots, a_d$ is A-regular. Let $z_1 - \tau_3 = a_1h + h'$ with $h \in A$ and $h' \in (a_2, a_3, \cdots, a_{d-1})^2$. Then since

$$a_1[a_d^2h - (\tau_1 + z_d)] = \tau_2 - a_d^2h' \in (a_2, a_3, \cdots, a_d)^3,$$

we have $a_d^2h - (\tau_1 + z_d) \in (a_2, a_3, \cdots, a_d)^3$, whence $a_d^2h \in I^3$.

We need the following.

Remark 8. $h \notin I$ but $h \in \tilde{I}$. Hence $\tilde{I} \neq I$.

Proof of Remark 8. If $h \in I$, then $a_1h \in QI$, so that $z_1 = a_1h + h' + \tau_3 \in QI$, whence $f_1 = 0$ in S (cf. Lemma 2 (2)), which is impossible. Let $1 \leq i \leq d-1$. Then

$$a_iz_1 - a_1z_i = a_i(a_1h + h' + \tau_3) - a_1z_i = a_1(a_ih - z_i) + a_i(h' + \tau_3) \in Q^2I.$$

Therefore, because $a_i(h' + \tau_3) \in Q^2I$, we get

$$a_1(a_ih - z_i) \in (a_1) \cap Q^2I.$$

Notice that

$$\begin{aligned} (a_1) \cap Q^2I &= (a_1) \cap [a_1QI + (a_2, a_3, \cdots, a_d)^2I] \\ &= a_1QI + [(a_1) \cap (a_2, a_3, \cdots, a_d)^2I] \\ &= a_1QI + a_1(a_2, a_3, \cdots, a_d)^2 \\ &= a_1QI \end{aligned}$$

and we have $a_ih - z_i \in QI$, whence $a_ih \in I^2$ for $1 \leq i \leq d-1$. Consequently $a_i^2h \in I^3$ for all $1 \leq i \leq d$, so that $h \in \tilde{I}$, whence $\tilde{I} \neq I$. $\square$

Because $\ell_A(\tilde{I}/I) \geq 1$, we have

$$\begin{aligned} e_1 &= e_0 - \ell_A(A/I) + 1 \\ &= \mathrm{e}_0(\tilde{I}) - \ell_A(A/\tilde{I}) + [1 - \ell_A(\tilde{I}/I)] \\ &\leq \mathrm{e}_0(\tilde{I}) - \ell_A(A/\tilde{I}) \\ &\leq \mathrm{e}_1(\tilde{I}) \\ &= e_1, \end{aligned}$$

where $\mathrm{e}_0(\tilde{I}) - \ell_A(A/\tilde{I}) \leq \mathrm{e}_1(\tilde{I})$ is the inequality of Northcott for the ideal $\tilde{I}$ (cf. Corollary 4). Hence $\ell_A(\tilde{I}/I) = 1$ and $\mathrm{e}_1(\tilde{I}) = \mathrm{e}_0(\tilde{I}) - \ell_A(A/\tilde{I})$, so that

$$\tilde{I} = I + (h) \quad \text{and} \quad \tilde{I}^2 = Q\tilde{I}$$

by Corollary 4 (recall that Q is a reduction of $\tilde{I}$ also). We then have, thanks to [2, Proposition 2.6], that $I^3 = QI^2$, which is a required contradiction. This completes the proof of Theorem 1 and that of Theorem 7 as well. $\square$

4. An example

Lastly we construct one example which satisfies condition (3) in Theorem 1. Our goal is the following. See [2, Section 5] for the detailed proofs.

Theorem 9. *Let $0 < c \le d$ be integers. Then there exists an $\mathfrak{m}$-primary ideal I in a Cohen-Macaulay local ring $(A, \mathfrak{m})$ such that*

$$d = \dim A, \quad \mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 1, \quad \text{and} \quad c = \ell_A(I^2/QI)$$

for some reduction $Q = (a_1, a_2, \cdots, a_d)$ of I.

To construct necessary examples we may assume that $c = d$.

Let m, $d > 0$ be integers. Let

$$U = k[\{X_j\}_{1 \le j \le m}, Y, \{V_i\}_{1 \le i \le d}, \{Z_i\}_{1 \le i \le d}]$$

be the polynomial ring with $m + 2d + 1$ indeterminates over an infinite field k and let

$$\begin{aligned} \mathfrak{b} \;=\; & [(X_j \mid 1 \le j \le m) + (Y)]\cdot[(X_j \mid 1 \le j \le m) + (Y) + (V_i \mid 1 \le i \le d)] \\ & +(V_iV_j \mid 1 \le i, j \le d,\, i \ne j) + (V_i^2 - Z_iY \mid 1 \le i \le d). \end{aligned}$$

We put $C = U/\mathfrak{b}$ and denote the images of X_j, Y, V_i, and Z_i in C by x_j, y, v_i, and a_i, respectively. Then $\dim C = d$, since $\sqrt{\mathfrak{b}} = (X_j \mid 1 \le j \le m) + (Y) + (V_i \mid 1 \le i \le d)$. Let $M = C_+ := (x_j \mid 1 \le j \le m) + (y) + (v_i \mid 1 \le i \le d) + (a_i \mid 1 \le i \le d)$ be the graded maximal ideal in C. Let Γ be a subset of $\{1, 2, \cdots, m\}$. We put

$$J = (a_i \mid 1 \le i \le d) + (x_\alpha \mid \alpha \in \Gamma) + (v_i \mid 1 \le i \le d) \quad \text{and} \quad \mathfrak{q} = (a_i \mid 1 \le i \le d).$$

Then $M^2 = \mathfrak{q}M$, $J^2 = \mathfrak{q}J + \mathfrak{q}y$, and $J^3 = \mathfrak{q}J^2$, whence $\mathfrak{q}$ is a reduction of both M and J, and $a_1, a_2, \cdots, a_d$ is a homogeneous system of parameters for the graded ring C.

Let $A = C_M$, $I = JA$, and $Q = \mathfrak{q}A$. We are now interested in the Hilbert coefficients $e_i's$ of the ideal I as well as the structure of the associated graded ring and the Sally module of I. We then have the following, which shows that the ideal I is a required example.

Theorem 10. *The following assertions hold true.*

(1) *A is a Cohen-Macaulay local ring with $\dim A = d$.*
(2) *$S \cong B_+$ as graded T-modules, whence $\ell_A(I^2/QI) = d$.*
(3) *$\mathrm{e}_0(I) = m + d + 2$ and $\mathrm{e}_1(I) = \sharp\Gamma + d + 1$.*
(4) *$\mathrm{e}_i(I) = 0$ for all $2 \le i \le d$.*
(5) *G is a Buchsbaum ring with $\operatorname{depth} G = 0$ and $\mathbb{I}(G) = d$.*

Proof. See [2, Theorem 5.2]. □

References

[1] S. Goto, K. Nishida, and K. Ozeki, *Sally modules of rank one*, Michigan Math. J., **57**, 2008, 359–381.
[2] S. Goto, K. Nishida, and K. Ozeki, *The structure of Sally modules of rank one*, Math. Les. Lett., **15**, 2008, 881–892.
[3] C. Huneke, *Hilbert functions and symbolic powers*, Michigan Math. J., **34**, 1987, 293–318.
[4] M. Narita, *A note on the coefficients of Hilbert characteristic functions in semi-regular rings*, Proc. Cambridge Philos. Soc., **59**, 1963, 269–275.
[5] D. G. Northcott, *A note on the coefficients of the abstract Hilbert function*, J. London Math. Soc., **35**, 1960, 209–214.

[6] L. J. Ratliff and D. Rush, *Two notes on reductions of ideals*, Indiana Univ. Math. J., **27**, 1978, 929-934.
[7] J. D. Sally, *Cohen-Macaulay local rings of maximal embedding dimension*, J. Algebra, **56**, 1979, 168–183.
[8] J. D. Sally, *Tangent cones at Gorenstein singularities*, Composito Math., **40**, 1980, 167–175.
[9] J. D. Sally, *Hilbert coefficients and reduction number 2*, J. Alg. Geo. and Sing., **1**, 1992, 325–333.
[10] W. V. Vasconcelos, *Hilbert Functions, Analytic Spread, and Koszul Homology*, Contemporary Mathematics, **159**, 1994, 410–422.

Meiji Institute for Advanced Study of Mathematical Sciences
Meiji University
1-1-1 Higashi-mita, Tama-ku, Kawasaki 214-8571, Japan
Email: kozeki@math.meiji.ac.jp

THICK SUBCATEGORIES OF THE STABLE CATEGORY OF COHEN-MACAULAY MODULES

RYO TAKAHASHI

Abstract. Various classification theorems of thick subcategories of a triangulated category have been obtained in many areas of mathematics. In this article, as a higher dimensional version of the classification theorem of thick subcategories of the stable category of finitely generated representations of a finite p-group due to Benson, Carlson and Rickard, we consider classifying thick subcategories of the stable category of Cohen-Macaulay modules over a Gorenstein local ring. The main result of this article yields a complete classification of the thick subcategories of the stable category of Cohen-Macaulay modules over a local hypersurface in terms of specialization-closed subsets of the prime ideal spectrum of the ring which are contained in its singular locus.

One of the principal approaches to the understanding of the structure of a given category is classifying its subcategories having a specific property. It has been studied in many areas of mathematics which include stable homotopy theory, ring theory, algebraic geometry and modular representation theory. A landmark result in this context was obtained in the definitive work due to Gabriel [21] in the early 1960s. He proved a classification theorem of the localizing subcategories of the category of modules over a commutative noetherian ring by making a one-to-one correspondence between the set of those subcategories and the set of specialization-closed subsets of the prime ideal spectrum of the ring. A lot of analogous classification results of subcategories of modules have been obtained by many authors; see [29, 39, 36, 22, 23, 24] for instance.

For a triangulated category, a high emphasis has been placed on classifying its *thick* subcategories, namely, full triangulated subcategories which are closed under taking direct summands. The first classification theorem was obtained in the deep work on stable homotopy theory due to Devinatz, Hopkins and Smith [18, 28]. They classified the thick subcategories of the category of compact objects in the p-local stable homotopy category. Hopkins [27] and Neeman [38] provided a corresponding classification result of the thick subcategories of the derived category of perfect complexes (i.e., bounded complexes of finitely generated projective modules) over a commutative noetherian ring by making a one-to-one correspondence between the set of those subcategories and the set of specialization-closed subsets of the prime ideal spectrum of the ring. Thomason [43] generalized the theorem of Hopkins and Neeman to quasi-compact and quasi-separated schemes, in particular, to arbitrary commutative rings and algebraic varieties. Recently, Avramov, Buchweitz, Christensen, Iyengar and Piepmeyer [4] gave a classification of the thick subcategories of the derived category of perfect differential modules over a commutative noetherian ring. On the other hand, Benson, Carlson and Rickard [9] classified the thick subcategories of the stable category of finitely generated representations of a finite p-group in terms of closed homogeneous subvarieties of the maximal ideal spectrum

The detailed version of this paper has been submitted for publication elsewhere.

of the group cohomology ring. Friedlander and Pevtsova [20] extended this classification theorem to finite group schemes. A recent work of Benson, Iyengar and Krause [11] gives a new proof of the theorem of Benson, Carlson and Rickard. A lot of other related results concerning thick subcategories of a triangulated category have been obtained. For example, see [6, 7, 8, 35, 14, 10, 31, 12, 19, 41].

Here we mention that in most of the classification theorems of subcategories stated above, the subcategories are classified in terms of certain sets of prime ideals. Each of them establishes an assignment corresponding each subcatgory to a set of prime ideals, which is (or should be) called the *support* of the subcategory.

In the present article, as a higher dimensional version of the work of Benson, Carlson and Rickard, we consider classifying thick subcategories of the stable category of Cohen-Macaulay modules over a Gorenstein local ring, through defining a suitable support for those subcategories. Over a hypersurface we shall give a complete classification of them in terms of specialization-closed subsets of the prime ideal spectrum of the base ring contained in its singular locus.

Convention. In the rest of this article, we assume that all rings are commutative and noetherian, and that all modules are finitely generated. Unless otherwise specified, let R be a local ring of Krull dimension d. The unique maximal ideal of R and the residue field of R are denoted by $\mathfrak{m}$ and k, respectively. By a *subcategory*, we always mean a full subcategory which is closed under isomorphism. (A full subcategory $\mathcal{X}$ of a category $\mathcal{C}$ is said to be closed under isomorphism provided that for two objects M, N of $\mathcal{C}$ if M belongs to $\mathcal{X}$ and N is isomorphic to M in $\mathcal{C}$, then N also belongs to $\mathcal{X}$.) Note that a subcategory in our sense is uniquely determined by the isomorphism classes of the objects in it.

We begin with recalling the definition of the syzygies of a module.

Definition 1. Let n be a nonnegative integer, and let M be an R-module. Let

$$\cdots \xrightarrow{\partial_{n+1}} F_n \xrightarrow{\partial_n} F_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \to M \to 0$$

be a minimal free resolution of M. The n^{th} *syzygy* of M is defined as the image of the map ∂_n, and we denote it by $\Omega^n M$. We simply write ΩM instead of $\Omega^1 M$. Note that the n^{th} syzygy of a given R-module is uniquely determined up to isomorphism because so is a minimal free resolution.

Next, we make a list of several closed properties of a subcategory.

Definition 2. (1) Let $\mathcal{C}$ be an additive category and $\mathcal{X}$ a subcategory of $\mathcal{C}$.

(i) We say that $\mathcal{X}$ is *closed under (finite) direct sums* provided that if $M_1, \dots, M_n$ are objects of $\mathcal{X}$, then the direct sum $M_1 \oplus \cdots \oplus M_n$ in $\mathcal{C}$ belongs to $\mathcal{X}$.

(ii) We say that $\mathcal{X}$ is *closed under direct summands* provided that if M is an object of $\mathcal{X}$ and N is a direct summand of M in $\mathcal{C}$, then N belongs to $\mathcal{X}$.

(2) Let $\mathcal{C}$ be a triangulated category and $\mathcal{X}$ a subcategory of $\mathcal{C}$. We say that $\mathcal{X}$ is *closed under triangles* provided that for each exact triangle $L \to M \to N \to \Sigma L$ in $\mathcal{C}$, if two of L, M, N belong to $\mathcal{X}$, then so does the third.

(3) We denote by $\operatorname{mod} R$ the category of finitely generated R-modules. Let $\mathcal{X}$ be a subcategory of $\operatorname{mod} R$.

(i) We say that $\mathcal{X}$ is *closed under extensions* provided that for each exact sequence $0 \to L \to M \to N \to 0$ in $\operatorname{mod} R$, if L and N belong to $\mathcal{X}$, then so does M.

(ii) We say that $\mathcal{X}$ is *closed under kernels of epimorphisms* provided that for each exact sequence $0 \to L \to M \to N \to 0$ in $\operatorname{mod} R$, if M and N belong to $\mathcal{X}$, then so does L.

(iii) We say that $\mathcal{X}$ is *closed under syzygies* provided that if M is an R-module in $\mathcal{X}$, then $\Omega^i M$ is also in $\mathcal{X}$ for all $i \geq 0$.

Let us make several definitions of subcategories.

Definition 3. (1) Let $\mathcal{C}$ be a category.

(i) We call the subcategory of $\mathcal{C}$ which has no object the *empty subcategory* of $\mathcal{C}$.

(ii) Suppose that $\mathcal{C}$ admits the zero object 0. We call the subcategory of $\mathcal{C}$ consisting of all objects that are isomorphic to 0 the *zero subcategory* of $\mathcal{C}$.

(2) A subcategory of a triangulated category is called *thick* if it is closed under direct summands and triangles.

(3) A subcategory of $\operatorname{mod} R$ is called *resolving* if it contains R and if it is closed under direct summands, extensions and kernels of epimorphisms.

Remark 4. (1) A resolving subcategory is a subcategory such that any two "minimal" resolutions of a module by modules in it have the same length; see [1, Lemma (3.12)].

(2) Every resolving subcategory of $\operatorname{mod} R$ contains all free R-modules.

(3) A subcategory of $\operatorname{mod} R$ is resolving if and only if it contains R and is closed under direct summands, extensions and syzygies.

The notion of a resolving subcategory was introduced by Auslander and Bridger [1] in the late 1960s. A lot of important subcategories of $\operatorname{mod} R$ are known to be resolving. To present examples of a resolving subcategory, let us recall here several definitions of modules. Let M be an R-module. We say that M is *bounded* if there exists an integer s such that $\beta_i^R(M) \leq s$ for all $i \geq 0$, where $\beta_i^R(M)$ denotes the i^{th} Betti number of M. We say that M has *complexity* c if c is the least nonnegative integer n such that there exists a real number r satisfying the inequality $\beta_i^R(M) \leq ri^{n-1}$ for $i \gg 0$. We call M *semidualizing* if the natural homomorphism $R \to \operatorname{Hom}_R(M, M)$ is an isomorphism and $\operatorname{Ext}_R^i(M, M) = 0$ for all $i > 0$. For a semidualizing R-module C, an R-module M is called *totally C-reflexive* if the natural homomorphism $M \to \operatorname{Hom}_R(\operatorname{Hom}_R(M, C), C)$ is an isomorphism and $\operatorname{Ext}_R^i(M, C) = 0 = \operatorname{Ext}_R^i(\operatorname{Hom}_R(M, C), C)$ for all $i > 0$. A totally R-reflexive R-module is simply called a *totally reflexive* R-module. For an ideal I of R we denote by $\operatorname{grade}(I, M)$ the infimum of the integers i with $\operatorname{Ext}_R^i(R/I, M) \neq 0$. We say that M has *lower complete intersection zero* if M is totally reflexive and has finite complexity. When R is a Cohen-Macaulay local ring, we say that M is *Cohen-Macaulay* if $\operatorname{depth} M = d$. Such a module is usually called maximal Cohen-Macaulay, but in this article, we call it just Cohen-Macaulay. We denote by $\operatorname{CM}(R)$ the subcategory of $\operatorname{mod} R$ consisting of all Cohen-Macaulay R-modules.

Example 5. Let n be a nonnegative integer, K an R-module, and I an ideal of R. The following R-modules form resolving subcategories of $\operatorname{mod} R$.

(1) The R-modules.
(2) The free R-modules.
(3) The Cohen-Macaulay R-modules, provided that R is Cohen-Macaulay.
(4) The totally C-reflexive R-modules, where C is a fixed semidualizing R-module.
(5) The R-modules M with $\operatorname{Tor}_i^R(M,K)=0$ for $i>n$ (respectively, $i \gg 0$).
(6) The R-modules M with $\operatorname{Ext}_R^i(M,K)=0$ for $i>n$ (respectively, $i \gg 0$).
(7) The R-modules M with $\operatorname{Ext}_R^i(K,M)=0$ for $i \gg 0$, provided that $\operatorname{Ext}_R^j(K,R)=0$ for $j \gg 0$.
(8) The R-modules M with $\operatorname{Ext}_R^i(K,M)=0$ for $i < \operatorname{grade} K (:= \operatorname{grade}(\operatorname{Ann} K, R))$.
(9) The R-modules M with $\operatorname{grade}(I,M) \geq \operatorname{grade}(I,R)$.
(10) The bounded R-modules.
(11) The R-modules having finite complexity.
(12) The R-modules of lower complete intersection dimension zero.

Next we recall the definitions of the nonfree loci of an R-module and a subcategory of $\operatorname{mod} R$.

Definition 6. (1) We denote by $\mathcal{V}(X)$ the *nonfree locus* of an R-module X, namely, the set of prime ideals $\mathfrak{p}$ of R such that $X_{\mathfrak{p}}$ is nonfree as an $R_{\mathfrak{p}}$-module.
(2) We denote by $\mathcal{V}(\mathcal{X})$ the *nonfree locus* of a subcategory $\mathcal{X}$ of $\operatorname{mod} R$, namely, the union of $\mathcal{V}(X)$ where X runs through all nonisomorphic R-modules in $\mathcal{X}$.

We denote by $\operatorname{Sing} R$ the *singular locus* of R, namely, the set of prime ideals $\mathfrak{p}$ of R such that $R_{\mathfrak{p}}$ is not a regular local ring. We denote by $\mathcal{S}(R)$ the set of prime ideals $\mathfrak{p}$ of R such that the local ring $R_{\mathfrak{p}}$ is not a field. Clearly, $\mathcal{S}(R)$ contains $\operatorname{Sing} R$. For each ideal I of R, we denote by $\mathrm{V}(I)$ the set of prime ideals of R containing I. Recall that a subset Z of $\operatorname{Spec} R$ is called *specialization-closed* provided that if $\mathfrak{p} \in Z$ and $\mathfrak{q} \in \mathrm{V}(\mathfrak{p})$ then $\mathfrak{q} \in Z$. Note that every closed subset of $\operatorname{Spec} R$ is specialization-closed. Let $\mathcal{C}$ be a category, and let $\mathbf{P}$ be a property of subcategories of $\mathcal{C}$. Let $\mathcal{X}$ be a subcategory of $\mathcal{C}$. A subcategory $\mathcal{Y}$ of $\mathcal{C}$ satisfying $\mathbf{P}$ is said to be *generated by* $\mathcal{X}$ if $\mathcal{Y}$ is the smallest subcategory of $\mathcal{C}$ satisfying $\mathbf{P}$ that contains $\mathcal{X}$. For a subset Φ of $\operatorname{Spec} R$, we denote by $\mathcal{V}^{-1}(\Phi)$ the subcategory of $\operatorname{mod} R$ consisting of all R-modules M such that $\mathcal{V}(M)$ is contained in Φ. The proposition below gives several basic properties of nonfree loci.

Proposition 7. (1) *Let R be a Cohen-Macaulay local ring. Then the nonfree locus $\mathcal{V}(\mathrm{CM}(R))$ coincides with the singular locus* $\operatorname{Sing} R$.
(2) *One has $\mathcal{V}(X) = \operatorname{Supp} \operatorname{Ext}^1(X, \Omega X)$ for every R-module X. In particular, the nonfree locus of an R-module is closed in $\operatorname{Spec} R$ in the Zariski topology. The nonfree locus of a subcategory of $\operatorname{mod} R$ is not necessarily closed but at least specialization-closed in $\operatorname{Spec} R$, and is contained in $\mathcal{S}(R)$.*
(3) *One has $\mathcal{V}(\mathcal{X}) = \mathcal{V}(\operatorname{res} \mathcal{X})$ for every subcategory $\mathcal{X}$ of $\operatorname{mod} R$, where $\operatorname{res} \mathcal{X}$ denotes the resolving subcategory of $\operatorname{mod} R$ generated by $\mathcal{X}$.*
(4) *Let N be a direct summand of an R-module M. Then one has $\mathcal{V}(N) \subseteq \mathcal{V}(M)$.*
(5) *Let $0 \to L \to M \to N \to 0$ be an exact sequence of R-modules. Then one has $\mathcal{V}(L) \subseteq \mathcal{V}(M) \cup \mathcal{V}(N)$ and $\mathcal{V}(M) \subseteq \mathcal{V}(L) \cup \mathcal{V}(N)$.*
(6) *For a subset Φ of $\operatorname{Spec} R$, the subcategory $\mathcal{V}^{-1}(\Phi)$ of $\operatorname{mod} R$ is resolving.*
(7) *For an ideal I of R, one has $\mathcal{V}_R(R/I) = \mathrm{V}(I + (0 : I))$.*

(8) *One has* $\mathcal{V}_R(R/\mathfrak{p}) = \mathrm{V}(\mathfrak{p})$ *for every* $\mathfrak{p} \in \mathcal{S}(R)$.
(9) *Let* Φ *be a specialization-closed subset of* $\operatorname{Spec} R$ *contained in* $\mathcal{S}(R)$. *Then one has* $R/\mathfrak{p} \in \mathcal{V}^{-1}(\Phi)$ *for every* $\mathfrak{p} \in \Phi$.

We recall the definition of the stable category of Cohen-Macaulay modules over a Cohen-Macaulay local ring.

Definition 8. (1) Let M, N be R-modules. We denote by $\mathcal{F}_R(M,N)$ the set of R-homomorphisms $M \to N$ factoring through free R-modules. It is easy to observe that $\mathcal{F}_R(M,N)$ is an R-submodule of $\mathrm{Hom}_R(M,N)$. We set $\underline{\mathrm{Hom}}_R(M,N) = \mathrm{Hom}_R(M,N)/\mathcal{F}_R(M,N)$.
(2) Let R be a Cohen-Macaulay local ring. The *stable category* of $\mathrm{CM}(R)$, which is denoted by $\underline{\mathrm{CM}}(R)$, is defined as follows.
  (i) $\mathrm{Ob}(\underline{\mathrm{CM}}(R)) = \mathrm{Ob}(\mathrm{CM}(R))$.
  (ii) $\mathrm{Hom}_{\underline{\mathrm{CM}}(R)}(M,N) = \underline{\mathrm{Hom}}_R(M,N)$ for $M, N \in \mathrm{Ob}(\underline{\mathrm{CM}}(R))$.

Remark 9. Let R be a Cohen-Macaulay local ring. Then $\underline{\mathrm{CM}}(R)$ is always an additive category. The direct sum of objects M and N in $\underline{\mathrm{CM}}(R)$ is the direct sum $M \oplus N$ of M and N as R-modules. Now, we consider the case where R is Gorenstein. Then $\mathrm{CM}(R)$ is a Frobenius category, and $\underline{\mathrm{CM}}(R)$ is a triangulated category. We recall in the following how to define an exact triangle in $\underline{\mathrm{CM}}(R)$. For the details, we refer to [26, Section 2 in Chapter I] or [15, Theorem 4.4.1]. Let M be an object of $\underline{\mathrm{CM}}(R)$. Then, since M is a Cohen-Macaulay R-module, there exists an exact sequence $0 \to M \to F \to N \to 0$ of Cohen-Macaulay R-modules with F free. Defining $\Sigma M = N$, we have an automorphism $\Sigma : \underline{\mathrm{CM}}(R) \to \underline{\mathrm{CM}}(R)$ of categories. This is the suspension functor. Let

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] \arrow[d, "f"'] & F \arrow[r] \arrow[d] & \Sigma L \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \arrow[r, "g"] & N \arrow[r, "h"] & \Sigma L \arrow[r] & 0 \end{tikzcd}$$

be a commutative diagram of Cohen-Macaulay R-modules with exact rows such that F is free. Then a sequence

$$L' \xrightarrow{f'} M' \xrightarrow{g'} N' \xrightarrow{h'} \Sigma L'$$

of morphisms in $\underline{\mathrm{CM}}(R)$ such that there exists a commutative diagram

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d, "\alpha"'] & M \arrow[r, "g"] \arrow[d, "\beta"'] & N \arrow[r, "h"] \arrow[d, "\gamma"'] & \Sigma L \arrow[d, "\Sigma\alpha"'] \\ L' \arrow[r, "f'"] & M' \arrow[r, "g'"] & N' \arrow[r, "h'"] & \Sigma L' \end{tikzcd}$$

in $\underline{\mathrm{CM}}(R)$ such that α, β, γ are isomorphisms is defined to be an exact triangle in $\underline{\mathrm{CM}}(R)$.

Now, we define the notion of a support for objects and subcategories of the stable category of Cohen-Macaulay modules.

Definition 10. Let R be a Cohen-Macaulay local ring.

(1) For an object M of $\underline{\mathrm{CM}}(R)$, we denote by $\underline{\mathrm{Supp}}\, M$ the set of prime ideals $\mathfrak{p}$ of R such that the localization $M_{\mathfrak{p}}$ is not isomorphic to the zero module 0 in the category $\underline{\mathrm{CM}}(R_{\mathfrak{p}})$. We call it the *stable support* of M.
(2) For a subcategory $\mathcal{Y}$ of $\underline{\mathrm{CM}}(R)$, we denote by $\underline{\mathrm{Supp}}\, \mathcal{Y}$ the union of $\underline{\mathrm{Supp}}\, M$ where M runs through all nonisomorphic objects in $\mathcal{Y}$. We call it the *stable support* of $\mathcal{Y}$.
(3) For a subset Φ of $\operatorname{Spec} R$, we denote by $\underline{\mathrm{Supp}}^{-1}\, \Phi$ the subcategory of $\underline{\mathrm{CM}}(R)$ consisting of all objects $M \in \underline{\mathrm{CM}}(R)$ such that $\underline{\mathrm{Supp}}\, M$ is contained in Φ.

The notion of a stable support is essentially the same thing as that of a nonfree locus.

Proposition 11. *Let R be a Cohen-Macaulay local ring.*

(1) *Let M be a Cohen-Macaulay R-module. Then one has $\underline{\mathrm{Supp}}\, M = \mathcal{V}(M)$.*
(2) *Let $\mathcal{X}$ be a subcategory of $\mathrm{CM}(R)$. Then one has $\underline{\mathrm{Supp}}\, \underline{\mathcal{X}} = \mathcal{V}(\mathcal{X})$.*
(3) *Let $\mathcal{Y}$ be a subcategory of $\underline{\mathrm{CM}}(R)$. Then one has $\underline{\mathrm{Supp}}\, \mathcal{Y} = \mathcal{V}(\overline{\mathcal{Y}})$.*
(4) *Let Φ be a subset of $\operatorname{Spec} R$. Then one has $\underline{\mathrm{Supp}}^{-1}\, \Phi = \underline{\mathcal{V}^{-1}(\Phi)}$.*

Here we recall the definitions of a hypersurface and an abstract hypersurface.

Definition 12. (1) A local ring R is called a *hypersurface* if there exist a regular local ring S and an element f of S such that R is isomorphic to $S/(f)$.
(2) A local ring R is called an *abstract hypersurface* if there exist a complete regular local ring S and an element f of S such that the completion $\widehat{R}$ of R in the $\mathfrak{m}$-adic topology is isomorphic to $S/(f)$.

Now we can state our main result.

Theorem 13. (1) *Let R be a local hypersurface. Then one has the following one-to-one correspondences:*

$$\begin{array}{c}
\{\textit{nonempty thick subcategories of } \underline{\mathrm{CM}}(R)\} \\
{\scriptstyle \underline{\mathrm{Supp}}}\big\downarrow \quad \big\uparrow{\scriptstyle \underline{\mathrm{Supp}}^{-1}} \\
\{\textit{specialization-closed subsets of } \operatorname{Spec} R \textit{ contained in } \operatorname{Sing} R\} \\
{\scriptstyle \mathcal{V}^{-1}}\big\downarrow \quad \big\uparrow{\scriptstyle \mathcal{V}} \\
\{\textit{resolving subcategories of } \operatorname{mod} R \textit{ contained in } \mathrm{CM}(R)\}.
\end{array}$$

(2) *Let R be a d-dimensional Gorenstein singular local ring with residue field k which is a hypersurface on the punctured spectrum. Then one has the following one-to-one correspondences:*

$$\begin{array}{c}
\{\textit{thick subcategories of } \underline{\mathrm{CM}}(R) \textit{ containing } \Omega^d k\} \\
{\scriptstyle \underline{\mathrm{Supp}}}\big\downarrow \quad \big\uparrow{\scriptstyle \underline{\mathrm{Supp}}^{-1}} \\
\{\textit{nonempty specialization-closed subsets of } \operatorname{Spec} R \textit{ contained in } \operatorname{Sing} R\} \\
{\scriptstyle \mathcal{V}^{-1}}\big\downarrow \quad \big\uparrow{\scriptstyle \mathcal{V}} \\
\{\textit{resolving subcategories of } \operatorname{mod} R \textit{ contained in } \mathrm{CM}(R) \textit{ containing } \Omega^d k\}.
\end{array}$$

Remark 14. Very recently, after the work in this article was completed, Iyengar announced in his lecture [32] that thick subcategories of the bounded derived category of finitely generated modules over a locally complete intersection which is essentially of finite type over a field are classified in terms of certain subsets of the prime ideal spectrum of the Hochschild cohomology ring. This provides a classification of thick subcategories of the stable category of Cohen-Macaulay modules over such a ring, which is a different classification from ours.

A singular local hypersurface and a Cohen-Macaulay singular local ring with an isolated sigularity are trivial examples of a ring which satisfies the assumption of Theorem 13(2). We make here some nontrivial examples.

Example 15. Let k be a field. The following rings R are Cohen-Macaulay singular local rings which are hypersurfaces on the punctured spectrums.

(1) Let $R = k[[x, y, z]]/(x^2, yz)$. Then R is a 1-dimensional local complete intersection which is neither a hypersurface nor with an isolated singularity. All the prime ideals of R are $\mathfrak{p} = (x, y)$, $\mathfrak{q} = (x, z)$ and $\mathfrak{m} = (x, y, z)$. It is easy to observe that both of the local rings $R_\mathfrak{p}$ and $R_\mathfrak{q}$ are hypersurfaces.

(2) Let $R = k[[x, y, z, w]]/(y^2-xz, yz-xw, z^2-yw, zw, w^2)$. Then R is a 1-dimensional Gorenstein local ring which is neither a complete intersection nor with an isolated singularity. All the prime ideals are $\mathfrak{p} = (y, z, w)$ and $\mathfrak{m} = (x, y, z, w)$. We easily see that $R_\mathfrak{p}$ is a hypersurface.

(3) Let $R = k[[x, y, z]]/(x^2, xy, yz)$. Then R is a 1-dimensional Cohen-Macaulay local ring which is neither Gorenstein nor with an isolated singularity. All the prime ideals are $\mathfrak{p} = (x, y)$, $\mathfrak{q} = (x, z)$ and $\mathfrak{m} = (x, y, z)$. We have that $R_\mathfrak{p}$ is a hypersurface and that $R_\mathfrak{q}$ is a field.

Applying Theorem 13(1), we observe that over a hypersurface R having an isolated singularity there are only trivial resolving subcategories of $\operatorname{mod} R$ contained in $\mathrm{CM}(R)$ and thick subcategories of $\mathrm{CM}(R)$.

Corollary 16. *Let R be a hypersurface with an isolated singularity.*

(1) *All resolving subcategories of* $\operatorname{mod} R$ *contained in* $\mathrm{CM}(R)$ *are* $\operatorname{add} R$ *and* $\mathrm{CM}(R)$.

(2) *All thick subcategories of* $\underline{\mathrm{CM}}(R)$ *are the empty subcategory, the zero subcategory, and* $\underline{\mathrm{CM}}(R)$.

Here let us consider an example of a hypersurface which does not have an isolated singularity, and an example of a Gorenstein local ring which is not a hypersurface but a hypersurface on the punctured spectrum.

Example 17. (1) Let $R = k[[x, y]]/(x^2)$ be a one-dimensional hypersurface over a field k. Then we have

$$\mathrm{CM}(R) = \operatorname{add}\{R, (x), (x, y^n) \mid n \geq 1\}$$

by [44, Example (6.5)] or [16, Proposition 4.1]. Set $\mathfrak{p} = (x)$ and $\mathfrak{m} = (x, y)$. We have $\operatorname{Sing} R = \operatorname{Spec} R = \{\mathfrak{p}, \mathfrak{m}\}$, hence all specialization-closed subsets of $\operatorname{Spec} R$ (contained in $\operatorname{Sing} R$) are $\emptyset$, $\{\mathfrak{m}\}$ and $\operatorname{Sing} R$. We have $\mathcal{V}^{-1}(\emptyset) = \operatorname{add} R$ and $\mathcal{V}^{-1}(\operatorname{Sing} R) = \mathrm{CM}(R)$. The subcategory $\mathcal{V}^{-1}(\{\mathfrak{m}\})$ of $\mathrm{CM}(R)$ consists of all Cohen-Macaulay modules that are free on the punctured spectrum of R, so it coincides with $\operatorname{add}\{R,(x,y^n) \mid n \geq 1\}$. Thus, by Theorem 13(1), all resolving subcategories of $\operatorname{mod} R$ contained in $\operatorname{CM}(R)$ are $\operatorname{add} R$, $\operatorname{add}\{R,(x,y^n) \mid n \geq 1\}$ and $\operatorname{CM}(R)$. All thick subcategories of $\underline{\operatorname{CM}}(R)$ are the empty subcategory, the zero subcategory, $\underline{\operatorname{add}\{(x,y^n) \mid n \geq 1\}}$ and $\underline{\operatorname{CM}}(R)$.

(2) Let $R = k[[x,y,z]]/(x^2,yz)$ be a one-dimensional complete intersection over a field k. Then R is neither a hypersurface nor with an isolated singularity. All prime ideals of R are $\mathfrak{p} = (x,y)$, $\mathfrak{q} = (x,z)$ and $\mathfrak{m} = (x,y,z)$. It is easy to see that both $R_\mathfrak{p}$ and $R_\mathfrak{q}$ are hypersurfaces. Note that all the nonempty specialization-closed subsets of $\operatorname{Spec} R$ (contained in $\operatorname{Sing} R$) are the following four sets:

$$\mathrm{V}(\mathfrak{p}),\ \mathrm{V}(\mathfrak{q}),\ \mathrm{V}(\mathfrak{p},\mathfrak{q}),\ \mathrm{V}(\mathfrak{p},\mathfrak{q},\mathfrak{m}).$$

Theorem 13(2) says that there exist just four thick subcategories of $\underline{\operatorname{CM}}(R)$ containing $\Omega^d k$, and exist just four resolving subcategories of $\operatorname{mod} R$ contained in $\operatorname{CM}(R)$ containing $\Omega^d k$.

Remark 18. Let R be a Gorenstein local ring. In the case where R has an isolated singularity, a thick subcategory of $\underline{\operatorname{CM}}(R)$ coincides with $\underline{\operatorname{CM}}(R)$ whenever it contains $\Omega^d k$. Example 17 especially says that this statement does not necessarily hold if one removes the assumption that R has an isolated singularity. Indeed, with the notation of Example 17(1), $\underline{\operatorname{add}\{(x,y^n) \mid n \geq 1\}}$ is a thick subcategory of $\underline{\operatorname{CM}}(R)$ containing $\Omega^d k = \mathfrak{m}$ which does not coincide with $\underline{\operatorname{CM}}(R)$. Example 17(2) also gives three such subcategories.

Using Theorem 13(1), we can show the following proposition. As it says, the subcategories of $\operatorname{CM}(R)$ and $\underline{\operatorname{CM}}(R)$ corresponding to a closed subset of $\operatorname{Spec} R$ are relatively "small."

Proposition 19. *Let R be a hypersurface. Then one has the following one-to-one correspondences:*

$$\begin{tikzcd}
\left\{\begin{matrix}\textit{thick subcategories of } \underline{\operatorname{CM}}(R)\\ \textit{generated by one object}\end{matrix}\right\} \arrow[r, "\subseteq"] \arrow[d, shift right, "\underline{\operatorname{Supp}}"'] & \left\{\begin{matrix}\textit{nonempty thick subcategories}\\ \textit{of } \underline{\operatorname{CM}}(R)\end{matrix}\right\} \arrow[d, shift right, "\underline{\operatorname{Supp}}"'] \\
\left\{\begin{matrix}\textit{closed subsets of } \operatorname{Spec} R\\ \textit{contained in } \operatorname{Sing} R\end{matrix}\right\} \arrow[r, "\subseteq"] \arrow[u, shift right, "\underline{\operatorname{Supp}}^{-1}"'] \arrow[d, shift right, "\mathcal{V}^{-1}"'] & \left\{\begin{matrix}\textit{specialization-closed subsets of } \operatorname{Spec} R\\ \textit{contained in } \operatorname{Sing} R\end{matrix}\right\} \arrow[u, shift right, "\underline{\operatorname{Supp}}^{-1}"'] \arrow[d, shift right, "\mathcal{V}^{-1}"'] \\
\left\{\begin{matrix}\textit{resolving subcategories of } \operatorname{mod} R\\ \textit{generated by one object in } \operatorname{CM}(R)\end{matrix}\right\} \arrow[r, "\subseteq"] \arrow[u, shift right, "\mathcal{V}"'] & \left\{\begin{matrix}\textit{resolving subcategories of } \operatorname{mod} R\\ \textit{contained in } \operatorname{CM}(R)\end{matrix}\right\} \arrow[u, shift right, "\mathcal{V}"']
\end{tikzcd}.$$

From now on, we make some applications of our Theorem 13. First, we have a vanishing result of homological and cohomological δ-functors from the category of finitely generated modules over a hypersurface.

Proposition 20. *Let R be a hypersurface and M an R-module. Let $\mathcal{A}$ be an abelian category.*

(1) *Let* $T : \operatorname{mod} R \to \mathcal{A}$ *be a covariant or contravariant homological* δ*-functor with* $T_i(R) = 0$ *for* $i \gg 0$*. If there exists an* R*-module* M *with* $\operatorname{pd}_R M = \infty$ *and* $T_i(M) = 0$ *for* $i \gg 0$*, then* $T_i(k) = 0$ *for* $i \gg 0$*.*
(2) *Let* $T : \operatorname{mod} R \to \mathcal{A}$ *be a covariant or contravariant cohomological* δ*-functor with* $T^i(R) = 0$ *for* $i \gg 0$*. If there exists an* R*-module* M *with* $\operatorname{pd}_R M = \infty$ *and* $T^i(M) = 0$ *for* $i \gg 0$*, then* $T^i(k) = 0$ *for* $i \gg 0$*.*

Proof. (1) First of all, note that each T_i preserves direct sums. We easily see that for any R-module N and any integers $n \geq 0$ and $i \gg 0$ we have

$$T_i(\Omega^n N) \cong \begin{cases} T_{i+n}(N) & \text{if } T \text{ is covariant,} \\ T_{i-n}(N) & \text{if } T \text{ is contravariant.} \end{cases}$$

Consider the subcategory $\mathcal{X}$ of $\mathrm{CM}(R)$ consisting of all Cohen-Macaulay R-modules X with $T_i(X) = 0$ for $i \gg 0$. Then it is easily observed that $\mathcal{X}$ is a thick subcategory of $\mathrm{CM}(R)$ containing R. Since $T_i(\Omega^d M)$ is isomorphic to $T_{i+d}(M)$ (respectively, $T_{i-d}(M)$) for $i \gg 0$ if T is covariant (respectively, contravariant), the nonfree Cohen-Macaulay R-module $\Omega^d M$ belongs to $\mathcal{X}$. Hence the maximal ideal $\mathfrak{m}$ belongs to $\mathcal{V}(\Omega^d M)$, which is contained in $\mathcal{V}(\mathcal{X})$, and we have $\mathcal{V}(\Omega^d k) \subseteq \{\mathfrak{m}\} \subseteq \mathcal{V}(\mathcal{X})$. Therefore $\Omega^d k$ belongs to $\mathcal{V}^{-1}(\mathcal{V}(\mathcal{X}))$, which coincides with $\mathcal{X}$ by Theorem 13(1). Thus we obtain $T_i(\Omega^d k) = 0$ for $i \gg 0$. Since $T_i(\Omega^d k)$ is isomorphic to $T_{i+d}(k)$ (respectively, $T_{i-d}(k)$) for $i \gg 0$ if T is covariant (respectively, contravariant), we have $T_i(k) = 0$ for $i \gg 0$, as desired.

(2) An analogous argument to the proof of (1) shows this assertion. □

As a corollary of Proposition 20, we obtain the following vanishing result of Tor and Ext modules.

Corollary 21. *Let* R *be an abstract hypersurface. Let* M *and* N *be* R*-modules.*

(1) *The following are equivalent:*
  (i) $\operatorname{Tor}_i^R(M, N) = 0$ *for* $i \gg 0$*;*
  (ii) *Either* $\operatorname{pd}_R M < \infty$ *or* $\operatorname{pd}_R N < \infty$*.*
(2) *The following are equivalent:*
  (i) $\operatorname{Ext}_R^i(M, N) = 0$ *for* $i \gg 0$*;*
  (ii) *Either* $\operatorname{pd}_R M < \infty$ *or* $\operatorname{id}_R N < \infty$*.*

The first assertion of Corollary 21 gives another proof of a theorem of Huneke and Wiegand [30, Theorem 1.9].

Corollary 22 (Huneke-Wiegand)**.** *Let* R *be an abstract hypersurface. Let* M *and* N *be* R*-modules. If* $\operatorname{Tor}_i^R(M, N) = \operatorname{Tor}_{i+1}^R(M, N) = 0$ *for some* $i \geq 0$*, then either* M *or* N *has finite projective dimension.*

Remark 23. Several generalizations of Corollaries 21(1) and 22 to complete intersections have been obtained by Jorgensen [33, 34], Miller [37] and Avramov and Buchweitz [3].

The assertions of Corollary 21 do not necessarily hold if the ring R is not an abstract hypersurface.

Example 24. Let k be a field. Consider the artinian complete intersection local ring $R = k[[x, y]]/(x^2, y^2)$. Then we can easily verify $\operatorname{Tor}_i^R(R/(x), R/(y)) = 0$ and

$\mathrm{Ext}^i_R(R/(x), R/(y)) = 0$ for all $i > 0$. But both $R/(x)$ and $R/(y)$ have infinite projective dimension, and infinite injective dimension by [13, Exercise 3.1.25].

Let $\mathbf{H}$ be a property of local rings. Let $\mathbf{H}\text{-}\dim_R$ be a numerical invariant for R-modules satisfying the following conditions.

(1) $\mathbf{H}\text{-}\dim_R R < \infty$.

(2) Let M be an R-module and N a direct summand of M. If $\mathbf{H}\text{-}\dim_R M < \infty$, then $\mathbf{H}\text{-}\dim_R N < \infty$.

(3) Let $0 \to L \to M \to N \to 0$ be an exact sequence of R-modules.
  - (i) If $\mathbf{H}\text{-}\dim_R L < \infty$ and $\mathbf{H}\text{-}\dim_R M < \infty$, then $\mathbf{H}\text{-}\dim_R N < \infty$.
  - (ii) If $\mathbf{H}\text{-}\dim_R L < \infty$ and $\mathbf{H}\text{-}\dim_R N < \infty$, then $\mathbf{H}\text{-}\dim_R M < \infty$.
  - (iii) If $\mathbf{H}\text{-}\dim_R M < \infty$ and $\mathbf{H}\text{-}\dim_R N < \infty$, then $\mathbf{H}\text{-}\dim_R L < \infty$.

(4) The following are equivalent:
  - (i) R satisfies $\mathbf{H}$;
  - (ii) $\mathbf{H}\text{-}\dim_R M < \infty$ for any R-module M;
  - (iii) $\mathbf{H}\text{-}\dim_R k < \infty$.

The conditions (1) and (3) imply the following condition:

(5) Let M be an R-module. If $\mathrm{pd}_R M < \infty$, then $\mathbf{H}\text{-}\dim_R M < \infty$.

Indeed, let M be an R-module with $\mathrm{pd}_R M < \infty$. Then there is an exact sequence

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

of R-modules with each F_i free. The conditions (1) and (3)(ii) imply that $\mathbf{H}\text{-}\dim_R F_i < \infty$ for all $0 \le i \le n$. Decomposing the above exact sequences into short exact sequences and applying the condition (3)(i), we have $\mathbf{H}\text{-}\dim_R M < \infty$, as required.

We call such a numerical invariant a *homological dimension*. A lot of homological dimensions are known. For example, projective dimension pd_R, complete intersection dimension $\mathrm{CI}\text{-}\dim_R$ (cf. [5]), Gorenstein dimension $\mathrm{G}\text{-}\dim_R$ (cf. [1, 17]) and Cohen-Macaulay dimension $\mathrm{CM}\text{-}\dim_R$ (cf. [25]) coincide with $\mathbf{H}\text{-}\dim_R$ where $\mathbf{H}$ is regular, complete intersection, Gorenstein and Cohen-Macaulay, respectively. Several other examples of a homological dimension can be found in [2]. A lot of studies of homological dimensions have been done so far. For each homological dimension $\mathbf{H}\text{-}\dim_R$, investigating R-modules M with $\mathbf{H}\text{-}\dim_R M < \infty$ but $\mathrm{pd}_R M = \infty$ is one of the most important problems in the studies of homological dimensions. In this sense, the following proposition says that a hypersurface does not admit a proper homological dimension.

Proposition 25. *With the above notation, let R be a hypersurface not satisfying the property $\mathbf{H}$. Let M be an R-module. Then $\mathbf{H}\text{-}\dim_R M < \infty$ if and only if $\mathrm{pd}_R M < \infty$.*

Proof. The condition (5) says that $\mathrm{pd}_R M < \infty$ implies $\mathbf{H}\text{-}\dim_R M < \infty$. Conversely, assume $\mathbf{H}\text{-}\dim_R M < \infty$. Let $\mathcal{X}$ be the subcategory of $\mathrm{CM}(R)$ consisting of all Cohen-Macaulay R-modules X satisfying $\mathbf{H}\text{-}\dim_R X < \infty$. It follows from the conditions (1), (2) and (3) that $\mathcal{X}$ is a thick subcategory of $\mathrm{CM}(R)$ containing R. Theorem 13(1) yields $\mathcal{X} = \mathcal{V}^{-1}(\mathcal{V}(\mathcal{X}))$. Suppose that $\mathrm{pd}_R M = \infty$. Then $\Omega^d M$ is a nonfree Cohen-Macaulay R-module. We have an exact sequence

$$0 \to \Omega^d M \to F_{d-1} \to \cdots \to F_0 \to M \to 0$$

of R-modules such that F_i is free for $0 \leq i \leq d-1$. Decomposing this into short exact sequences and using the conditions (1) and (3), we see that $\Omega^d M$ belongs to $\mathcal{X}$. Hence the maximal ideal $\mathfrak{m}$ of R is in $\mathcal{V}(\mathcal{X})$, and we obtain $\mathcal{V}(\Omega^d k) \subseteq \{\mathfrak{m}\} \subseteq \mathcal{V}(\mathcal{X})$. Therefore $\Omega^d k$ belongs to $\mathcal{V}^{-1}(\mathcal{V}(\mathcal{X})) = \mathcal{X}$, namely, $\mathbf{H}\text{-}\dim_R(\Omega^d k) < \infty$. There is an exact sequence

$$0 \to \Omega^d k \to G_{d-1} \to \cdots \to G_1 \to G_0 \to k \to 0$$

of R-modules with each G_i free. Decomposing this into short exact sequences and using the conditions (1) and (3), we get $\mathbf{H}\text{-}\dim_R k < \infty$. Thus the condition (4) implies that R satisfies the property $\mathbf{H}$, which contradicts our assumption. Consequently, we must have $\operatorname{pd}_R M < \infty$. $\square$

References

[1] M. Auslander; M. Bridger, Stable module theory, Memoirs of the American Mathematical Society, No. 94, *American Mathematical Society, Providence, R.I.*, 1969.

[2] L. L. Avramov, Homological dimensions and related invariants of modules over local rings, *Representations of algebra, Vol. I, II*, 1–39, *Beijing Norm. Univ. Press, Beijing*, 2002.

[3] L. L. Avramov; R.-O. Buchweitz, Support varieties and cohomology over complete intersections, *Invent. Math.* **142** (2000), no. 2, 285–318.

[4] L. L. Avramov; R.-O. Buchweitz; L. W. Christensen; S. B. Iyengar; G. Piepmeyer, Differential modules over commutative rings, in preparation.

[5] L. L. Avramov; V. N. Gasharov; I. V. Peeva, Complete intersection dimension, *Inst. Hautes Études Sci. Publ. Math.* No. 86 (1997), 67–114 (1998).

[6] P. Balmer, Presheaves of triangulated categories and reconstruction of schemes, *Math. Ann.* **324** (2002), no. 3, 557–580.

[7] P. Balmer, The spectrum of prime ideals in tensor triangulated categories, *J. Reine Angew. Math.* **588** (2005), 149–168.

[8] P. Balmer, Supports and filtrations in algebraic geometry and modular representation theory, *Amer. J. Math.* **129** (2007), no. 5, 1227–1250.

[9] D. J. Benson; J. F. Carlson; J. Rickard, Thick subcategories of the stable module category, *Fund. Math.* **153** (1997), no. 1, 59–80.

[10] D. Benson; S. B. Iyengar; H. Krause, Local cohomology and support for triangulated categories, *Ann. Sci. Éc. Norm. Supér. (4)* **41** (2008), no. 4, 573–619.

[11] D. Benson; S. B. Iyengar; H. Krause, Stratifying modular representations of finite groups, Preprint (2008), `http://arxiv.org/abs/0810.1339`.

[12] K. Brüning, Thick subcategories of the derived category of a hereditary algebra, *Homology, Homotopy Appl.* **9** (2007), no. 2, 165–176.

[13] W. Bruns; J. Herzog, Cohen-Macaulay rings, revised edition, Cambridge Studies in Advanced Mathematics, 39, *Cambridge University Press, Cambridge*, 1998.

[14] A. B. Buan; H. Krause; Ø. Solberg, Support varieties: an ideal approach, *Homology, Homotopy Appl.* **9** (2007), no. 1, 45–74.

[15] R.-O. Buchweitz, Maximal Cohen-Macaulay modules and Tate-cohomology over Gorenstein rings, Preprint (1986), `http://hdl.handle.net/1807/16682`.

[16] R.-O. Buchweitz; G.-M. Greuel; F.-O. Schreyer, Cohen-Macaulay modules on hypersurface singularities II, *Invent. Math.* **88** (1987), no. 1, 165–182.

[17] L. W. Christensen, Gorenstein dimensions, Lecture Notes in Mathematics, 1747, *Springer-Verlag, Berlin*, 2000.

[18] E. S. Devinatz; M. J. Hopkins; J. H. Smith, Nilpotence and stable homotopy theory, I, *Ann. of Math. (2)* **128** (1988), no. 2, 207–241.

[19] W. Dwyer; J. P. C. Greenlees; S. Iyengar, Finiteness in derived categories of local rings, *Comment. Math. Helv.* **81** (2006), no. 2, 383–432.

[20] E. M. Friedlander; J. Pevtsova, Π-supports for modules for finite group schemes, *Duke Math. J.* **139** (2007), no. 2, 317–368.
[21] P. Gabriel, Des catégories abéliennes, *Bull. Soc. Math. France* **90** (1962), 323–448.
[22] G. Garkusha; M. Prest, Classifying Serre subcategories of finitely presented modules, *Proc. Amer. Math. Soc.* **136** (2008), no. 3, 761–770.
[23] G. Garkusha; M. Prest, Torsion classes of finite type and spectra, pp. 393-412 in *K-theory and Noncommutative Geometry*, European Math. Soc., 2008.
[24] G. Garkusha; M. Prest, Reconstructing projective schemes from Serre subcategories, *J. Algebra* **319** (2008), no. 3, 1132–1153.
[25] A. A. Gerko, On homological dimensions, (Russian) *Mat. Sb.* **192** (2001), no. 8, 79–94; translation in *Sb. Math.* **192** (2001), no. 7-8, 1165–1179.
[26] D. Happel, Triangulated categories in the representation theory of finite-dimensional algebras, London Mathematical Society Lecture Note Series, 119, *Cambridge University Press, Cambridge*, 1988.
[27] M. J. Hopkins, Global methods in homotopy theory, *Homotopy theory (Durham, 1985)*, 73–96, London Math. Soc. Lecture Note Ser., 117, *Cambridge Univ. Press, Cambridge*, 1987.
[28] M. J. Hopkins; J. H. Smith, Nilpotence and stable homotopy theory, II, *Ann. of Math. (2)* **148** (1998), no. 1, 1–49.
[29] M. Hovey, Classifying subcategories of modules, *Trans. Amer. Math. Soc.* **353** (2001), no. 8, 3181–3191.
[30] C. Huneke; R. Wiegand, Tensor products of modules, rigidity and local cohomology, *Math. Scand.* **81** (1997), no. 2, 161–183.
[31] S. B. Iyengar, Stratifying the derived category of a complete intersection, Workshop: Kommutative Algebra, Oberwolfach Report **22** (2009), 41–43.
[32] S. B. Iyengar, Stratifying derived categories associated to finite groups and commutative rings, *Kyoto RIMS Workshop on Algebraic Triangulated Categories and Related Topics*, `http://www.mi.s.osakafu-u.ac.jp/~kiriko/seminar/09JulRIMS.html`.
[33] D. A. Jorgensen, Tor and torsion on a complete intersection, *J. Algebra* **195** (1997), no. 2, 526–537.
[34] D. A. Jorgensen, Complexity and Tor on a complete intersection, *J. Algebra* **211** (1999), no. 2, 578–598.
[35] H. Krause, Decomposing thick subcategories of the stable module category, *Math. Ann.* **313** (1999), no. 1, 95–108.
[36] H. Krause, Thick subcategories of modules over commutative Noetherian rings (with an appendix by Srikanth Iyengar), *Math. Ann.* **340** (2008), no. 4, 733–747.
[37] C. Miller, Complexity of tensor products of modules and a theorem of Huneke-Wiegand, *Proc. Amer. Math. Soc.* **126** (1998), no. 1, 53–60.
[38] A. Neeman, The chromatic tower for $D(R)$, With an appendix by Marcel Bökstedt, *Topology* **31** (1992), no. 3, 519–532.
[39] R. Takahashi, Classifying subcategories of modules over a commutative Noetherian ring, *J. Lond. Math. Soc. (2)* **78** (2008), no. 3, 767–782.
[40] R. Takahashi, Modules in resolving subcategories which are free on the punctured spectrum, *Pacific J. Math.* **241** (2009), no. 2, 347–367.
[41] R. Takahashi, On localizing subcategories of derived categories, to appear in *J. Math. Kyoto Univ.*, `http://arxiv.org/abs/0712.1874`.
[42] R. Takahashi, Classifying thick subcategories of the stable category of Cohen-Macaulay modules, Preprint (2009), `http://arxiv.org/abs/0908.0107`.
[43] R. W. Thomason, The classification of triangulated subcategories, *Compositio Math.* **105** (1997), no. 1, 1–27.
[44] Y. Yoshino, Cohen-Macaulay modules over Cohen-Macaulay rings, London Mathematical Society Lecture Note Series, 146, *Cambridge University Press, Cambridge*, 1990.

Department of Mathematical Sciences
Faculty of Science
Shinshu University
3-1-1 Asahi, Matsumoto, Nagano 390-8621, Japan
E-mail address: takahasi@math.shinshu-u.ac.jp

NONCOMMUTATIVE ALGEBRAIC GEOMETRY : A SURVEY OF THE APPROACH VIA SHEAVES ON NONCOMMUTATIVE SPACES

FREDDY M. J. VAN OYSTAEYEN

0. Introduction

To me noncommutative algebraic geometry came from the consideration of noncommutative spaces defined in terms of notions like : noncommutative valuations and pseudovaluations, primes in algebras, prime ideals of Noetherian rings or prime torsion theories for rings or categories. The root of the theory was in the theory of the Brauer group of a field via suitable subrings of central simple algebras, therefore at first rings satisfying polynomial identities played a dominating role. For such a ring R the noncommutative space prompting itself is Spec$R$, the prime ideal spectrum with its Zariski topology; in [93], a structure sheaf over Spec$R$ for a noncommutative ring R had been first constructed. In the case of rings with polynomial identities this could be tied to arithmetical pseudo-valuation theory and a corresponding divisor theory leading to a noncommutative version of a Riemann-Roch theorem for central simple algebras over curves (see [130], [137] which turned out to be an extension of some idea of E. Witt (see the book by M. Deuring, *Algebra*), This combined in the concept of noncommutative geometry in the P.I. case, the subject being first called that in the publication [137]. Also this theory connected well with maximal orders and Azumaya algebras and it developed into a branch related to the Brauer group of schemes and varieties. Now the localization theory was well established for abelian categories, see P. Gabriel [47], while on the other hand a result of Van Oystaeyen, Verschoren stated that $Br\mathrm{Proj}C = Br(C, K_+)$-gr where C is a commutative positively graded ring and (C, K_+)-gr is the quotient category of finitely generated graded C-modules for the torsion theory κ_+ associated to the positive cone $C_+ = C_1 \oplus, \ldots, \oplus C_n \oplus, \ldots$ of C. Deleting Br in the formula suggests that Proj$C$ is "identified" with that quotient category. The J. P. Serre's global section theorem does relate the quasi-coherent sheaves over Proj$C$ to that quotient category, in fact when $C_o = k$, a field, and $C = C_0[C_1]$, then the quotient category is just finitely generated graded C-modules modulo finite length modules. So assuming that a noncommutative version of J.P. Serre's result exists, the noncommutative geometry of Proj$R$ should be approachable via the homological algebraic theory of the category (R, κ_+)-gr. It turned out that a noncommutative versions of the global section theorem is available only in case one introduces a noncommutative topology on the localizations spectrum allowing compositions of localizations that are not again localizations. This leads to the definition of schematic algebras and it was checked that a very large class of noncommutative rings are schematic, inducing all interesting quantized algebras and other rings appearing in recent literature.

The paper is in a final form and no version of it will be submitted for publication elsewhere.

Rings appearing in nature were given by generators and relations and as such they inherited the filtration defined by the grading filtrations on the free algebra. The definition of Zariskian filtration, introduced in [79] and the use of the Rees ring (blow-up ring) then allowed the interplay between algebraic geometry and its projective version much as in the commutative case. The filter-graded transfer of homological properties and of the schematic condition provided for a fruitful technical framework to study many interesting examples, e.g. generalized Weyl algebras, generalized gauge algebra containing E. Witten's gauge algebra for gauge theory of slU_2, etc... Using Auslander's regularity condition it was possible to extend regularity from Azumaya algebras over regular center to more general noncommutative rings, not necessarily finite over the center; the filter-graded transfer for Auslander regularity provided many interesting examples of noncommutative regular algebras (schemes). The study of regular algebras and their classification in low dimension became a fruitful research direction, recently developing into the direction of Calabi-Yau algebras (see [26]) etc...

Let us point out that a good version of geometric product may be found in the general twisted product of algebras, cf. [83]; its good behaviour with respect to connections provides a link with the work of A. Connes. The noncommutative geometry developed by A. Connes after the 1980s was more based in operator theory and C^*-algebras, one could call it noncommutative differential geometry. The space in this geometry remains virtual and one imagines the noncommutative algebra as a ring of "functions" defined on the virtual variety. There are several contact points between both versions of noncommutative geometry, it remains to be seen whether the phrasing in terms of noncommutative spaces in algebraic geometry is feasable and useful in the other case.

Contents

In an arbitrary ring S a couple (P, S') where S' is a subring of S and P is a prime ideal of S' is called a **prime of** S if $xS'y \subset P$ with $x, y \in S$ yields that x of y is in P. The map $S' \to S'/P$ is a **pseudoplace** of S. A couple (P, S') **dominates** (P_1, S_1) if $S' \supset S_1$ and $P \cap S_1 = P_1$, a **dominating prime** is one that is maximal with respect to domination. The set $\mathrm{Prim}(S)$ of all primes of S has a topology with basis $D(F) = \{P \in \mathrm{Prim}(S), P \cap F = \emptyset\}$ for F a finite subset of S. For example if K is a field and A a K- central simple algebra, O_v a valuation ring of K then any maximal O_v-order Λ in A yields a dominating prime $(J(\Lambda), \Lambda)$ where $J(\Lambda)$ is the Jacobson radical of Λ.

Consider a prime P.I.-ring S with quotient ring $(Q(S)$, which is then a central simple algebra. A **fractional ideal** I of S is a twosided S-submodule of $Q(S)$ such that $cI \subset S$ for some nonzero $c \in Z(S)$ (the centre of S). S is an arithmetical ring when fractional ideals commute for the product in $Q(S)$. Let $F(S)$ be the set of fractional ideals of S. Consider a totally ordered semigroup Γ, a **pseudovaluation** v on $F(S)$ is a function $v : F(S) \to \Gamma$, satisfying :

i) $v(IJ) \geq v(I)v(J)$ for $I, JvF(S)$
ii) $v(I + J) \geq \inf\{v(I), v(J)\}$ for $I, J \in F(S)$
iii) $v(S) = 0$ and $v(o) = \infty$
iv) If $I \subset J$ then $v(I) \geq v(J)$ for $I, J \in F(J)$

If moreover we have : (v)$v(IJ) = v(I) + v(J)$, we say v is an **arithmetical pseudovaluation** (a.p.v.).

Any a.p.v. on Q defines a prime (P, Q^P) where Q^P is the idealizer of P in Q and $P = \{q \in Q, v(SqS) \subset 0\}$. Conversely any prime (P, Q^P) where $S \subset Q^P$ defines an a.p.v., v say, such that $P = \{q \in Q, v(SqS) \supset o\}$.

If the value semi-group of an a.v.p. is a group then the corresponding prime is dominating. Any prime (P, Q^P) of a central simple algebra is said to be **discrete** if Q^P contains an arithmetical ring S and satisfies the a.c.c. on ideals while P is the unique maximal ideal of Q such that $P = \pi Q^P$ for some invertible π in Q. In the discrete case $\Gamma \cong \mathbb{Z}$ and Q^P is itself arithmetical. In particular any maximal order in Q over a dicrete valuation ring of $K = Z(Q)$ is a discrete prime. A set of discrete primes inducing inequivalent valuations on K is said to be **divisorial** if for $q \in Q$ we have $v(q) = 0$ for almost all a.p.v. associated to the discrete primes in the set, this condition has to be checked only for $q \in Z(Q)$! The elements of a divisorial set Q are called **prime divisors**. A divisorial set Q is associated to be chosen fixed : a **divisor** δ of Q associated to Q is a formal product $\prod_{v \in Q} v^{\tau_v}$ with $\tau_v \in \mathbb{Z}$ and $\tau_v = 0$ for almost all $v \in Q$, the exponent τ_v is called the **order** $\tau_v = \mathrm{ord}_v \delta$.

Consider a subfield k_o of an algebraically closed field k. In [137] we consider an affine curve over k_o as a k_o-quasivariety $\Omega(R)$ for some prime affine P.I. algebra R over k_o having Krull dimension 1. By a result of L. Small such an algebra is a finite module over its centre. If $n = p.i.\mathrm{deg}R$ then $M \in \Omega(R)_n$ correspond to $m \in \Omega(C)$, $L = Z(R)$, such that $R \otimes_k k_C(m) = M_n(k)$. For $k_o \neq k$ it is still true that $k_R(M) = R \otimes_{k_o} k_o(m)$ is a central simple algebra and $P \in \Omega_n(R)$ if and only if $k_R(P)$ has degree n (dimension : n^2) if and only if $P \cap C$ is non split.

An algebraic function field K in one variable over k_o is an extension K of k_o such that k_o is algebraically closed in K and K is separable of t.d. over k_o.

A **function algebra** in one variable over k_o is a central simple K-algebra A. For an affine prime P.I. algebra over k_o there is equivalence between $\Omega(R)$ (the space of maximal ideals) being an affine k_o-curve and $Q(R)$ being a function algebra in one variable. The complement of unramified points in $Z(R)$ is the ramification divisor of $\Omega(C)$ for Q, these correspond to the maximal ideals of C that are split in R.

Let $C_{k_o}(R)$ be the set of all k_o-valuation rings O_v of K (those are discrete). For every $O_v \in C_{k_o}(K)$ we choose and fix a maximal order Λ_v over O_v and write $C_{k_o}(Q)$ for this set. This choice can'be made such that almost all Λ_v contain a suitable Azumaya algebra (obtained as $\cap_{P\in\Omega_n(R)}R_P$ for some R ascending the curve). Write D_Q for the group of divisors generated by $C_{k_o}(Q)$. The degree of a divisor $\delta \in D_Q, \delta = \Sigma f_v \text{ord}_v\delta$, where f_v is the absolute residue class degree u.e. $f_v = \dim_{k0} k_v$, k_v the residue field of O_v. We say that $\delta_1|\delta_2$ if for all $v \in Q$, $\text{ord}_v\delta_1 \leq \text{ord}_v\delta_2$.

1.1 Lemma. *If $\delta_1|\delta_2$ then :*

$$\dim_{k_o}(\Gamma(\delta_1(S))/\Gamma(\delta_2(S)) = \deg\, \delta_2 - \deg\, \delta_1$$

where for any finite subset S of the algebra of valuation vectors $V_Q, \Gamma(\delta|S) = \{a \in R, v(a) \geq \text{ord}_v\delta, \text{all } v \in S\}$ (cf. [130], [137]).

In particular if $S = V_A$ then we define $L(S)$ as $\Gamma(\delta|V_A)$ and $l(\delta) = \dim_{k_o} L(\delta)$. Valuation forms can now also be defined in the noncommutative case and by using the reduced trace map for Q every valuation form is of the form $w(Tr(a-))$ for some $a \in Q$ and fixed valuation form w.

1.2 Theorem. Riemann-Roch for n.c. curves *Let $\beta \in D_Q$ be arbitrary and δ "canonical" (see Proposition XI.3.9. p. 376 of [137]), then :*

$$\deg\beta + l(\beta) = l(\beta^{-1}\delta^{-1}) + 1 - g_Q$$

where g_Q is a constant, called the **genus** *of Q.*

The ring $l = \cap\{\Lambda_v, \Lambda_v \in C_{k_o}(Q)\}$ is the ring of k_o-constants it is algebraic over k_o and a central simple algebra finite dimensional over K_0 (XI.2.14 of [137]).

1.3 Corollary.

i) $\ell(\delta^{-1}) = n - 1 + g_Q, n = \dim_{k_o}\ell$.

ii) $\deg(\delta^{-1}) = 2 - 2y_Q$

iii) $g_Q = N_{g_K} - N + 1 + \frac{1}{2}\Sigma f_v(r_v - 1)$, *where f_v is the residual degree r_v the ramification index of v,* $\mathbb{N} = \dim[Q : K]$.

1.4 Theorem. *Let $k = k_o$ and $X = \Omega(R)$ an affine k-curve with central curve $Y = \Omega(Z(R))$ then : $g_X = Ng_Y - N + 1$ (since k is algebraically closed $Q = Q(R) = M_n(K)$ by Tsen's theorem.*

1.5 Remark. If $Q = M_r(\Delta)$ then $g_Q = r^2 g_\Delta - r^2 + 1$. The Brauer group of K (not trivial if k_o is not algebraically closed) yields invariants g_Q for every $[Q] \in BrK$. What is the relation between the commutative geometry of the central curve and these invariants ?

1.A. Project : Noncommutative Invariants of Varieties

After [130], [137], Van den Bergh, Van Geel obtained a cohomological Riemann-Roch result for higher dimensional noncommutative varieties. The foregoing question may be generalized to this higher dimensional situation using the ingredients (invariants) stemming from the Riemann Roch theorem.

In dimension more than two there are noncommutative invariants stemming from the Brauer group of the function field that is now not trivial even in the case where k_o is algebraically closed. There is some work of M. Artin about maximal orders over surfaces (see [9]) but a complete noncommutative version of the work of O. Zariski on surfaces remains to be developed. In general the set of discrete primes of a central simple algebra provides us with something like s noncommutative Riemann surface. The theory of a.p.v's works well if some arithmetical ring is given but it should be extended to more genertal situations using rings in which ideals do not commute and noncommutative (totally ordered) value groups.

1.B. Project : Valuations of Weyl Algebras, Enveloping Algebras etc..

The theory of valuations also extends to the non P.I. case; O. Schilling (cf. [117]) already introduced noncommutative valuations on skewfields not necessarily finite dimensional over the centre. However, the valuation theory for most quantized algebras nowadays popular remains unexplored. In a paper with L. Willaert, I investigated valuations of the Weyl skewfield and this led to the discovery of a subring of the Weyl skewfield having it as a ring of fractions (therefore in some sense birational to the Weyl algebra $K[X][\frac{\partial}{\partial X}] \cong K < X, Y > /(YX - XY - 1)$) and being a kind of antipode for the Weyl algebra. This ring appearing as the intersection of noncommutative valuation rings is a "duo ring" i.e. each one sided ideal is two sided and localizations at prime ideals correspond to valuation over rings. A divisor theory for the Weyl field remained to be worked out. Up to a particiular application related to Sklyanin algebras, the noncommutative valuation theory remains to be applied. For example, it is an unpublished consequence of some results in the Ph. D. thesis of L. Hellström (Lund T. U., Sweden) that one may construct large families of noncommutative valuations of the skewfield appearing as the ring of fractions of the enveloping algebra of a finite dimensional Lie algebra. Further characterization of these n.c. valuations and calculations similar to a divisor calculus should be undertaken and these results should have meaning in the structure theory of Lie algebras or at least in the noncommutative geometry of their enveloping algebras. In particular some rings appearing as intersections of families of n.c. valuation rings could shed new light on the algebraic structure ?

2. Schematic Algebras and Noncommutative Schemes

Algebraic geometry is built upon the correspondence between quotients of polynomial rings and varieties embedded in affine (projective) spaces. In noncommutative algebra the generic algebra i.e. the free algebra, is not too well behaved and the formation of products (tensor products) is also somewhat problematic. Is it possible to fix a class of algebras such that most operations from scheme theory may be performed whilst keeping a good duality with noncommuative algebra constructions ? For a given noncommutative algebra one may of course try to extend the algebraic techniques appearing in commutative algebraic geometry to it without trying to associate a "geometric" space to it. This works to some extent in several cases but it is perhaps not guaranteed that one is really studying a noncommutative geometry, it is noncommutative algebra in disguise. I always wanted some kind of topological space and (coherent) sheaves on it to correspond to the module of some ring of functions via some noncommutative version of J.P. Serre's global section theorem. This led to the introduction of noncommutative topology and schematic algebras.

2.1 Noncommutative Spaces and Localization

Perhaps a few historical remarks concerning the development of this subject. During my stay at Cambridge University in 1972-73 I worked with D. Murdoch (Vancouver University B. C.) on localization theory and we constructed the first structure sheaf for a noncommutative ring yielding the ring as global sections, cf. [93]. For me this was connected to the primes or pseudoprimes I introduced in my thesis and I combined the ideas into a theory of prime spectra for noncommutative rings in [134] where I also started the projective theory by constructing Proj for a noncommutative positively graded ring. This was also related to my search for an answer to a question J. Murre (University of Leiden) asked me concering a purely algebraic description of the Brauer group of a projective variety during our stay at Cambridge. Since maximal orders were at the centre of all these problems I started a seminar on this topic at the University of Antwerp (UIA) which attracted many students and visitors. With J. Van Geel, E. Nauwelaerts and visitor H. Marubayashi and later L. Willaert we continued in the direction of primes of noncommutative algebras; with A. Verschoren and L. Le Bruyn in the direction of localization and noncommutative schemes, with L. Le Bruyn, E. Jespers, P. Wauters in the direction of graded orders and later with M. Van den Bergh in projective noncommutative geometry. The work with A. Verschoren (resulting in the first book with the title "Noncommutative Geometry", cf. [137]) was noticed by M. Artin and after a stay of A. Verschoren at the M.I.T. there was a growing group of people involved in the development, including M. Artin, W. Schelter etc... Starting from regularity conditions from homological algebra, M. Van den Bergh then cooperated with M. Artin, J. Tate (cf. [6],[7]) and later with T. Stafford, P. Smith and many more, specially on low dimensional noncommutative varieties. On the other hand the graded constructions in the constructions of proj created a cooperation with C. Năstăsescu on graded ring theory, cfr. [97], [95], [94]. Meanwhile it turned out that the answer to the question of J. Murre fitted completely in the framework of graded localization. After I introduced the graded Brauer group of a $\mathbb{Z}$-graded ring, A. Verschoren and I described the Brauer group of a projective variety algebraically as the Brauer group of the category (in modern language) appearing as a quotient category of the finitely generated graded modules modulo those of finite length i.e. the graded quotient category associated to the graded localization at the positive cone of the positively graded coordinate ring. This continued in work on the cohomology of graded rings with S. Caenepeel [32] and extended to Brauer groups of other actions and conditions leading to the Brauer group of a quantun group, cf. [33], [34], and later with Y. H. Zhang to a theory of Brauer groups of braided categories, cf. [142], [143]. After the beginning of the interest in graded rings, filtrations also became interesting, particularly because of the use of the Rees (blow up) ring; this makes for a transfer between graded and filtered ring theory allowing several nice applications to for example rings of differential operators and (generalized) Weyl algebras cfr. work with Li Huishi [79], [80], and later with V. Bavula [17] [18]. The connection with representation theory was also explored at several places and developed mainly by L. Le Bruyn and M. Van den Bergh e.g. in the geometry of path algebras for quivers cf. [69],[68]. This shows how the original ideas concerning a kind of noncommutative geometry has branched into many directions that have achieved nowadays a good level of popularity.

So originally we considered a noncommutative variety or scheme as a structure sheaf on the prime spectrum, that prime spectrum was either determined in terms of prime ideals (Murdoch, Van Oystaeyen) or prime torsion theories (J. Golan, J. Raynand, F. Van Oystaeyen cf. [47]). However in the noncommutative case, the construction was not functorial (I remember to have proved, unpublished, that functionality forces commutativity) but it was possible to view Spec as a (localization) functor on the category of modules and to relate a ring morphism to a natural transformation of the Spec functor. This has convinced me that the construction of a (noncommutative) topology was more essential than the choice of points, in fact one could work with a pointless topology and sheaf theory over that. This gave rise to the construction of virtual topology and functor geometry, a very abstract framework for categorical algebraic geometry, cf. [135].

A noncommutative ring R is said to be (affine) **schematic** if there exists a finite set of nontrivial Ore sets $S_1, \ldots, S_n$ such that for every choice of $s_i \in S_i, i = 1, \ldots, n$ we have that $\sum_{i=1}^{n} Rs_i = R$ or equivalently $\cap_i \mathcal{L}(S_i) = \{R\}$ where $\mathcal{L}(S_i)$ is the Gabriel filter of S_i. Recall that a left **Ore set** S of R is a multiplicatively closed subset of R, $0 \notin S, 1 \in S$, such that for given $r \in R, s \in S$ there exists $r' \in R, s' \in S$ such that : $s'r = r's$ and moreover if $rs = 0$ then there is an $s'' \in S$ such that $s''r = 0$. The right version is defined symmetrically. For an Ore set S the ring of fractions $S^{-1}R$ exists and in this case the left localization at S and the right localization coincide. For an R-module M the S-torsion part of M is $M_s = \{m \in M, sm = 0 \text{ for some } s \in S\}$ and $S^{-1}M = S^{-1}R \otimes_R M$ is the (left) localization of M at S. Clearly $M/t_S M$ is S-torsion free i.e. $t_S(M/t_S M) = 0$ and we have the standard localization morphism in $j_S : M \to S^{-1}M$ with $\mathrm{ker} j_S = t_S M$ and $\mathrm{Im} j_S \cong M/t_S M$. In case R is the free algebra it only has trivial Ore sets i.e. contained in the ground field and hence already invested in the ring. So free algebras, the generic algebras in the associative situation, are not schematic. On the other hand all rings frequently encountered in noncommutative algebra seem to be schematic. For example the ring of generic matrices, the Weyl algebras, the coordinate ring of quantum 2×2-matrices $Q_q(M_2(\mathbb{C}))$ is schematic (1.2.11 of [131]), quantum Weyl algebras $A_n(q)$ (1.2.14 of [131]), rings that are finite modules over their centre, the Sklyanin algebra $S_K(a, b, c)$ (1.2.17 of [131]), E. Witten's gauge algebras $W(\mathbb{C})$, (1.2.21 of [131]), quantum sl_2 (1.2.23 of [131]). Let A be a K-algebra and positvely graded such that $A = K \oplus A_2 \oplus \ldots$ we write A_+ for $A_1 \oplus A_2 \oplus \ldots$ and K_+ for the torsion theory with Gabriel filter $\mathcal{L}(K_+) = \{L \text{ left ideal of } A, L \supset A_+^n \text{ for some } n \in \mathbb{N}\}$. We say A is **schematic** (projective) if there exists a finite set of homogeneous Ore sets, say $\mathcal{I}$, such that for every $S \in \mathcal{I}$, $S \cap A_+ \neq \emptyset$ and such that for any $s_i \in S_i, i \in \mathcal{I}$, there exists an $m \in \mathbb{N}$ such that $(A_+)^m \subset \sum_{i\in\mathcal{I}} As_i$ (or equivalently : $\cap\ \mathcal{L}(S_i) = \mathcal{L}(K_+)$ where $\mathcal{L}(S_i) = \{L \subset A, L \supset As_i$ for some $s_i \in S_i\}$, or equivalently $\kappa_+ = \Lambda_{i\in\mathcal{I}}\mathcal{L}(S_i)$ holds in the lattice of torsion theories on A-gr, the category of graded A-modules). A schematic positively graded K-algebra need not be affine schematic, we have a weaker notion **weakly affine schematic** defined as (projective) schematic plus the fact that the $S_i \in \mathcal{I}$ are such that $A = \cap_{i\in\mathcal{I}} S_i^{-1}A$.

If we have a positively filtered K-algebra A with filtration ... $\subset F_{n-1}A \subset F_nA \subset \ldots \subset A$ then the associated graded algebra is $G(A) = \oplus_{n\in\mathbb{N}} F_nA|F_{n-1}$
$A = K \oplus F_1(A)|K \oplus \ldots$ and the Rees algebra (or the blow-up algebra of FA) is $\widetilde{A} \cong \sum_n F_n(A)T^n \subset A[T]$. It is easy to see that $G(A) = \widetilde{A}|\widetilde{A}T, A = \widetilde{A}|(T-1)\widetilde{A}$ and T is a central regular element homogeneous of degree 1 in $\widetilde{A}$. In the positively filtered situation (this is a discrete filtration) the filtration will be Zariskian in the sense of [79] exactly when $\widetilde{A}$ is Noetherian which in this case is equivalent to $G(A)$ being Noetherian.

2.1.1 Theorem. *If FA is a positive Zariskian filtration on A such that $F_0A = K$, then if $G(A)$ is schematic it follows that $\widetilde{A}$ is schematic too.*

2.1.2 Corollary. *If in the situation of the theorem $G(A)$ is commutative then $\widetilde{A}$ is schematic. It follows from this that rings of differential operators on varieties (non-singular) and enveloping algebras of Lie algebras have schematic Rees rings.*

When trying to introduce a scheme theory on Proj$A = Y$ for some positively graded noncommutative K-algebra $A = K \oplus A_1 \oplus A_2 \oplus \ldots$..., a good idea could be to replace an affine open, something like $Y(f)$ in the commutative case, by a homogeneous Ore set S of A and the ring of sections (in the commutative case $(A_f)_o$) by $(S^{-1}A)_o$. If A is schematic then we have covered Y by "opens" corresponding to the $S_i, i \in \mathcal{I}$. For commutative A if $Y(f_i)$ cover Y then $Y(ff_i)$ cover $Y(f)$ and for modules of sections we have $M_f = \varprojlim_i \{M_{ff_i}, i\}$ where M_f stand for the localization at the multiplicative set $\{1, f, f^2, \ldots\}$. The straightforward generalization of this property would require that the canonical map :

$$(*) \qquad Q\kappa_{S_1} \wedge \ldots \wedge \kappa_{S_d}(M) \longrightarrow \varprojlim \left(\begin{array}{ccc} Q_{S_i}(M) & & \\ & \searrow & \\ \cdots & & Q_{S_i \vee S_j}(M) \\ & \nearrow & \\ Q_{S_j}(M) & & \end{array} \right)$$

has to be an isomorphism for all $M \in A$-gr. Looking at just two Ore sets S and T, (*) will be an isomorphism if and only if Q_S and Q_T commute, i.e. if and only if : $\kappa_S Q_T = Q_T \kappa_S$ and $\kappa_T Q_S = Q_S \kappa_T$. This compatibility does not always hold and the solution is to introduce more "open sets" i.e. to define a suitable noncommutative Grothendieck topology defined in terms of localization functors on a suitable category. For Proj$A$ the category on which the scheme structure is defined is A-gr localized at κ_T, i.e. (A, κ_+)-gr or the finitely generated objects in this. Let us write $\mathcal{O}(A)$ for the set of homogeneous left Ore sets S of A such that $1 \in S, 0 \notin S$ and $S \cap A_+ \neq \emptyset$. The free monoid on $\mathcal{O}(A)$ is denoted by $\mathcal{D}(A)$. If $W = S_1, \ldots, S_n \in \mathcal{W}(A)$ then we write $w \in W$ meaning that w is of the form $s_1 \ldots s_n$ with $s_i \in S_i, i = 1, \ldots, n$. The category $\underline{\mathcal{W}}$ is defined by taking the elements of $W(A)$ for the objects while for words $W = S_1 \ldots S_n, W' = T_1 \ldots T_m$ we define : $\mathrm{Hom}(W', W) = \{W' \to W\}$ or $\emptyset$ depending on whether there exists a strictly increasing map $\alpha : \{1, \ldots, n\} \to \{1, \ldots, m\}$ for which $S_i = T_{\alpha(i)}$ or not. So $\mathrm{Hom}(W'W)$ is a singleton if it is not empty. Put $Q_W(M) = (Q_{S_n} \circ \ldots \circ Q_{S_1})(M) = Q_{S_n}(A) \otimes_A \ldots \otimes Q_{S_1}(A) \oplus_A M$. To W we associate a filter of left ideals of A, $\mathcal{L}(W) = \{L, w \in L \text{ for some } w \in W\}$. For $w, w' \in W$ there are $a, b \in A$ such that : $aw = bw' = w'' \in W$, also for $w \in W, a \in A$ there are $w' \in W, b \in A$ such that $w'a = bw$. For $M \in A$-mod, $\kappa_W(M) = \{x \in M, wx = 0 \text{ for some } w \in W\}$; this κ_W is an exact preradical on A-mod and it is not necessarily idempotent. $\mathcal{L}(W)$ has a cofinal system of graded left ideals so it induces on exact preradical of A-gr. If $W' \to W$ in $\underline{W}$ then $\mathcal{L}(W) \subset \mathcal{L}(W')$ and for every $V \in \underline{W}, W'W \to WV$, as well as $VW' \to VW$, are morphisms in $\underline{W}$. A global cover of $Y = \mathrm{Proj}A$ is just a finite subset $\{W_i, i \in \mathcal{I}\}$ of objects of $\underline{W}$ such that $\bigcap_{i \in \mathcal{I}} \mathcal{L}(W_i) = \mathcal{L}(\kappa_+)$; the existence of at least one global cover given by words consisting of one letter, is ensured by the schematic constitution for A. For $W \in \underline{W}$ we let $\mathrm{cov}(W)$ be $\{W_iW \to W, i \in \mathcal{W}\}$. The category $\underline{W}$ together with the sets $\mathrm{cov}(W)$ form a noncommutative Grothendieck topology. Global covers induce covers because of :

2.1.3 Lemma. *If $\{W_i, i \in \mathcal{I}\}$ is a global cover then for all $V \in \underline{W}$ we have that $\mathcal{L}(V) = \bigcap_{i \in \mathcal{I}} \mathcal{L}(W_iV)$.*

A presheaf Q on $\underline{W}$ is now a contravariant functor from $\underline{W}$ to A-gr such that for all $w \in \underline{W}$ the sections $Q(W)$ of Q over W form a graded $Q_S(R)$-module where S is the last letter of W. For $W = 1$ we demand $Q(1)$ to be a $Q_{\kappa_+}(A)$-module, we write $\Gamma_*(Q) = Q(1)$. It is straightforward to define sheaves by introducing separatedness and glueing conditions.

For any graded A-module M we obtain a structure presheaf $\underline{O}^g_M$ associating to W the $Q_W(M)$.

2.1.4 Theorem. *For any graded A-module M, A being a schematic K-algebra, the structure presheaves $\underline{O}^g_M$ and $\underline{O}_M = (\underline{O}^g_M)_o$ are in fact sheaves !*

The affine-like properties follows from :

2.1.5 Proposition. *Let A be a schematic K-algebra and suppose that $A = K[A_1]$. For every homogeneous Ore set $S \in \mathcal{O}(A)$ (thus $S \cap A_+ \neq \emptyset$) the ring $S^{-1}A = Q_S(A)$ is strongly graded.*

Recall that a R**-graded ring** is said to be strongly graded if $R_nR_{-n} = R_0 = R_{-n}R_n$ for all n, or equivalently $R_1R_{-1} = R_0 = R_{-1}R_1$. For a strongly graded ring $R - \mathrm{gr} \cong R_0$-mod. On the basic opens $Q_S(A)$ is a (strongly) graded ring and $Q_S(M)$ is a graded $Q_S(A)$-module ! This need not hold with respect to Q_W for general W ! To $S \in \mathcal{O}(A)$ we associate a basic open $Y(S)$ given by $Q_S(A)$-gr equivalent to $Q_S(A)_o$-fgmod, (fg stands for finitely generated) the latter may be viewed as "Spec$Q_S(A)_o$".

2.1.6 Definition. A noncommutative projective scheme Proj$A$ is defined by $(A, \kappa_+)_{fg}$-gr with a non-commutative Grothendieck topology $\underline{W}$ with affines $Y(S_i)$ generating the topology by intersections. We may view $Y(S)$ as Spec$A_{(S)}$, where $A_{(S)} = (S^{-1}A)_o$, defined in a categorical way.

2.2 Noncommutative Topology and Categorical Theory

The correspondence between coherent sheaves and module categories over the ring of global sections is in the commutative case given by J. P. Serre's fundamental global sections theorem. In the noncommutative case a sheaf $\mathcal{F}$ on $\underline{W}$ is **quasi-coherent** if there is an affine cover $\{T_i, i \in J\}$ for $Y = $ Proj$A$ together with graded $Q_{T_i}(A)$-modules M_i such that for any morphism $V \to W$ in $\underline{W}$ we obtain a commutative diagram, the vertical arrows representing isomorphisms in A-gr.

$$\begin{tikzcd} \mathcal{F}(T_iW) \arrow[r] \arrow[d] & \mathcal{F}(T_iV) \arrow[d] \\ Q_W(M_i) \arrow[r] & Q_V(M_i) \end{tikzcd}$$

A quasi-coherent $\mathcal{F}$ is said to be coherent if all M_i, $i \in J$, one finitely generated $Q_{T_i}(A)$-modules.

2.2.1 Theorem. *If $\mathcal{F}$ is a quasi-coherent sheaf on $\underline{W}$ and $\Gamma_*(\mathcal{F})(= \mathcal{F}(1))$ denotes its global section A-module then $\mathcal{F}$ is sheaf isomorphic to the structure sheaf of $\Gamma_*(\mathcal{F})$.*

2.2.2 Theorem. (Noncommutative version of J. P. Serre's global section theorem) *For a schematic K-algebra A, the category of quasi-coherent sheaves on $\underline{W}$ is equivalent to (A, κ_+)-gr. The category of coherent scheaves on $\underline{W}$ is equivalent to* Proj(A), *i.e. $(A, \kappa_+)_{fg}$-gr.*

These results are due to L. Willaert, F. Van Oystaeyen, see [141] or also Theorem 2.1.5. in [131].

The Rees ring $\widetilde{A}$ of a Noetherian positively filtered K-algebra A is isomorphic to $\sum F_nAT^n \subset A[T]$ and inverting the central homogeneous element of degree $1, T$, we obtain $\widetilde{A}_T = A[T, T^{-1}]$. We may view $Y(T)$ in $Y = $ Proj$\widetilde{A}$ with sections $A[T, T^{-1}]_{fg}$-gr $= A$-mod$_{fg} \simeq$ Spec$A$.

A filtered K-algebra A as above such that $G(A)$ is a schematic domain has an $\widetilde{A}$ which is again a schematic domain; let $\pi : \widetilde{A} \to \widetilde{A}/T\widetilde{A} \cong G(A)$ be the canonical epimorphism. The Ore sets $S_1, \ldots, S_n$ defining the schematic property for $G(A)$ yield $T_i = \mathcal{L}\pi^{-1}(S_i)$ plus the special Ore set $< T >= S_T$ central in $\widetilde{A}$ and thus compatible to all the $T_i, i = 1, \ldots, n$. The images $\overline{T}_i$ in A via $\widetilde{A} \to A = \widetilde{A}/(T-1), \widetilde{A}$ are saturated Ore sets such that $\sigma(\overline{T}_i) = S_i$ where $\sigma : A \to G(A)$ is the principal symbol map sending $a \in F_nA - F_{n-1}A$ to $\overline{a} = a$ mod$F_{n-1}A$ in $G(A)_n = F_nA/F_{n-1}A$. The Rees ring is the homogenization of A with respect to FA, geometrically this means :

2.2.3 Proposition. *For A as above,* Proj$G(A)$ *is a closed subscheme of $Y = $* Proj$\widetilde{A}$ *and* Proj$\widetilde{A} = $ Proj$G(A) \cup$ Spec$A$ *(cf. Proposition 2.1.10 of [131]).*

So we may think of Proj$G(A)$ as the part at ∞ for the projective closure of Spec$A$. The part Spec$A$ corresponds to the T-torsionfree class of objects from $(A, \kappa_+)_{fg}$-gr, the part Proj$G(R)$ corresponds to the T-torsion objects. Using microlocalizations of filtered rings one may define quantum section, in [131] Section 2.3. many examples of quantum sections are calculated and given by generators and relations, e.g. for the Weyl algebra $\mathbb{A}_1(\mathbb{C})$, enveloping algebras, colour Lie superalgebras, quantized Weyl algebras. We may look at almost commutative geometry by studying filtered rings A as before, but with $G(A)$ an affine commutative algebra generated by homogeneous elements of degree one. For such rings microlocalization functors do commute and sheaf theory becomes more easy, Section 2.4. in [131].

The latter results provide us with more hints that a completely categorical version of noncommutative geometrical may be possible in terms of arbitrary localizations (torsion theories or quotient categories) and a formally defined P noncommutative topology. This was the aim of [135]. We consider a poset Λ with $0, 1$ and take operations $\wedge, \vee$ on Λ satisfying :

A.1. For $x, y \in \Lambda$, $x \wedge y \leq y$

A.2. For $x \in \Lambda, x \wedge 1 = 1 \wedge x, 0 \wedge x = x \wedge 0 = 0$, moreover $x \wedge \ldots \wedge x = 0$ if and only if $x = 0$

A.3. For $x, y, z \in \Lambda, x \wedge y \wedge z = (x \wedge y) \wedge z = x \wedge (y \wedge z)$

A.4. For $a \leq b$ in Λ and $x, y \in \Lambda$ we obtain : $x \wedge a \leq x \wedge b, a \wedge y \leq b \wedge y$ (it follows that $x \wedge y \leq x$ too !)

A.5. For $x, y \in \Lambda, y \leq x \vee y$

A.6. For $x \in \Lambda$, $1 \vee x = x \vee 1 = 1$, $x \vee 0 = x = 0 \vee x$, moreover $x \vee \ldots \vee x = 1$ if and only if $x = 1$

A.7. For $x, y, z \in \Lambda, x \vee (y \vee z) = x \vee y \vee z = (x \vee y) \vee z$.

A.8. For $a \leq b$ in Λ and $x, y \in \Lambda$ we obtain : $x \vee a \leq x \vee b$, $a \vee y \leq b \vee y$ (it follows that $x, y \leq x \vee y$).

A.9. (weak modularity). Let $i_\wedge(\Lambda)$ be the $\wedge$-idempotent elements i.e. the $x \in \Lambda$ such that $x \wedge x = x$, then for $x \in i_\wedge(\Lambda)$ and $x \leq z$ in Λ we have :

$$x \vee (x \wedge z) \leq (x \vee x) \wedge z$$
$$x \vee (z \wedge x) \leq (z \vee z) \wedge z$$

(if Λ satisfies A.1.... A.9., then $i_1(\Lambda) \subset i_\vee(\Lambda)$ where $i_\vee(\Lambda)$ consists of $z \in \Lambda$ such that $z \vee z = z$).

A.10. For any global cover $1 = \lambda_1 \vee \ldots \vee \lambda_n$ and any $z \in \Lambda$ we have : $(x \wedge \lambda_1) \vee \ldots \vee (x \wedge \lambda_n) = x$. The presheaves on Λ with values in a Grothendieck (abelian) category is again a Grothentieck (abelian) category but this fails for the category of sheaves (defined suitably), this category is not a topos. If $x \wedge x = x$ for all x in Λ then Λ is an abelian operation in Λ so the noncommutativity of the topology is exactly characterized by the existence of nontrivial selfintersection.

The definition of a noncommutative Grothendieck topology may be given by "symmetrizaion" of the classical definition. A category $\underline{\mathcal{C}}$ such that for each object U of $\underline{\mathcal{C}}$ a set Cov(U) is given, consisting of subsets of morphisms with common target U, is a noncommutative Grothendieck topology if it satisfies the following properties :

G.1. $\{U \to U\} \in \mathrm{Cov}(U)$

G.2. If $\{U_i \to U, i \in I) \in \mathrm{Cov}(U)$ and $\{U_{ij} \to U_i, j \in J\} \in \mathrm{Cov}(U)$ for all $i \in I$, then $\{U_{ij} \to U, i \in I, \in J\} \in \mathrm{Cov}(U)$.

G.3. For given $U' \to U$ and $\{U_i \to U, i \in I\} \in \mathrm{Cov}(U)$ there is a cover $\{U' \times_U U_i \to U', i \in I\}$ satisfying : for $V \to U_i$, $V \to U'$ and $T \to U_i, T \to U'$ there exist $V \wedge T \to U' X_U U_i$ and $T \wedge V \to U' \times_U U_i$ fitting in the commutative diagram :

$$\begin{tikzcd}
 & & & & U & & & & \\
 & & U_i \arrow[urr] & & & & U' \arrow[ull] & & \\
T \arrow[urr] \arrow[urrrrrr] & & & & U' \times_U U_i \arrow[ull] \arrow[urr] & & & & V \arrow[ullllll] \arrow[ull] \\
 & & T \wedge V \arrow[ull] \arrow[urr] & & & & V \wedge T \arrow[ull] \arrow[urr] & &
\end{tikzcd}$$

Taking $T = V$ in the foregoing, one obtains the obvious non-idempotent versions of the pullback property reducing to G.3. in case $T = T \wedge T$.

2.2.4 Example. Any modular lattice satisfies A.1. ...A.9. A distributive lattice satisfies A.1....A.10. The lattice of all torsion theories on R-mod for a associative ring R, say R-tors, is a complete modular lattice; we shall look at the torsion theories by their kernel functors. If the idempotent kernel functors σ, τ are given by their Gabriel filters $\mathcal{L}(\sigma), \mathcal{L}(\tau)$ resp. then $\sigma \wedge \tau$ and $\sigma \vee \tau$ are defined by $\mathcal{L}(\sigma \wedge \tau), \mathcal{L}(\sigma \vee \tau)$ resp. Define $W(R)$ is the set of filters obtained by evaluating expressions involving products and intersections of filters corresponding to elements of R-tors. For $w, w' \in W(R)$ put $w \leq w'$ if and only if $\mathcal{L}(w') \subset \mathcal{L}(w)$. We define $w \vee w'$ by $\mathcal{L}(w) \cap \mathcal{L}(w')$, hence $\vee$ is a commutative operation here. Put $\mathcal{L}(ww')$ equal to $\{L \in R, L \supset J'J, J' \in \mathcal{L}(w'), J \in \mathcal{L}(w)\}$, this defines $w \wedge w'$ and the corresponding function $Q_w Q_{w'}$.

2.2.5 Proposition. *$W(R)$ consists of exact preradicals and it is a noncommutative topology with respect to the structures defined above.*

A categorical version of noncommutative algebraic geometry can now be developed, cf. [135]; there are many open questions in this theory, I refer to loc. cit. for many exercises and research projects. The example obtained from Ore sets has some interesting applications using Çech-cohomology on the noncommutative topology one may calculate a moduli space for the left ideals of the Weyl algebra (work of L. Willaert, F. Van Oystaeyen, recovering a result of L. Le Bruyn). This technique may very probably be applied to several other quantized algebras where we can calculate enough Ore localizations.

3. Regularity and Filter-Graded Transfer

3.1 Graded Homological Algebra and Regularity

A ling is **left regular** if every finitely generated R-moduli has finite projective dimension. For a graded ring R **left*gr*-regularity** is defined in terms of objects of R-gr. For a left

Noetherien R we have $\mathrm{gldim} R[X_1, \ldots, X_n] = n + \mathrm{gldim} R$ and Auslander's theorem learns that for a Noetherian R, $\mathrm{rgldim} = \mathrm{lgldim} R$. For graded rings the graded versions of several dimensions can be defined (and used) in the obvious way. For example if R is a graded Noetherian ring, then the left and right (graded) global dimensions coincide.

3.1.1 Theorem. *Let R be a Zariski filtered ring (in the positive case $G(R), R, \widetilde{R}$ are Noetherian rings) then :*

1. *If $G(R)$ is left gr-regular then $\widetilde{R}$ is left regular*
2. *We have :*

$$\mathrm{grgldim}\widetilde{R} \leq 1 + \mathrm{grgldim} G(R)$$
$$\mathrm{gldim}\widetilde{R} \leq 1 + \mathrm{gldim} G(R)$$

and equalities hold in case $G(R)$ has finite (gr-)global dimension.

It is now possible to obtain left regularity of a.o. the following rings : $A[X, \sigma.\delta]$ where δ is a σ-derivation of the left regular A and σ automorphism of A, the crossed product $A * G$ where A is left regular and G is poly-infinite cyclic, the crossed product $A * U(g)$ where A is a left regular K-algebra and $U(g)$ the K-enveloping algebra of a finite dimensional Lie algebra $g, \ldots$. For a survey on $GK\mathrm{dim}$ and a new dimension, the schematic dimension Sdim we refer to [131] Section 3.1.

For a left Noetherian R and a finitely generated R-module M we have $\mathrm{pdim}_R M = n < \infty$ if and only if $\mathrm{Ext}_R^{n+1}(M, N) = 0$ for all finitely generated R-modules N, consequently $\mathrm{Ext}_R^n(M, R) \neq 0$. For any R-module M the **grade number** $j_R(M)$ is the unique smallest integer such that $\mathrm{Ext}_R^{j_R(M)}(M, R) \neq 0$; if such integer does not exist then we put $j_R(M) = \infty$. We say that M satisfies the Auslander condition if for $k \geq 0$ and any R-submodule N of $\mathrm{Ext}_R^k(M, R)$ it follows that $j_R(N) \geq k$. If we have an exact sequence of R-modules :

$$0 \to M' \to M \to M'' \to 0$$

then if M', M'' satisfy the Auslander condition so does M. In case M satisfies the Auslander condition then $j_R(M) = \inf\{j_R(M'), j_R(M'')\}$. A left and right Noetherian ring R of finite global dimension is **Auslander regular** if every finitely generated left or right R-module satisfies the Auslander condition.

3.1.2 Theorem. (Li Huishi, F. Van Oystaeyen, cf. [131] Theorem 5) *If R is a (left and right) Zariski filtered ring such that $G(R)$ is Auslander regular then R is Auslander regular. The theorem yields Auslander regularity of the following rings : $U(g)$ for a finite dimensional Lie algebra g, the n-th Weyl algebra $\mathbb{A}_n(K)$, the ring $\mathcal{D}(V)$ of $\mathbb{C}$-linear differential operators on irreducible smooth subvarieties V of affine n-space, the ring $\mathcal{D}_1$ of O-linear differential operators on the reular local ring O_n of convergent power series in n-variables over $\mathbb{C}$, the stalks $\mathcal{E}_P$ of the sheaf of microlocal differential operators,. . .*

3.1.3 Theorem. *Let R be a Zariski filtered ring with $G(R)$ Auslander regular then for every filtered R-module M with good filtration we have $j_R(M) = j_{G(R)}(G(M)) = j_{\widetilde{R}}(\widetilde{M})$ where $\widetilde{M}$ is the Rees module of M with respect to FM.*

Concerning Auslander regularity of the Rees ring we obtain the following result :

3.1.4 Theorem. *If R is a filtered ring such that $\widetilde{R}$ is Noetherian then Auslander regularity of R and $G(R)$ implies Auslander regularity of $\widetilde{R}$. If moroever R is Zariski filtered and $G(R)$ is Auslander regular then $\widetilde{R}$ is Auslander regular.*

Applying the foregoing and some corollaries of it one arrives at the following examples of Auslander regular rings constructed over an Auslander regular A : $A[X,\sigma]$ σ an automorphism of A, $A[[X,\sigma]]$ the X-completion of $A[X,\sigma]$, $A[X,\sigma,\delta]$ where δ is a σ-derivation of A, the crossed product $A*G$ where G is the poly-infinite cyclic group, $A*U(g)$ where A is a K-algebra and g a Lie algebra of finite K-dimension. In particular one finds that the following rings are Auslander regular too : coordinate ring of quantum 2×2 matrices, quantum Weyl algebras $\mathbb{A}_n(q)$, Witten gauge algebras $W(\mathbb{C})$ and "quantum sl_2" $W_q(\mathrm{sl}_2)$.

Using injective resolutions and injective dimension instead of projective resolutions and projective dimension one obtains a similar theory with respect to so-called Auslander-Gorenstein regularity.

A lot of work has gone into the classification of low dimensional algebras e.g. M. Artin, W. Schelter [5]. All 3-dimensional regular algebras have been classified by P. R. Stephenson. Using Cohen-Macauley modules point and line modules over a 3-dimensional quadratic algebra are classified by their homological properties. If R is graded and $R = R_0[R_1]$ then a (left) point module is a cyclic graded R-module $M = \oplus_{n\geq 0} M_n$ such that $M = RM_0$ and the Hilbert series $H_M(t)$ is $(1-t)^{-1}$. A (left) line module is as before but with $H_M(t) = (1-t)^{-2}$.

The point modules, force the Hibert series to look as in the commutative case, maybe too commutative in spirit to yield a good tool in noncommutative geometry. In fact there exist higher dimensional regular algebras with finitely many, say 20, point modules, even there are some without points. Some of these nice algebras, having very few point modules are graded (generic) Clifford algebras.

Put $C = \mathbb{C}[Y_1,\dots,Y_n], \alpha \in M_n(C)$ a symmetric matrix (α_{ij}) where each α_{ij} is a homogeneous linear polynomial. The **Clifford algebra** $A(\alpha)$ associated to α is defined as the K-algebra with generators $\{X_1,\dots,X_n,Y_1,\dots,Y_n\}$ and defining relations :

$$\begin{cases} X_iX_j + X_jX_i = \alpha_{ij}, \text{ for } i,j = 1,\dots,n \\ [Y_i,X_j] = 0 = [Y_i,Y_j] \text{ for } i,j, = 1\dots n \end{cases}$$

The gradation of $A(\alpha)$ is defined by putting $X_i \in A(\alpha)_1$ for $i = 1,\dots,n$ and $Y_j \in A(\alpha)_2$ for $j = 1,\dots,n$. Expanding $\alpha = \alpha_1 Y_1 + \dots + \alpha_n Y_n$ where $\alpha_1,\dots,\alpha_n$ are symmetric matrices in $M_n(K)$ we associate to α an n-dimensional linear system of quadrics $Q_1,\dots,Q_n \subset \mathbb{P}^{n-1}(K), \underline{Q} = KQ_1 + \dots + KQ_n$ where $Q_i = \{z \in \mathbb{P}^{n-1}, z^t\alpha z = 0\}$. A base point of $\underline{Q}$ is any point in the intersection of $Q_1,\dots,Q_n$.

3.1.5 Proposition. (L. Le Bruyn, J. of Algebra 177, 1995)*A Clifford algebra $A(\alpha)$ is a quadratic Auslander regular algebra of dimension n if and only if $\underline{Q}$ has no base points.*

For $n = 4$, M. Van den bergh proved that $A(\alpha)$ has exactly 20 point modules for generic α, an explicit construction of such algebra was given by M. Van Cliff, K. Van Rompay, L. Willaert.

Several references provide a good starting point for reading about low dimensional regular algebras. Results now exist up to dimension 4 as fas as I know. There are interesting research problems here e.g. the relation between graded deformations and so-called rolled up Rees rings (cf. [68]).

3.2 Examples

Several examples of well-known rings popular nowadays have been referred to with reference to the literature. As an appendix I include some of them with full definition.

4. Applications and other Directions

4.1 Cayley Smooth Orders

As a consequence of project 1.1. there always was a tendency to try to relate noncommutative information to the commutative theory (via the centre of the algebras used). This is the case too with the theory related to canonical resolutions of quotient singularities. For a finite group G acting on the vector space $\mathbb{C}^d$ (freely away from the origin), the quotient spqce $\mathbb{C}^d/G$ is an isolated singularity and resolutions $Y \twoheadrightarrow \mathbb{C}^d|G$ were constructed using the skew groupring $\mathbb{C}[X_1, \ldots, X_d] * G$ which is an order having the fix-ring $\mathbb{C}[X_1, \ldots, X_d]^G$ for its centre. In case $d = 2$ we are in the situation of Kleinian singularities this yields minimal resolutions. In case $d = 3$ the skew groupring appears via the superpotential and commuting matrices (in Physics) or via the McKay quiver. For abelian G the study leads to "crepant" resolutions, for general G one obtains partial resolutions with remaining manifold singularities. In [69] L. Le Bruyn obtains lists of types of singularities contained in partial resolutions of the quotient variety $\mathbb{C}^d/G$.

Smoothness of R-orders , R a commutative ring e.g. the coordinate ring of some (quotient) variety, is defined in two ways :

(1) J. P. Serre smoothness i.e. the R-order A has finite global dimension plus Auslander regularity and the Cohen-Macauley property
(2) Cayley-smooth of the corresponding G-variety is smooth. The Zariski and étale covers are used.

Cayley-Hamilton algebras are introduced as algebras with a nice trace map (Definition 1.4. in [69]) and every R-order in a central simple algebra is a CH-algebra. C. Procesi proved the reconstruction of orders and their centers from the G-equivariant geometry of the quotient variety in case $G = PGL_n$. The category of CH-algebras of degree n with trace preserving morphisms constitutes a version of noncommutative geometry. A Cayley smooth algebra A is an object of the foregoing category with a lifting property i.e. if R is in the category and I is a nilpotent ideal of B such that $B|I$ is in the category and the natural $B \to B|I$ preserves the trace than any trace preserving $\emptyset : A \to BI$ lifts to $A \to B$. These Cayley smooth algebras correspond to smooth PGL_n-varieties. The noncommutative structure sheaf of an R-order is then used as the noncommutative geometry, as explained in the first part of this survey. Via the representation theory (marked) quiver settings are associated to the orders which connect the Zariski and étale structure to quivers (Theorem 7.9. of [69]. This leads to quiver-recognition of isolated singularities (Theorem 1.12. loc. cit) and noncommutative desingularizations. We refer to loc. cit. for complete detail.

4.2 Hopf Algebras and twisted Algebras

The effect of a Hopf algebra structure on the noncommutative geometry remains largely to be studied. Of course one may consider braided categories and localizations thereof but this does not connect nicely to something like a noncommutative algebraic group. For P.I. rings we know that the assumption of a group variety structure on its prime spectrum makes it into a commutative variety, perhaps one should look for a theory of noncommutative algebraic semigroups ? In the direction of valuation theory there has been some work by Aly Farahat, F. Van Oystaeyen on Hopf valuations and related Hopf orders. An interesting consequence of this theory (cf. [44]) is the appearance of new maximal orders over specific number rings leading to very concrete examples.

On the other hand, a replacement of the geometric product may be found by using the twisted product of algebras. A general theory of twisting algebras appeared in the paper [83] by X. Lopez, F. Panaite, F. Van Oystaeyen. An example is given by A. Connes quantum space that turns out to be a twisted product of quantum planes. The twisted product can be iterated under some pentagonal diagramme condition, cf. [89]. The algebraic properties of general twisted products of low dimensional algebras (e.g. with the quaternions $\mathbb{H}$ over the reals) should be further investigated. Since connections behave well with respect to twisted products some further relations with A. Conne's noncommutative geometry remain to be investigated.

4.3 Simple Modules

The classification of simple (left) modules of a noncommutative algebra is a basic problem relating to representation theory on one side and to some kind of noncommutative geometry on the other side. For algebras of quantized type (deformations) not many cases have been completely solved. For example the case of the second Weyl algebras remained open for a while till V. Bavula, F. Van Oystaeyen obtained a classification by pairs of elements in twisted Laurent polynomials in [17]. They continued this for rings of differential operators on surfaces that are products of curves in [18]. The techniques make use of a gradation and graded module theory as well as G/K-dimension.

5. Appendix : Some Examples

5.1 Quantum 2×2-matrices

The $\mathbb{C}$-algebra generated by a, b, c, d with defining relations :

$$\begin{aligned} &ba = q^{-2}ab, ca = q^{-2}ac, bc = cb, \\ &db = q^{-2}bd, dc = q^{-2}cd, ad - da = (q^2 - g^{-2})bc \end{aligned}$$

is called the algebra of quantum 2×2-matrices $M_q(2)$. Then $M_q(2)$ is a schematic algebra and a Noetherian domain as it is an iterated Ore extension of a nice kind :

$R_1 = \mathbb{C}[a]$
$R_2 = \mathbb{C}[a, b]/(ba - q^{-2}ab)$

$R_3 = \mathbb{C}[a,b,c]/(ba - q^{-2}ab, ca - q^{-2}ac, bc - cb)$
$R_2 = R_1[b, \rho_1]$ where the automorphism ρ_1 of R_1 is determined by $\rho_1(a) = q^{-2}a$.
$R_3 = R_2[c, \rho_2]$ where $\rho_2(a) = q^{-2}a$, $\rho_2(b) = b$
Finally $M_q(2) = R_3[d, \rho_3, \delta]$ where the automorphism ρ_3 is determined by $\rho_3(a) = a, \rho_3(b) = q^{-2}b, \rho_3(c) = q^{-2}c$ and the ρ_3-derivation δ is given by $\delta(a) = (q^2 - q^{-2})bc$ and $\delta(b) - \delta(c) = 0$.

5.2 Quantum Weyl Algebras

Look at $(\lambda_{ij}) \in M_n(k)$ with $\lambda_{ij} \in k^*$, together with a row $(q_1, \ldots, q_n), q_i \in k^*$. The **quantum Weyl algebra** $A_n(\overline{q}, \Lambda)$ in the R-algebra generated by $x_1, \ldots, x_n, y_1, \ldots, y_n$ with defining relations : (putting $\mu_{ij} = \lambda_{ij}q_i$), for $i \subset j$:

$$\begin{aligned} x_ix_j &= \mu_{ij}x_jx_i \\ x_iy_j &= \lambda_{ji}y_jx_i \\ y_jy_i &= \lambda_{ji}y_iy_j \\ x_jy_i &= \mu_{ij}y_ix_j \\ x_jy_j &= q_jy_ix_j + 1 + \textstyle\sum_{i<j}(q_i - 1)y_ix_i \end{aligned}$$

We may again establish that $A_n(\overline{q}, \Lambda)$ is an iterated Ore extension by adding the variables in the order : $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$. The associated graded rings with respect to the standard filtrations may be calculated and one obtains the fact that $A_n(\overline{q}, \Lambda)$ is affine schematic (and its Rees rings too) and also schematic.

5.3 The Sklyanin Algebra

The 3-dimensional algebra generated over k by three homogeneous elements of degree $1, X, Y, Z$ say, with defining relations :

$$\begin{aligned} aXY + bYX + cZ^2 &= 0 \\ aYZ + bZY + cX^2 &= 0 \\ aZX + bXZ + cY^2 &= 0 \end{aligned}$$

$(a, b, x \in k)$ is said to be the Sklyanin algebra $S_k(a, b, c)$. This algebra is schematic.

5.4 Color Lie Superalgebras

Consider an abelian group Γ and $\epsilon : \Gamma \times \Gamma \to \mathbb{C}^*$ satisfying : $\epsilon(\alpha, \beta)\ \epsilon(\beta, \alpha) = 1,\ \epsilon(\alpha, \beta + \gamma) = \epsilon(\alpha, \beta)\ \epsilon(\alpha, \gamma),\ \epsilon(\alpha + \beta, \gamma) = \epsilon(\alpha, \gamma)\ \epsilon(\beta, \gamma)$.

Let $L = \oplus_{\gamma \in \Gamma} L_\gamma$ be a Γ-graded vector space together with a graded bilinear mapping $< \ldots, \ldots >$ satisfying for $a \in L_\alpha, b \in L_\beta, c \in L_\gamma, \alpha, \beta, \gamma \in \Gamma, < a, b >= -\epsilon(\alpha, \beta) < b, a >$
$0 = \epsilon(\gamma, \alpha) < a, < b, c >> + \epsilon(\alpha, \beta) < b, < c, a >> + \epsilon(\beta, \gamma) < c, < a, b >>$

Consider the tensor algebra $T(L)$ and let $J(L)$ be the ideal generated by all

$$a \otimes b - \epsilon(\alpha, \beta) b \otimes a - < a, b >$$

for $a \in L_\alpha, b \in L_\beta$. The algebra $T(L)/J(L)$ is the universal enveloping algebra of L, it is a Γ-graded ring and it has also a positive filtration by taking $F_nU_K(L)$ to be the image of $T(K)_n$. ¿From the generalized Poincaré-Birkhoff-Witt theorem it follows that the associated $\mathbb{Z}$-graded algebra $G(U(L))$ is a $\mathbb{Z} \times \Gamma$-graded algebra isomorphic to $T(L)$ modulo the ideal generated by all $a \otimes b - \epsilon(\alpha, \beta) b \otimes a$, for $a \in L_\alpha, b \in L_\beta$. We have established (Theorem 1.2.19 in [131]) that $G(U(L))$ is schematic, $U(L)$ is weakly affine schematic and the Rees ring $U(L)^{\sim}$ is schematic.

5.5 Witten's Gauge Algebras

Consider the $\mathbb{C}$-algebra W generated by X, Y, Z subjected to the relations :

$$\begin{aligned}&XY + \alpha YX + \beta Y = 0\\&YZ + \gamma ZY + \delta X^2 + \epsilon X = 0\\&ZX + \xi XZ + \eta Z = 0\end{aligned}$$

Total degree on X, Y, Z defines the standard filtration on W. It is not hard to verify that $G(W)$ is defined by the relations :

$$\begin{aligned}&XY + \alpha YX = 0\\&YZ + \gamma ZY + \delta X^2 = 0\\&ZX + \xi XZ = 0\end{aligned}$$

The algebra $G(W)$ is quadratic and represents a quantum space in the sense of Y. Manin. The algebra $W(\mathbb{C})$ is weakly affine schematic, $G(W)$ and the Rees ring $\widetilde{W}$ are schematic.

5.6 Quantum sl_2 (Woronowicz)

Let $W_q(\mathrm{sl}_2)$ be the $\mathbb{C}$-algebra generated by X, Y, Z subjected to the following defining relations :

$$\begin{aligned}&\sqrt{q}XZ - \sqrt{q}^{-1}ZX = \sqrt{q+q^{-1}}Z\\&\sqrt{q}^{-1}XY = \sqrt{q}YX = -\sqrt{q+q^{-1}}Y\\&YZ - ZY = (\sqrt{q} - \sqrt{q}^{-1})X^2 - \sqrt{q+q^{-1}}X\end{aligned}$$

(classically $q = \exp\left(\frac{2\pi i}{k+2}\right)$ and k is the Chern coupling constant.

In $W_q(\mathrm{sl}_2)$ there is a central quadratic element, the deformed Casimir operator $C = \sqrt{q}^{-1}ZY + \sqrt{q}YZ + X^2$. Put $A = 1 - C(\sqrt{q} - \sqrt{q}^{-1})(\sqrt{q+q^{-1}})$ and write :

$$\begin{aligned}&x = (X - (\sqrt{q} - \sqrt{q}^{-1})\sqrt{q+q^{-1}}^{-1}c)\sqrt{q+q^{-1}}A^{-1}\\&y = Y(\sqrt{q+q^{-1}}^{-1})\sqrt{A}^{-1}\\&z = Z(\sqrt{q+q^{-1}}^{-1})\sqrt{A}^{-1}\end{aligned}$$

which is posible **up to inverting the central element** A ! The relations rewrite in the new arguments x, y, z as

$$\begin{aligned}&\sqrt{q}\;xz - \sqrt{q}^{-1}zx = z\\&\sqrt{q}^{-1}xy - \sqrt{q}\;yx = y\\&q^{-1}zy - qyz = x\end{aligned}$$

One calculates from this the relations for the associated graded rings in the standard filtration :

$$\begin{aligned}&\sqrt{q}\;xz - \sqrt{q}^{-1}zx = 0\\&\sqrt{q}^{-1}xy - \sqrt{q}\;yx = 0\\&q^{-1}zy - qyz = 0\end{aligned}$$

The Rees ring $W_q(\mathrm{sl}_2)^{\sim}$ can be written by homogenizing the relations between the x, y, z.

Looking at the Witten algebra W defined by putting $\delta = 0$ and making obvious choices for the $\alpha, \beta, \gamma, \ldots$ it is clear that the special Witten algebra then obtained contains A^{-1} as a normalizing element. This means that $W_q(\mathrm{sl}_2)$ and the special Witten algebra are birational in the noncommutative sense (up to inverting a central element in the first and a normalizing element in the second they yield the same localization but up to the quadratic extension obtained by adding $\sqrt{A}$. Again $G(W_q(\mathrm{sl}_2))$ and the Rees ring $W_q(\mathrm{sl}_2)^{\sim}$ are schematic.

All foregoing examples are Auslander regular (3.2.17. of [131]).

References

[1] E. Abe, *Hopf Algebras*, Cambridge Univ. Press, Cambridge, 1977.

[2] M. Artin, *Geometry of Quantum Planes*, in : Azumaya Algebras, Actions and Groups, (eds. D. Haile and J. Osterberg, Contemp. Math. Vol 124, Amer. Math. Soc., Providence, 1992.

[3] M. Artin, *Grothendiek Topologies*, Harvard Univ. Lect. Notes, Cambridge Mass. 1962.

[4] M. Artin, A. Grotendieck, J.-L. Verdier, *Théorie des topos et cohomologie étale des schémas*, Lect. Notes in Math. 269, 270, 305, Springer Verlag, Berlin, 1972-73.

[5] M. Artin, W. Schelter, *Graded algebras of global dimension 3*, Adv. Math. **66** (1987), 171-216.

[6] M. Artin, J. Tate, M. Van den Bergh, *Modules over Regular Algebras of Dimension 3*, Invent. Math. **106** (1991), 335-388.

[7] M. Artin, J. Tate, M. Van den Bergh, *Some algebras associated to automorphisms of curves*, in The Grothendieck Festschrift (Eds. P. Cartier et al), Birkhauser, Basel, 1990.

[8] M. Artin, J. Zhang, *Noncommutative Projective Schemes*, Adv. in Math. **109** (1994), 228-287

[9] M. Artin, *Local Structure of Maximal Orders on Surfaces* in : LNM 917, Springer Verlag, Berlin 1982, pp. 146-181.

[10] M.J. Asensio, M. Van den Bergh, F. Van Oystaeyen, *A new algebraic approach to microlocalization of filtered rings*, Trans. Amer. Math. Soc.**316**(2) (1989), 537-555.

[11] B. Auslander, *The Brauer Group of a Ringed Space*, J. Alg. **4** (1966), 220-273.

[12] M. Auslander, O. Goldman, *The Brauer Group of a Commutative Ring*, Trans. Amer. Math. Soc. **97** (1960), 367-409

[13] M. Auslander, *On the dimension of modules and algebras III*, Nagoya Math. J. **9** (1955), 67-77.

[14] M. Awami, F. Van Oystaeyen, *On filtered rings with Noetherian associated graded rings, in Proceedings of Ring Theory Meeting*, Granada 1986, Springer Verlag, LNM.

[15] G. Azumaya, *On Maximally Central Algebras*, Nagoya Math. J. **2** (1951), 119-150.

[16] H. Bass, *Algebraic K-Theory*, Benjamin, New York, 1968.

[17] V. Bavula, F. Van Oystaeyen, *Simple Holonomic Modules over the Second Weyl Algebra*, Adv. Math. **150**(1) (2000), 80-116.

[18] V. Bavula, F. Van Oystaeyen, *Simple Holonomic Modules over Rings of Differential Operators with Regular Coefficients of Krull Dimension 2*, Trans. Amer. Math. Soc. **353**(6) (2001), 2193-2214.

[19] G. Bergman, *Everybody Knows What a Hopf Algebra is.* Contemp. Math. **43** (1985), 25-48.

[20] I.N. Bernstein, *Algebraic theory of $\mathcal{D}$-modules*, Preprint, 1982.

[21] I.N. Bernstein, *On the dimension of modules and algebras III*, Direct limits, Nagoya Math. J. **13** (1958), 83-84.

[22] J.-E. Björk, *The Auslander condition on Noetherian rings*, Proc. Sem. Malliavin, LNM, 1404, Springer Verlag, 1988-173.

[23] J.-E. Björk, *Filtered Noetherian rings*, Math. Surveys and Monographs, No. 24, 1987, 59-97.

[24] J.-E. Björk, *Rings of differential operators*, Math. Library 21, North Holland, Amsterdam, 1979.

[25] R. Bocklandt, L. Le Bruyn, G. Van de Weyer, *Smooth Order Singularities*, J. of Algebra, Appl. **2**(4) (2003), 365-395.

[26] R. Bocklandt, *Graded Calabi-Yau Algebras of Dimension 3*, J. Pure Appl. Algebra **212**(1) (2008), 14-32.

[27] M. Boratinski, *A change of rings theorem and the Artin Rees property*, Proc. Amer. Math Soc. **53** (1975), 307-320.

[28] A. Borel, *Algebraic $\mathcal{D}$-modules*, Academic press, London-New York, 1987.

[29] W. Borho, J. L. Brylinski, *Differential Operators in Homogeneous Spaces Irreducibility of the associated Variety*, Invent. Math. **69** (1982), 437-476.

[30] D. Bulacu, F. Panaite, *A Generalization of the Quasi-Hopf Algebra $D^\omega(G)$*, Comm. Alg. , to appear.

[31] S. Caenepeel, *Brauer Groups, Hopf Algebras and Galois Theory*, Kluwer Academic Publ., Dordrecht, 1998.

[32] S. Caenepeel, F. Van Oystaeyen, *Brauer Groups and the Cohomology of Graded Rings*, Monographs. vol. 121, M. Dekker Publ., New York, 1989

[33] S. Caenepeel, F. Van Oystaeyen, Y. H. Zhang, *The Brauer Group of Yetter-Drinfel'd Module Algebras*, Trans. Amer. Math. Soc. **349** (1997), 3737-3771.

[34] S. Caenepeel, F. Van Oystaeyen, Y. H. Zhang, *Quantum Yang-Baxter Module Algebras*, K-Theory **8** (1994), 231-255.

[35] H. Cartan, S. Eilenberg, *Homological Algebra*, Princeton Univ. Press., Princeton 1956.

[36] A. Connes, *C^*-Algébres et Géometrie Differentielle*, C.R. Acad. Sc. Paris **290** (1980), 599-604.

[37] P. Deligne, J. S. Milne, *Tannakian Categories*, Lect. Notes in Math. 900, Springer-Verlag, Berlin 1982.

[38] F. De Meyer, E. Ingraham, *Separable Algebras over Commutative Rings*, Lect. Notes in Math. 181, Springer Verlag, Berlin 1971.

[39] R. Dijkgraaf, V. Pasquier, P. Roche, *Quasi-Hopf Algebras*, Group Cohomology and Orbifold Models, Nuclear Phys. B. Proc. Suppl. 18 B, 1990, 60-72.

[40] J. Dixmier, *Enveloping algebras*, Math. Library, North Holland, Vol. 14, Amsterdam, 1977.

[41] V.G. Drinfel'd, *Quantum Groups*, in Proc. ICM, Amer. Math. Soc., Providence, 1987, 798-820

[42] V.G. Drinfel'd, *Quasi-Hopf Algebras*, Leningrad Math. J. **1** (1990), 1419-1457.

[43] V.G. Drinfel'd, *Quasi-Hopf Algebras*, Algebra i Amaliz **1** (1989), 114-148.

[44] A. Farahat, F. Van Oystaeyen, *Hopf Filtrations and Larson-type Orders in Hopf Algebras*, J. of Algebra **267**(2) (2003), 756-772.

[45] R. M. Fossum, A. Griffith, I. Reiten, *Trivial extensions of abelian categories, homological algebra of trivial extensions of abelian categories with applications to ring theory*, LNM. 456, Springer-Verlag, 1975.

[46] O. Gabber, *The integrability of characteristic varieties*, Amer. J. Math. **103** (1981), 445-468.

[47] P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France **90** (1962), 323-448.

[48] P. Gabriel, R. Rentschler, *Sur la dimension des anneaux et ensembles ordonnées*, C. R. Acad. Sci. Paris **265** (1967), 712-715.

[49] V. Ginsburg, *Characteristic varieties and vanishing cycles*, Invent. Math. **84** (1986), 327-402.

[50] J. Golan, *Localization of noncommutative rings*, M. Dekker, New York, 1975.

[51] O. Goldman, *Rings and Modules of Quotients*, J. of Algebra, **13** (1969), 10-47.

[52] R. Gordon, J. C. Robson, *Krull Dimension, Memoirs of the AMS*, No. 133, 1973.

[53] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Math. 52, Springer Verlag, Berlin, 1977.

[54] M. Hussein, F. Van Oystaeyen, *Discrete Valuations Extend to Certain Algebras of Quantum Type*, Comm. in Algebra **24**(8) (1996), 2551-2566.

[55] M. Jimbo, *A q-difference Analog of $U(g)$ and the Yang-Baxter Equation*, Lett. Math. Phys. **10** (1985), 63-69.

[56] D.A. Jordan, *Iterated skew Polynomial Rings and Quantum Groups*, J. Alg. **156** (1993), 194-218.

[57] A. Joyal, R. Street, *Braided Monoidal Categories*, Adv. Math. **102** (1993), 20-78.

[58] A. Joyal, R. Street, *Tortile Yang-Baxter Operators in Tensor Categories*, J. Pure Appl. Algebra **71** (1991), 43-51.

[59] M. Kashiwara, *B-function and holonomic systems*, Invent. Math. **38** (1976), 33-53.

[60] C. Kassel, *Quantum Groups*, Springer-Verlag, Berlin, 1995.

[61] C. Kassel, V. Turaev, *Double Construction for Monoidal Categories*, Acta Math. **175** (1995), 1-48.

[62] M. Kontsevich, *Triangulated Categories and Geometry*, Course a É cole Norm. Sup., Paris, 1998.

[63] G. R. Krause, T. H. Lenagan, *Growth of algebras and Gelfand-Kirillov dimension*, Research Notes in Mathematics, 116, Pitman Advanced Publishing Program, 1985.

[64] R. G. Larson, J. Towber, *Two Dual Classes of Bialgebras Related to the Concepts of Quantum Groups and Quantum Lie Algebras*, Comm. Algebra **19** (1991), 3295-3345.

[65] L. Le Bruyn, *Conformal sl_2 Enveloping Algebras*, Comm. Alg. **23**(4), (1995), 1325-1362.

[66] L. Le Bruyn, *Homological Properties of Braided Matrices*, J. of Alg. **170**(2) (1994), 596-607.

[67] L. Le Bruyn, F. Van Oystaeyen, *Quantum Sections and Gauge Algebras*, Publ. Math. **36**(2A) (1992), 693-714.

[68] L. Le Bruyn, M. Van den Bergh, F. Van Oystaeyen, *Graded Orders*, Birkhauser Verlag, 1988.

[69] L. Le Bruyn, *Nomcommutative Geometry and Cayley-smooth Orders*, in Pure and Applied Mathematics, Chapman & Hall, Lect. Notes Vol. 290, Boca Raton, 2006.

[70] Huishi Li, *Note on microlocalization of filtered rings and the embedding of rings in skew fields*, Bull. Math. Soc. Belgique, (serie A) **43** (1991), 49-57.

[71] Huishi Li, *Note on pure module theory over Zariskian filtered ring and the generalized Roos theorem*, Comm. Alg. **19** (1991), 843-862.

[72] Huishi Li, *Structure Sheaves on Almost Commutative Algebras*, Comm. in Algebra, to appear.

[73] Huishi Li, M. Van den Bergh, F. Van Oystaeyen, *The global Dimension and Regularity of Rees Rings for Non-Zariskian Filtration*, Comm. Alg. **18** (1990), 3195-3208.

[74] Huishi Li, F. Van Oystaeyen, *Dehomogenization of Gradings to Zariskian Filtrations and Applications to Invertible Ideals*, Proc. Amer. Math. Soc. **115**(1) (1992), 1-11.

[75] Huishi Li, F. Van Oystaeyen, *Filtrations on Simple Artinian Rings*, J. Alg. **132** (1990), 361-376.

[76] Huishi Li, F. Van Oystaeyen, *Global dimension and Auslander regularity of graded rings*, Bull. Math. Soc. Belgique, (serie A) **43** (1991), 59-87.

[77] Huishi Li, F. Van Oystaeyen, *Strongly filtered rings, applied to Gabber's integrability theorem and modules with regular singularities*, Proc. Sem. Malliavin, LNM. 1404, Springer-Verlag, 1988.

[78] Huishi Li, F. Van Oystaeyen, E. Wexler-Kreindler, *Zariski Rings and Flatness of Completion*, J. Alg. **138** (1991), 327-339.

[79] Huishi Li, F. Van Oystaeyen, *Zariskian Filtrations*, Kluwer Monographs, Vol. 2, Kluwer Acad. Publ., 1995.

[80] Huishi Li, F. Van Oystaeyen, *Zariskian Filtrations*, Comm. Alg. **17** (1989), 2945-2970.

[81] F. W. Long, *The Brauer Group of Dimodule Algebras*, J. Alg. **31** (1974), 559-601.

[82] F.W. Long, *Generalization of the Brauer Group of Graded Algebras*, Proc. London Math. Soc. **29** (1974), 237-256.

[83] X. Lopez, F. Panaite, F. Van Oystaeyen, *General Twisting of Algebras*, Adv. Math. **212**(1) (2007), 315-337.

[84] S. Maclane, *Categories for the Working Mathematician*, Springer-Verlag, New York, 1974.

[85] S. Majid, *Algebras and Hopf Algebras in Braided Categories*, Lect. Notes in Pure Applied Math. 158, 1994, 55-105.

[86] S. Majid, *Doubles of Quasitriangular Hopf Algebras*, Comm. Alg. **19** (1991), 3061-3073.

[87] S. Majid, *Tannaka-Krein Theorem for Quasi-Hopf Algebras and Other Results*, Contemp. Math. **134** (1992), 219-232.

[88] Yu I Manin, *Quantum Groups and Noncommutative Geometry*, Publ. Centre Math. Montreal, 1988.

[89] P. J. Martinez, J. Lopez, F. Panaite, F. Van Oystaeyen, *On Iterated Twisted Tensor Products of Algebra*, Internat. J. Math. **19**(9) (2008), 1053-1101.

[90] J. C. McConnell, *The K-theory of filtered rings and skew Laurent extensions*, LNM. 1146, Springer-Verlag, 1987.

[91] J.C. McConnell, J.C. Robson, *Noncommutative Noetherian rings*, J. Wiley, London, 1987.

[92] S. Montgomery, *Hopf Algebras and Their Actions on Rings*, Amer. Math. Soc., Providence, 1993.

[93] D. Murdoch, F. Van Oystaeyen, *Noncommutative Localization and Sheaves*, J. of Algebra **35** (1975), 500-515.

[94] C. Năstăsescu, E. Nauwelaerts, F. Van Oystaeyen, *Arithmetically Graded Rings Revisited*, Comm. Alg. **14**(10) (1986), 1991-2017.

[95] C. Năstăsescu, M. Van den Bergh, F. Van Oystaeyen, *Separable Functors Applied to Graded Rings*, J. Alg. **123** (1989), 397-413.
[96] C. Năstăsescu, F. Van Oystaeyen, *Dimensions of Ring Theory*, D. Reidel Publ. Co., 1987.
[97] C. Năstăsescu, F. Van Oystaeyen, *Graded Ring Theory*, Library of Math. 28, North-Holland, Amsterdam 1982.
[98] D.G. Northcott, *An Introduction to Homological Algebra*, Cambridge Univ. Press, London and New York, 1960.
[99] D. G. Northcott, *Lessons on Rings, Modules and Multiplicities*, Cambridge Univ. Press, London and New York, 1968.
[100] A. V. Odeskii, B. L. Feigin, *Elliptic Sklyanin Algebras*, (in Russian), Funkt. Anal. i Priloz **23** (1989), 45-54
[101] F. Panaite, F. Van Oystaeyen, *Quasi-Hopf Algebras and the Centre of a Tensor Category*, Preprint 1998.
[102] B. Pareigis, *The Brauer Group of a Symmetric Monoidal Category*, in Lect. Notes in Math. 549, Springer Verlag, Berlin, 1976.
[103] B. Pareigis, *Non-additive Ring and Module Theory IV, The Brauer Group of a Symmetric Monoidal Category*, Lect. Notes in Math. 549, 1976, 112-133.
[104] B. Pareigis, *Non-additive Ring and Module Theory I, III.* Publ. Math. Debrecen **24** (1977), 189-204, Publ. Math. Debrecen **25** (1978), 177-186.
[105] N. Popescu, *Abelian Categories with Applications to Rings and Modules*, Academic Press, London, 1973.
[106] C. Procesi, *Rings with Polynomial Identities*, Marcel Dekker INC. , New York, 1977.
[107] D. Radford, *The Structure of Hopf Algebras with a Projection*, J. Alg. **92** (1987), 322-347.
[108] A. Radwan, F. Van Oystaeyen, *Coherent Sheaves over Microstructure Sheaves.*, Bull. Soc. Math. Belg, 1993.
[109] A. Radwan, F. Van Oystaeyen, *Microstructure Sheaves and Quantum Sections over Projective Schemes* in : Hermann, Paris 1995.
[110] R. Rentschler, P. Gabriel, *Sur la Dimension des Anneaux et Ensembles Ordonnés*, C.R. Acad. Sci. Paris **265** (1967), 712-715.
[111] J.-E. Roos, *Bidualité et structure des foncteurs dérivés d'un anneau régulier*, C. R. Acad. Sci. Paris **254** 1962.
[112] J.-E. Roos, *Compléments à l' étude des quotients primitifs des algèbres enveloppantes des algébres de Lie sémi-simple*, C. R. Acad. Sci. Paris. Sér. A-B.
[113] A. Rosenberg, *Noncommutative Geometry*, Reidel Publ. G., Amsterdam, 1997.
[114] J. J. Rotman, *An introduction to homological algebra*, New York, 1979.
[115] R. Sallam, F. Van Oystaeyen, *A Microstructure Sheaf and Quantum Section over a Projective Scheme*, J. Alg. **158**(1) (1993), 201-225.
[116] P. Schapira, *Microdifferential Systems in the Complex Domain*, Springer Verlag, Berlin, 1985.
[117] O. Schilling, *The Theory of Valuations*, AMS. Math. Suur. IV, 1950.
[118] J.P. Serre, *Algébre Locale, Multiplicités*, LNM. 11, Springer-Verlag, 1975.
[119] J.P. Serre, *Faisceaux Algébriques Cohérents*, Ann. Math. **61** (1955), 197-278.
[120] G. Sjödin, *On filtered modules and their associated graded modules*, Math. Scand. **33** (1973), 229-240.
[121] E. K. Sklyanin, *Some Algebraic Structures connected to the Yang-Baxter Equation*, Func. Anal. Appl. **16** (1982), 27-34
[122] T. Springer, *Microlocalization algébrique*, Sem. Algèbre Dubreil- Malliavin, LNM, Springer Verlag, Berlin, 1984.
[123] J. T. Stafford, *Non-holonomic modules over Weyl algebras and enveloping algebras*, Invent. Math. **79** (1984), 619-638.
[124] M. E. Sweedler, *Hopf Algebras*, Benjamin, N.Y., 1969.
[125] B. R. Tennison, *Sheaf Theory*, LMS Lect. Notes Series, vol. 20, Cambridge Univ. Press, 1975.
[126] K.H. Ulbrich, *On Hopf Algebras and Rigid Monoidal Categories*, Isp. J. of Math. **72** (1990), 252-256.

[127] M. Van den Bergh, F. Van Oystaeyen, *Lifting maximal orders*, Comm. Alg. **2**(2) (1989), 341-349.
[128] M. Van den Bergh, *Noncommutative Crepant Resolutions*, Springer verlag, Berlin, pp. 749-770, 2004.
[129] A. Van den Essen, *Modules with regular singularities over filtered rings*, P.R.I. for Mathematical Science, Kyoto Univ., Vol. 22, No. 5, 1986.
[130] J. Van Geel, *Places and valuations in noncommutative ring theory*, Lecture notes, Vol. 71, Marcel Dekker, New York, 1981.
[131] F. Van Oystaeyen, *Algebraic Geometry for Associative Algebras*, Monographs, Pure and Applied Math., Vol. 232, M. Dekker, New York, 2000.
[132] F. Van Oystaeyen, *Graded P.I. Rings and Generalized Crossed Product Azumaya Algebras*, Chin. Ann. Math. B **8**(1) (1987), 13-21.
[133] F. Van Oystaeyen, *A Note on Graded P. I. Rings*, Bull. Soc. Math. Belgique **32** (1980), 22-28.
[134] F. Van Oystaeyen, *Prime Spectra in Noncommutative Algebra*, Lect. Notes in Math. 444, Springer-Verlag, Berlin, 1978.
[135] F. Van Oystaeyen, *Virtual Topology and Functor Geometry*, Lect. Notes in Pure and Applied Math. Vol. 256, Chapman & Hall, Boca Raton, 2008.
[136] F. Van Oystaeyen, R. Sallam, *A Microstructure Sheaf and Quantum Sections over a Projective Scheme*, J. of Algebra **158** (1993), 201-225.
[137] F. Van Oystaeyen, A. Verschoren, *Non-commutative Algebraic Geometry*, Lect. Notes in Math. vol. 887, Springer Verlag, Berlin, 1981.
[138] F. Van Oystaeyen, A. Verschoren, *Relative Invariants of Rings*, I and II. Monographs in Pure Appl. Math. vol 79 and vol. 86, M. Dekker, New York, 1983 and 1984.
[139] F. Van Oystaeyen, L. Willaert, *Cohomology of Schematic Algebras*, J. of Algebra **183**(2) (1996), 359-364.
[140] F. Van Oystaeyen, L. Willaert, *Examples and Quantum Sections of Schematic Algebras*, J. Pure Applied Algebra **120** (1997), 195-211.
[141] F. Van Oystaeyen, L. Willaert, *Grothendieck Topology, Coherent Sheaves and Serre's Theorem for Schematic Algebras*, J. Pure Applied Algebra **104** (1995), 109-122.
[142] F. Van Oystaeyen, J. H. Zhang, *The Brauer Group of a Braided Monoidal Category*, J. Alg. **202** (1998), 96-128.
[143] F. Van Oystaeyen, Y. H. Zhang, *Brauer Groups of Actions, in "Rings, Hopf Algebras and Brauer Groups*, Vol. 197 of Lecture Notes in Pure and Applied Math., S. Caenepeel and A. Verschoren (eds.), Marcel Dekker, 1998, 299-311.
[144] F. Van Oystaeyen, Y. H. Zhang,*Embedding the Hopf Automorphism Group Into the Brauer Group of a Hopf Algebra*, Bull. Canad. Math. Soc. to appear.
[145] C.T.C. Wall, *Graded Brauer Groups*, J. Reine, Angew Math. **213** (1964), 187-199.
[146] W.C. Waterhouse, *Introduction to Affine Group Schemes*, Graduate Texts in Math. 66, Springer Verlag, Berlin 1979.
[147] L. Willaert, *Schematic Algebras and the Auslander Gorenstein Property*, UIA preprint Soc. Math. France, 149-156.
[148] A. Yekuteli, J. Zhang, *Serre Duality for Noncommutative Projective Schemes*, preprint.
[149] D. N. Yetter, *Quantum Groups and Representations of Monoidal Categories*, Math. Proc. Cambridge Philos Soc. **108** (1990), 261-290.

Dept. of Mathematics and Computer Science
University of Antwerp
Campus Middelheim
Middelheimlaan 1, 2020 Antwerpen, Belgium
E-mail address: fred.vanoystaeyen@ua.ac.be

HOMOLOGICAL CONJECTURES AND RADICAL-FULL EXTENSIONS

CHANGCHANG XI

Abstract. This survey paper is based on my lectures giving at the '42nd Symposium on Ring Theory and Representation Theory' held at Osaka Kyoiku University, Japan, 10-12 October 2009. In this paper, we consider the finitistic dimension and the strong no loop conjectures (and related other homological conjectures). We approach these conjectures by the so-called radical-full extensions, and reduce the verification of these conjectures to the following question: Suppose that $B \subseteq A$ is a radical-full extension such that the radical of B is a left ideal in A, and that one of these conjectures is true for A, is it possible to prove that the same conjecture is true for B ? We shall provide basic definitions and examples, and report current results on the two conjectures in this direction.

Key Words: Algebra, homological conjecture, module, radical-full extension, syzygy.

2000 *Mathematics Subject Classification*: Primary 18G20, 16G10; Secondary 16G70, 18E30.

Acknowledgements

The author would like to thank the organizers of the '42nd Symposium on Ring Theory and Representation Theory' held at Osaka Kyoiku University for invitation to speak at this conference. He is very grateful to Prof. S. Ariki and the RIMS at Kyoto University for providing this wonderful chance to visit Japan, and to many of his Japanese colleagues for their warm and friendly hospitality, in particular, to H.Abe(Tsukuba), S.Ariki(Kyoto), H.Asashiba(Shizuoka), O.Iyama(Nagoya), M.Sato(Yamanashi) and K.Yamagata(Tokyo). He enjoys his stay in Japan.

1. Introduction to two homological conjectures

In the modern representation theory of algebras, homological methods are used quite often to describe algebraic invariants and properties of modules and algebras. These homological aspects nowadays become interesting topics, and stimulate many deep investigations in different directions. It has turned out that many homological conjectures on algebras and modules arise (see [1]). Among them are the finitistic dimension and the strong no loop conjectures, on which we will concentrate in the present paper. In this section, we shall give the precise statements of the conjectures, and mention other related conjectures; In Section 2, we propose a new idea to understand these two conjectures, namely we want to approach the conjectures by algebra extensions, in this way, one may use external information of an algebra with simple representation theory to investigate homological conjectures for another algebra with usually complicated representation theory, and show that this new method may be useful for attacking the conjectures. In Section

The detailed version of this paper has been submitted for publication elsewhere.

3, we introduce a special extension of algebras, namely the radical-full extension, and reduce the consideration of our homological conjectures to questions related to radical-full extensions of algebras. We shall give two kinds of examples for obtaining radical-full extensions. In Section 4 and Section 5, we summarize current results on the finitistic dimension and the strong no loop conjectures under our setting, respectively.

Let us fix some notations. Let A be a finite-dimensional k-algebra over a field k. By a module we mean a finitely generated left module, and by A-mod we denote the category of all A-modules. For a module $M \in A$-mod, we denote by $\mathrm{pd}({}_AM)$ (respectively, $\mathrm{id}({}_AM)$) the projective (respectively, injective) dimension of M, and by $\mathrm{gl.dim}(A)$ the global dimension of A. The finitistic dimension of A is defined as

$$\mathrm{fin.dim}(A) = \sup\{\mathrm{pd}({}_AM) \mid M \in A\text{-mod},\ \mathrm{pd}({}_AM) < \infty\}$$

The following question on finitistic dimension was mentioned in a paper [2] of H.Bass in 1960, which now becomes a conjecture (see [1]), and will be called the finitistic dimension conjecture in this paper.

Finitistic dimension conjecture: For a finite-dimensional k-algebra A, $\mathrm{fin.dim}(A)$ is finite.

As is known, this conjecture is related to many other homological conjectures in homological algebra and in the representation theory of Artin algebras. Among them are the following:

- **Wakamatsu tilting conjecture**: Suppose that T is a Wakamatsu tilting A-module over a finite-dimensional algebra A, If $\mathrm{pd}({}_AT) < \infty$, then T is a tilting A-module.

  Recall that an A-module T is called a *Wakamatsu tilting module* if $\mathrm{Ext}^n_A(T,T) = 0$ for all $n > 0$, and there is an exact sequence

  $$0 \to {}_AA \to T_0 \xrightarrow{f_0} T_1 \xrightarrow{f_1} \cdots \longrightarrow T_n \xrightarrow{f_n} T_{n+1} \to \cdots$$

  in A-mod with $T_i \in \mathrm{add}(T)$ such that $\mathrm{Ext}^1_A(T, \mathrm{Im}(f_i)) = 0$ for all $i \geq 0$, where $\mathrm{add}(T)$ stands for the additive subcategory of A-mod generated by T, and $\mathrm{Im}(f_i)$ denotes the image of f_i.

- **Tilting complement conjecture**: An almost tilting A-module has only finitely many non-isomorphic indecomposable tiling complements.

  Recall that an A-module T is called an *almost tilting module* if $\mathrm{pd}({}_AT) < \infty$, $\mathrm{Ext}^i_A(T,T) = 0$ for all $i > 0$, and the number of non-isomorphic indecomposable summands of T is equal to the number of non-isomorphic simple A-modules minus 1. Given an almost tilting module T, an indecomposable A-module M is called a *tilting complement* to T if $T \oplus M$ is a tilting module.

- **Nakayama Conjecture**: If all injective A-modules I_j in a minimal injective resolution $0 \to {}_AA \to I_0 \to I_1 \to \cdots$ of A are projective, then A is self-injective, that is, ${}_AA$ is an injective A-module.

- **General Nakayama conjecture**: Every indecomposable injective A-module is isomorphic to a direct summand of some I_j in a minimal injective resolution of A: $0 \to {}_AA \to I_0 \to I_1 \to \cdots$.

- **Strong Nakayama conjecture**: If M is a non-zero A-module, then there is an $n \geq 0$ such that $\mathrm{Ext}^n_A(M, A) \neq 0$.
- **Gorenstein symmetry conjecture**: For an algebra A, if $\mathrm{id}({}_AA) < \infty$, then $\mathrm{id}(A_A) < \infty$.

The relationship between these conjectures is that if the finitistic dimension conjecture is true for all Artin algebras, then each of these other conjectures is true for all Artin algebras.

For further discussion on the links between these conjectures, we refer the reader to [17].

Now, we turn to introducing the strong no loop conjecture. In the papers [10] and [6], it was shown that if the global dimension of a finite-dimensional algebra A is finite, then $\mathrm{Ext}^1_A(S, S) = 0$ for all simple A-modules S. Thus, in this case, the quiver of the algebra A has no loops. In [6], a strong version of this result was proposed:

Strong no loop conjecture: If a simple A-module S satisfies $\mathrm{Ext}^1_A(S, S) \neq 0$, then $\mathrm{pd}({}_AS) = \infty$.

We notice that all conjectures listed here are still open.

2. Main ideas and questions

To understand the finitistic dimension and the strong no loop conjectures, we will use certain extensions of algebras. Our idea is to employ external information of bigger algebras A with relatively simple representation theory to investigate the conjectures for subalgebras B with, usually, a relatively complicated representation theory. In this way, we may work out a method for understanding these conjectures, which is applicable to general finite-dimension algebras instead of a special class of algebras.

If A and B are algebras such that B is a subalgebra of A with the same identity, then we say that A is an extension of B. In this case, we also say that $B \subseteq A$ is an extension of algebras.

We consider the following question:

Let $B \subseteq A$ be an extension of algebras. Suppose that a conjecture is true for A, is it possible to show that the same conjecture is true for B ?

Clearly, for an arbitrary extension, we could not say much about this question. So we confront immediately with the following questions that we have to think about:

(a) What kind of extensions should we choose ?
(b) What kind of A should be considered ?
(c) Does such an idea make sense ?

To question (c): On the one hand, every finite-dimensional algebra can be embedded into a full matrix algebra, this experience tells us that a bigger algebra may have a relatively simple representation theory and homological property. On the other hand, for any algebra A given by quiver and relations, if the quiver contains at least two arrows, then A contains a subalgebra of infinite global dimension and of infinite representation type. This means that in general subalgebras of an algebra may be more complicated than the algebra itself. Also, the content of the finitistic dimension conjecture itself does not tell us any information or indication about algebras and modules that we are concerning, so some external information for looking at this "black box" may be needed. From these points of view, our idea may make sense.

To question (b): Transparently, we should choose algebras for which the conjectures hold true. Moreover, we would like to replace the bigger algebras A by some algebras that are "equivalent" to A. For equivalences we here choose stable equivalences of Morita type and derived equivalences since the finiteness of finitistic dimension is preserved under these two kinds of equivalences (see [11]). In fact, it is easy to see that stable equivalences of Morita type even preserve finitistic dimension. This leads us to considering the invariants and constructions of these equivalences, a topic which we shall not touch in this paper.

To question (a): Of course, we cannot choose arbitrary extensions since they do not provide us desired information. So we would like to choose certain idealized extensions and the so-called radical-full extensions, both of which involve the Jacobson radicals of algebras. This topic will be discussed in the next section.

3. Radical-full extensions

In literature, there are many types of extensions, for example, separable extension, semisimple extension, H-separable extension, Frobenius extension, and so on. For our purpose, we shall introduce an extension related to the Jacobson radicals of algebras (see [12] and [13], for example).

An extension $B \subseteq A$ of Artin algebras is called *radical-idealized* if $\mathrm{rad}(B)$ is a left ideal in A, and *radical-full* if $\mathrm{rad}(_BA) = \mathrm{rad}(_AA)$, that is, $\mathrm{rad}(A) = \mathrm{rad}(B)A$. A special case of a radical-full extension is the radical-equal extension, that is, an extension $B \subseteq A$ with $\mathrm{rad}(B) = \mathrm{rad}(A)$. Similarly, one can define a right version of these notions by using right modules.

The following propositions show that our approach to the finitistic dimension and the strong no loop conjectures by radical-full extensions may be useful.

Proposition 1. *Let k be a perfect field. Then the following are equivalent:*

(1) *For all k-algebras A,* $\mathrm{fin.dim}(A) < \infty$.

(2) *For any radical-idealized, radical-full extension $C \subseteq B$ of k-algebras, if* $\mathrm{fin.dim}(B) < \infty$*, then* $\mathrm{fin.dim}(C) < \infty$.

(3) *For any radical-idealized extension $C \subseteq B$ of k-algebras, if* $\mathrm{fin.dim}(B) < \infty$*, then* $\mathrm{fin.dim}(C) < \infty$.

(4) *For any extension $C \subseteq B$ of k-algebras such that* $\mathrm{rad}(C)$ *is an ideal in B, if* $\mathrm{fin.dim}(B) < \infty$*, then* $\mathrm{fin.dim}(C) < \infty$.

Similarly, for the strong no loop conjecture, we have the following equivalent conditions. Note that when we say that the strong no loop conjecture is true for an algebra A, we mean that for every simple A-module S with $\mathrm{Ext}^1_A(S,S) \neq 0$, we have $\mathrm{pd}(_AS) = \infty$.

Proposition 2. *Let k be a perfect field. Then the following are equivalent:*

(1) *The strong no loop conjecture is true for all k-algebras A.*

(2) *For any radical-idealized, radical-full extension $C \subseteq B$ of k-algebras, if the strong no loop conjecture is true for B, then so is it for C.*

(3) *For any radical-idealized extension $C \subseteq B$ of k-algebras, if the strong no loop conjecture is true for B, then so is it for C.*

(4) *For any radical-idealized extension $C \subseteq B$ of k-algebras such that* $\operatorname{rad}(C)$ *is an ideal in B, if the strong no loop conjecture is true for B, then so is it for C.*

Thus, from the above two propositions, it is sufficient to investigate the question in Section 2 for radical-idealized and radical-full extensions. An immediate question is how to get such extensions.

Now let us give three constructions of radical-full extensions.

Suppose that $A = kQ/I$ is a finite-dimensional algebra (over a field k) presented by a quiver $Q = (Q_0, Q_1)$ with relations, where I is an admissible ideal in the path algebra kQ of Q. Note that the composition of two arrows $\alpha, \beta \in Q_1$ is written as $\alpha\beta$, where α comes first and then β follows. As usual, for $i \in Q_0$, we denote by e_i the primitive idempotent element in A corresponding to the vertex i.

(1) **Gluing vertices**

Suppose we are given a partition of the vertex set Q_0, say $Q_0 = \cup_{j=1}^{m} I_j$. Let $f_j = \sum_{i \in I_j} e_i$ for $j = 1, 2, \cdots, m$. Let B be the subalgebra of A generated by $f_1, f_2, \cdots, f_m$ and $\operatorname{rad}(A)$. Then we see that $B \subseteq A$ is a radical-equal extension. The quiver of B is obtained from that of A by gluing all vertices in I_j together for every I_j.

(2) **Unifying arrows**

Let $\{1, 2, \cdots, n\}$ be a subset of Q_0, and let α_i be n distinct arrows in Q_1 such that α_i has the terminus i and that all α_j have a common starting vertex. We define $\overline{Q}_0 = Q_0 \setminus \{1, 2, \cdots, n\}$, $\overline{Q}_1 = Q_1 \setminus \{\alpha_i \mid i = 1, 2, \cdots, n\}$, $e = \sum_{i=1}^{n} e_i$, and $\alpha = \sum_{i=1}^{n} \alpha_i$. Let B be the subalgebra of A generated by the idempotent elements e, e_j, with $j \in \overline{Q}_0$ and the arrows α, β, with $\beta \in \overline{Q}_1$. Note that if α_n is a loop in A, then we have $\alpha_n \alpha = \alpha^2$. It is not hard to see that $\operatorname{rad}(B)$ is a left ideal in A and $\operatorname{rad}(A) = \operatorname{rad}(B)S = \operatorname{rad}(B)A$, where S is the maximal semisimple subalgebra of A generated by all e_i with $i \in Q_0$. Thus the extension $B \subseteq A$ is radical-idealized and radical-full. The quiver of B is obtained from that of A by gluing all vertices in $\{1, 2, \cdots, n\}$ together into one vertex, and unifying all arrows $\alpha_1, \alpha_2, \cdots, \alpha_n$ into one arrow.

(3) **Triangulation**

Suppose that we are given an algebra B with a decomposition $B = S \oplus \operatorname{rad}(B)$, where S is a maximal semisimple subalgebra of B. Let n be the nilpotency of $\operatorname{rad}(B)$. We define $\bar{B} = B/\operatorname{rad}^{n-1}(B)$, and

$$A = \begin{pmatrix} S & 0 \\ \operatorname{rad}(B) & \bar{B} \end{pmatrix}.$$

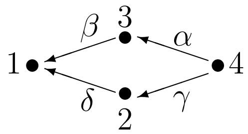
Then there is an embedding of B into A such that this extension is radical-idealized and radical-full, namely

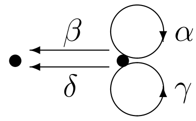
$$B \subseteq A, \qquad b = s + r \mapsto \begin{pmatrix} s & 0 \\ r & \bar{b} \end{pmatrix},$$

where $\bar{b}$ is the image of $b \in B$ under the canonical surjection from B to $\bar{B}$.

Now we display two concrete examples to illustrate the first two constructions.

Example 1. Let A and B be the following two algebras presented by quivers with relations, respectively:

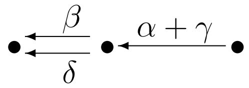
$A:$ 

$B:$ 

$\alpha\beta = \gamma\delta.$ $\qquad$ $\alpha\beta = \gamma\delta, \alpha\delta = \gamma\beta = \alpha^2 = \gamma^2 = \gamma\alpha = \alpha\gamma = 0.$

We can see that B is obtained by gluing the vertices 2, 3 and 4 in the quiver of A. Thus the extension $B \subseteq A$ is radical-equal. Note that A is representation-finite and has finite global dimension, while the subalgebra B of A is representation-infinite and of infinite global dimension.

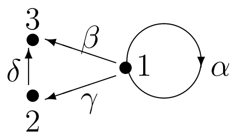
If we unify the arrows α and γ in the quiver of A, then we get the following subalgebra C of A:

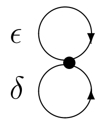
$C:$ 

$(\alpha + \gamma)\beta = (\alpha + \gamma)\delta.$

Thus $C \subseteq A$ is a radical-full extension. Again, the subalgebra C of A is representation-infinite. Clearly, the radical of C is properly contained in the radical of A.

Example 2. Let A and B be algebras presented by the following quivers with relations, respectively:

$A:$ 

$B:$ 

$\alpha\beta - \alpha\gamma\delta = \alpha^2 = 0.$ $\qquad$ $\delta^2 = \epsilon^3 = \delta\epsilon = 0.$

Clearly, we see that B can be obtained from A by unifying the arrows α, β and γ into one arrow $\epsilon = (\alpha + \beta + \gamma)$. In this procedure, the arrow δ in the quiver of A becomes a loop in the quiver of B.

Finally, we mention some facts on radical-full extensions from [12] and [15].

Assume that $B \subseteq A$ is a radical-idealized extension of Artin algebras. Then

(1) for any B-module ${}_BX$, the B-module $\Omega_B^j(X)$ is an A-module for $j \geq 2$, where Ω_B^i is the i-th syzygy operator of B.

(2) For each A-module Y, we have $\Omega_A(A \otimes_B Y) \simeq \Omega_B(Y)$ as A-modules.

(3) If the extension is radical-full, then $\mathrm{add}\big({}_B(A/\mathrm{rad}(A))\big) = \mathrm{add}(B/\mathrm{rad}(B))$. Thus every simple B-module is a direct summand of the restriction of a simple A-module to B.

A direct consequence of the facts (1) and (2) is the following proposition.

Proposition 3. *Suppose that $B \subseteq A$ is a radical-idealized extension of Artin algebras. If $\mathrm{pd}(A_B) < \infty$, then $\mathrm{fin.dim}(B) \leq \mathrm{fin.dim}(A) + \mathrm{pd}(A_B) + 2$.*

Proof. Let $n = \mathrm{pd}(A_B)$. Pick a B-module X, define $Y := \Omega_B^{n+2}(X)$, which is an A-module by (1), and consider a minimal projective resolution of ${}_BY$:

$$0 \to P_m \to \cdots \to P_1 \to P_0 \to Y \to 0.$$

By tensoring this sequence, we get a sequence:

$$(*) \quad 0 \to A \otimes_B P_m \to \cdots \to A \otimes_B P_1 \to A \otimes_B P_0 \to A \otimes_B Y \to 0.$$

Since $\mathrm{Tor}_j^B(A_B, Y) = \mathrm{Tor}_j^B(A_B, \Omega_B^{n+2}(X)) = \mathrm{Tor}_{n+2+j}^B(A_B, X) = 0$ for $j \geq 1$, we see that this sequence is exact. Furthermore, we can show by the fact (2) that the sequence is also a minimal projective resolution of the A-module $A \otimes_B Y$. Thus $\mathrm{pd}({}_BY) = \mathrm{pd}({}_AA \otimes_B Y) \leq \mathrm{fin.dim}(A)$, and therefore we have the estimation in the proposition.

4. Recent results on the finitistic dimension conjecture

In this section we present some results along the idea of algebra extensions.

In [12], we showed the following result.

Theorem 4. *Let $C \subseteq B \subseteq A$ be three Artin algebras with the same identity such that both $C \subseteq B$ and $B \subseteq A$ are radical-idealized. If A is representation-finite, then C has finite finitistic dimension.*

An open question is to extend this result to a chain containing four or more than four algebras. A positive answer to this question for finite chain of algebras would solve the finitistic dimension conjecture [12]. The next result involves global dimension [13].

Theorem 5. *Let $B \subseteq A$ be a radical-idealized, radical-full extension of Artin algebras. If* $\mathrm{gl.dim}(A) \leq 4$, *then* $\mathrm{fin.\ dim}(B) < \infty$.

The case of $\mathrm{gl.dim}(A) \geq 5$ is open. It would be interesting to generalize this result.

When considering an extension, we may automatically think of the notion of relatively projective modules, and the one of relative global dimension.

Recall that, given an extension $B \subseteq A$ of algebras, an A-module X is called *relatively projective* if the multiplication map $\mu : {}_AA \otimes_B X \longrightarrow {}_AX$ of A-modules is a split-epimorphism, that is, there is a homomorphism $\varphi : X \longrightarrow A \otimes_B X$ of A-modules such that $\varphi\mu$ is the identity map on X. In this case we also say that X is (A, B)-projective. A short exact sequence of A-modules is called (A, B)-exact if it splits as an exact sequence of B-modules. The relative projective dimension of an A-module can be defined by (A, B)-projective modules and exact (A, B)-sequences. We leave the precise formulation of this notion to the reader. We denote by $\mathrm{gl.dm}(A, B)$ the relative global dimension of the extension $B \subseteq A$. For more details on relative homological algebra one may look at the paper [4].

It is known that $\mathrm{gl.dim}(A, B) = 0$ if and only if the extension $B \subseteq A$ is semisimple, that is, every A-module is (A, B)-projective. Examples of semisimple extension are radical-equal extensions.

Related to (A, B)-projective modules, we have the following results in [15].

Theorem 6. *Let $B \subseteq A$ be a radical-idealized extension of Artin algebra. Suppose the category of all finitely generated (A, B)-projective A-modules is closed under taking A-syzygies (for example, the extension is semisimple, or A_B is projective). If* $\mathrm{fin.dim}(A) < \infty$, *then* $\mathrm{fin.dim}(B) < \infty$.

In [14] there is another approach to finitistic dimension conjecture, namely we use the pair $eAe \subseteq A$ with $e^2 = e \in A$, and try to understand the finitistic dimension of eAe by that of A. For details we refer to the paper [14]. Recently, Huard, Lanzilotta and Mendoza use socle or top layers of a module to approach the finitistic dimension conjecture. Again, I refer the details to the paper [5].

5. Recent results on the strong no loop conjecture

Concerning the strong no loop conjecture, not much is known. There are only a few papers dealing with this conjecture in literature. It was verified for monomial algebras [6], quasi-monomial algebras [3], special biserial algebras and quasi-stratified algebras [8, 9], and algebras (over an algebraically closed field) of radical-cube-zero with two simple modules [7].

Along the approach by extensions, we have the following result in [16].

Theorem 7. *Let $B \subseteq A$ be a radical-idealized, radical-full extension of Artin algebras. If* $\mathrm{gl.dim}(A) \leq 2$*, then the strong no loop conjecture is true for B.*

If we stress the condition on extension, we have the following result.

Theorem 8. *Let $B \subseteq A$ be a radical-idealized extension of Artin algebras with gl.dim$(A, B) = 0$. If the strong no loop conjecture is true for A, then it is true for B.*

Thus, if we glue vertices from an algebra A given by quiver and relations, then we get a new algebra B for which the strong no loop conjecture is true. Moreover, if we start with algebra of global dimension at most 2 (for example, with an Auslander algebra), and unify arrows, then the strong no loop conjecture is true for the new algebra.

Finally, we remark that $\mathrm{gl.dim}(A, B) \leq 1$ for any radical-idealized, radical-full extension $B \subseteq A$ of Artin algebras. Thus, if we could extend Theorem 6 and Theorem 8 to the case of $\mathrm{gl.dim}(A, B) \leq 1$, we would prove both the finitistic dimension conjecture and the strong no loop conjecture.

References

[1] M. Auslander, I. Reiten and S. Smalø, *Represenattion theory of artin algebras*, Cambridge Studies in Advanced Mathematics **36**, Cambridge University Press, 1995.

[2] H. Bass, *Finitistic dimension and a homological generalization of semiprimary rings*, Trans. Amer. Math. Soc. **95** (1960), 466–488.

[3] L. Diracca and S. König, *Cohomological reduction by split pairs*, J. Pure Appl. Algebra **212**(2008), 471-485.

[4] G. Hochschild, *Relative homological algebra*, Trans. Amer. Math. Soc. **82**(1956), 246–269.

[5] F. Huand, M. Lanzilotta and O. Mendoza, *Finitistic dimension through infinite projective dimension*, Bull. London Math. Soc. **41**(2)(2009), 367-376

[6] K. Igusa, *Notes on the no loop conjecture*, J. Pure Appl. Algebra **69**(1990), 161-176.

[7] B.T. Jensen, *Strong no loop conjecture for algebras with two simples and radical cube zero*, Colloquium Math. **102**(2008), 1-7.

[8] S.P. Liu and J.-P. Morin, *The strong no loop conjecture for special biserial algebras*, Proc. Amer. Math. Soc. **132**(12)(2004), 3513-3523.

[9] ——— and Ch. Paquette, *Some homological conjectures for quasi-stratified algebras*, J. Algebra **301**(1)(2006), 240-255.

[10] H. Lenzing, *Nilpotente Elemente in Ringen von endlicher globaler Dimension*, Math. Z. **108**(1969), 313-324.
[11] S.Y. Pan and C.C. Xi, *Finiteness of finitistic dimension is invariant under derived equivalences*, J. Algebra **322** (2009), 21-24.
[12] C.C. Xi, *On the finitistic dimension conjecture, I. Related to representation-finite algebras*, J. Pure Appl. Algbera **193** (2004), 1287–305. *Erratum to "On the finitistic dimension conjecture, I. "*, J. Pure Appl. Algbera **202(1-3)** (2005), 325–328.
[13] ———, *On the finitistic dimension conjecture, II. Related to finite global dimension*, Adv. Math. **201** (2006), 116–142.
[14] ———, *On the finitistic dimension conjecture, III. Related to the pair $eAe \subseteq A$*, J. Algebra. **319** (2008), 3666–3688.
[15] C.C. Xi and D.M. Xu, *The finitistic dimension conjecture and relatively projective modules*, Preprint, 2007, available at: http://math.bnu.edu.cn/~ccxi/.
[16] ———,*The strong no loop conjecture and radical-full extensions*, Preprint, 2009.
[17] K. Yamagata, *Frobenius Algebras*, In: Handbook of Algebra. Vol.1 (1996), 841-887.

School of Mathematical Sciences
Laboratory of Mathematics and Complex Systems,
Beijing Normal University
Beijing 100875,CHINA
E-mail address: xicc@bnu.edu.cn

THE CLASSIFICATION OF TILTING MODULES OVER HARADA ALGEBRAS

KOTA YAMAURA

Abstract. In the 1980s, Harada introduced a class of algebras now called Harada algebras, which give a common generalization of quasi-Frobenius algebras and Nakayama algebras. In this paper, we classify tilting modules over Harada algebras by giving a bijection between tilting modules over Harada algebras and tilting modules over direct products of upper triangular matrix algebras over K. A combinatorial description of tilting modules over upper triangular matrix algebras over K is known. These facts allow us to classify tilting modules over a given Harada algebra.

1. Introduction

Two classes of algebras have been studied for a long time. The first is Nakayama algebras and the second is quasi-Frobenius algebras. In the 1980s, Harada introduced a class of algebras now called Harada algebras, which give a common generalization of quasi-Frobenius algebras and Nakayama algebras. Many authors have studied the structure of Harada algebras (e.g. [7, 8, 17, 18, 19, 20, 21, 22]). Now let us recall that left Harada algebras as defined from a structural point of view as follows.

Definition 1. Let R be a basic algebra and $\mathrm{Pi}(R)$ be a complete set of orthogonal primitive idempotents of R. We call R a *left Harada algebra* if $\mathrm{Pi}(R)$ can be arranged such that $\mathrm{Pi}(R) = \{e_{ij}\}_{i=1,}^{m}{}_{j=1}^{n_i}$ where

(a) $e_{i1}R$ is an injective R-module for any $i = 1, \cdots, m$,
(b) $e_{ij}R \simeq e_{i,j-1}J$ for any $i = 1, \cdots, m$, $j = 2, \cdots, n_i$.

Here J is the Jacobson radical of R.

Then we put

$$P_{ij} := e_{i1}J^{j-1} \simeq e_{ij}R \qquad (1 \leq i \leq m,\ 1 \leq j \leq n_i) \tag{1.1}$$

for simplicity. By the above conditions (1) and (2), we have a chain

$$P_{i1} \supset P_{i2} \supset \cdots \supset P_{in_i}$$

of indecomposable projective R-modules.

It follows from definition that left Harada algebras satisfy the property QF-3 which is the condition that the injective hull of the algebra is projective. This property is called 1-Gorenstein by Auslander (and dominant dimension at least one by Tachikawa) [5, 12, 14, 15, 24], and often plays an important role in the representation theory. Left Harada algebras form a class of 1-Gorenstein algebras, and their indecomposable projective modules have "nice" structure.

The detailed version of this paper will be submitted for publication elsewhere.

In this paper, we classify tilting modules over left Harada algebras. Tilting modules provide a powerful tool in the representation theory of algebras and are due to [4, 9, 10].

Definition 2. Let R be an algebra. An R-module T is called a *partial tilting module* if it satisfies the following conditions.

(1) $\mathrm{proj.dim}T \leq 1$.
(2) $\mathrm{Ext}^1_R(T,T) = 0$.

A partial tilting R-module T is called a *tilting module* if it satisfies the following condition.

(3) There exists an exact sequence

$$0 \longrightarrow R_R \longrightarrow T_0 \longrightarrow T_1 \longrightarrow 0$$

where T_0, $T_1 \in \mathrm{add}T$.

We can see from the above definition that tilting modules are a generalization of progenerators. Morita theory shows that any progenerator P over an algebra R induces a categorical equivalence between $\mathrm{mod}R$ and $\mathrm{mod}(\mathrm{End}_R(P))$. This result is generalized by Brenner-Butler. It says that any tilting module T over an algebra R induces two categorical equivalences between certain full subcategories of $\mathrm{mod}R$ and of $\mathrm{mod}(\mathrm{End}_R(T))$. As a consequence, R and $\mathrm{End}_R(T)$ share a lot of homological properties (e.g. finiteness of global dimension). By this reason, tilting modules are important for the study of algebras and finding a classification of tilting modules over a given algebra is an important problem in representation theory.

Now we give notion which gives an essential class of tilting modules.

Definition 3. Let T be a module over an algebra R and $T \simeq \oplus_{i=1}^n T_i$ an indecomposable decomposition of T. Then we call T *basic* if T_i and T_j are not isomorphic to each other for any $i \neq j$.

Thanks to Morita theory, it is enough to consider basic tilting modules. We denote by $\mathrm{tilt}(R)$ the set of isomorphism classes of basic tilting modules over an algebra R.

The aim of this paper is to give a classification of tilting modules over a left Harada algebra. We present our main theorem which return the classification of tilting modueles over left Harada algebras to that of tilting modules over upper triangular matrix algebras over K. We dente by $\mathrm{T}_n(K)$ an $n \times n$ upper triangular matrix algebra over K.

Theorem 4. *Let R be a left Harada algebra as in Definition 1. Then there is a bijection*

$$\mathrm{tilt}(R) \longrightarrow \mathrm{tilt}(\mathrm{T}_{n_1}(K)) \times \mathrm{tilt}(\mathrm{T}_{n_2}(K)) \times \cdots \times \mathrm{tilt}(\mathrm{T}_{n_m}(K)).$$

We will construct the above bijection in Section 2, and give outline of the proof in Section 3.

In Section 4, we give a description of tilting $\mathrm{T}_n(K)$-modules by using non-crossing partitions of regular polygons. Then we can completely classify tilting modules over a given left Harada algebra.

In Section 5, we show an example of the classification of tilting modules over left Harada algebras.

Throughout this paper, an algebra means a finite dimensional associative algebra over an algebraically closed field K. We always deal with finitely generated right modules over algebras. We denote by J the Jacobson radical of an algebra R.

2. Main results

In this section, let R be a left Harada algebra as in Definition 1. We use the notation (1.1). We consider a factor algebra $\overline{R} = R/I$ of R which is isomorphic to direct product of upper triangular matrix algebras over K. $\overline{R}$ contains important information of R which is seen in Lemma 10 and Proposition 11. After introducing $\overline{R}$, we define a functor $F : \mathrm{mod}R \longrightarrow \mathrm{mod}\overline{R}$ which induces the bijection of Theorem 4, and give the precise statement of Thorem 4.

We start by giving the ideal I of R. We put

$$e_{ij}R \supset I_{ij} := e_{ij}J^{n_i - j+1} \ (1 \le i \le m, 1 \le j \le n_i),$$

$$R \supset I := \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n_i} I_{ij}.$$

Obviously I is a right ideal of R. But it can be seen that I is also a left ideal of R. Thus we have the following lemma.

Lemma 5. *I is an ideal of R.*

By Lemma 5, we can consider a factor algebra

$$\overline{R} := R/I.$$

We show that $\overline{R}$ is isomorphic to direct product of upper triangular matrix algebras over K. To show this, we describe all indecomposable projective $\overline{R}$-modules as factor modules of indecomposable projective R-modules. Since I is contained in J,

$$\{\overline{e}_{ij} := e_{ij} + I \mid 1 \le i \le m, \ 1 \le j \le n_i\}$$

is a complete set of orthogonal primitive idempotents of $\overline{R}$. Thus

$$\{\overline{e}_{ij}\overline{R} \mid 1 \le i \le m, \ 1 \le j \le n_i\}$$

is a complete set of indecomposable projective $\overline{R}$-modules. Obviously we have

$$\overline{e}_{ij}\overline{R} \simeq e_{ij}R/e_{in_i}J \simeq P_{ij}/(P_{in_i}J).$$

By the structure of R in Definition 1, indecomposable projective $\overline{R}$-modules have the following unique composition series.

$$\begin{array}{ccccccc}
P_{11}/(P_{1n_1}J) & \supset & P_{12}/(P_{1n_1}J) & \supset \cdots\cdots \supset & P_{1n_1}/(P_{in_i}J) & \supset & 0 \\
P_{21}/(P_{2n_2}J) & \supset & P_{22}/(P_{2n_2}J) & \supset \cdots\cdots \supset & P_{2n_2}/(P_{2n_2}J) & \supset & 0 \\
\vdots & & & & & & \\
P_{m1}/(P_{mn_m}J) & \supset & P_{m2}/(P_{mn_m}J) & \supset \cdots\cdots \supset & P_{mn_m}/(P_{mn_m}J) & \supset & 0
\end{array}$$

We note that composition factors of the above composition series are not isomorphic to each other.

We put

$$\overline{e}_i := \overline{e}_{i1} + \overline{e}_{i2} + \cdots + \overline{e}_{in_i}$$

for any $1 \le i \le m$. Then by the above argument, we have the following result.

Proposition 6. *We have the following algebra isomorphisms.*

(1) $\overline{e}_i\overline{R}\,\overline{e}_j \simeq \mathrm{Hom}_{\overline{R}}(\overline{e}_j\overline{R}, \overline{e}_i\overline{R}) \simeq \begin{cases} \mathrm{T}_{n_i}(K) & (i = j), \\ 0 & (i \neq j). \end{cases}$

(2) $\overline{R} \simeq \mathrm{T}_{n_1}(K) \times \mathrm{T}_{n_2}(K) \times \cdots \times \mathrm{T}_{n_m}(K)$.

Next we consider a functor

$$F := - \otimes_R \overline{R} : \mathrm{mod}R \longrightarrow \mathrm{mod}\overline{R}.$$

This functor plays a key role for our main theorem.

Now we state a theorem which gives a bijection between $\mathrm{tilt}(R)$ and $\mathrm{tilt}(\overline{R})$ by using the functor F.

Theorem 7. *We have a bijection*

$$F : \mathrm{tilt}(R) \ni T \longmapsto F(T) \in \mathrm{tilt}(\overline{R}).$$

As a consequence of Theorem 7, we have the following result immediately.

Corollary 8. *We have a bijection*

$$\mathrm{tilt}(R) \ni T \longmapsto (F(T)e_1, \cdots, F(T)e_m) \in \mathrm{tilt}(\overline{R}e_1) \times \cdots \times \mathrm{tilt}(\overline{R}e_m).$$

Hence by Proposition 6, we have Theorem 4.

3. Proof of Theorem 7

In this section, we keep the notations from the previous section. We show outline of the proof of Theorem 7.

First we give a more stronger result than our main theorem. Namely we classify indecomposable R-modules whose projective dimension is equal to one. Obviously projective dimension of P_{ik}/P_{il} is equal to one for any $1 \leq i \leq m$, $1 \leq k < l \leq n_i$. The following theorem shows that the converse holds.

Theorem 9. *A complete set of isomorphism classes of indecomposable R-modules whose projective dimension is equal to one is given as follows.*

$$\{P_{ik}/P_{il} \mid 1 \leq i \leq m,\ 1 \leq k < l \leq n_i\}.$$

Next we consider the restriction on F to full subcategories $\mathcal{P}$ or $\mathcal{P}_i$ of $\mathrm{mod}R$ which are defined by

$$\mathcal{P} := \{M \in \mathrm{mod}R \mid \mathrm{proj.dim}M \leq 1\}$$

and

$$\mathcal{P}_i := \mathrm{add}\{P_{ij},\ P_{ik}/P_{il} \mid 1 \leq j \leq n_i,\ 1 \leq k < l \leq n_i\}$$

for any $1 \leq i \leq m$. By Theorem 9, we have

$$\mathcal{P} = \mathrm{add}(\mathcal{P}_1 \cup \mathcal{P}_2 \cup \cdots \cup \mathcal{P}_m).$$

The restriction on F to $\mathcal{P}$ has two important properties. First property is the following lemma which is proved by easy calculations.

Lemma 10. *The following hold.*

(1) *The restriction on F to $\mathcal{P}$ induces a bijection from isomorphism classes of $\mathcal{P}$ to that of* $\mathrm{mod}\overline{R}$.

(2) *The restriction on F to $\mathcal{P}_i$ induces a bijection from isomorphism classes of $\mathcal{P}_i$ to that of* $\mathrm{mod}(\overline{R}\overline{e}_i)$.

We remark that the restriction on F to $\mathcal{P}$ is not faithful in general, in particular, it is not an equivalence.

Second property is that F preserves vanishing property of first extension group on $\mathcal{P}$.

Proposition 11. *For any M, $N \in \mathcal{P}$, $\mathrm{Ext}^1_R(M, N) = 0$ if and only if $\mathrm{Ext}^1_{\overline{R}}(F(M), F(N)) = 0$.*

Finally by using the following well-known charactarization of tilting module, we can prove Theorem 7.

Proposition 12. [3] *Let R be a general algebra. Let T be a partial tilting module. Then the following are equivalent.*

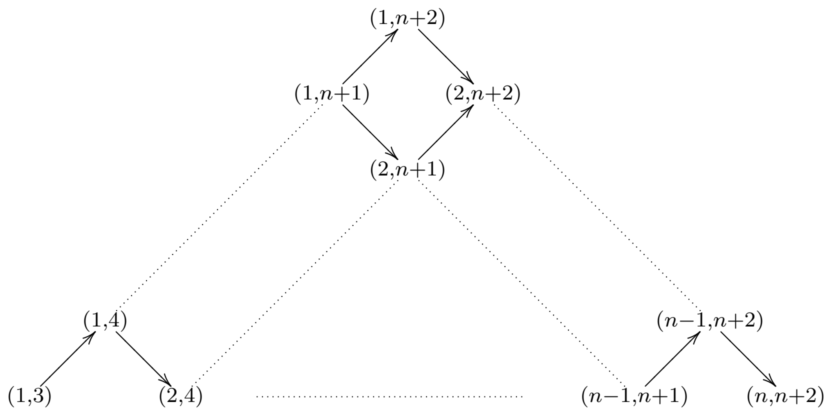
(1) *T is a tilting module.*
(2) *The number of pairwise nonisomorphic indecomposable direct summands of T is equal to that of pairwise nonisomorphic simple R-modules.*

Now we prove Theorem 7. Let T be a basic tilting R-module. It is enough to show that $F(T)$ is a basic tilting $\overline{R}$-module. First by proj.dim$T \leq 1$, we have $T \in \mathcal{P}$. Next by $\mathrm{Ext}^1_R(T, T) = 0$ and Proposition 11, we have $\mathrm{Ext}^1_{\overline{R}}(F(T), F(T)) = 0$. Therefore $F(T)$ is a basic partial tilting $\overline{R}$-module. Finally by Lemma 10 and Proposition 12, we can see that the number of pairwise nonisomorphic indecomposable direct summands of $F(T)$ is equal to that of pairwise nonisomorphic simple $\overline{R}$-modules. Consequently by Proposition 12, $F(T)$ is a basic tilting $\overline{R}$-module. ☐

4. Combinatorial description of tilting $\mathrm{T}_n(K)$-modules

In this section, we show a classification of basic tilting $\mathrm{T}_n(K)$-modules by constructing a bijection between tilt($\mathrm{T}_n(K)$) and the set of non-crossing partitions of the regular $(n+2)$-polygon. We remark that our classification should be well-known for experts [2, 11, 16, 23].

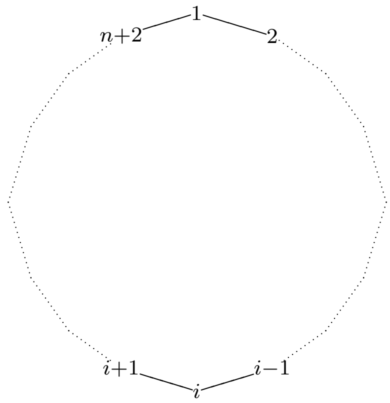
First we introduce coordinates in the AR-quiver of $\mathrm{T}_n(K)$ as follows.



We remark that the vertex (i, j) corresponds the $\mathrm{T}_n(K)$-module

$$M_{ij} = \left(0 \cdots 0\ \overset{j-2}{K} \cdots \overset{1}{K}\right) \big/ \left(0 \cdots 0\ \overset{i}{K} \cdots \overset{1}{K}\right) = \left(0 \cdots 0\ \overset{j-2}{K} \cdots \overset{i}{K}\ 0 \cdots \overset{1}{0}\right).$$

Next we consider a regular $(n+2)$-polygon R_{n+2} whose vertices are numbered as follows.



We denote by $D(R_{n+2})$ the set of all diagonals of R_{n+2} except edges of R_{n+2}. We call a subset S of $D(R_{n+2})$ a *non-crossing partition* of R_{n+2} if S satisfies the following conditions.

(1) Any two distinct diagonals in S do not cross except at their endpoints.
(2) R_{n+2} is divided into triangles by diagonals in S.

We denote by $\mathcal{P}_{n+2}$ the set of an non-crossing partitions of R_{n+2}.

Now we construct the correspondence Φ from $\mathcal{P}_{n+2}$ to $\mathrm{tilt}(\mathrm{T}_n(K))$. We take $S \in \mathcal{P}_{n+2}$. We remark that non-crossing partition of R_{n+2} consists of $n-1$ diagonals. We denote by (i,j) the diagonal between i and j for $i<j$ and put

$$S = \{(i_1, j_1),\ (i_2, j_2),\ \cdots,\ (i_{n-1}, j_{n-1})\}.$$

Then we define

$$\Phi(S) := M_{1,n+2} \oplus \left(\bigoplus_{k=1}^{n-1} M_{i_k, j_k} \right).$$

It is shown that this is a basic tilting $\mathrm{T}_n(K)$-module.

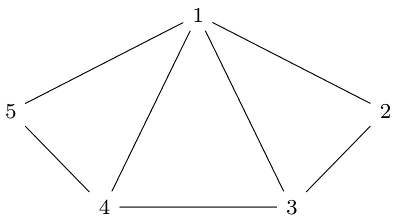
Then the following hold.

Theorem 13. *The above correspondence Φ is a bijection.*

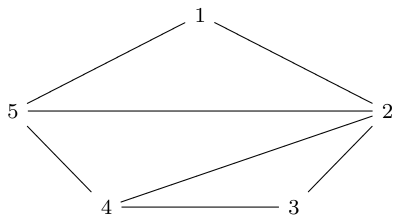
Theorem 13 gives a constructive bijection.

Example 14. We consider $n=3$ case. We classify basic tilting $\mathrm{T}_3(K)$-modules by using Theorem 13. The partitions of the regular pentagon into triangles are given as follows.

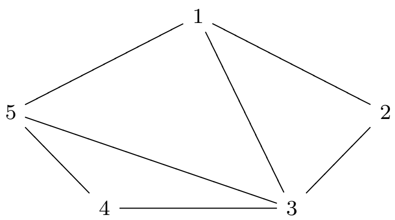
(1)



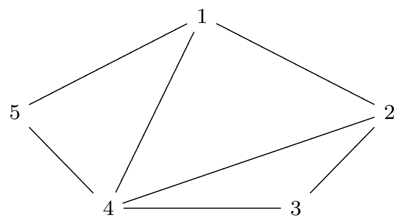
(2)



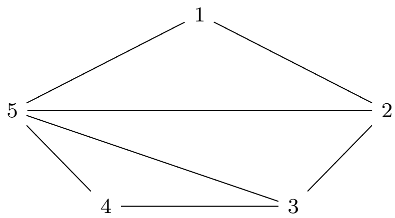
(3)



(4)



(5)



Therefore the number of basic tilting $\mathrm{T}_3(K)$-modules is equal to 5 and all of basic tilting $\mathrm{T}_3(K)$-modules are given as follows.

(1) $\left(\begin{smallmatrix} K & K & K \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} 0 & K & K \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} 0 & 0 & K \end{smallmatrix}\right)$,
(2) $\left(\begin{smallmatrix} K & K & K \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} K & K & 0 \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} 0 & K & 0 \end{smallmatrix}\right)$,
(3) $\left(\begin{smallmatrix} K & K & K \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} K & 0 & 0 \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} 0 & 0 & K \end{smallmatrix}\right)$,
(4) $\left(\begin{smallmatrix} K & K & K \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} 0 & K & K \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} 0 & K & 0 \end{smallmatrix}\right)$,
(5) $\left(\begin{smallmatrix} K & K & K \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} K & K & 0 \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} K & 0 & 0 \end{smallmatrix}\right)$.

5. Example

In this section, we show an example of the classifications of tilting modules over Harada algebras.

Example 15. Let R be a basic QF-algebra whose complete set of orthogonal primitive idempotents is given by $\{e, f\}$. Then we can represent R as the following matrix form.

$$R \simeq \begin{pmatrix} eRe & eRf \\ fRe & fRf \end{pmatrix} =: \begin{pmatrix} Q & A \\ B & W \end{pmatrix}.$$

Now we consider the *block extension* (c.f. [8, 22])

$$R(n_1, n_2) := \left(\begin{array}{ccc|ccc} Q & \cdots & Q & A & \cdots & A \\ & \ddots & \vdots & \vdots & & \vdots \\ J(Q) & & Q & A & \cdots & A \\ \hline B & \cdots & B & W & \cdots & W \\ \vdots & & \vdots & & \ddots & \vdots \\ B & \cdots & B & J(W) & & W \end{array}\right)$$

for $n_1, n_2 \in \mathbb{N}$ of R which is a subalgebra of $\mathrm{End}_R((eR)^{n_1} \oplus (fR)^{n_2})$. We can show that

(a) the first and $(n_1 + 1)$-th rows are injective modules,
(b) the i-th row is the Jacobson radical of the $(i-1)$-th row for $2 \leq i \leq n$ and $n+2 \leq i \leq n+m$.

In particular $R(n_1, n_2)$ is a left Harada algebra with $m = 2$ in Definition 1.

We classify basic tilting $R(n_1, n_2)$-modules. By easy calculation, we can see that the ideal I which is defined in Section 2 of $R(n_1, n_2)$ is given by

$$I = \left(\begin{array}{ccc|ccc} J(Q) & \cdots & J(Q) & A & \cdots & A \\ \vdots & & \vdots & \vdots & & \vdots \\ J(Q) & \cdots & J(Q) & A & \cdots & A \\ \hline B & \cdots & B & J(W) & \cdots & J(W) \\ \vdots & & \vdots & \vdots & & \vdots \\ B & \cdots & B & J(W) & \cdots & J(W) \end{array}\right).$$

Hence we have

$$\overline{R} = R/I = \left(\begin{array}{ccc|ccc} Q/J(Q) & \cdots & Q/J(Q) & 0 & \cdots & 0 \\ & \ddots & \vdots & \vdots & & \vdots \\ 0 & & Q/J(Q) & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & W/J(W) & \cdots & W/J(W) \\ \vdots & & \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & & W/J(W) \end{array}\right) \simeq \mathrm{T}_{n_1}(K) \times \mathrm{T}_{n_2}(K).$$

By Theorem 7, The functor

$$F = - \otimes \overline{R} : \mathrm{mod}R \longrightarrow \mathrm{mod}\overline{R}$$

induces a bijection

$$\mathrm{tilt}(R(n_1, n_2)) \longrightarrow \mathrm{tilt}(\mathrm{T}_{n_1}(K)) \times \mathrm{tilt}(\mathrm{T}_{n_2}(K)).$$

We can obtain all basic tilting $R(n_1, n_2)$-modules from the above bijection and Theorem 13.

References

[1] F. W. Anderson, K. R. Fuller, *Rings and Categories of Modules (second edition)*, Graduate Texts in Math. 13, Springer-Verlag, Heidelberg/New York/Berlin (1991).

[2] T. Araya, *Exceptional Sequences over path algebras of type A_n and Non-crossing Spanning Trees*, arXiv: 0904. 2831.

[3] I. Assem, D. Simson, A. Skowronski, *Elements of the Representation Theory of Associative Algebras*, London Mathematical Society Student Texts 65, Cambridge university press (2006).

[4] M. Auslander, M. I. Platzeck, I. Reiten, *Coxeter functors without diagrams.* Trans. Amer. Math. Soc. 250 (1979), 1–46

[5] M. Auslander, I. Reiten, *k-Gorenstein algebras and syzygy modules*, J. Pure Appl. Algebra 92 (1994), 1–27.

[6] M. Auslander, I. Reiten, S. Smalo, *Representation Theory of Artin Algebras*, Cambridge Studies in Advanced Mathematics 36, Cambridge university press (1995).

[7] Y. Baba and K. Iwase, *On quasi-Harada rings*, J. Algebra 155 (1996), 415–434.

[8] Y. Baba and K. Oshiro, *Classical Artinian Rings and Related Topics*, World Scientific (2009).

[9] S. Brenner, M. C. R. Butler, *Generalizations of the Bernstein Gelfand Ponomarev reflection functors*, Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979), Lecture Notes in Math., 832, Springer, Berlin-New York (1980), 103–169.

[10] I. N. Bernstein, I. M. Gelfand, V. A. Ponomarev, *Coxeter functors and Gabriel's theorem*, (Russian) Uspehi Mat. Nauk 28 no. 2(170) (1973), 19–33.

[11] P. Caldero, F. Chapoton, R. Schiffler, *Quivers with relations arising from clusters (A_n case)*, Trans. Amer. Math. Soc. 358 no. 3 (2006), 1347–1364.

[12] R. Fossum, P. Griffith, I. Reiten, *Trivial extensions of abelian categories*, Lecture Notes in Mathematics Vol. 456. Springer-Verlag, Berlin-New York, (1975).

[13] M. Harada, *Non-small modules and non-cosmall modules*, Ring Theory. Proceedings of 1978 Antwerp Conference, New York (1979), 669–690.

[14] Z. Huang, O. Iyama, *Auslander-type conditions and cotorsion pairs*, J. Algebra 318 no. 1 (2007), 93–100.

[15] K. Igusa, S. O. Smalo, G. Todorov, *Finite projectivity and contravariant finiteness*. Proc. Amer. Math. Soc. 109 no. 4 (1990), 937–941.

[16] O. Iyama, *Higher-dimensional Auslander-Reiten theory on maximal orthogonal subcategories*, Adv. Math. 210 no. 1 (2007), 22–50.

[17] K. Koike, *Almost self-duality and Harada rings*, J. Algebra 254 (2002), 336–361.

[18] K. Oshiro, *Lifting modules, extending modules and their applications to QF-rings*, Hokkaido Math. J. 13 (1984), 310–338.

[19] ———, *Lifting modules, extending modules and their applications to generalized uniserial rings*, Hokkaido Math. J. 13 (1984), 339–346.

[20] ———, *On Harada rings I*, Math. J. Okayama Univ. 31 (1989), 161–178.

[21] ———, *On Harada rings II*, Math. J. Okayama Univ. 31 (1989), 169–188.

[22] ———, *On Harada rings III*, Math. J. Okayama Univ. 32 (1990), 111–118.

[23] C. Riedtmann, *Representation-finite self-injective algebras of class A_n*, Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979), Lecture Notes in Math., 832, Springer, Berlin (1980), 449–520.

[24] H. Tachikawa, *Quasi-Frobenius rings and generalizations*, Lecture Notes in Mathematics Vol. 351. Springer-Verlag, Berlin-New York, (1973).

[25] Thrall, R. M., *Some generalization of quasi-Frobenius algebras*, Trans. Amer. Math. Soc. 64 (1948), 173–183.

Graduate School of Mathematics
Nagoya University
Frocho, Chikusaku, Nagoya 464-8602 Japan
E-mail address: m07052d@math.nagoya-u.ac.jp

HOMOLOGICAL APPROACH TO THE FACE RING OF A SIMPLICIAL POSET

KOHJI YANAGAWA*

ABSTRACT. A finite poset P is called *simplicial*, if it has the smallest element $\hat{0}$, and every interval $[\hat{0}, x]$ is a boolean algebra. The face poset of a simplicial complex is a typical example. Generalizing the Stanley-Reisner ring of a simplicial complex, Stanley assigned the graded ring A_P to P. This ring has been studied from both combinatorial and topological perspective. In this paper, we will give a concise description of a dualizing complex of A_P and some related results.

1. Introduction

All posets (partially ordered sets) in this paper will be assumed to be finite. By the order given by inclusion, the power set of a finite set can be seen as a poset, and it is called a *boolean algebra*. We say a poset P is *simplicial*, if it admits the smallest element $\hat{0}$, and the interval $[\hat{0}, x] := \{ y \in P \mid y \leq x \}$ is isomorphic to a boolean algebra for all $x \in P$. For the simplicity, we denote $\mathrm{rank}(x)$ of $x \in P$ just by $\rho(x)$. If P is simplicial and $\rho(x) = m$, then $[\hat{0}, x]$ is isomorphic to the boolean algebra $2^{\{1,\dots,m\}}$.

Let Δ be a finite simplicial complex (with $\emptyset \in \Delta$). The face poset (i.e., the set of the faces of Δ with order given by inclusion) is a simplicial poset. Any simplicial poset P is obtained as the face poset of a regular cell complex, which we denote by $\Gamma(P)$. For $\hat{0} \neq x \in P$, $c(x) \in \Gamma(P)$ denotes the open cell corresponds to x. Clearly, $\dim c(x) = \rho(x) - 1$. While the closure $\overline{c(x)}$ of $c(x)$ is always a simplex, the intersection $\overline{c(x)} \cap \overline{c(y)}$ for $x, y \in P$ is not necessarily a simplex. For example, if two d-simplices are glued along their boundaries, then it is not a simplicial complex, but gives a simplicial poset.

For $x, y \in P$, set

$$[x \vee y] := \text{the set of the minimal elements of } \{ z \in P \mid z \geq x, y \}.$$

More generally, for $x_1, \dots, x_m \in P$, $[x_1 \vee \cdots \vee x_m]$ denotes the set of the minimal elements of the common upper bounds of $x_1, \dots, x_m$.

Set $\{ y \in P \mid \rho(y) = 1 \} = \{y_1, \dots, y_n\}$. For $U \subset [n] := \{1, \dots, n\}$, we simply denote $[\bigvee_{i \in U} y_i]$ by $[U]$. If $x \in [U]$, then $\rho(x) = \#U$. For each $x \in P$, there exists a unique U such that $x \in [U]$. Let $x, x' \in P$ with $x \geq x'$ and $\rho(x) = \rho(x') + 1$, and take $U, U' \subset [n]$ such that $x \in [U]$ and $x' \in [U']$. Since $U = U' \coprod \{i\}$ for some i in this case, we can set

$$\alpha(i, U) := \#\{ j \in U \mid j < i \} \qquad \text{and} \qquad \epsilon(x, x') := (-1)^{\alpha(i,U)}.$$

This article is basically a digest version of the preprint [11], which has been submitted to a journal.
* Partially supported by Grant-in-Aid for Scientific Research (c) (no.19540028).

Then ϵ gives an incidence function of the cell complex $\Gamma(P)$, that is, for all $x, y \in P$ with $x > y$ and $\rho(x) = \rho(y) + 2$, we have $\epsilon(x, z) \cdot \epsilon(z, y) + \epsilon(x, z') \cdot \epsilon(z', y) = 0$, where $\{z, z'\} = \{\, w \in P \mid x > w > y \,\}$.

As is well-known, the Stanley-Reisner ring of a finite simplicial complex is a powerful tool for combinatorics. Generalizing this idea, Stanley [6] assigned the commutative ring A_P to a simplicial poset P. For the definition of A_P, we remark that if $[x \vee y] \neq \emptyset$ then $\{\, z \in P \mid z \leq x, y \,\}$ has the largest element $x \wedge y$. Let $\Bbbk$ be a field, and $S := \Bbbk[\, t_x \mid x \in P \,]$ the polynomial ring in the variables t_x. Consider the ideal

$$I_P := (\, t_x t_y - t_{x \wedge y} \sum_{z \in [x \vee y]} t_z \mid x, y \in P \,) + (\, t_{\hat{0}} - 1 \,)$$

of S (if $[x \vee y] = \emptyset$, we interpret that $\sum_{z \in [x \vee y]} t_z = 0$), and set

$$A_P := S/I_P.$$

We denote A_P just by A, if there is no danger of confusion. Clearly, $\dim A_P = \operatorname{rank} P = \dim \Gamma(P) + 1$. For a rank 1 element $y_i \in P$, set $t_i := t_{y_i}$. If $\{x\} = [U]$ for some $U \subset [n]$ with $\# U \geq 2$, then $t_x = \prod_{i \in U} t_i$ in A, and t_x is a "dummy variable". Clearly, A is a graded ring with $\deg(t_x) = \rho(x)$. If $\Gamma(P)$ is a simplicial complex, then A_P is generated by degree 1 elements, and coincides with the Stanley-Reisner ring of $\Gamma(P)$.

Note that A also has a $\mathbb{Z}^n$-grading such that $\deg t_i \in \mathbb{N}^n$ is the ith unit vector. For each $x \in P$, the ideal

$$\mathfrak{p}_x := (t_z \mid z \not\leq x)$$

of A is a prime ideal with $\dim A/\mathfrak{p}_x = \rho(x)$, since $A/\mathfrak{p}_x \cong \Bbbk[\, t_i \mid y_i \leq x \,]$.

Recently, M. Masuda and his coworkers studied A_P with a view from *toric topology*, since the *equivariant cohomology* ring of a torus manifold is of the form A_P (cf. [4, 5]). In this paper, we will introduce another approach.

Let R be a noetherian commutative ring, $\operatorname{Mod} R$ the category of R-modules, and $\operatorname{mod} R$ its full subcategory consisting of finitely generated modules. The *dualizing complex* $D_R^\bullet$ of R gives the important duality $\mathbf{R}\operatorname{Hom}_R(-, D_R^\bullet)$ on the bounded derived category $\mathsf{D}^b(\operatorname{mod} R)$. If R is a (graded) local ring with the maximal ideal $\mathfrak{m}$, then the (graded) Matlis dual of $H^{-i}(D_R^\bullet)$ is the local cohomology $H_{\mathfrak{m}}^i(R)$.

We have a concise description of the dualizing complex A_P as follows.

Theorem 1. *Let P be a simplicial poset with $d = \operatorname{rank} P$, and set $A := A_P$. The complex*

$$I_A^\bullet : 0 \to I_A^{-d} \to I_A^{-d+1} \to \cdots \to I_A^0 \to 0,$$

given by

$$I_A^{-i} := \bigoplus_{\substack{x \in P, \\ \rho(x) = i}} A/\mathfrak{p}_x,$$

and

$$\partial_{I_A^\bullet}^{-i} : I_A^{-i} \supset A/\mathfrak{p}_x \ni 1_{A/\mathfrak{p}_x} \longmapsto \sum_{\substack{\rho(y) = i-1, \\ y \leq x}} \epsilon(x, y) \cdot 1_{A/\mathfrak{p}_y} \in \bigoplus_{\substack{\rho(y) = i-1, \\ y \leq x}} A/\mathfrak{p}_y \subset I_A^{-i+1}$$

is isomorphic to the dualizing complex $D_A^\bullet$ of A in $\mathsf{D}^b(\operatorname{Mod} A)$.

To prove this, it might be possible to use the description of $H^i_{\mathfrak{m}}(A)$ by Duval ([1]). However, we will take more conceptual approach. In [8], the author defined a *squarefree module* over a polynomial ring, and many applications have been found. (For example, regarding A as a squarefree module over the polynomial ring $\operatorname{Sym} A_1$, Duval's formula of $H^i_{\mathfrak{m}}(A)$ mentioned above can be proved quickly. See Remark 15.) We will extend this notion to modules over A, and use it in the proof of Theorem 1.

The category $\operatorname{Sq} A$ of square free A-modules is an abelian category with enough injectives, and $A/\mathfrak{p}_x$ is an injective object. Hence $I^\bullet_A$ is a complex in $\operatorname{Sq} A$, and $\mathbb{D}(-) := \underline{\operatorname{Hom}}^\bullet_A(-, I^\bullet_A)$ gives a duality on $\mathsf{K}^b(\text{Inj-Sq})\,(\cong \mathsf{D}^b(\operatorname{Sq} A))$. Moreover, via the forgetful functor $\operatorname{Sq} A \to \operatorname{Mod} A$, $\mathbb{D}$ coincides with the duality $\mathbf{R}\operatorname{Hom}_A(-, D^\bullet_A)$ on $\mathsf{D}^b(\operatorname{mod} A)$.

As [9, 10], we can assign a squarefree A-module M the constructible sheaf M^+ on (the underlying space of) $\Gamma(P)$. In this context, the duality $\mathbb{D}$ corresponds to the Poincaré-Verdier duality for the constructible sheaves on X up to translation. In particular, the sheafification of the complex $I^\bullet_A[-1]$ coincides with the Verdier dualizing complex of X with the coefficients in $\Bbbk$, where $[-1]$ represents a translation by -1.

Using this argument, we can show the following. At least for the Cohen-Macaulay case, it has been shown in Duval [1]. However our proof gives new perspective.

Corollary 2 (see, Theorem 16)**.** *The Cohen-Macaulay (resp. Gorenstein*, Buchsbaum properties) and Serre's condition (S_i) of A_P are topological properties of the underlying space of $\Gamma(P)$. Here we say A_P is Gorenstein*, if A_P is Gorenstein and the $\mathbb{Z}$-graded canonical module ω_{A_P} is generated by its degree 0 part.*

2. Preparation

In the rest of the paper, P is a simplicial poset with $\operatorname{rank} P = d$. As in the preceding section, we use the convention that $A = A_P$, $\{\, y \in P \mid \rho(y) = 1 \,\} = \{y_1, \dots, y_n\}$, and $t_i := t_{y_i} \in A$.

For a subset $U \subset [n] = \{\, 1, \dots, n \,\}$, A_U denotes the localization of A by the multiplicatively closed set $\{\, \prod_{i \in U} t_i^{a_i} \mid a_i \geq 0 \,\}$. If $[U] = \emptyset$, then $A_U = 0$. For $x \in [U]$,

$$u_x := \frac{t_x}{\prod_{i \in U} t_i} \in A_U$$

is an idempotent. Moreover, $u_x \cdot u_{x'} = 0$ for $x, x' \in [U]$ with $x \neq x'$, and $1_{A_U} = \sum_{x \in [U]} u_x$. Hence we have a $\mathbb{Z}^n$-graded direct sum decomposition

$$A_U = \bigoplus_{x \in [U]} A_U \cdot u_x.$$

Let $\operatorname{Gr} A$ be the category of $\mathbb{Z}^n$-graded A-modules, and $\operatorname{gr} A$ its full subcategory consisting of finitely generated modules. Here a morphism $f : M \to N$ in $\operatorname{Gr} A$ is an A-homomorphism with $f(M_{\mathbf{a}}) \subset N_{\mathbf{a}}$ for all $\mathbf{a} \in \mathbb{Z}^n$. As usual, for M and $\mathbf{a} \in \mathbb{Z}^n$, $M(\mathbf{a})$ denotes the shifted module of M with $M(\mathbf{a})_{\mathbf{b}} = M_{\mathbf{a}+\mathbf{b}}$. For $M, N \in \operatorname{Gr} A$,

$$\underline{\operatorname{Hom}}_A(M, N) := \bigoplus_{\mathbf{a} \in \mathbb{Z}^n} \operatorname{Hom}_{\operatorname{Gr} A}(M, N(\mathbf{a}))$$

has a $\mathbb{Z}^n$-graded A-module structure. Similarly, $\underline{\mathrm{Ext}}^i_A(M,N) \in \mathrm{Gr}\,A$ can be defined. If $M \in \mathrm{gr}\,A$, the underlying module of $\underline{\mathrm{Hom}}_A(M,N)$ is isomorphic to $\mathrm{Hom}_A(M,N)$, and the same is true for $\underline{\mathrm{Ext}}^i_A(M,N)$.

If $M \in \mathrm{Gr}\,A$, then $M^\vee := \bigoplus_{\mathbf{a}\in\mathbb{Z}^n} \mathrm{Hom}_{\Bbbk}(M_{-\mathbf{a}}, \Bbbk)$ can be regarded as a $\mathbb{Z}^n$-graded A-module, and $(-)^\vee$ gives an exact contravariant functor from $\mathrm{Gr}\,A$ to itself, which is called the *graded Matlis duality functor*.

Lemma 3. *(1) $E_A(x) := (A_U \cdot u_x)^\vee$ is injective in $\mathrm{Gr}\,A$. Conversely, any indecomposable injective in $\mathrm{Gr}\,A$ is isomorphic to $E_A(x)(\mathbf{a})$ for some $x \in P$ and $\mathbf{a} \in \mathbb{Z}^n$.*

(2) For $M \in \mathrm{Gr}\,A$, set $M_{\geq \mathbf{0}} := \bigoplus_{\mathbf{a}\in\mathbb{N}^n} M_{\mathbf{a}}$. Then we have a canonical isomorphism

$$\phi_x : A/\mathfrak{p}_x \xrightarrow{\cong} E_A(x)_{\geq \mathbf{0}}.$$

The Cěch complex $C^\bullet$ of A with respect to $t_1, \dots, t_n$ is of the form

$$0 \to C^0 \to C^1 \to \cdots \to C^d \to 0 \quad \text{with} \quad C^i = \bigoplus_{\substack{U \subset [n] \\ \#U = i}} A_U$$

(note that if $\#U > d$ then $A_U = 0$). The differential map is given by

$$C^i \supset A_U \ni a \longmapsto \sum_{\substack{U' \supset U \\ \#U' = i+1}} (-1)^{\alpha(U' \setminus U, U)} f_{U',U}(a) \in \bigoplus_{\substack{U' \supset U \\ \#U' = i+1}} A_{U'} \subset C^{i+1},$$

where $f_{U',U} : A_U \to A_{U'}$ is the natural map.

Since the radical of the ideal $(t_1, \dots, t_n)$ is the maximal ideal $\mathfrak{m} := (t_x \mid \hat{0} \neq x \in P)$, the cohomology $H^i(C^\bullet)$ of $C^\bullet$ is isomorphic to the local cohomology $H^i_{\mathfrak{m}}(A)$. Moreover, $C^\bullet$ is isomorphic to $\mathbf{R}\Gamma_{\mathfrak{m}}(A)$ in $\mathsf{D}^b(\mathrm{Mod}\,A)$. Here $\mathbf{R}\Gamma_{\mathfrak{m}}$ is the right derived functor of $\Gamma_{\mathfrak{m}} : \mathrm{Mod}\,A \to \mathrm{Mod}\,A$ given by $\Gamma_{\mathfrak{m}}(M) = \{ s \in M \mid \mathfrak{m}^i s = 0 \text{ for } i \gg 0 \}$. The same is true in the $\mathbb{Z}^n$-graded context. We may regard $\Gamma_{\mathfrak{m}}$ as a functor from $\mathrm{Gr}\,A$ to itself, and let ${}^*\mathbf{R}\Gamma_{\mathfrak{m}}$ be its right derived functor. Then $C^\bullet \cong {}^*\mathbf{R}\Gamma_{\mathfrak{m}}(A)$ in $\mathsf{D}^b(\mathrm{Gr}\,A)$.

Let ${}^*D^\bullet_A$ be a $\mathbb{Z}^n$-graded normalized dualizing complex of A. By the $\mathbb{Z}^n$-graded version of the local duality theorem [2, Theorem V.6.2], we have a quasi-isomorphism $({}^*D^\bullet_A)^\vee \longrightarrow {}^*\mathbf{R}\Gamma_{\mathfrak{m}}(A)$. Taking the Matlis dual, we get a quasi-isomorphism ${}^*\mathbf{R}\Gamma_{\mathfrak{m}}(A)^\vee \longrightarrow {}^*D^\bullet_A$. Hence

$${}^*D^\bullet_A \cong {}^*\mathbf{R}\Gamma_{\mathfrak{m}}(A)^\vee \cong (C^\bullet)^\vee$$

in $\mathsf{D}^b(\mathrm{Gr}\,A)$. Since

$$(C^i)^\vee \cong \bigoplus_{\substack{x \in P \\ \rho(x) = i}} E_A(x)$$

and each $E_A(x)$ is injective in $\mathrm{Gr}\,A$, $(C^\bullet)^\vee$ actually coincides with ${}^*D^\bullet_A$. Hence ${}^*D^\bullet_A$ is of the form

$$0 \to \bigoplus_{\substack{x \in P \\ \rho(x) = d}} E_A(x) \to \bigoplus_{\substack{x \in P \\ \rho(x) = d-1}} E_A(x) \to \cdots \to E_A(\hat{0}) \to 0,$$

where the cohomological degree is given by the same way to $I^\bullet_A$. We will show that this ϕ is a quasi-isomorphism.

For each $i \in \mathbb{Z}$, we have an injection $\phi^i : I^i_A \to {}^*D^i_A$ given by the injection $\phi_x : A/\mathfrak{p}_x \to E_A(x)$ of Lemma 3. Then $\phi := (\phi_i)_{i\in\mathbb{Z}}$ is a chain map $I^\bullet_A \hookrightarrow {}^*D^\bullet$.

Since $\mathfrak{p}_x$ is a $\mathbb{Z}^n$-graded ideal, ${}^*D^\bullet_{A/\mathfrak{p}_x} := \underline{\mathrm{Hom}}^\bullet_A(A/\mathfrak{p}_x, {}^*D^\bullet_A)$ is a $\mathbb{Z}^n$-graded (or $\mathbb{Z}^{\rho(x)}$-graded) dualizing complex of $A/\mathfrak{p}_x$, and quasi-isomorphic to its non-negative part $I^\bullet_{A/\mathfrak{p}_x} := ({}^*D^\bullet_{A/\mathfrak{p}_x})_{\geq \mathbf{0}}$ (the latter statement is the polynomial ring case of Theorem 1, and it is a well-known result). We have the following.

Lemma 4. *For all $x \in P$, $\phi : I^\bullet_A \to {}^*D^\bullet_A$ induces a quasi-isomorphism*
$$I^\bullet_{A/\mathfrak{p}_x} = \underline{\mathrm{Hom}}^\bullet_A(A/\mathfrak{p}_x, I^\bullet_A) \longrightarrow \underline{\mathrm{Hom}}^\bullet_A(A/\mathfrak{p}_x, {}^*D^\bullet_A) = {}^*D^\bullet_{A/\mathfrak{p}_x},$$

3. Squarefree Modules over A_P, and The Proof of Theorem 1

Let $R = \Bbbk[x_1, \dots, x_n]$ be a polynomial ring, and regard it as a $\mathbb{Z}^n$-graded ring. For $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$, set $\mathrm{supp}(\mathbf{a}) := \{\, i \mid a_i \neq 0 \,\} \subset [n]$, and let $x^{\mathbf{a}}$ denote the monomial $\prod x_i^{a_i} \in R$.

Definition 5 ([8])**.** With the above notation, a $\mathbb{Z}^n$-graded R-module M is called *squarefree*, if it is finitely generated, $\mathbb{N}^n$-graded (i.e., $M = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} M_{\mathbf{a}}$), and the multiplication map $M_{\mathbf{a}} \ni s \longmapsto x^{\mathbf{b}} s \in M_{\mathbf{a}+\mathbf{b}}$ is bijective for all $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$ with $\mathrm{supp}(\mathbf{a}) \supset \mathrm{supp}(\mathbf{b})$.

To define a squarefree module over the face ring $A = A_P$ of a simplicial poset P, we equip A with a finer "grading", where the index set is no longer a monoid.

Recall the convention that $\{\, y \in P \mid \rho(y) = 1 \,\} = \{y_1, \dots, y_n\}$ and $t_i = t_{y_i} \in A$. For each $x \in P$, set
$$\mathbb{M}(x) := \bigoplus_{y_i \leq x} \mathbb{N}\, \mathbf{e}_i^x,$$
where $\mathbf{e}_i^x$ is a basis element. So $\mathbb{M}(x) \cong \mathbb{N}^{\rho(x)}$ as additive monoids. For x, z with $x \leq z$, we have an injection $\iota_{z,x} : \mathbb{M}(x) \ni \mathbf{e}_i^x \longmapsto \mathbf{e}_i^z \in \mathbb{M}(z)$ of monoids. Set
$$\mathbb{M} := \varinjlim_{x \in P} \mathbb{M}(x),$$
where the direct limit is taken with respect to $\iota_{z,x} : \mathbb{M}(x) \to \mathbb{M}(z)$ for $x, z \in P$ with $x \leq z$. Note that $\mathbb{M}$ is no longer a monoid, just a set. Since all $\iota_{z,x}$ is an injection, we can regard $\mathbb{M}(x)$ as a subset of $\mathbb{M}$. For each $\underline{\mathbf{a}} \in \mathbb{M}$, $\{x \in P \mid \underline{\mathbf{a}} \in \mathbb{M}(x)\}$ has the smallest element, which is denoted by $\sigma(\underline{\mathbf{a}})$.

We say a monomial $\mathsf{m} = \prod_{x \in P} t_x^{n_x} \in A$, $n_x \in \mathbb{N}$, is *standard*, if $\{\, x \in P \mid n_x \neq 0 \,\}$ is a totally ordered set. The set of the standard monomials forms a $\Bbbk$-basis of A. Let $\underline{\mathbf{a}}, \underline{\mathbf{b}} \in \mathbb{M}$. If $[\sigma(\underline{\mathbf{a}}) \vee \sigma(\underline{\mathbf{b}})] \neq \emptyset$, then we can take the sum $\underline{\mathbf{a}} + \underline{\mathbf{b}} \in \mathbb{M}(x)$ for each $x \in [\sigma(\underline{\mathbf{a}}) \vee \sigma(\underline{\mathbf{b}})]$. Unless $[\sigma(\underline{\mathbf{a}}) \vee \sigma(\underline{\mathbf{b}})]$ consists of a single element, we cannot define $\underline{\mathbf{a}} + \underline{\mathbf{b}} \in \mathbb{M}$. Hence we denote each $\underline{\mathbf{a}} + \underline{\mathbf{b}} \in \mathbb{M}(x)$ by $(\underline{\mathbf{a}} + \underline{\mathbf{b}})|x$.

Definition 6. $M \in \mathrm{Mod}\, A$ is said to be $\mathbb{M}$-*graded* if the following are satisfied;

(1) $M = \bigoplus_{\underline{\mathbf{a}} \in \mathbb{M}} M_{\underline{\mathbf{a}}}$ as $\Bbbk$-vector spaces;

(2) For $\underline{\mathbf{a}}, \underline{\mathbf{b}} \in \mathbb{M}$, we have
$$t^{\underline{\mathbf{a}}} M_{\underline{\mathbf{b}}} \subset \bigoplus_{x \in [\sigma(\underline{\mathbf{a}}) \vee \sigma(\underline{\mathbf{b}})]} M_{(\underline{\mathbf{a}}+\underline{\mathbf{b}})|x}.$$

Hence, if $[\sigma(\underline{\mathbf{a}}) \vee \sigma(\underline{\mathbf{b}})] = \emptyset$, then $t^{\underline{\mathbf{a}}} M_{\underline{\mathbf{b}}} = 0$.

Clearly, A itself is an $\mathbb{M}$-graded module with $A_{\underline{\mathbf{a}}} = \Bbbk\, t^{\underline{\mathbf{a}}}$. Since there is a natural map $\mathbb{M} \to \mathbb{N}^n$, an $\mathbb{M}$-graded module can be see as a $\mathbb{Z}^n$-graded module.

If M is an $\mathbb{M}$-graded A-module, then

$$M_{\not\leq x} := \bigoplus_{\underline{\mathbf{a}} \notin \mathbb{M}(x)} M_{\underline{\mathbf{a}}}$$

is an $\mathbb{M}$-graded submodule for all $x \in P$, and

$$M_{\leq x} := M / M_{\not\leq x}$$

is a $\mathbb{Z}^{\rho(x)}$-graded module over $A/\mathfrak{p}_x \cong \Bbbk[\, t_i \mid y_i \leq x \,]$.

Definition 7. We say an $\mathbb{M}$-graded A-module M is *squarefree*, if $M_{\leq x}$ is a squarefree module over the polynomial ring $A/\mathfrak{p}_x \cong \Bbbk[\, t_i \mid y_i \leq x \,]$ for all $x \in P$.

Clearly, A itself, $\mathfrak{p}_x$ and $A/\mathfrak{p}_x$ for $x \in P$, are squarefree. Let $\operatorname{Sq} A$ be the category of squarefree A-modules and their A-homomorphisms $f : M \to M'$ with $f(M_{\underline{\mathbf{a}}}) \subset M'_{\underline{\mathbf{a}}}$ for all $\underline{\mathbf{a}} \in \mathbb{M}$. For example, $I_A^\bullet$ is a complex in $\operatorname{Sq} A$.

The *incidence algebra* Λ of P over $\Bbbk$ is a finite dimensional associative $\Bbbk$-algebra with basis $\{\, e_{x,y} \mid x, y \in P,\ x \geq y \,\}$ whose multiplication is defined by

$$e_{x,y} \cdot e_{z,w} = \begin{cases} e_{x,w} & \text{if } y = z; \\ 0 & \text{otherwise.} \end{cases}$$

Let $\operatorname{mod} \Lambda$ be the category of finitely generated left Λ-modules.

Proposition 8. *We have* $\operatorname{Sq} A \cong \operatorname{mod} \Lambda$. *Hence* $\operatorname{Sq} A$ *is an abelian category with enough injectives and the injective dimension of each object is at most d. An object $M \in \operatorname{Sq} A$ is an indecomposable injective if and only if $M \cong A/\mathfrak{p}_x$ for some $x \in P$.*

Let Inj-Sq be the full subcategory of $\operatorname{Sq} A$ consisting of all injective objects, that is, finite direct sums of $A/\mathfrak{p}_x$ for various $x \in P$. As is well-known, the bounded homotopy category $\mathsf{K}^b(\text{Inj-Sq})$ is equivalent to $\mathsf{D}^b(\operatorname{Sq} A)$. Since

$$\underline{\operatorname{Hom}}_A(A/\mathfrak{p}_x, A/\mathfrak{p}_y) = \begin{cases} A/\mathfrak{p}_y & \text{if } x \geq y, \\ 0 & \text{otherwise,} \end{cases}$$

we have $\underline{\operatorname{Hom}}_A^\bullet(J^\bullet, I_A^\bullet) \in \mathsf{K}^b(\text{Inj-Sq})$ for all $J^\bullet \in \mathsf{K}^b(M^\bullet)$. Moreover, $\underline{\operatorname{Hom}}_A^\bullet(-, I_A^\bullet)$ preserves homotopy equivalences, and gives a functor $\mathbb{D} : \mathsf{K}^b(\text{Inj-Sq}) \to \mathsf{K}^b(\text{Inj-Sq})^{\mathsf{op}}$.

On the other hand, $M^\bullet \longmapsto \underline{\operatorname{Hom}}_A^\bullet(M^\bullet, {}^*D_A^\bullet)$ gives the functor $\mathbf{R}\underline{\operatorname{Hom}}_A(-, {}^*D_A^\bullet) : \mathsf{D}^b(\operatorname{gr} A) \to \mathsf{D}^b(\operatorname{gr} A)^{\mathsf{op}}$ under the identification $\mathsf{D}^b_{\operatorname{gr} A}(\operatorname{Gr} A) \cong \mathsf{D}^b(\operatorname{gr} A)$. Combining $\mathbb{U} : \mathsf{K}^b(\text{Inj-Sq}) \xrightarrow{\cong} \mathsf{D}^b(\operatorname{Sq} A) \longrightarrow \mathsf{D}^b(\operatorname{gr} A)$ given by the forgetful functor $\operatorname{Sq} A \to \operatorname{gr} A$, we have the two functors $\mathbb{U} \circ \mathbb{D}$ and $\mathbf{R}\underline{\operatorname{Hom}}_A(-, {}^*D_A^\bullet) \circ \mathbb{U}$.

$$\begin{tikzcd}
(\mathsf{D}^b(\operatorname{Sq} A) \cong)\ \mathsf{K}^b(\text{Inj-Sq}) \arrow[r, "\mathbb{U}"] \arrow[d, "\mathbb{D}"'] & \mathsf{D}^b(\operatorname{gr} A) \arrow[d, "\mathbf{R}\underline{\operatorname{Hom}}_A(-, {}^*D_A^\bullet)"] \\
\mathsf{K}^b(\text{Inj-Sq})^{\mathsf{op}} \arrow[r, "\mathbb{U}"'] & \mathsf{D}^b(\operatorname{gr} A)^{\mathsf{op}}
\end{tikzcd}$$

By the chain map $\phi : I_A^\bullet \to {}^*D_A^\bullet$ constructed in the end of the preceding section, we have a natural transformation $\Phi : \mathbb{U} \circ \mathbb{D} \to \mathbf{R}\underline{\mathrm{Hom}}_A(-, {}^*D_A^\bullet) \circ \mathbb{U}$.

Proposition 9. *Φ is a natural isomorphism. Hence $\mathbb{U} \circ \mathbb{D} \cong \mathbf{R}\underline{\mathrm{Hom}}_A(-, {}^*D_A^\bullet) \circ \mathbb{U}$.*

Proof. For $x \in P$, $\Phi(A/\mathfrak{p}_x)$ is the chain map $\underline{\mathrm{Hom}}_A(A/\mathfrak{p}_x, \phi) : \underline{\mathrm{Hom}}_A^\bullet(A/\mathfrak{p}_x, I_A^\bullet) \to \underline{\mathrm{Hom}}_A^\bullet(A/\mathfrak{p}_x, {}^*D_A^\bullet)$, which is a quasi-isomorphism as shown in Lemma 4. Since any indecomposable injectives in $\mathrm{Sq}\, A$ is isomorphic to $A/\mathfrak{p}_x$ for some $x \in P$, Φ is a natural isomorphism by [2, Proposition 7.1]. □

The proof of Theorem 1. Since $A \in \mathrm{Sq}\, A$, we have

$$I_A^\bullet = \mathbb{D}(A) \cong \mathbf{R}\underline{\mathrm{Hom}}_A(A, {}^*D_A^\bullet) = {}^*D_A^\bullet$$

by Proposition 9, where the isomorphism in the center is given by $\Phi(A)$. If we forget the $\mathbb{Z}^n$-grading, ${}^*D_A^\bullet$ is quasi-isomorphic to the usual (non-graded) dualizing complex $D_A^\bullet$. Hence $I_A^\bullet \cong D_A^\bullet$ in $\mathsf{D}^b(\mathrm{Mod}\, A)$. □

Remark 10. For $x \in P$ with $r = \rho(x)$, set $\underline{\mathbf{a}}(x) := (r, r, \dots, r) \in \mathbb{N}^r \cong \mathbb{M}(x) \subset \mathbb{M}$. If $x \geq y$, then there is a degree $\underline{\mathbf{a}}(x) - \underline{\mathbf{a}}(y) \in \mathbb{M}$ such that $t^{\underline{\mathbf{a}}(x)-\underline{\mathbf{a}}(y)} \cdot t^{\underline{\mathbf{a}}(y)} = t^{\underline{\mathbf{a}}(x)}$.

By $\mathsf{K}^b(\text{Inj-Sq}) \cong \mathsf{D}^b(\mathrm{Sq}\, A)$, $\mathbb{D}$ can be regarded as a duality on $\mathsf{D}^b(\mathrm{Sq}\, A)$. Then, through the equivalence $\mathrm{Sq}\, R \cong \mathrm{mod}\, \Lambda$, $\mathbb{D}$ coincides with the duality functor $\mathbf{D}$ on $\mathsf{D}^b(\mathrm{mod}\, \Lambda)$ defined in [10] up to translation. Hence, for $M^\bullet \in \mathsf{D}^b(\mathrm{Sq}\, A)$, the complex $\mathbb{D}(M^\bullet)$ has the following description: The term of cohomological degree p is

$$\mathbb{D}(M^\bullet)^p := \bigoplus_{i+\rho(x)=-p} (M^i_{\underline{\mathbf{a}}(x)})^* \otimes_{\Bbbk} A/\mathfrak{p}_x,$$

where $(-)^*$ denotes the $\Bbbk$-dual. The differential is given by

$$(M^i_{\underline{\mathbf{a}}(x)})^* \otimes_{\Bbbk} A/\mathfrak{p}_x \ni f \otimes 1_{A/\mathfrak{p}_x} \longmapsto \sum_{\substack{y \leq x, \\ \rho(y)=\rho(x)-1}} \epsilon(x,y) \cdot f_y \otimes 1_{A/\mathfrak{p}_y} + (-1)^p \cdot f \circ \partial^{i-1}_{M^\bullet} \otimes 1_{A/\mathfrak{p}_x},$$

where $f_y \in (M_{\underline{\mathbf{a}}(y)})^*$ denotes $M_{\underline{\mathbf{a}}(y)} \ni s \mapsto f(t^{\underline{\mathbf{a}}(x)-\underline{\mathbf{a}}(y)} \cdot s) \in \Bbbk$, and $\epsilon(x,y)$ is the incidence function.

Since $H^{-i}(\mathbb{D}(M)) \cong \underline{\mathrm{Ext}}_A^{-i}(M, {}^*D_A^\bullet) \cong H^i_{\mathfrak{m}}(M)^\vee$ in $\mathrm{Gr}\, A$, we have the following.

Corollary 11. *If $M \in \mathrm{Sq}\, A$, then the local cohomology $H^i_{\mathfrak{m}}(M)^\vee$ can be seen as a squarefree module.*

4. Sheaves and Poincaré-Verdier duality

The results in this section are parallel to those in [9, 10]. Recall that a simplicial poset P gives a regular cell complex $\Gamma(P)$. Let X be the underlying space of $\Gamma(P)$, and $c(x)$ the open cell corresponding to $\hat{0} \neq x \in P$. Hence, for each $x \in P$ with $\rho(x) \geq 2$, $c(x)$ is an open subset of X homeomorphic to $\mathbb{R}^{\rho(x)-1}$ (if $\rho(x) = 1$, then $c(x)$ is a single point), and X is the disjoint union of the cells $c(x)$. Moreover, $x \geq y$ if and only if $\overline{c(x)} \supset c(y)$.

As in the preceding section, let Λ be the incidence algebra of P. In [10], we assigned the constructible sheaf $N^\dagger$ on X to $N \in \mathrm{mod}\, \Lambda$. Through $\mathrm{Sq}\, A \cong \mathrm{mod}\, \Lambda$, we have the constructible sheaf M^+ on X corresponding to $M \in \operatorname{Sq} A$. Here we give a precise construction for the reader's convenience. For the sheaf theory, consult [3].

For $M \in \operatorname{Sq} A$, set

$$\operatorname{Spé}(M) := \bigcup_{\hat{0} \neq x \in P} c(x) \times M_{\underline{\mathbf{a}}(x)},$$

where $\underline{\mathbf{a}}(x) \in \mathbb{M}(x) \subset \mathbb{M}$ is the one defined in Remark 10. Let $\pi : \operatorname{Spé}(M) \to X$ be the projection map which sends $(p, m) \in c(x) \times M_{\underline{\mathbf{a}}(x)} \subset \operatorname{Spé}(M)$ to $p \in c(x) \subset X$. For an open subset $U \subset X$ and a map $s : U \to \operatorname{Spé}(M)$, we will consider the following conditions:

- $(*)$ $\pi \circ s = \mathrm{id}_U$ and $s_p = t^{\underline{\mathbf{a}}(x) - \underline{\mathbf{a}}(y)} \cdot s_q$ for all $p \in c(x) \cap U$, $q \in c(y) \cap U$ with $x \geq y$. Here $s_p \in M_{\underline{\mathbf{a}}(x)}$ (resp. $s_q \in M_{\underline{\mathbf{a}}(y)}$) with $s(p) = (p, s_p)$ (resp. $s(q) = (q, s_q)$).
- $(**)$ There is an open covering $U = \bigcup_{i \in I} U_i$ such that the restriction of s to U_i satisfies $(*)$ for all $i \in I$.

Now we define a sheaf M^+ on X as follows: For an open set $U \subset X$, set

$$M^+(U) := \{\, s \mid s : U \to \operatorname{Spé}(M) \text{ is a map satisfying } (**) \,\}$$

and the restriction map $M^+(U) \to M^+(V)$ for $U \supset V$ is the natural one. It is easy to see that M^+ is a constructible sheaf with respect to the cell decomposition $\Gamma(P)$. For example, A^+ is the $\Bbbk$-constant sheaf $\underline{\Bbbk}_X$ on X, and $(A/\mathfrak{p}_x)^+$ is (the extension to X of) the $\Bbbk$-constant sheaf on the closed cell $\overline{c(x)}$.

Let $\operatorname{Sh}(X)$ be the category of sheaves of $\Bbbk$-vector spaces on X. Since the stalk $(M^+)_p$ at $p \in c(x) \subset X$ is isomorphic to $M_{\underline{\mathbf{a}}(x)}$, the functor $(-)^+ : \operatorname{Sq} A \to \operatorname{Sh}(X)$ is exact.

As mentioned in the previous section, $\mathbb{D} : \mathsf{D}^b(\operatorname{Sq} A) \to \mathsf{D}^b(\operatorname{Sq} A)^{\mathsf{op}}$ corresponds to $\mathbf{T} \circ \mathbf{D}$: $\mathsf{D}^b(\operatorname{mod} \Lambda) \to \mathsf{D}^b(\operatorname{mod} \Lambda)^{\mathsf{op}}$, where $\mathbf{D}$ is the one defined in [10], and $\mathbf{T}$ is the translation functor (i.e., $\mathbf{T}(M^\bullet)^i = M^{i+1}$). Through $(-)^\dagger : \operatorname{mod} \Lambda \to \operatorname{Sh}(X)$, $\mathbf{D}$ gives the Poincaré-Verdier duality on $\mathsf{D}^b(\operatorname{Sh}(X))$, so we have the following.

Theorem 12. *For $M^\bullet \in \mathsf{D}^b(\operatorname{Sq} A)$, we have*

$$\mathbf{T}^{-1} \circ \mathbb{D}(M^\bullet)^+ \cong \mathbf{R}\mathcal{H}om((M^\bullet)^+, \mathcal{D}_X^\bullet)$$

in $\mathsf{D}^b(\operatorname{Sh}(X))$. In particular, $\mathbf{T}^{-1}((I_A^\bullet)^+) \cong \mathcal{D}_X^\bullet$, where $I_A^\bullet$ is the complex constructed in Theorem 1, and $\mathcal{D}_X^\bullet$ is the Verdier dualizing complex of X with the coefficients in $\Bbbk$.

The next result follows from results in [10].

Theorem 13. *For $M \in \operatorname{Sq} A$, we have the decomposition $H_{\mathfrak{m}}^i(M) = \bigoplus_{\underline{\mathbf{a}} \in \mathbb{M}} H_{\mathfrak{m}}^i(M)_{-\underline{\mathbf{a}}}$ by Corollary 11. The the following hold.*

(a) *There is an isomorphism*

$$H^i(X, M^+) \cong H_{\mathfrak{m}}^{i+1}(M)_{\mathbf{0}} \quad \text{for all } i \geq 1,$$

and an exact sequence

$$0 \to H_{\mathfrak{m}}^0(M)_{\mathbf{0}} \to M_{\mathbf{0}} \to H^0(X, M^+) \to H_{\mathfrak{m}}^1(M)_{\mathbf{0}} \to 0.$$

(b) *If $0 \neq \underline{\mathbf{a}} \in \mathbb{M}$ with $x = \sigma(\underline{\mathbf{a}})$, then*

$$H_{\mathfrak{m}}^i(M)_{-\underline{\mathbf{a}}} \cong H_c^{i-1}(U_x, M^+|_{U_x})$$

for all $i \geq 0$. Here $U_x = \bigcup_{z \geq x} c(z)$ is an open set of X, and $H_c^\bullet(-)$ stands for the cohomology with compact support.

Let $\tilde{H}^i(X;\Bbbk)$ denote the ith *reduced cohomology* of X with coefficients in $\Bbbk$. That is, $\tilde{H}^i(X;\Bbbk) \cong H^i(X;\Bbbk)$ for all $i \geq 1$, and $\tilde{H}^0(X;\Bbbk) \oplus \Bbbk \cong H^0(X;\Bbbk)$, where $H^i(X;\Bbbk)$ is the usual cohomology of X. Recall that $H^i(X;\Bbbk)$ is isomorphic to the sheaf cohomology $H^i(X, \underline{\Bbbk}_X)$. In the Stanley-Reisner ring case, (the latter half of) the next result is nothing other than a famous formula of Hochster.

Corollary 14 (Duval [1, Theorem 5.9]). *We have*

$$[H^i_{\mathfrak{m}}(A)]_{\mathbf{0}} \cong \tilde{H}^{i-1}(X;\Bbbk) \quad \textit{and} \quad [H^i_{\mathfrak{m}}(A)]_{-\underline{\mathbf{a}}} \cong H^{i-1}_c(U_x;\Bbbk)$$

for all $i \geq 0$ and all $\mathbf{0} \neq \underline{\mathbf{a}} \in \mathbb{M}$ with $x = \sigma(\underline{\mathbf{a}})$.

Here, $[H^i_{\mathfrak{m}}(A)]_{-\underline{\mathbf{a}}}$ is also isomorphic to the ith cohomology of the cochain complex

$$K^\bullet_x : 0 \to K^{\rho(x)}_x \to K^{\rho(x)+1}_x \to \cdots \to K^d_x \to 0 \quad \textit{with} \quad K^i_x = \bigoplus_{\substack{z \geq x \\ \rho(z)=i}} \Bbbk\, b_z$$

(b_z is a basis element) whose differential map is given by

$$b_z \longmapsto \sum_{\substack{w \geq z \\ \rho(w)=\rho(z)+1}} \epsilon(w,z)\, b_w.$$

For this description, $\underline{\mathbf{a}}$ can be $\mathbf{0} \in \mathbb{M}$. In this case, $x = \hat{0}$.

Duval uses the latter description, and he denotes $H^i(K^\bullet_x)$ by $H^{i-\rho(x)-1}(\mathrm{lk}_P\, x)$.

Proof. The former half follows from Theorem 13. The latter part follows from that $H^i_{\mathfrak{m}}(A) \cong H^{-i}(\mathbb{D}(A))^\vee$ and that $(\mathbb{D}(A)^\vee)_{-\underline{\mathbf{a}}} = K^\bullet_x$ as complexes of $\Bbbk$-vector spaces by Remark 10. □

Remark 15. Consider the polynomial ring $T := \mathrm{Sym}\, A_1 \cong \Bbbk[t_1, \ldots, t_n]$ (note that T is *not* a subring of A). Since A is a squarefree module over T, the $\mathbb{Z}^n$-graded Hilbert function of $H^i_{\mathfrak{m}}(A)$ can be computed by [8, Theorem 2.10], and [1, Theorem 5.9] (essentially, the latter half of Corollary 14) follows rather quickly.

Similarly, we can easily describe $\mathbb{D}_T(A) \cong \mathbf{R}\underline{\mathrm{Hom}}_T(A, D^\bullet_T)$, and it coincides with $I^\bullet_A$ as a complex of T-modules. That is, the dualizing complex $D^\bullet_A$ becomes much easier if we regard it as a complex of T-modules.

Theorem 16 (c.f. Duval [1]). *Set $d := \mathrm{rank}\, P = \dim X + 1$. Then we have the following.*

(a) *A is Cohen-Macaulay if and only if $\mathcal{H}^i(\mathcal{D}^\bullet_X) = 0$ for all $i \neq -d+1$, and $\tilde{H}^i(X;\Bbbk) = 0$ for all $i \neq d-1$.*

(b) *Assume that A is Cohen-Macaulay and $d \geq 2$. Then A is Gorenstein*, if and only if $\mathcal{H}^{-d+1}(\mathcal{D}^\bullet_X) \cong \underline{\Bbbk}_X$. (When $d = 1$, A is Gorenstein* if and only if X consists of exactly two points.)*

(c) *A is Buchsbaum if and only if $\mathcal{H}^i(\mathcal{D}^\bullet_X) = 0$ for all $i \neq -d+1$.*

(d) *Set*

$$d_j := \begin{cases} \dim(\mathrm{supp}\, \mathcal{H}^{-j}(\mathcal{D}^\bullet_X)) & \textit{if } \mathcal{H}^{-j}(\mathcal{D}^\bullet_X) \neq 0, \\ -1 & \textit{if } \mathcal{H}^{-j}(\mathcal{D}^\bullet_X) = 0 \textit{ and } \tilde{H}^j(X;\Bbbk) \neq 0, \\ -\infty & \textit{if } \mathcal{H}^{-j}(\mathcal{D}^\bullet_X) = 0 \textit{ and } \tilde{H}^j(X;\Bbbk) = 0. \end{cases}$$

Here $\operatorname{supp}\mathcal{F} = \{\, p \in X \mid \mathcal{F}_p \neq 0 \,\}$ *for a sheaf* $\mathcal{F}$ *on* X*. Then, for* $2 \leq i < d$*,* A *satisfies Serre's condition* (S_i) *if and only if* $d_j \leq j - i$ *for all* $j < d - 1$*.*

Hence, Cohen-Macaulay (resp. Gorenstein, Buchsbaum) property and Serre's condition* (S_i) *of* A *are topological properties of* X*, while it may depend on* $\operatorname{char}(\Bbbk)$*.*

References

[1] A.M. Duval, *Free resolutions of simplicial posets*, J. Algebra **188** (1997), 363–399.

[2] R. Hartshorne, Residues and duality, Lecture notes in Mathematics **20**, Springer, 1966.

[3] B. Iversen, Cohomology of sheaves, Springer-Verlag, 1986.

[4] M. Masuda, *h-vectors of Gorenstein* simplicial posets*, Adv. Math. **194** (2005), 332–344.

[5] M. Masuda, T. Panov, *On the cohomology of torus manifolds*, Osaka J. Math. **43** (2006) 711–746.

[6] R. Stanley, *f-vectors and h-vectors of simplicial posets*, J. Pure Appl. Algebra **71** (1991), 319–331.

[7] R. Stanley, Combinatorics and commutative algebra, 2nd ed. Birkhäuser 1996.

[8] K. Yanagawa, *Alexander duality for Stanley-Reisner rings and squarefree* $\mathbb{N}^n$*-graded modules*, J. Algebra 225 (2000), 630–645.

[9] K. Yanagawa, *Stanley-Reisner rings, sheaves, and Poincaré-Verdier duality*, Math. Res. Lett. **10** (2003), 635–650.

[10] K. Yanagawa, *Dualizing complex of the incidence algebra of a finite regular cell complex*, Illinois J. Math. **49** (2005), 1221–1243.

[11] K. Yanagawa, *Dualizing complex of the face ring of a simplicial poset*, preprint, `arXiv:0910.1498`.

Department of Mathematics
Kansai University
Suita 564-8680, JAPAN

E-mail address: `yanagawa@ipcku.kansai-u.ac.jp`