HOCHSCHILD COHOMOLOGY OF BRAUER ALGEBRAS

HIROSHI NAGASE

ABSTRACT. Suppose B is an algebra with a stratifying ideal BeB generated by an idempotent e. We will establish long exact sequences relating the Hochschild cohomology groups of the three algebras B, B/BeB and eBe. This provides a common generalization of various known results, all of which extend Happel's long exact sequence for one-point extensions. Applying one of these sequences to Hochschild cohomology algebras modulo the ideal generated by homogeneous nilpotent elements, we obtain, in some cases, that these algebras are finitely generated.

1. INTRODUCTION

Let B be an algebra with a stratifying ideal BeB generated by an idempotent e and $\mathrm{HH}^n(B)$ the nth Hochschild cohomology group of B. In [11], we obtain a long exact sequence

$$\cdots \to \mathrm{Ext}^n_{B^e}(B/BeB, BeB) \to \mathrm{HH}^n(B) \to \mathrm{HH}^n(B/BeB) \oplus \mathrm{HH}^n(eBe) \to \cdots,$$

which is a generalization of Happel's long exact sequence in [9]. Moreover this is a generalization of various known long exact sequences in the case of triangular matrix algebras by Michelena and Platzeck in [13], Green and Solberg in [8] and Cibils, Marcos, Redondo and Solotar in [2], and in the case of algebras with heredity ideals by de la Peña and Xi in [14].

For any finite dimensional algebra B with a stratifying ideal BeB, we will apply our long exact sequence to the quotient of the Hochschild cohomology algebra $\mathrm{HH}^*(B)$ modulo the ideal $\mathcal{N}_B$ generated by homogeneous nilpotent elements. We denote by $\overline{\mathrm{HH}}^*(B)$ the graded factor algebra $\mathrm{HH}^*(B)/\mathcal{N}_B$.

In [16], for any finite dimensional algebra A, Snashall and Solberg studied support variety by using Hocschild cohomology algebra $\mathrm{HH}^*(A)$ and conjectured that $\overline{\mathrm{HH}}^*(A)$ is a finitely generated algebra. Green, Snashall and Solberg have shown the conjecture to hold true for self-injective algebras of finite representation type [6] and for monomial algebras [7]. Recently Xu has shown that there exists a counter example to the conjecture in [17]. We are, however, interested in the condition when $\overline{\mathrm{HH}}^*(A)$ is finitely generated.

Applying the long exact sequence above to Brauer algebra $B_k(n,\delta)$, we obtain an embedding

$$\overline{\mathrm{HH}}^*(B_k(n,\delta)) \hookrightarrow \overline{\mathrm{HH}}^*(k\Sigma_n) \times \overline{\mathrm{HH}}^*(k\Sigma_{n-2}) \times \cdots \times \overline{\mathrm{HH}}^*(k\Sigma_t)$$

where Σ_m is the symmetric group on m letters, $k\Sigma_0 = k$ and t is 0 or 1 depending on whether n is even or odd (see Proposition 5). By using this embedding, we obtain the result that $\overline{\mathrm{HH}}^*(B_k(n,\delta))$ is finitely generated in some cases.

The detailed version of this paper will be submitted for publication elsewhere.

2. Stratifying ideals

In this section we recall some results about Hochschild cohomology groups of algebras with stratifying ideals in [11]. The following definition is due to Cline, Parshall and Scott ([3], 2.1.1 and 2.1.2), who work with finite dimensional algebras over fields. We keep our general setup of algebras projective over a commutative noetherian ring.

Definition 1. Let B be an algebra and e an idempotent. The two-sided ideal BeB generated by e is called a *stratifying ideal* if the following equivalent conditions (A) and (B) are satisfied:
(A) (a) The multiplication map $Be \otimes_{eBe} eB \to BeB$ is an isomorphism.
(b) For all $n > 0$, $\mathrm{Tor}_n^{eBe}(Be, eB) = 0$.
(B) The epimorphism $B \to A := B/BeB$ induces isomorphisms

$$\mathrm{Ext}_A^*(X, Y) \simeq \mathrm{Ext}_B^*(X, Y)$$

for all A-modules X and Y.

The following remark can be used to check if an ideal is stratifying.

Remark 2. Let e be an idempotent element in B. Then BeB is projective as a left (resp. right) B-module if and only if eB (resp. Be) is projective as a left (respectively right) eBe-module and the multiplication map $Be \otimes_{eBe} eB \to BeB$ is an isomorphism.

Heredity ideals are examples of stratifying ideals, thus our results will extend results obtained in [14]. On the other hand, for any triangulated algebra B has an idempotent e such that BeB is projective. By Remark 2, BeB is a stratifying ideal. Thus our results also will extend results of [2, 8, 13]. There are, however, plenty of other examples. Stratifying ideals and stratified algebras occur frequently in applications, for example in algebraic Lie theory in the context of Schur algebras and of blocks of the Bernstein-Gelfand-Gelfand category of a semisimple complex Lie algebra.

From now on, we assume that BeB is a stratifying ideal of B and we put $A := B/BeB$.

Theorem 3. *There are long exact sequences as follows:*
(1) $\cdots \to \mathrm{Ext}^n_{B^e}(B, BeB) \to \mathrm{HH}^n(B) \to \mathrm{HH}^n(A) \to \cdots$;
(2) $\cdots \to \mathrm{Ext}^n_{B^e}(A, B) \to \mathrm{HH}^n(B) \to \mathrm{HH}^n(eBe) \to \cdots$; *and*
(3) $\cdots \to \mathrm{Ext}^n_{B^e}(A, BeB) \to \mathrm{HH}^n(B) \xrightarrow{f} \mathrm{HH}^n(A) \oplus \mathrm{HH}^n(eBe) \to \cdots$.

We remark that by using the partial recollement of bounded below derived categories

$$D^+(\mathrm{mod}\, A) \rightleftarrows D^+(\mathrm{mod}\, B) \rightleftarrows D^+(\mathrm{mod}\, eBe),$$

we also can obtain the long exact sequence (3).

We also note that Suarez-Alvarez [15] independently has obtained the first long exact sequence in Theorem 3 above by using different methods based on spectral sequences.

Recall the notation that $\mathcal{N}_B$ is the ideal of $\mathrm{HH}^*(B)$ which is generated by homogeneous nilpotent elements, and $\overline{\mathrm{HH}}^*(B)$ is the factor algebra $\mathrm{HH}^*(B)/\mathcal{N}_B$.

Corollary 4.
(1) *Let* $f : \mathrm{HH}^*(B) \to \mathrm{HH}^*(A) \times \mathrm{HH}^*(eBe)$ *be the graded algebra homomorphism in sequence (3) above. Then* $(\mathrm{Ker}\, f)^2$ *vanishes.*

(2) *The induced homomorphism* $\overline{f} : \overline{\mathrm{HH}}^*(B) \to \overline{\mathrm{HH}}^*(A) \times \overline{\mathrm{HH}}^*(eBe)$ *is injective.*

We note that the the graded algebra homomorphism in Corollary above was studied in the case of a one point extension by Green, Marcos and Snashall [5].

3. Brauer algebras

Finally we give an example of an algebra occurring in algebraic Lie theory, see for instance [12] or [10] for the properties of Brauer algebras used in this example. We denote by Σ_n the symmetric group on n letters and k an algebraically closed field. For any natural number n and any δ in k, we denote by $B_k(n,\delta)$ the Brauer algebra.

Proposition 5. *If δ is not 0 or n is odd, then there is an injective graded algebra homomorhism*

$$\overline{\mathrm{HH}}^*(B_k(n,\delta)) \hookrightarrow \overline{\mathrm{HH}}^*(k\Sigma_n) \times \overline{\mathrm{HH}}^*(k\Sigma_{n-2}) \times \cdots \times \overline{\mathrm{HH}}^*(k\Sigma_t)$$

where $k\Sigma_0 = k$ and t is 0 or 1 depending on whether n is even or odd.

Proof. For any Brauer algebra $B_k(n,\delta)$, if δ is not 0 or n is odd, then there is a filtration

$$0 < I_t < I_{t+2} < \cdots < I_{n-2} < I_n = B_k(n,\delta)$$

such that the subquotient I_s/I_{s-2} is a stratifying ideal of $B_s = B_k(n,\delta)/I_{s-2}$,where t is 0 or 1 depending on whether n is even or odd and $I_s = 0$ if $s < 0$ (see [10]). Moreover there is an idempotent e_s in B_s such that $I_s/I_{s-2} = B_s e_s B_s$, $e_s B_s e_s \cong k\Sigma_s$ and e_n is the identity of B_n (see [4]). By Corollary 4, there exists an injective graded algebra homomorphism

$$\overline{\mathrm{HH}}^*(B_s) \hookrightarrow \overline{\mathrm{HH}}^*(B_{s+2}) \times \overline{\mathrm{HH}}^*(k\Sigma_s).$$

Since $B_t = B_k(n,\delta)$ and $B_n \cong k\Sigma_n$, the claim follows. □

Corollary 6. *Suppose that δ is not 0 or n is odd. If the characteristic of k is either zero or bigger than n, then $\overline{\mathrm{HH}}^*(B_k(n,\delta))$ is a finitely generated algebra.*

Proof. If the characteristic of k is either zero or bigger than n, then for any $s < n$, $k\Sigma_s$ is semisimple and $\overline{\mathrm{HH}}^*(k\Sigma_s) \cong k^m$ where m is the number of the blocks of $k\Sigma_s$. By Proposition 5, $\overline{\mathrm{HH}}^*(B_k(n,\delta))$ is a finitely generated algebra. □

Corollary 7. $\overline{\mathrm{HH}}^*(B_k(2,\delta))$ *and* $\overline{\mathrm{HH}}^*(B_k(3,\delta))$ *are finitely generated algebras.*

Proof. By Proposition 5, there exists an embedding

$$\overline{\mathrm{HH}}^*(B_k(3,\delta)) \hookrightarrow \overline{\mathrm{HH}}^*(k\Sigma_3) \times \overline{\mathrm{HH}}^*(k\Sigma_1)$$

as a graded algebra homomorphism. Since $k\Sigma_1 \cong k$ and $k\Sigma_3$ is a self-injective algebra of finite representation type, $\overline{\mathrm{HH}}^*(k\Sigma_3) \times \overline{\mathrm{HH}}^*(k\Sigma_1)$ is isomorphic to a product of some polynomial algebras in one variable $k[x]$ and some copies of the ground field k (see [6]). Because any graded subalgebra of a product of some polynomial algebras with one variable $k[x]$ is a finitely generated algebra, we obtain the result that $\overline{\mathrm{HH}}^*(B_k(3,\delta))$ is a finitely generated algebra.

By Proposition 5, if δ is not zero, then there exists an embedding

$$\overline{\mathrm{HH}}^*(B_k(2,\delta)) \hookrightarrow \overline{\mathrm{HH}}^*(k\Sigma_2) \times \overline{\mathrm{HH}}^*(k\Sigma_0)$$

as a graded algebra homomorphism. Since $k\Sigma_0 = k$ and $k\Sigma_2$ is a self-injective algebra of finite representation type, $\overline{\mathrm{HH}}^*(B_k(2,\delta))$ is a finitely generated algebra by the same argument above. If $\delta = 0$, then $B_k(2,\delta)$ is isomorphic to

$$k \times k[x]/x^2 \ (\mathrm{char} k \neq 2) \text{ or } k[x,y]/(x^2, xy, y^2) \ (\mathrm{char} k = 2).$$

Since both are radical square zero algebras, $\overline{\mathrm{HH}}^*(B_k(2,\delta))$ is a finitely generated algebra (see [1] or [7]). □

References

[1] C. Cibils: *Hochschild cohomology algebra of radical square zero algebras*, Algebras and modules II. In: CMS Conference Proceedings. 24(1998), 93-101.

[2] C. Cibils, E. Marcos, M. J. Redondo and A. Solotar: *Cohomology of split algebras and of trivial extensions*, Glasgow Math. J. **45** (2003), no. 1, 21-40.

[3] E. Cline, B. Parshall and L. Scott: *Stratifying endomorphism algebras*, Memoir AMS **591** (1996), i-vi, 1-119.

[4] L. Diracca and S. Koenig: *Cohomological reduction by split pairs*, J. Pure. Appl. Algebra, **212** (2008), no. 3, 471-668.

[5] E. L. Green, E. N. Marcos and N. Snashall: *The Hochschild cohomology ring of a one point extension*, Comm. Algebra **31** (2003), 357-379.

[6] E. L. Green, N. Snashall and Ø. Solberg: *The Hochschild cohomology ring of a selfinjective algebra of finite representation type*, Proc. Amer. Math. Soc. **131** (2003), no. 11, 3387-3393.

[7] E. L. Green, N. Snashall and Ø. Solberg: *The Hochschild cohomology ring modulo nilpotence of a monomial algebra*, J. Algebra Appl. **5** (2006), no. 2, 153–192.

[8] E. L. Green and Ø. Solberg: *Hochschild cohomology rings and triangular rings*, Proceedings of the Ninth International Conference, Beijing, 2000, eds. D. Happel and Y. B. Zhang, Beijing Normal University Press, vol. II, (2002), 192-200.

[9] D. Happel: *Hochschild cohomology of finite-dimensional algebras*, Lecture Notes in Math., **1404**, Springer, Berlin (1989), 108-126

[10] R. Hartmann, A. Henke, S. Koenig and R. Paget: *Cohomological stratification of diagram algebras*, preprint.

[11] S. Koenig and H. Nagase: *Hochschild cohomology and stratifying ideals*, J. Pure. Appl. Algebra, to appear.

[12] S. Koenig and C. C. Xi: *A characteristic free approach to Brauer algebras*, Trans. Amer. Math. Soc. **353** (2001), no. 4, 1489–1505.

[13] S. Michelena and M. I. Platzeck: *Hochschild cohomology of triangular matrix algebras*, J. Algebra, **233** (2000), 502-525.

[14] J. A. de la Peña and C. C. Xi: *Hochschild cohomology of algebras with homological ideals*, Tsukuba J. Math. **30** (2006), no. 1, 61-79.

[15] M. Suarez-Alvarez: *Applications of the change-of-rings spectral sequence to the computation of Hochschild cohomology*, arXiv:0707.3210.

[16] N. Snashall and Ø. Solberg: *Support varieties and Hochschild cohomology rings*, Proc. London Math. Soc. (3) **88** (2004), no. 3, 705-732.

[17] F. Xu: *Hochschild and ordinary cohomology rings of small categories*, Adv. Math. **219** (2008), 1872-1893.

Nara National College of Technology
22 Yata-cho, Yamatokoriyama,
Nara, 639-1080, JAPAN

E-mail address: nagase@libe.nara-k.ac.jp