ON THE STRUCTURE OF SALLY MODULES OF RANK ONE

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Abstract. A complete structure theorem of Sally modules of $\mathfrak{m}$-primary ideals I in a Cohen-Macaulay local ring $(A, \mathfrak{m})$ satisfying the equality $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 1$ is given, where $\mathrm{e}_0(I)$ and $\mathrm{e}_1(I)$ denote the first two Hilbert coefficients of I.

Key Words: commutative algebra, Cohen-Macaulay local ring, associated graded ring, Rees algebra, Hilbert coefficient.

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1. Introduction

This is based on a joint work with Shiro Goto and Koji Nishida.

Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring with $d = \dim A > 0$ and assume that the residue class field $k = A/\mathfrak{m}$ of A is infinite. Let I be an $\mathfrak{m}$-primary ideal in A and choose a minimal reduction $Q = (a_1, a_2, \cdots, a_d)$ of I. Let

$$\begin{array}{rcl} R &=& \mathrm{R}(I) := A[It] \subseteq A[t], \\ T &=& \mathrm{R}(Q) := A[Qt], \\ R' &=& \mathrm{R}'(I) := A[It, t^{-1}] \subseteq A[t, t^{-1}], \\ G &=& \mathrm{G}(I) := R'/t^{-1}R' \ \cong \ \bigoplus_{n \geq 0} I^n/I^{n+1} \end{array}$$

denote, respectively, the Rees algebras of I and Q, the extended Rees algebras of I and the associated graded ring of I, where t stands for an indeterminate over A.

Let $B = T/\mathfrak{m}T \cong k[X_1, X_2, \cdots, X_d]$, which is the polynomial ring with d indeterminates over the field k. Following W. V. Vasconcelos [10], we then define

$$S = \mathrm{S}_Q(I) = IR/IT$$

and call it the Sally module of I with respect to Q. We notice that the Sally module S is a finitely generated graded T-module, since R is a module-finite extension of the graded ring T.

Let $\ell_A(*)$ stand for the length. Then we have integers $\{\mathrm{e}_i(I)\}_{0 \leq i \leq d}$ such that the equality

$$\ell_A(A/I^{n+1}) = \mathrm{e}_0(I)\binom{n+d}{d} - \mathrm{e}_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d \mathrm{e}_d(I)$$

holds true for all $n \gg 0$. For each integers $0 \leq i \leq d$, we call $e_i = \mathrm{e}_i(I)$ the i-th Hilbert coefficients of I.

The contents of this article are based on [1, 2]. Refer to them for the details.

The Sally module S was introduced by W. V. Vasconcelos [10], where he gave an elegant review, in terms of his *Sally* module, of the works [7, 8, 9] of J. Sally about the structure of $\mathfrak{m}$-primary ideals I with interaction to the structure of the graded ring G and the Hilbert coefficients e_i's of I.

As is well-known, we have the inequality ([5])

$$e_1 \geq e_0 - \ell_A(A/I)$$

and C. Huneke [3] showed that $e_1 = e_0 - \ell_A(A/I)$ if and only if $I^2 = QI$ (cf. Corollary 4). When this is the case, both the graded rings G and $\mathrm{F}(I) = \bigoplus_{n \geq 0} I^n/\mathfrak{m}I^n$ are Cohen-Macaulay, and the Rees algebra R of I is also a Cohen-Macaulay ring, provided $d \geq 2$. Thus, the ideals I with $e_1 = e_0 - \ell_A(A/I)$ enjoy very nice properties.

J. Sally firstly investigated the second border, that is the ideals I satisfying the equality $e_1 = e_0 - \ell_A(A/I) + 1$ but $e_2 \neq 0$ (cf. [9, 10]). The present research is a continuation of [9, 10] and aims to give a complete structure theorem of the Sally module of an $\mathfrak{m}$-primary ideal I satisfying the equality $e_1 = e_0 - \ell_A(A/I) + 1$.

The main result of this paper is the following Theorem 1. Our contribution in Theorem 1 is the implication (1) $\Rightarrow$ (3), the proof of which is based on the new result that the equality $I^3 = QI^2$ holds true if $e_1 = e_0 - \ell_A(A/I) + 1$ (cf. Theorem 7).

Theorem 1. *The following three conditions are equivalent to each other.*

(1) $e_1 = e_0 - \ell_A(A/I) + 1$.

(2) $\mathfrak{m}S = (0)$ *and* $\operatorname{rank}_B S = 1$.

(3) $S \cong (X_1, X_2, \cdots, X_c)B$ *as graded T-modules for some* $0 < c \leq d$, *where* $\{X_i\}_{1 \leq i \leq c}$ *are linearly independent linear forms of the polynomial ring B.*

When this is the case, $c = \ell_A(I^2/QI)$ *and* $I^3 = QI^2$, *and the following assertions hold true.*

(i) $\operatorname{depth} G \geq d - c$ *and* $\operatorname{depth}_T S = d - c + 1$.

(ii) $\operatorname{depth} G = d - c$, *if* $c \geq 2$.

(iii) *Suppose* $c < d$. *Then*

$$\ell_A(A/I^{n+1}) = e_0\binom{n+d}{d} - e_1\binom{n+d-1}{d-1} + \binom{n+d-c-1}{d-c-1}$$

for all $n \geq 0$. *Hence*

$$e_i = \begin{cases} 0 & \text{if } i \neq c+1, \\ (-1)^{c+1} & \text{if } i = c+1 \end{cases}$$

for $2 \leq i \leq d$.

(iv) *Suppose* $c = d$. *Then*

$$\ell_A(A/I^{n+1}) = e_0\binom{n+d}{d} - e_1\binom{n+d-1}{d-1}$$

for all $n \geq 1$. *Hence* $e_i = 0$ *for* $2 \leq i \leq d$.

Thus Theorem 1 settles a long standing problem, although the structure of ideals I with $e_1 = e_0 - \ell_A(A/I) + 2$ or the structure of Sally modules S with $\mathfrak{m}S = (0)$ and $\operatorname{rank}_B S = 2$ remains unknown.

Let us now briefly explain how this paper is organized. We shall prove Theorem 1 in Section 3. In Section 2 we will pick up from the paper [1] some auxiliary results on Sally modules, all of which are known, but let us note them for the sake of the reader's convenience. In Section 4 we will construct one example in order to see the ubiquity of ideals I which satisfy condition (3) in Theorem 1.

In what follows, unless otherwise specified, let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring with $d = \dim A > 0$. We assume that the field $k = A/\mathfrak{m}$ is infinite. Let I be an $\mathfrak{m}$-primary ideal in A and let S be the Sally module of I with respect to a minimal reduction $Q = (a_1, a_2, \cdots, a_d)$ of I. We put $R = A[It], T = A[Qt], R' = A[It, t^{-1}]$, and $G = R'/t^{-1}R'$. Let

$$\tilde{I} = \bigcup_{n \geq 1} [I^{n+1} :_A I^n] = \bigcup_{n \geq 1} [I^{n+1} :_A (a_1^n, a_2^n, \cdots, a_d^n)]$$

denote the Ratliff-Rush closure of I, which is the largest $\mathfrak{m}$-primary ideal in A such that $I \subseteq \tilde{I}$ and $\mathrm{e}_i(\tilde{I}) = e_i$ for all $0 \leq i \leq d$ (cf. [6]). We denote by $\mu_A(*)$ the number of generators.

2. Auxiliary results

In this section let us firstly summarize some known results on Sally modules, which we need throughout this paper. See [1] and [10] for the detailed proofs.

The first two results are basic facts on Sally modules developed by Vasconcelos [10].

Lemma 2. *The following assertions hold true.*

(1) $\mathfrak{m}^\ell S = (0)$ *for integers* $\ell \gg 0$.

(2) *The homogeneous components* $\{S_n\}_{n \in \mathbb{Z}}$ *of the graded* T*-module* S *are given by*

$$S_n \cong \begin{cases} (0) & \text{if } n \leq 0, \\ I^{n+1}/IQ^n & \text{if } n \geq 1. \end{cases}$$

(3) $S = (0)$ *if and only if* $I^2 = QI$.

(4) *Suppose that* $S \neq (0)$ *and put* $V = S/MS$, *where* $M = \mathfrak{m}T + T_+$ *is the graded maximal ideal in* T. *Let* V_n $(n \in \mathbb{Z})$ *denote the homogeneous component of the finite-dimensional graded* T/M*-space* V *with degree* n *and put* $\Lambda = \{n \in \mathbb{Z} \mid V_n \neq (0)\}$. *Let* $q = \max \Lambda$. *Then we have* $\Lambda = \{1, 2, \cdots, q\}$ *and* $\mathrm{r}_Q(I) = q + 1$, *where* $\mathrm{r}_Q(I) = \min\{n \in \mathbb{Z} \mid I^{n+1} = QI^n\}$ *stands for the reduction number of* I *with respect to* Q.

(5) $S = TS_1$ *if and only if* $I^3 = QI^2$.

Proof. See [1, Lemma 2.1]. □

Proposition 3. *Let* $\mathfrak{p} = \mathfrak{m}T$. *Then the following assertions hold true.*

(1) $\mathrm{Ass}_T S \subseteq \{\mathfrak{p}\}$. *Hence* $\dim_T S = d$, *if* $S \neq (0)$.

(2) $\ell_A(A/I^{n+1}) = e_0\binom{n+d}{d} - (e_0 - \ell_A(A/I)) \cdot \binom{n+d-1}{d-1} - \ell_A(S_n)$ *for all* $n \geq 0$.

(3) *We have* $e_1 = e_0 - \ell_A(A/I) + \ell_{T_\mathfrak{p}}(S_\mathfrak{p})$. *Hence* $e_1 = e_0 - \ell_A(A/I) + 1$ *if and only if* $\mathfrak{m}S = (0)$ *and* $\mathrm{rank}_B\, S = 1$.

(4) *Suppose that* $S \neq (0)$. *Let* $s = \mathrm{depth}_T\, S$. *Then* $\mathrm{depth}\, G = s - 1$ *if* $s < d$. S *is a Cohen-Macaulay* T*-module if and only if* $\mathrm{depth}\, G \geq d - 1$.

Proof. See [1, Proposition 2.2]. $\square$

Combining Lemma 2 (3) and Proposition 3, we readily get the following results of Northcott [5] and Huneke [3].

Corollary 4 ([3, 5])**.** *We have $e_1 \geq e_0 - \ell_A(A/I)$. The equality $e_1 = e_0 - \ell_A(A/I)$ holds true if and only if $I^2 = QI$. When this is the case, $e_i = 0$ for all $2 \leq i \leq d$.*

The following result is one of the keys for our proof of Theorem 1.

Theorem 5. *The following conditions are equivalent.*

(1) $e_1 = e_0 - \ell_A(A/I) + 1$.

(2) *$S \cong \mathfrak{a}$ as graded T-modules for some graded ideal $\mathfrak{a}$ ($\neq B$) of B.*

Proof. We have only to show (1) $\Rightarrow$ (2). We have $\mathfrak{m}S = (0)$ and $\mathrm{rank}_B S = 1$ by Proposition 3 (3). Because $S_1 \neq (0)$ and $S = \sum_{n \geq 1} S_n$ by Lemma 2, we have $S \cong B(-1)$ as graded B-modules once S is B-free.

Suppose that S is not B-free. The B-module S is torsionfree, since $\mathrm{Ass}_T S = \{\mathfrak{m}T\}$ by Proposition 3 (1). Therefore, since $\mathrm{rank}_B S = 1$, we see $d \geq 2$ and $S \cong \mathfrak{a}(m)$ as graded B-modules for some integer m and some graded ideal $\mathfrak{a}$ ($\neq B$) in B, so that we get the exact sequence

$$0 \to S(-m) \to B \to B/\mathfrak{a} \to 0$$

of graded B-modules. We may assume that $\mathrm{ht}_B\, \mathfrak{a} \geq 2$, since $B = k[X_1, X_2, \cdots, X_d]$ is the polynomial ring over the field $k = A/\mathfrak{m}$. We then have $m \geq 0$, since $\mathfrak{a}_{m+1} = [\mathfrak{a}(m)]_1 \cong S_1 \neq (0)$ and $\mathfrak{a}_0 = (0)$. We want to show $m = 0$.

Because $\dim B/\mathfrak{a} \leq d-2$, the Hilbert polynomial of $B/\mathfrak{a}$ has degree at most $d-3$. Hence

$$\begin{aligned} \ell_A(S_n) &= \ell_A(B_{m+n}) - \ell_A([B/\mathfrak{a}]_{m+n}) \\ &= \binom{m+n+d-1}{d-1} - \ell_A([B/\mathfrak{a}]_{m+n}) \\ &= \binom{n+d-1}{d-1} + m\binom{n+d-2}{d-2} + (\text{lower terms}) \end{aligned}$$

for $n \gg 0$. Consequently

$$\begin{aligned} \ell_A(A/I^{n+1}) &= e_0\binom{n+d}{d} - (e_0 - \ell_A(A/I))\cdot\binom{n+d-1}{d-1} - \ell_A(S_n) \\ &= e_0\binom{n+d}{d} - (e_0 - \ell_A(A/I) + 1)\cdot\binom{n+d-1}{d-1} - m\binom{n+d-2}{d-2} \\ &\quad + (\text{lower terms}) \end{aligned}$$

by Proposition 3 (2), so that we get $e_2 = -m$. Thus $m = 0$, because $e_2 \geq 0$ by Narita's theorem ([4]). $\square$

The following result will enable us to reduce the proof of Theorem 1 to the proof of the fact that $I^3 = QI^2$ if $e_1 = e_0 - \ell_A(A/I) + 1$.

Proposition 6. *Suppose $e_1 = e_0 - \ell_A(A/I) + 1$ and $I^3 = QI^2$. Let $c = \ell_A(I^2/QI)$. Then the following assertions hold true.*

(1) $0 < c \leq d$ *and* $\mu_B(S) = c$.
(2) $\operatorname{depth} G \geq d - c$ *and* $\operatorname{depth}_B S = d - c + 1$.
(3) $\operatorname{depth} G = d - c$, *if* $c \geq 2$.
(4) *Suppose $c < d$. Then $\ell_A(A/I^{n+1}) = e_0\binom{n+d}{d} - e_1\binom{n+d-1}{d-1} + \binom{n+d-c-1}{d-c-1}$ for all $n \geq 0$. Hence*
$$e_i = \begin{cases} 0 & \text{if } i \neq c+1 \\ (-1)^{c+1} & \text{if } i = c+1 \end{cases}$$
for $2 \leq i \leq d$.
(5) *Suppose $c = d$. Then $\ell_A(A/I^{n+1}) = e_0\binom{n+d}{d} - e_1\binom{n+d-1}{d-1}$ for all $n \geq 1$. Hence $e_i = 0$ for $2 \leq i \leq d$.*

Proof. We have $\mathfrak{m}S = (0)$ and $\operatorname{rank}_B S = 1$ by Proposition 3 (3), while $S = TS_1$ since $I^3 = QI^2$ (cf. Lemma 2 (5)). Therefore by Theorem 5 we have $S \cong \mathfrak{a}$ as graded B-modules where $\mathfrak{a} = (X_1, X_2, \cdots, X_c)B$ is an ideal in B generated by linear forms $\{X_i\}_{1 \leq i \leq c}$. Hence $0 < c \leq d$, $\mu_B(S) = c$, and $\operatorname{depth}_B S = d - c + 1$, so that assertions (1), (2), and (3) follow (cf. Proposition 3 (4)). Considering the exact sequence
$$0 \to S \to B \to B/\mathfrak{a} \to 0$$
of graded B-modules, we get
$$\begin{aligned} \ell_A(S_n) &= \ell_A(B_n) - \ell_A([B/\mathfrak{a}]_n) \\ &= \binom{n+d-1}{d-1} - \binom{n+d-c-1}{d-c-1} \end{aligned}$$
for all $n \geq 0$ (resp. $n \geq 1$), if $c < d$ (resp. $c = d$). Thus assertions (4) and (5) follow (cf. Proposition 3 (2)). $\square$

3. Proof of Theorem 1

The purpose of this section is to prove Theorem 1. See Proposition 3 (3) for the equivalence of conditions (1) and (2) in Theorem 1. The implication (3) $\Rightarrow$ (2) is clear. So, we must show the implication (1) $\Rightarrow$ (3) together with the last assertions in Theorem 1. Suppose that $e_1 = e_0 - \ell_A(A/I) + 1$. Then, thanks to Theorem 5, we get an isomorphism
$$\varphi : S \to \mathfrak{a}$$
of graded B-modules, where $\mathfrak{a} \subsetneq B$ is a graded ideal of B. Notice that once we are able to show $I^3 = QI^2$, the last assertions of Theorem 1 readily follow from Proposition 6. On the other hand, since $\mathfrak{a} \cong S = BS_1$ (cf. Lemma 2 (5)), the ideal $\mathfrak{a}$ of B is generated by linearly independent linear forms $\{X_i\}_{1 \leq i \leq c}$ ($0 < c \leq d$) of B and so, the implication (1) $\Rightarrow$ (3) in Theorem 1 follows. We have $c = \ell_A(I^2/QI)$, because $\mathfrak{a}_1 \cong S_1 = I^2/QI$ (cf. Lemma 2 (2)). Thus our Theorem 1 has been proven modulo the following theorem.

Theorem 7. *Suppose that $e_1 = e_0 - \ell_A(A/I) + 1$. Then $I^3 = QI^2$.*

Proof. We proceed by induction on d. Suppose that $d = 1$. Then S is B-free of rank one (recall that the B-module S is torsionfree; cf. Proposition 3 (1)) and so, since $S_1 \neq (0)$ (cf. Lemma 2 (3)), $S \cong B(-1)$ as graded B-modules. Thus $I^3 = QI^2$ by Lemma 2 (5).

Let us assume that $d \geq 2$ and that our assertion holds true for $d-1$. Since the field $k = A/\mathfrak{m}$ is infinite, without loss of generality we may assume that a_1 is a superficial element of I. Let

$$\overline{A} = A/(a_1), \quad \overline{I} = I/(a_1), \text{ and } \quad \overline{Q} = Q/(a_1).$$

We then have $\mathrm{e}_i(\overline{I}) = e_i$ for all $0 \leq i \leq d-1$, whence

$$\mathrm{e}_1(\overline{I}) = \mathrm{e}_0(\overline{I}) - \ell_{\overline{A}}(\overline{A}/\overline{I}) + 1.$$

Therefore the hypothesis of induction on d yields $\overline{I}^3 = \overline{Q}\,\overline{I}^2$. Hence, because the element a_1t is a nonzerodivisor on G if $\operatorname{depth} G > 0$, we have $I^3 = QI^2$ in that case.

Assume that $\operatorname{depth} G = 0$. Then, thanks to Sally's technique ([9]), we also have $\operatorname{depth} \mathrm{G}(\overline{I}) = 0$. Hence $\ell_{\overline{A}}(\overline{I}^2/\overline{Q}\,\overline{I}) = d-1$ by Proposition 6 (2), because $\mathrm{e}_1(\overline{I}) = \mathrm{e}_0(\overline{I}) - \ell_{\overline{A}}(\overline{A}/\overline{I}) + 1$. Consequently, $\ell_A(S_1) = \ell_A(I^2/QI) \geq d-1$, because $\overline{I}^2/\overline{Q}\,\overline{I}$ is a homomorphic image of I^2/QI. Let us take an isomorphism

$$\varphi : S \to \mathfrak{a}$$

of graded B-modules, where $\mathfrak{a} \subsetneq B$ is a graded ideal of B. Then, since

$$\ell_A(\mathfrak{a}_1) = \ell_A(S_1) \geq d-1,$$

the ideal $\mathfrak{a}$ contains $d-1$ linearly independent linear forms, say $X_1, X_2, \cdots, X_{d-1}$ of B, which we enlarge to a basis $X_1, \cdots, X_{d-1}, X_d$ of B_1. Hence

$$B = k[X_1, X_2, \cdots, X_d],$$

so that the ideal $\mathfrak{a}/(X_1, X_2, \cdots, X_{d-1})B$ in the polynomial ring

$$B/(X_1, X_2, \cdots, X_{d-1})B = k[X_d]$$

is principal. If $\mathfrak{a} = (X_1, X_2, \cdots, X_{d-1})B$, then $I^3 = QI^2$ by Lemma 2 (5), since $S = BS_1$. However, because $\ell_A(I^2/QI) = \ell_A(\mathfrak{a}_1) = d-1$, we have $\operatorname{depth} G \geq 1$ by Proposition 6 (2), which is impossible. Therefore $\mathfrak{a}/(X_1, X_2, \cdots, X_{d-1})B \neq (0)$, so that we have

$$\mathfrak{a} = (X_1, X_2, \cdots, X_{d-1}, X_d^{\alpha})B$$

for some $\alpha \geq 1$. Notice that $\alpha = 1$ or $\alpha = 2$ by Lemma 2 (4). We must show that $\alpha = 1$.

Assume that $\alpha = 2$. Let us write, for each $1 \leq j \leq d$, $X_j = \overline{a_jt}$ with $a_j \in Q$, where $\overline{a_jt}$ denotes the image of $a_it \in T$ in $B = T/\mathfrak{m}T$. Then $\mathfrak{a} = (\overline{a_1t}, \overline{a_2t}, \cdots, \overline{a_{d-1}t}, \overline{(a_dt)^2})$. We now choose elements $f_i \in S_1$ for $1 \leq i \leq d-1$ and $f_d \in S_2$ so that $\varphi(f_i) = X_i$ for $1 \leq i \leq d-1$ and $\varphi(f_d) = X_d^2$. Let $z_i \in I^2$ for $1 \leq i \leq d-1$ and $z_d \in I^3$ such that $\{f_i\}_{1 \leq i \leq d-1}$ and f_d are, respectively, the images of $\{z_it\}_{1 \leq i \leq d-1}$ and z_dt^2 in S. We now consider the relations $X_if_1 = X_1f_i$ in S for $1 \leq i \leq d-1$ and $X_d^2f_1 = X_1f_d$, that is

$$a_iz_1 - a_1z_i \in Q^2I$$

for $1 \leq i \leq d-1$ and

$$a_d^2z_1 - a_1z_d \in Q^3I.$$

Notice that
$$Q^3 = a_1Q^2 + (a_2, a_3, \cdots, a_{d-1})^2{\cdot}(a_2, a_3, \cdots, a_d) + a_d^2 Q$$
and write
$$a_d^2 z_1 - a_1 z_d = a_1\tau_1 + \tau_2 + a_d^2\tau_3$$
with $\tau_1 \in Q^2I$, $\tau_2 \in (a_2, a_3, \cdots, a_{d-1})^2{\cdot}(a_2, a_3, \cdots, a_d)I$, and $\tau_3 \in QI$. Then
$$a_d^2(z_1 - \tau_3) = a_1(\tau_1 + z_d) + \tau_2 \in (a_1) + (a_2, a_3, \cdots, a_{d-1})^2.$$
Hence $z_1 - \tau_3 \in (a_1) + (a_2, a_3, \cdots, a_{d-1})^2$, because the sequence $a_1, a_2, \cdots, a_d$ is A-regular. Let $z_1 - \tau_3 = a_1h + h'$ with $h \in A$ and $h' \in (a_2, a_3, \cdots, a_{d-1})^2$. Then since
$$a_1[a_d^2 h - (\tau_1 + z_d)] = \tau_2 - a_d^2 h' \in (a_2, a_3, \cdots, a_d)^3,$$
we have $a_d^2 h - (\tau_1 + z_d) \in (a_2, a_3, \cdots, a_d)^3$, whence $a_d^2 h \in I^3$.

We need the following.

Remark 8. $h \notin I$ but $h \in \tilde{I}$. Hence $\tilde{I} \neq I$.

Proof of Remark 8. If $h \in I$, then $a_1h \in QI$, so that $z_1 = a_1h + h' + \tau_3 \in QI$, whence $f_1 = 0$ in S (cf. Lemma 2 (2)), which is impossible. Let $1 \le i \le d-1$. Then
$$a_iz_1 - a_1z_i = a_i(a_1h + h' + \tau_3) - a_1z_i = a_1(a_ih - z_i) + a_i(h' + \tau_3) \in Q^2I.$$
Therefore, because $a_i(h' + \tau_3) \in Q^2I$, we get
$$a_1(a_ih - z_i) \in (a_1) \cap Q^2I.$$
Notice that
$$\begin{aligned} (a_1) \cap Q^2I &= (a_1) \cap [a_1QI + (a_2, a_3, \cdots, a_d)^2I] \\ &= a_1QI + [(a_1) \cap (a_2, a_3, \cdots, a_d)^2I] \\ &= a_1QI + a_1(a_2, a_3, \cdots, a_d)^2 \\ &= a_1QI \end{aligned}$$
and we have $a_ih - z_i \in QI$, whence $a_ih \in I^2$ for $1 \le i \le d-1$. Consequently $a_i^2h \in I^3$ for all $1 \le i \le d$, so that $h \in \tilde{I}$, whence $\tilde{I} \neq I$. □

Because $\ell_A(\tilde{I}/I) \ge 1$, we have
$$\begin{aligned} e_1 &= e_0 - \ell_A(A/I) + 1 \\ &= \mathrm{e}_0(\tilde{I}) - \ell_A(A/\tilde{I}) + [1 - \ell_A(\tilde{I}/I)] \\ &\le \mathrm{e}_0(\tilde{I}) - \ell_A(A/\tilde{I}) \\ &\le \mathrm{e}_1(\tilde{I}) \\ &= e_1, \end{aligned}$$
where $\mathrm{e}_0(\tilde{I}) - \ell_A(A/\tilde{I}) \le \mathrm{e}_1(\tilde{I})$ is the inequality of Northcott for the ideal $\tilde{I}$ (cf. Corollary 4). Hence $\ell_A(\tilde{I}/I) = 1$ and $\mathrm{e}_1(\tilde{I}) = \mathrm{e}_0(\tilde{I}) - \ell_A(A/\tilde{I})$, so that
$$\tilde{I} = I + (h) \quad \text{and} \quad \tilde{I}^2 = Q\tilde{I}$$
by Corollary 4 (recall that Q is a reduction of $\tilde{I}$ also). We then have, thanks to [2, Proposition 2.6], that $I^3 = QI^2$, which is a required contradiction. This completes the proof of Theorem 1 and that of Theorem 7 as well. □

4. An example

Lastly we construct one example which satisfies condition (3) in Theorem 1. Our goal is the following. See [2, Section 5] for the detailed proofs.

Theorem 9. *Let $0 < c \leq d$ be integers. Then there exists an $\mathfrak{m}$-primary ideal I in a Cohen-Macaulay local ring $(A, \mathfrak{m})$ such that*

$$d = \dim A, \quad \mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 1, \quad \text{and} \quad c = \ell_A(I^2/QI)$$

for some reduction $Q = (a_1, a_2, \cdots, a_d)$ of I.

To construct necessary examples we may assume that $c = d$.

Let m, $d > 0$ be integers. Let

$$U = k[\{X_j\}_{1 \leq j \leq m}, Y, \{V_i\}_{1 \leq i \leq d}, \{Z_i\}_{1 \leq i \leq d}]$$

be the polynomial ring with $m + 2d + 1$ indeterminates over an infinite field k and let

$$\begin{aligned} \mathfrak{b} \;=\; & [(X_j \mid 1 \leq j \leq m) + (Y)]{\cdot}[(X_j \mid 1 \leq j \leq m) + (Y) + (V_i \mid 1 \leq i \leq d)] \\ & +(V_iV_j \mid 1 \leq i, j \leq d,\, i \neq j) + (V_i^2 - Z_iY \mid 1 \leq i \leq d). \end{aligned}$$

We put $C = U/\mathfrak{b}$ and denote the images of X_j, Y, V_i, and Z_i in C by x_j, y, v_i, and a_i, respectively. Then $\dim C = d$, since $\sqrt{\mathfrak{b}} = (X_j \mid 1 \leq j \leq m) + (Y) + (V_i \mid 1 \leq i \leq d)$. Let $M = C_+ := (x_j \mid 1 \leq j \leq m) + (y) + (v_i \mid 1 \leq i \leq d) + (a_i \mid 1 \leq i \leq d)$ be the graded maximal ideal in C. Let Γ be a subset of $\{1, 2, \cdots, m\}$. We put

$$J = (a_i \mid 1 \leq i \leq d) + (x_\alpha \mid \alpha \in \Gamma) + (v_i \mid 1 \leq i \leq d) \quad \text{and} \quad \mathfrak{q} = (a_i \mid 1 \leq i \leq d).$$

Then $M^2 = \mathfrak{q}M$, $J^2 = \mathfrak{q}J + \mathfrak{q}y$, and $J^3 = \mathfrak{q}J^2$, whence $\mathfrak{q}$ is a reduction of both M and J, and $a_1, a_2, \cdots, a_d$ is a homogeneous system of parameters for the graded ring C.

Let $A = C_M$, $I = JA$, and $Q = \mathfrak{q}A$. We are now interested in the Hilbert coefficients $e_i's$ of the ideal I as well as the structure of the associated graded ring and the Sally module of I. We then have the following, which shows that the ideal I is a required example.

Theorem 10. *The following assertions hold true.*

(1) *A is a Cohen-Macaulay local ring with* $\dim A = d$.
(2) *$S \cong B_+$ as graded T-modules, whence $\ell_A(I^2/QI) = d$.*
(3) *$\mathrm{e}_0(I) = m + d + 2$ and $\mathrm{e}_1(I) = \sharp\Gamma + d + 1$.*
(4) *$\mathrm{e}_i(I) = 0$ for all $2 \leq i \leq d$.*
(5) *G is a Buchsbaum ring with* $\operatorname{depth} G = 0$ *and $\mathbb{I}(G) = d$.*

Proof. See [2, Theorem 5.2]. □

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