THICK SUBCATEGORIES OF THE STABLE CATEGORY OF COHEN-MACAULAY MODULES

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Abstract. Various classification theorems of thick subcategories of a triangulated category have been obtained in many areas of mathematics. In this article, as a higher dimensional version of the classification theorem of thick subcategories of the stable category of finitely generated representations of a finite p-group due to Benson, Carlson and Rickard, we consider classifying thick subcategories of the stable category of Cohen-Macaulay modules over a Gorenstein local ring. The main result of this article yields a complete classification of the thick subcategories of the stable category of Cohen-Macaulay modules over a local hypersurface in terms of specialization-closed subsets of the prime ideal spectrum of the ring which are contained in its singular locus.

One of the principal approaches to the understanding of the structure of a given category is classifying its subcategories having a specific property. It has been studied in many areas of mathematics which include stable homotopy theory, ring theory, algebraic geometry and modular representation theory. A landmark result in this context was obtained in the definitive work due to Gabriel [21] in the early 1960s. He proved a classification theorem of the localizing subcategories of the category of modules over a commutative noetherian ring by making a one-to-one correspondence between the set of those subcategories and the set of specialization-closed subsets of the prime ideal spectrum of the ring. A lot of analogous classification results of subcategories of modules have been obtained by many authors; see [29, 39, 36, 22, 23, 24] for instance.

For a triangulated category, a high emphasis has been placed on classifying its *thick* subcategories, namely, full triangulated subcategories which are closed under taking direct summands. The first classification theorem was obtained in the deep work on stable homotopy theory due to Devinatz, Hopkins and Smith [18, 28]. They classified the thick subcategories of the category of compact objects in the p-local stable homotopy category. Hopkins [27] and Neeman [38] provided a corresponding classification result of the thick subcategories of the derived category of perfect complexes (i.e., bounded complexes of finitely generated projective modules) over a commutative noetherian ring by making a one-to-one correspondence between the set of those subcategories and the set of specialization-closed subsets of the prime ideal spectrum of the ring. Thomason [43] generalized the theorem of Hopkins and Neeman to quasi-compact and quasi-separated schemes, in particular, to arbitrary commutative rings and algebraic varieties. Recently, Avramov, Buchweitz, Christensen, Iyengar and Piepmeyer [4] gave a classification of the thick subcategories of the derived category of perfect differential modules over a commutative noetherian ring. On the other hand, Benson, Carlson and Rickard [9] classified the thick subcategories of the stable category of finitely generated representations of a finite p-group in terms of closed homogeneous subvarieties of the maximal ideal spectrum

The detailed version of this paper has been submitted for publication elsewhere.

of the group cohomology ring. Friedlander and Pevtsova [20] extended this classification theorem to finite group schemes. A recent work of Benson, Iyengar and Krause [11] gives a new proof of the theorem of Benson, Carlson and Rickard. A lot of other related results concerning thick subcategories of a triangulated category have been obtained. For example, see [6, 7, 8, 35, 14, 10, 31, 12, 19, 41].

Here we mention that in most of the classification theorems of subcategories stated above, the subcategories are classified in terms of certain sets of prime ideals. Each of them establishes an assignment corresponding each subcatgory to a set of prime ideals, which is (or should be) called the *support* of the subcategory.

In the present article, as a higher dimensional version of the work of Benson, Carlson and Rickard, we consider classifying thick subcategories of the stable category of Cohen-Macaulay modules over a Gorenstein local ring, through defining a suitable support for those subcategories. Over a hypersurface we shall give a complete classification of them in terms of specialization-closed subsets of the prime ideal spectrum of the base ring contained in its singular locus.

Convention. In the rest of this article, we assume that all rings are commutative and noetherian, and that all modules are finitely generated. Unless otherwise specified, let R be a local ring of Krull dimension d. The unique maximal ideal of R and the residue field of R are denoted by $\mathfrak{m}$ and k, respectively. By a *subcategory*, we always mean a full subcategory which is closed under isomorphism. (A full subcategory $\mathcal{X}$ of a category $\mathcal{C}$ is said to be closed under isomorphism provided that for two objects M, N of $\mathcal{C}$ if M belongs to $\mathcal{X}$ and N is isomorphic to M in $\mathcal{C}$, then N also belongs to $\mathcal{X}$.) Note that a subcategory in our sense is uniquely determined by the isomorphism classes of the objects in it.

We begin with recalling the definition of the syzygies of a module.

Definition 1. Let n be a nonnegative integer, and let M be an R-module. Let

$$\cdots \xrightarrow{\partial_{n+1}} F_n \xrightarrow{\partial_n} F_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \to M \to 0$$

be a minimal free resolution of M. The n^{th} *syzygy* of M is defined as the image of the map ∂_n, and we denote it by $\Omega^n M$. We simply write ΩM instead of $\Omega^1 M$. Note that the n^{th} syzygy of a given R-module is uniquely determined up to isomorphism because so is a minimal free resolution.

Next, we make a list of several closed properties of a subcategory.

Definition 2. (1) Let $\mathcal{C}$ be an additive category and $\mathcal{X}$ a subcategory of $\mathcal{C}$.

(i) We say that $\mathcal{X}$ is *closed under (finite) direct sums* provided that if $M_1, \dots, M_n$ are objects of $\mathcal{X}$, then the direct sum $M_1 \oplus \cdots \oplus M_n$ in $\mathcal{C}$ belongs to $\mathcal{X}$.

(ii) We say that $\mathcal{X}$ is *closed under direct summands* provided that if M is an object of $\mathcal{X}$ and N is a direct summand of M in $\mathcal{C}$, then N belongs to $\mathcal{X}$.

(2) Let $\mathcal{C}$ be a triangulated category and $\mathcal{X}$ a subcategory of $\mathcal{C}$. We say that $\mathcal{X}$ is *closed under triangles* provided that for each exact triangle $L \to M \to N \to \Sigma L$ in $\mathcal{C}$, if two of L, M, N belong to $\mathcal{X}$, then so does the third.

(3) We denote by $\operatorname{mod} R$ the category of finitely generated R-modules. Let $\mathcal{X}$ be a subcategory of $\operatorname{mod} R$.

(i) We say that $\mathcal{X}$ is *closed under extensions* provided that for each exact sequence $0 \to L \to M \to N \to 0$ in $\operatorname{mod} R$, if L and N belong to $\mathcal{X}$, then so does M.
(ii) We say that $\mathcal{X}$ is *closed under kernels of epimorphisms* provided that for each exact sequence $0 \to L \to M \to N \to 0$ in $\operatorname{mod} R$, if M and N belong to $\mathcal{X}$, then so does L.
(iii) We say that $\mathcal{X}$ is *closed under syzygies* provided that if M is an R-module in $\mathcal{X}$, then $\Omega^i M$ is also in $\mathcal{X}$ for all $i \geq 0$.

Let us make several definitions of subcategories.

Definition 3. (1) Let $\mathcal{C}$ be a category.
(i) We call the subcategory of $\mathcal{C}$ which has no object the *empty subcategory* of $\mathcal{C}$.
(ii) Suppose that $\mathcal{C}$ admits the zero object 0. We call the subcategory of $\mathcal{C}$ consisting of all objects that are isomorphic to 0 the *zero subcategory* of $\mathcal{C}$.
(2) A subcategory of a triangulated category is called *thick* if it is closed under direct summands and triangles.
(3) A subcategory of $\operatorname{mod} R$ is called *resolving* if it contains R and if it is closed under direct summands, extensions and kernels of epimorphisms.

Remark 4. (1) A resolving subcategory is a subcategory such that any two "minimal" resolutions of a module by modules in it have the same length; see [1, Lemma (3.12)].
(2) Every resolving subcategory of $\operatorname{mod} R$ contains all free R-modules.
(3) A subcategory of $\operatorname{mod} R$ is resolving if and only if it contains R and is closed under direct summands, extensions and syzygies.

The notion of a resolving subcategory was introduced by Auslander and Bridger [1] in the late 1960s. A lot of important subcategories of $\operatorname{mod} R$ are known to be resolving. To present examples of a resolving subcategory, let us recall here several definitions of modules. Let M be an R-module. We say that M is *bounded* if there exists an integer s such that $\beta_i^R(M) \leq s$ for all $i \geq 0$, where $\beta_i^R(M)$ denotes the i^{th} Betti number of M. We say that M has *complexity* c if c is the least nonnegative integer n such that there exists a real number r satisfying the inequality $\beta_i^R(M) \leq ri^{n-1}$ for $i \gg 0$. We call M *semidualizing* if the natural homomorphism $R \to \operatorname{Hom}_R(M, M)$ is an isomorphism and $\operatorname{Ext}_R^i(M, M) = 0$ for all $i > 0$. For a semidualizing R-module C, an R-module M is called *totally C-reflexive* if the natural homomorphism $M \to \operatorname{Hom}_R(\operatorname{Hom}_R(M, C), C)$ is an isomorphism and $\operatorname{Ext}_R^i(M, C) = 0 = \operatorname{Ext}_R^i(\operatorname{Hom}_R(M, C), C)$ for all $i > 0$. A totally R-reflexive R-module is simply called a *totally reflexive* R-module. For an ideal I of R we denote by $\operatorname{grade}(I, M)$ the infimum of the integers i with $\operatorname{Ext}_R^i(R/I, M) \neq 0$. We say that M has *lower complete intersection zero* if M is totally reflexive and has finite complexity. When R is a Cohen-Macaulay local ring, we say that M is *Cohen-Macaulay* if $\operatorname{depth} M = d$. Such a module is usually called maximal Cohen-Macaulay, but in this article, we call it just Cohen-Macaulay. We denote by $\operatorname{CM}(R)$ the subcategory of $\operatorname{mod} R$ consisting of all Cohen-Macaulay R-modules.

Example 5. Let n be a nonnegative integer, K an R-module, and I an ideal of R. The following R-modules form resolving subcategories of $\operatorname{mod} R$.

(1) The R-modules.
(2) The free R-modules.
(3) The Cohen-Macaulay R-modules, provided that R is Cohen-Macaulay.
(4) The totally C-reflexive R-modules, where C is a fixed semidualizing R-module.
(5) The R-modules M with $\operatorname{Tor}_i^R(M,K)=0$ for $i>n$ (respectively, $i\gg 0$).
(6) The R-modules M with $\operatorname{Ext}_R^i(M,K)=0$ for $i>n$ (respectively, $i\gg 0$).
(7) The R-modules M with $\operatorname{Ext}_R^i(K,M)=0$ for $i\gg 0$, provided that $\operatorname{Ext}_R^j(K,R)=0$ for $j\gg 0$.
(8) The R-modules M with $\operatorname{Ext}_R^i(K,M)=0$ for $i<\operatorname{grade} K (:= \operatorname{grade}(\operatorname{Ann} K, R))$.
(9) The R-modules M with $\operatorname{grade}(I,M)\geq \operatorname{grade}(I,R)$.
(10) The bounded R-modules.
(11) The R-modules having finite complexity.
(12) The R-modules of lower complete intersection dimension zero.

Next we recall the definitions of the nonfree loci of an R-module and a subcategory of $\operatorname{mod} R$.

Definition 6. (1) We denote by $\mathcal{V}(X)$ the *nonfree locus* of an R-module X, namely, the set of prime ideals $\mathfrak{p}$ of R such that $X_\mathfrak{p}$ is nonfree as an $R_\mathfrak{p}$-module.
(2) We denote by $\mathcal{V}(\mathcal{X})$ the *nonfree locus* of a subcategory $\mathcal{X}$ of $\operatorname{mod} R$, namely, the union of $\mathcal{V}(X)$ where X runs through all nonisomorphic R-modules in $\mathcal{X}$.

We denote by $\operatorname{Sing} R$ the *singular locus* of R, namely, the set of prime ideals $\mathfrak{p}$ of R such that $R_\mathfrak{p}$ is not a regular local ring. We denote by $\mathcal{S}(R)$ the set of prime ideals $\mathfrak{p}$ of R such that the local ring $R_\mathfrak{p}$ is not a field. Clearly, $\mathcal{S}(R)$ contains $\operatorname{Sing} R$. For each ideal I of R, we denote by $\mathrm{V}(I)$ the set of prime ideals of R containing I. Recall that a subset Z of $\operatorname{Spec} R$ is called *specialization-closed* provided that if $\mathfrak{p}\in Z$ and $\mathfrak{q}\in \mathrm{V}(\mathfrak{p})$ then $\mathfrak{q}\in Z$. Note that every closed subset of $\operatorname{Spec} R$ is specialization-closed. Let $\mathcal{C}$ be a category, and let $\mathbf{P}$ be a property of subcategories of $\mathcal{C}$. Let $\mathcal{X}$ be a subcategory of $\mathcal{C}$. A subcategory $\mathcal{Y}$ of $\mathcal{C}$ satisfying $\mathbf{P}$ is said to be *generated by* $\mathcal{X}$ if $\mathcal{Y}$ is the smallest subcategory of $\mathcal{C}$ satisfying $\mathbf{P}$ that contains $\mathcal{X}$. For a subset Φ of $\operatorname{Spec} R$, we denote by $\mathcal{V}^{-1}(\Phi)$ the subcategory of $\operatorname{mod} R$ consisting of all R-modules M such that $\mathcal{V}(M)$ is contained in Φ. The proposition below gives several basic properties of nonfree loci.

Proposition 7. (1) *Let R be a Cohen-Macaulay local ring. Then the nonfree locus $\mathcal{V}(\mathrm{CM}(R))$ coincides with the singular locus $\operatorname{Sing} R$.*
(2) *One has $\mathcal{V}(X)=\operatorname{Supp}\operatorname{Ext}^1(X,\Omega X)$ for every R-module X. In particular, the nonfree locus of an R-module is closed in $\operatorname{Spec} R$ in the Zariski topology. The nonfree locus of a subcategory of $\operatorname{mod} R$ is not necessarily closed but at least specialization-closed in $\operatorname{Spec} R$, and is contained in $\mathcal{S}(R)$.*
(3) *One has $\mathcal{V}(\mathcal{X})=\mathcal{V}(\operatorname{res}\mathcal{X})$ for every subcategory $\mathcal{X}$ of $\operatorname{mod} R$, where $\operatorname{res}\mathcal{X}$ denotes the resolving subcategory of $\operatorname{mod} R$ generated by $\mathcal{X}$.*
(4) *Let N be a direct summand of an R-module M. Then one has $\mathcal{V}(N)\subseteq\mathcal{V}(M)$.*
(5) *Let $0\to L\to M\to N\to 0$ be an exact sequence of R-modules. Then one has $\mathcal{V}(L)\subseteq\mathcal{V}(M)\cup\mathcal{V}(N)$ and $\mathcal{V}(M)\subseteq\mathcal{V}(L)\cup\mathcal{V}(N)$.*
(6) *For a subset Φ of $\operatorname{Spec} R$, the subcategory $\mathcal{V}^{-1}(\Phi)$ of $\operatorname{mod} R$ is resolving.*
(7) *For an ideal I of R, one has $\mathcal{V}_R(R/I)=\mathrm{V}(I+(0:I))$.*

(8) *One has* $\mathcal{V}_R(R/\mathfrak{p}) = \mathrm{V}(\mathfrak{p})$ *for every* $\mathfrak{p} \in \mathcal{S}(R)$.
(9) *Let* Φ *be a specialization-closed subset of* $\operatorname{Spec} R$ *contained in* $\mathcal{S}(R)$. *Then one has* $R/\mathfrak{p} \in \mathcal{V}^{-1}(\Phi)$ *for every* $\mathfrak{p} \in \Phi$.

We recall the definition of the stable category of Cohen-Macaulay modules over a Cohen-Macaulay local ring.

Definition 8. (1) Let M, N be R-modules. We denote by $\mathcal{F}_R(M, N)$ the set of R-homomorphisms $M \to N$ factoring through free R-modules. It is easy to observe that $\mathcal{F}_R(M, N)$ is an R-submodule of $\mathrm{Hom}_R(M, N)$. We set $\underline{\mathrm{Hom}}_R(M, N) = \mathrm{Hom}_R(M, N)/\mathcal{F}_R(M, N)$.
(2) Let R be a Cohen-Macaulay local ring. The *stable category* of $\mathrm{CM}(R)$, which is denoted by $\underline{\mathrm{CM}}(R)$, is defined as follows.
(i) $\mathrm{Ob}(\underline{\mathrm{CM}}(R)) = \mathrm{Ob}(\mathrm{CM}(R))$.
(ii) $\mathrm{Hom}_{\underline{\mathrm{CM}}(R)}(M, N) = \underline{\mathrm{Hom}}_R(M, N)$ for $M, N \in \mathrm{Ob}(\underline{\mathrm{CM}}(R))$.

Remark 9. Let R be a Cohen-Macaulay local ring. Then $\underline{\mathrm{CM}}(R)$ is always an additive category. The direct sum of objects M and N in $\underline{\mathrm{CM}}(R)$ is the direct sum $M \oplus N$ of M and N as R-modules. Now, we consider the case where R is Gorenstein. Then $\mathrm{CM}(R)$ is a Frobenius category, and $\underline{\mathrm{CM}}(R)$ is a triangulated category. We recall in the following how to define an exact triangle in $\underline{\mathrm{CM}}(R)$. For the details, we refer to [26, Section 2 in Chapter I] or [15, Theorem 4.4.1]. Let M be an object of $\underline{\mathrm{CM}}(R)$. Then, since M is a Cohen-Macaulay R-module, there exists an exact sequence $0 \to M \to F \to N \to 0$ of Cohen-Macaulay R-modules with F free. Defining $\Sigma M = N$, we have an automorphism $\Sigma : \underline{\mathrm{CM}}(R) \to \underline{\mathrm{CM}}(R)$ of categories. This is the suspension functor. Let

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d, "f"'] & F \arrow[r] \arrow[d] & \Sigma L \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r, "g"] & N \arrow[r, "h"] & \Sigma L \arrow[r] & 0
\end{tikzcd}$$

be a commutative diagram of Cohen-Macaulay R-modules with exact rows such that F is free. Then a sequence

$$L' \xrightarrow{f'} M' \xrightarrow{g'} N' \xrightarrow{h'} \Sigma L'$$

of morphisms in $\underline{\mathrm{CM}}(R)$ such that there exists a commutative diagram

$$\begin{tikzcd}
L \arrow[r, "f"] \arrow[d, "\alpha"'] & M \arrow[r, "g"] \arrow[d, "\beta"'] & N \arrow[r, "h"] \arrow[d, "\gamma"'] & \Sigma L \arrow[d, "\Sigma\alpha"'] \\
L' \arrow[r, "f'"] & M' \arrow[r, "g'"] & N' \arrow[r, "h'"] & \Sigma L'
\end{tikzcd}$$

in $\underline{\mathrm{CM}}(R)$ such that α, β, γ are isomorphisms is defined to be an exact triangle in $\underline{\mathrm{CM}}(R)$.

Now, we define the notion of a support for objects and subcategories of the stable category of Cohen-Macaulay modules.

Definition 10. Let R be a Cohen-Macaulay local ring.

(1) For an object M of $\underline{\mathrm{CM}}(R)$, we denote by $\underline{\mathrm{Supp}}\, M$ the set of prime ideals $\mathfrak{p}$ of R such that the localization $M_{\mathfrak{p}}$ is not isomorphic to the zero module 0 in the category $\underline{\mathrm{CM}}(R_{\mathfrak{p}})$. We call it the *stable support* of M.
(2) For a subcategory $\mathcal{Y}$ of $\underline{\mathrm{CM}}(R)$, we denote by $\underline{\mathrm{Supp}}\, \mathcal{Y}$ the union of $\underline{\mathrm{Supp}}\, M$ where M runs through all nonisomorphic objects in $\mathcal{Y}$. We call it the *stable support* of $\mathcal{Y}$.
(3) For a subset Φ of $\mathrm{Spec}\, R$, we denote by $\underline{\mathrm{Supp}}^{-1}\, \Phi$ the subcategory of $\underline{\mathrm{CM}}(R)$ consisting of all objects $M \in \underline{\mathrm{CM}}(R)$ such that $\underline{\mathrm{Supp}}\, M$ is contained in Φ.

The notion of a stable support is essentially the same thing as that of a nonfree locus.

Proposition 11. *Let R be a Cohen-Macaulay local ring.*

(1) *Let M be a Cohen-Macaulay R-module. Then one has* $\underline{\mathrm{Supp}}\, M = \mathcal{V}(M)$.
(2) *Let $\mathcal{X}$ be a subcategory of* $\mathrm{CM}(R)$. *Then one has* $\underline{\mathrm{Supp}}\, \underline{\mathcal{X}} = \mathcal{V}(\mathcal{X})$.
(3) *Let $\mathcal{Y}$ be a subcategory of* $\underline{\mathrm{CM}}(R)$. *Then one has* $\underline{\mathrm{Supp}}\, \mathcal{Y} = \mathcal{V}(\overline{\mathcal{Y}})$.
(4) *Let Φ be a subset of* $\mathrm{Spec}\, R$. *Then one has* $\underline{\mathrm{Supp}}^{-1}\, \Phi = \underline{\mathcal{V}^{-1}(\Phi)}$.

Here we recall the definitions of a hypersurface and an abstract hypersurface.

Definition 12. (1) A local ring R is called a *hypersurface* if there exist a regular local ring S and an element f of S such that R is isomorphic to $S/(f)$.
(2) A local ring R is called an *abstract hypersurface* if there exist a complete regular local ring S and an element f of S such that the completion $\widehat{R}$ of R in the $\mathfrak{m}$-adic topology is isomorphic to $S/(f)$.

Now we can state our main result.

Theorem 13. (1) *Let R be a local hypersurface. Then one has the following one-to-one correspondences:*

$$\begin{array}{c}
\{\textit{nonempty thick subcategories of } \underline{\mathrm{CM}}(R)\} \\
\underline{\mathrm{Supp}}\downarrow \quad \uparrow \underline{\mathrm{Supp}}^{-1} \\
\{\textit{specialization-closed subsets of } \mathrm{Spec}\, R \textit{ contained in } \mathrm{Sing}\, R\} \\
\mathcal{V}^{-1}\downarrow \quad \uparrow \mathcal{V} \\
\{\textit{resolving subcategories of } \mathrm{mod}\, R \textit{ contained in } \mathrm{CM}(R)\}.
\end{array}$$

(2) *Let R be a d-dimensional Gorenstein singular local ring with residue field k which is a hypersurface on the punctured spectrum. Then one has the following one-to-one correspondences:*

$$\begin{array}{c}
\{\textit{thick subcategories of } \underline{\mathrm{CM}}(R) \textit{ containing } \Omega^d k\} \\
\underline{\mathrm{Supp}}\downarrow \quad \uparrow \underline{\mathrm{Supp}}^{-1} \\
\{\textit{nonempty specialization-closed subsets of } \mathrm{Spec}\, R \textit{ contained in } \mathrm{Sing}\, R\} \\
\mathcal{V}^{-1}\downarrow \quad \uparrow \mathcal{V} \\
\{\textit{resolving subcategories of } \mathrm{mod}\, R \textit{ contained in } \mathrm{CM}(R) \textit{ containing } \Omega^d k\}.
\end{array}$$

Remark 14. Very recently, after the work in this article was completed, Iyengar announced in his lecture [32] that thick subcategories of the bounded derived category of finitely generated modules over a locally complete intersection which is essentially of finite type over a field are classified in terms of certain subsets of the prime ideal spectrum of the Hochschild cohomology ring. This provides a classification of thick subcategories of the stable category of Cohen-Macaulay modules over such a ring, which is a different classification from ours.

A singular local hypersurface and a Cohen-Macaulay singular local ring with an isolated sigularity are trivial examples of a ring which satisfies the assumption of Theorem 13(2). We make here some nontrivial examples.

Example 15. Let k be a field. The following rings R are Cohen-Macaulay singular local rings which are hypersurfaces on the punctured spectrums.

(1) Let $R = k[[x, y, z]]/(x^2, yz)$. Then R is a 1-dimensional local complete intersection which is neither a hypersurface nor with an isolated singularity. All the prime ideals of R are $\mathfrak{p} = (x, y)$, $\mathfrak{q} = (x, z)$ and $\mathfrak{m} = (x, y, z)$. It is easy to observe that both of the local rings $R_\mathfrak{p}$ and $R_\mathfrak{q}$ are hypersurfaces.

(2) Let $R = k[[x, y, z, w]]/(y^2-xz, yz-xw, z^2-yw, zw, w^2)$. Then R is a 1-dimensional Gorenstein local ring which is neither a complete intersection nor with an isolated singularity. All the prime ideals are $\mathfrak{p} = (y, z, w)$ and $\mathfrak{m} = (x, y, z, w)$. We easily see that $R_\mathfrak{p}$ is a hypersurface.

(3) Let $R = k[[x, y, z]]/(x^2, xy, yz)$. Then R is a 1-dimensional Cohen-Macaulay local ring which is neither Gorenstein nor with an isolated singularity. All the prime ideals are $\mathfrak{p} = (x, y)$, $\mathfrak{q} = (x, z)$ and $\mathfrak{m} = (x, y, z)$. We have that $R_\mathfrak{p}$ is a hypersurface and that $R_\mathfrak{q}$ is a field.

Applying Theorem 13(1), we observe that over a hypersurface R having an isolated singularity there are only trivial resolving subcategories of $\operatorname{mod} R$ contained in $\mathrm{CM}(R)$ and thick subcategories of $\mathrm{CM}(R)$.

Corollary 16. *Let R be a hypersurface with an isolated singularity.*

(1) *All resolving subcategories of* $\operatorname{mod} R$ *contained in* $\mathrm{CM}(R)$ *are* $\operatorname{add} R$ *and* $\mathrm{CM}(R)$.

(2) *All thick subcategories of* $\underline{\mathrm{CM}}(R)$ *are the empty subcategory, the zero subcategory, and* $\underline{\mathrm{CM}}(R)$.

Here let us consider an example of a hypersurface which does not have an isolated singularity, and an example of a Gorenstein local ring which is not a hypersurface but a hypersurface on the punctured spectrum.

Example 17. (1) Let $R = k[[x, y]]/(x^2)$ be a one-dimensional hypersurface over a field k. Then we have

$$\mathrm{CM}(R) = \operatorname{add}\{R, (x), (x, y^n) \mid n \geq 1\}$$

by [44, Example (6.5)] or [16, Proposition 4.1]. Set $\mathfrak{p} = (x)$ and $\mathfrak{m} = (x, y)$. We have $\operatorname{Sing} R = \operatorname{Spec} R = \{\mathfrak{p}, \mathfrak{m}\}$, hence all specialization-closed subsets of $\operatorname{Spec} R$ (contained in $\operatorname{Sing} R$) are $\emptyset$, $\{\mathfrak{m}\}$ and $\operatorname{Sing} R$. We have $\mathcal{V}^{-1}(\emptyset) = \operatorname{add} R$ and $\mathcal{V}^{-1}(\operatorname{Sing} R) = \mathrm{CM}(R)$. The subcategory $\mathcal{V}^{-1}(\{\mathfrak{m}\})$ of $\mathrm{CM}(R)$ consists of all Cohen-Macaulay modules that are free on the punctured spectrum of R, so it coincides with $\mathrm{add}\{R, (x, y^n) \mid n \geq 1\}$. Thus, by Theorem 13(1), all resolving subcategories of $\mathrm{mod}\, R$ contained in $\mathrm{CM}(R)$ are $\mathrm{add}\, R$, $\mathrm{add}\{R, (x, y^n) \mid n \geq 1\}$ and $\mathrm{CM}(R)$. All thick subcategories of $\underline{\mathrm{CM}}(R)$ are the empty subcategory, the zero subcategory, $\underline{\mathrm{add}\{(x, y^n) \mid n \geq 1\}}$ and $\underline{\mathrm{CM}}(R)$.

(2) Let $R = k[[x, y, z]]/(x^2, yz)$ be a one-dimensional complete intersection over a field k. Then R is neither a hypersurface nor with an isolated singularity. All prime ideals of R are $\mathfrak{p} = (x, y)$, $\mathfrak{q} = (x, z)$ and $\mathfrak{m} = (x, y, z)$. It is easy to see that both $R_\mathfrak{p}$ and $R_\mathfrak{q}$ are hypersurfaces. Note that all the nonempty specialization-closed subsets of $\mathrm{Spec}\, R$ (contained in $\mathrm{Sing}\, R$) are the following four sets:

$$\mathrm{V}(\mathfrak{p}),\ \mathrm{V}(\mathfrak{q}),\ \mathrm{V}(\mathfrak{p}, \mathfrak{q}),\ \mathrm{V}(\mathfrak{p}, \mathfrak{q}, \mathfrak{m}).$$

Theorem 13(2) says that there exist just four thick subcategories of $\underline{\mathrm{CM}}(R)$ containing $\Omega^d k$, and exist just four resolving subcategories of $\mathrm{mod}\, R$ contained in $\mathrm{CM}(R)$ containing $\Omega^d k$.

Remark 18. Let R be a Gorenstein local ring. In the case where R has an isolated singularity, a thick subcategory of $\underline{\mathrm{CM}}(R)$ coincides with $\underline{\mathrm{CM}}(R)$ whenever it contains $\Omega^d k$. Example 17 especially says that this statement does not necessarily hold if one removes the assumption that R has an isolated singularity. Indeed, with the notation of Example 17(1), $\underline{\mathrm{add}\{(x, y^n) \mid n \geq 1\}}$ is a thick subcategory of $\underline{\mathrm{CM}}(R)$ containing $\Omega^d k = \mathfrak{m}$ which does not coincide with $\underline{\mathrm{CM}}(R)$. Example 17(2) also gives three such subcategories.

Using Theorem 13(1), we can show the following proposition. As it says, the subcategories of $\mathrm{CM}(R)$ and $\underline{\mathrm{CM}}(R)$ corresponding to a closed subset of $\mathrm{Spec}\, R$ are relatively "small."

Proposition 19. *Let R be a hypersurface. Then one has the following one-to-one correspondences:*

$$\begin{array}{ccc}
\left\{\begin{array}{c}\textit{thick subcategories of } \underline{\mathrm{CM}}(R) \\ \textit{generated by one object}\end{array}\right\} & \xrightarrow{\subseteq} & \left\{\begin{array}{c}\textit{nonempty thick subcategories} \\ \textit{of } \underline{\mathrm{CM}}(R)\end{array}\right\} \\
\underline{\mathrm{Supp}}\downarrow \quad \uparrow \underline{\mathrm{Supp}}^{-1} & & \underline{\mathrm{Supp}}\downarrow \quad \uparrow \underline{\mathrm{Supp}}^{-1} \\
\left\{\begin{array}{c}\textit{closed subsets of } \mathrm{Spec}\, R \\ \textit{contained in } \mathrm{Sing}\, R\end{array}\right\} & \xrightarrow{\subseteq} & \left\{\begin{array}{c}\textit{specialization-closed subsets of } \mathrm{Spec}\, R \\ \textit{contained in } \mathrm{Sing}\, R\end{array}\right\} \\
\mathcal{V}^{-1}\downarrow \quad \uparrow \mathcal{V} & & \mathcal{V}^{-1}\downarrow \quad \uparrow \mathcal{V} \\
\left\{\begin{array}{c}\textit{resolving subcategories of } \mathrm{mod}\, R \\ \textit{generated by one object in } \mathrm{CM}(R)\end{array}\right\} & \xrightarrow{\subseteq} & \left\{\begin{array}{c}\textit{resolving subcategories of } \mathrm{mod}\, R \\ \textit{contained in } \mathrm{CM}(R)\end{array}\right\}.
\end{array}$$

From now on, we make some applications of our Theorem 13. First, we have a vanishing result of homological and cohomological δ-functors from the category of finitely generated modules over a hypersurface.

Proposition 20. *Let R be a hypersurface and M an R-module. Let $\mathcal{A}$ be an abelian category.*

(1) *Let $T : \operatorname{mod} R \to \mathcal{A}$ be a covariant or contravariant homological δ-functor with $T_i(R) = 0$ for $i \gg 0$. If there exists an R-module M with $\operatorname{pd}_R M = \infty$ and $T_i(M) = 0$ for $i \gg 0$, then $T_i(k) = 0$ for $i \gg 0$.*
(2) *Let $T : \operatorname{mod} R \to \mathcal{A}$ be a covariant or contravariant cohomological δ-functor with $T^i(R) = 0$ for $i \gg 0$. If there exists an R-module M with $\operatorname{pd}_R M = \infty$ and $T^i(M) = 0$ for $i \gg 0$, then $T^i(k) = 0$ for $i \gg 0$.*

Proof. (1) First of all, note that each T_i preserves direct sums. We easily see that for any R-module N and any integers $n \geq 0$ and $i \gg 0$ we have

$$T_i(\Omega^n N) \cong \begin{cases} T_{i+n}(N) & \text{if } T \text{ is covariant,} \\ T_{i-n}(N) & \text{if } T \text{ is contravariant.} \end{cases}$$

Consider the subcategory $\mathcal{X}$ of $\mathrm{CM}(R)$ consisting of all Cohen-Macaulay R-modules X with $T_i(X) = 0$ for $i \gg 0$. Then it is easily observed that $\mathcal{X}$ is a thick subcategory of $\mathrm{CM}(R)$ containing R. Since $T_i(\Omega^d M)$ is isomorphic to $T_{i+d}(M)$ (respectively, $T_{i-d}(M)$) for $i \gg 0$ if T is covariant (respectively, contravariant), the nonfree Cohen-Macaulay R-module $\Omega^d M$ belongs to $\mathcal{X}$. Hence the maximal ideal $\mathfrak{m}$ belongs to $\mathcal{V}(\Omega^d M)$, which is contained in $\mathcal{V}(\mathcal{X})$, and we have $\mathcal{V}(\Omega^d k) \subseteq \{\mathfrak{m}\} \subseteq \mathcal{V}(\mathcal{X})$. Therefore $\Omega^d k$ belongs to $\mathcal{V}^{-1}(\mathcal{V}(\mathcal{X}))$, which coincides with $\mathcal{X}$ by Theorem 13(1). Thus we obtain $T_i(\Omega^d k) = 0$ for $i \gg 0$. Since $T_i(\Omega^d k)$ is isomorphic to $T_{i+d}(k)$ (respectively, $T_{i-d}(k)$) for $i \gg 0$ if T is covariant (respectively, contravariant), we have $T_i(k) = 0$ for $i \gg 0$, as desired.

(2) An analogous argument to the proof of (1) shows this assertion. $\square$

As a corollary of Proposition 20, we obtain the following vanishing result of Tor and Ext modules.

Corollary 21. *Let R be an abstract hypersurface. Let M and N be R-modules.*

(1) *The following are equivalent:*
  (i) $\operatorname{Tor}_i^R(M, N) = 0$ *for* $i \gg 0$*;*
  (ii) *Either* $\operatorname{pd}_R M < \infty$ *or* $\operatorname{pd}_R N < \infty$*.*
(2) *The following are equivalent:*
  (i) $\operatorname{Ext}_R^i(M, N) = 0$ *for* $i \gg 0$*;*
  (ii) *Either* $\operatorname{pd}_R M < \infty$ *or* $\operatorname{id}_R N < \infty$*.*

The first assertion of Corollary 21 gives another proof of a theorem of Huneke and Wiegand [30, Theorem 1.9].

Corollary 22 (Huneke-Wiegand)**.** *Let R be an abstract hypersurface. Let M and N be R-modules. If $\operatorname{Tor}_i^R(M, N) = \operatorname{Tor}_{i+1}^R(M, N) = 0$ for some $i \geq 0$, then either M or N has finite projective dimension.*

Remark 23. Several generalizations of Corollaries 21(1) and 22 to complete intersections have been obtained by Jorgensen [33, 34], Miller [37] and Avramov and Buchweitz [3].

The assertions of Corollary 21 do not necessarily hold if the ring R is not an abstract hypersurface.

Example 24. Let k be a field. Consider the artinian complete intersection local ring $R = k[[x, y]]/(x^2, y^2)$. Then we can easily verify $\operatorname{Tor}_i^R(R/(x), R/(y)) = 0$ and

$\mathrm{Ext}^i_R(R/(x), R/(y)) = 0$ for all $i > 0$. But both $R/(x)$ and $R/(y)$ have infinite projective dimension, and infinite injective dimension by [13, Exercise 3.1.25].

Let $\mathbf{H}$ be a property of local rings. Let $\mathbf{H}\text{-}\dim_R$ be a numerical invariant for R-modules satisfying the following conditions.

(1) $\mathbf{H}\text{-}\dim_R R < \infty$.
(2) Let M be an R-module and N a direct summand of M. If $\mathbf{H}\text{-}\dim_R M < \infty$, then $\mathbf{H}\text{-}\dim_R N < \infty$.
(3) Let $0 \to L \to M \to N \to 0$ be an exact sequence of R-modules.
  (i) If $\mathbf{H}\text{-}\dim_R L < \infty$ and $\mathbf{H}\text{-}\dim_R M < \infty$, then $\mathbf{H}\text{-}\dim_R N < \infty$.
  (ii) If $\mathbf{H}\text{-}\dim_R L < \infty$ and $\mathbf{H}\text{-}\dim_R N < \infty$, then $\mathbf{H}\text{-}\dim_R M < \infty$.
  (iii) If $\mathbf{H}\text{-}\dim_R M < \infty$ and $\mathbf{H}\text{-}\dim_R N < \infty$, then $\mathbf{H}\text{-}\dim_R L < \infty$.
(4) The following are equivalent:
  (i) R satisfies $\mathbf{H}$;
  (ii) $\mathbf{H}\text{-}\dim_R M < \infty$ for any R-module M;
  (iii) $\mathbf{H}\text{-}\dim_R k < \infty$.

The conditions (1) and (3) imply the following condition:

(5) Let M be an R-module. If $\mathrm{pd}_R M < \infty$, then $\mathbf{H}\text{-}\dim_R M < \infty$.

Indeed, let M be an R-module with $\mathrm{pd}_R M < \infty$. Then there is an exact sequence

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

of R-modules with each F_i free. The conditions (1) and (3)(ii) imply that $\mathbf{H}\text{-}\dim_R F_i < \infty$ for all $0 \le i \le n$. Decomposing the above exact sequences into short exact sequences and applying the condition (3)(i), we have $\mathbf{H}\text{-}\dim_R M < \infty$, as required.

We call such a numerical invariant a *homological dimension*. A lot of homological dimensions are known. For example, projective dimension pd_R, complete intersection dimension $\mathrm{CI}\text{-}\dim_R$ (cf. [5]), Gorenstein dimension $\mathrm{G}\text{-}\dim_R$ (cf. [1, 17]) and Cohen-Macaulay dimension $\mathrm{CM}\text{-}\dim_R$ (cf. [25]) coincide with $\mathbf{H}\text{-}\dim_R$ where $\mathbf{H}$ is regular, complete intersection, Gorenstein and Cohen-Macaulay, respectively. Several other examples of a homological dimension can be found in [2]. A lot of studies of homological dimensions have been done so far. For each homological dimension $\mathbf{H}\text{-}\dim_R$, investigating R-modules M with $\mathbf{H}\text{-}\dim_R M < \infty$ but $\mathrm{pd}_R M = \infty$ is one of the most important problems in the studies of homological dimensions. In this sense, the following proposition says that a hypersurface does not admit a proper homological dimension.

Proposition 25. *With the above notation, let R be a hypersurface not satisfying the property $\mathbf{H}$. Let M be an R-module. Then $\mathbf{H}\text{-}\dim_R M < \infty$ if and only if $\mathrm{pd}_R M < \infty$.*

Proof. The condition (5) says that $\mathrm{pd}_R M < \infty$ implies $\mathbf{H}\text{-}\dim_R M < \infty$. Conversely, assume $\mathbf{H}\text{-}\dim_R M < \infty$. Let $\mathcal{X}$ be the subcategory of $\mathrm{CM}(R)$ consisting of all Cohen-Macaulay R-modules X satisfying $\mathbf{H}\text{-}\dim_R X < \infty$. It follows from the conditions (1), (2) and (3) that $\mathcal{X}$ is a thick subcategory of $\mathrm{CM}(R)$ containing R. Theorem 13(1) yields $\mathcal{X} = \mathcal{V}^{-1}(\mathcal{V}(\mathcal{X}))$. Suppose that $\mathrm{pd}_R M = \infty$. Then $\Omega^d M$ is a nonfree Cohen-Macaulay R-module. We have an exact sequence

$$0 \to \Omega^d M \to F_{d-1} \to \cdots \to F_0 \to M \to 0$$

of R-modules such that F_i is free for $0 \leq i \leq d-1$. Decomposing this into short exact sequences and using the conditions (1) and (3), we see that $\Omega^d M$ belongs to $\mathcal{X}$. Hence the maximal ideal $\mathfrak{m}$ of R is in $\mathcal{V}(\mathcal{X})$, and we obtain $\mathcal{V}(\Omega^d k) \subseteq \{\mathfrak{m}\} \subseteq \mathcal{V}(\mathcal{X})$. Therefore $\Omega^d k$ belongs to $\mathcal{V}^{-1}(\mathcal{V}(\mathcal{X})) = \mathcal{X}$, namely, $\mathbf{H}\text{-}\dim_R(\Omega^d k) < \infty$. There is an exact sequence

$$0 \to \Omega^d k \to G_{d-1} \to \cdots \to G_1 \to G_0 \to k \to 0$$

of R-modules with each G_i free. Decomposing this into short exact sequences and using the conditions (1) and (3), we get $\mathbf{H}\text{-}\dim_R k < \infty$. Thus the condition (4) implies that R satisfies the property $\mathbf{H}$, which contradicts our assumption. Consequently, we must have $\operatorname{pd}_R M < \infty$. $\square$

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