POLYNOMIAL INVARIANTS OF FINITE-DIMENSIONAL HOPF ALGEBRAS DERIVED FROM BRAIDING STRUCTURES

MICHIHISA WAKUI (和久井道久)

Abstract. We introduce invariants of a finite-dimensional semisimple and cosemisimple Hopf algebra A over a field $\boldsymbol{k}$ by using the braiding structures of A. The invariants are given in the form of polynomials. The polynomials have integral coefficients under some condition, and become stable by taking some suitable extension of the base field. Furthermore, the polynomials give invariants of the representation category of a finite-dimensional semisimple and cosemisimple Hopf algebra under $\boldsymbol{k}$-linear tensor equivalence. By using the polynomials, we can find some pairs of Hopf algebras, whose representation rings are same, but representation categories are different.

1. Introduction

Given a quantum group, namely, a Hopf algebra with a braiding structure, we have a topological invariant of low-dimensional manifolds, for example, (framed) knots and links. Such an invariant is so-called a quantum invariant. It is well-known that quantum invariants are not only powerful tool for investigating topologies of low-dimensional manifolds, but also closely related to mathematical physics as well as other areas, for example, number theory, gauge theory, and so on.

Although in many investigations on quantum invariants, topological problems of low-dimensional manifolds are studied under a fixed Hopf algebra, in this research, we fix a framed knot or link, and study on representation categories of Hopf algebras. In this article, by using quantum invariants of the unknot with (+1)-framing, for a finite-dimensional semisimple and cosemisimple Hopf algebra A over a field $\boldsymbol{k}$, polynomials $P_A^{(d)}(x)$ $(d = 1, 2, \cdots)$ are introduced as invariants of A, and properties of them are studied. That polynomials are defined as in the following form thanks to some results of Etingof and Gelaki[5] (for detail see Section 2):

$$P_A^{(d)}(x) = \prod_{i=1}^{t} \prod_{R\,:\,\text{braidings of } A} \Bigl(x - \frac{\underline{\dim}_R M_i}{\dim M_i} \Bigr) \quad \in \quad \boldsymbol{k}[x],$$

where $\{M_1, \cdots, M_t\}$ is a full set of non-isomorphic absolutely simple left A-modules with dimension d (so, $\dim M_i = d$ for all i), and $\underline{\dim}_R M_i$ is the quantum invariant of unknot with (+1)-framing and colored by M_i. In algebraic language, $\underline{\dim}_R M_i$ is the category-theoretic rank of M_i in the left rigid braided monoidal category $({}_A\mathbb{M}^{\text{f.d.}}, c_R)$ [10], where ${}_A\mathbb{M}^{\text{f.d.}}$ is the monoidal category of finite-dimensional left A-modules and A-homomorphisms, and c_R is the braiding of ${}_A\mathbb{M}^{\text{f.d.}}$ determined by R.

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Each polynomial $P^{(d)}_A(x)$ has the following properties. All coefficients of the polynomial are integers if $\boldsymbol{k}$ is a finite Galois extension of the rational number field $\mathbb{Q}$, and A coincides with the scalar extension of some finite-dimensional semisimple Hopf algebra over $\mathbb{Q}$. The polynomial becomes also stable by taking some suitable extension of the base field, more precisely, there is a finite separable field extension $L/\boldsymbol{k}$ so that $P^{(d)}_{A^E}(x) = P^{(d)}_{A^L}(x)$ for any field extension E/L.

It is more interesting to note that our polynomial invariants give an invariant of representation categories of Hopf algebras, that is, if representation categories of finite-dimensional semisimple and cosemisimple Hopf algebras A and B are equivalent as $\boldsymbol{k}$-linear tensor categories, then $P^{(d)}_A(x) = P^{(d)}_B(x)$. In general, if representation categories of two finite-dimensional semisimple Hopf algebras A and B over an algebraically field $\boldsymbol{k}$ of characteristic 0 are equivalent as $\boldsymbol{k}$-linear tensor categories, then their representation rings are isomorphic as rings (with $*$-structure)[13, 15]. However, the converse is not true. For example, by Tambara and Yamagami[18], it was proved that three non-commutative and semisimple Hopf algebras $\mathbb{C}[D_8], \mathbb{C}[Q_8], K_8$ of dimension 8 over the complex number field $\mathbb{C}$ have the same representation ring, but their representation categories are not mutually equivalent, where D_8 is the dihedral group of order 8, Q_8 is the quaternion group, and K_8 is the Kac-Paljutkin algebra[6, 11]. This result is generalized by Masuoka[12] in the case where the base field of Hopf algebras is an algebraically closed field of characteristic 0 or $p > 2$. In this article, we give an another proof of Tambara and Yamagami's result by using our polynomial invariants, and furthermore, give other examples of pairs of Hopf algebras, whose representation rings are same, but representation categories are mutually different (see the final section).

Throughout this article, we use the notation $\otimes$ instead of $\otimes_{\boldsymbol{k}}$, and denote by $\mathrm{ch}(\boldsymbol{k})$ the characteristic of the field $\boldsymbol{k}$.

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2. Definition of polynomial invariants

In this section, we introduce invariants of a semisimple and cosemisimple Hopf algebra of finite dimension over an arbitrary field. They are given by polynomials derived from the quasitriangular structures of the Hopf algebra, and become invariants under tensor equivalence of representation categories of Hopf algebras.

Let us recall the definition of a quasitriangular Hopf algebra [3]. Let A be a Hopf algebra and $R \in A \otimes A$ an invertible element. The pair (A, R) is said to be a *quasitriangular Hopf algebra*, and R is said to be a *universal R-matrix* of A, if the following three conditions are satisfied:

(i) $\Delta^{\mathrm{cop}}(a) = R \cdot \Delta(a) \cdot R^{-1}$ for all $a \in A$,
(ii) $(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}$,
(iii) $(\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}$.

Here $\Delta^{\mathrm{cop}} = T \circ \Delta$, $T : A \otimes A \longrightarrow A \otimes A$, $T(a \otimes b) = b \otimes a$, and $R_{ij} \in A \otimes A \otimes A$ is given by $R_{12} = R \otimes 1$, $R_{23} = 1 \otimes R$, $R_{13} = (T \otimes \mathrm{id})(R_{23}) = (\mathrm{id} \otimes T)(R_{12})$.

If $R = \sum_i \alpha_i \otimes \beta_i$ is a universal R-matrix of A, then the element $u = \sum_i S(\beta_i)\alpha_i$ of A is invertible, and has the following properties:

(i) $S^2(a) = uau^{-1}$ for all $a \in A$,

(ii) $S(u) = \sum_i \alpha_i S(\beta_i)$.

The above element u is called the *Drinfel'd element* associated to R. If the characteristic of $\boldsymbol{k}$ is 0, and A is semisimple or cosemisimple of finite dimension, then the Drinfel'd element u belongs to the center of A by the property (i) and $S^2 = \mathrm{id}_A$ [8].

Let (A, R) be a quasitriangular Hopf algebra over a field $\boldsymbol{k}$ and u the Drinfel'd element associated to R. For a finite-dimensional left A-module M, we denote by $\underline{\dim}_R M$ the trace of the left action of u on M, and call it the *R-dimension* of M.

To define polynomial invariants, we use the following result on a semisimple and cosemisimple Hopf algebra of finite dimension due to Etingof and Gelaki [5, Corollary 3.2(ii), Corollary 1.5].

Theorem 1 (Etingof-Gelaki). *Let A be a semisimple and cosemisimple Hopf algebra of finite dimension over a field $\boldsymbol{k}$. Then*

(1) $(\dim M)1_{\boldsymbol{k}} \neq 0$ for any absolutely simple left A-module M,

(2) the set of universal R-matrices $\underline{\mathrm{Braid}}(A)$ is finite. □

Let $A = (A, \Delta, \varepsilon, S)$ be a semisimple and cosemisimple Hopf algebra of finite dimension over a field $\boldsymbol{k}$. For a finite-dimensional left A-module M with $(\dim M)1_{\boldsymbol{k}} \neq 0$, we set

$$P_{A,M}(x) := \prod_{R \in \underline{\mathrm{Braid}}(A)} \Big(x - \frac{\underline{\dim}_R M}{\dim M} \Big).$$

This is a polynomial in $\boldsymbol{k}[x]$. Furthermore, for each positive integer d we define a polynomial $P_A^{(d)}(x)$ in $\boldsymbol{k}[x]$ by

$$P_A^{(d)}(x) := \prod_{i=1}^{t} P_{A,M_i}(x),$$

where $\{M_1, \cdots, M_t\}$ is a full set of non-isomorphic absolutely simple left A-modules with dimension d. If there is no absolutely simple left A-module, then we set $P_A^{(d)}(x) := 1$.

Example 2. Let G be the cyclic group of order m, and $\boldsymbol{k}$ a field of $\mathrm{ch}(\boldsymbol{k}) \nmid m$ which contains a primitive m-th root of unity. Then, the polynomial invariant $P_{\boldsymbol{k}[G]}^{(1)}(x)$ of the group Hopf algebra $\boldsymbol{k}[G]$ is given by the formula

$$P_{\boldsymbol{k}[G]}^{(1)}(x) = \prod_{d,j=0}^{m-1} (x - \omega^{dj^2}) = \prod_{j=0}^{m-1} (x^{\frac{m}{\gcd(j^2,m)}} - 1)^{\gcd(j^2,m)}.$$

For a $\boldsymbol{k}$-bialgebra A we write ${}_A\mathbb{M}$ for the $\boldsymbol{k}$-linear monoidal category whose objects are left A-modules and morphisms are left A-homomorphisms. Two bialgebras A and B

over $\boldsymbol{k}$ are called *monoidally Morita equivalent* if monoidal categories ${}_A\mathbb{M}$ and ${}_B\mathbb{M}$ are equivalent as $\boldsymbol{k}$-linear monoidal categories.

Lemma 3. *Let A and B two Hopf algebras of finite dimension over $\boldsymbol{k}$. If a $\boldsymbol{k}$-linear monoidal functor $F : {}_A\mathbb{M} \longrightarrow {}_B\mathbb{M}$ gives an equivalence between monoidal categories, then* $\dim M = \dim F(M)$ *for a finite-dimensional left A-module M, and there is a bijection $\Phi : \underline{\mathrm{Braid}}(A) \longrightarrow \underline{\mathrm{Braid}}(B)$ such that $\underline{\dim}_R M = \underline{\dim}_{\Phi(R)} F(M)$ for a finite-dimensional left A-module M and a universal R-matrix $R \in \underline{\mathrm{Braid}}(A)$.* □

From the above lemma we have the following theorem immediately.

Theorem 4. *Let A and B be semisimple and cosemisimple Hopf algebras of finite dimension over $\boldsymbol{k}$. If A and B are monoidally Morita equivalent, then $P_A^{(d)}(x) = P_B^{(d)}(x)$ for any positive integer d.* □

3. Properties of polynomial invariants

In this section, we describe properties of polynomial invariants $P_A^{(d)}(x)$ defined in Section 2.

Lemma 5. *Let (A, R) be a quasitriangular Hopf algebra over a field $\boldsymbol{k}$ and u the Drinfel'd element associated to R. If A is semisimple and cosemisimple, then $u^{(\dim A)^3} = 1$.*

Proof. Let us consider the following sub-Hopf algebras B and H of A :

$$B := \{\ (\alpha \otimes \mathrm{id})(R) \mid \alpha \in A^*\ \},$$
$$H := \{\ (\mathrm{id} \otimes \alpha)(R) \mid \alpha \in A^*\ \}.$$

By [14, Proposition 2], the Hopf algebra B is isomorphic to the Hopf algebra $H^{*\mathrm{cop}}$. Let $(D(H), \mathcal{R})$ be the Drinfel'd double of H. By [14, Theorem 2], there is a homomorphism $F : (D(H), \mathcal{R}) \longrightarrow (A, R)$ of quasitriangular Hopf algebras. It follows that the Drinfel'd element $\tilde{u}$ associated to $(D(H), \mathcal{R})$ satisfies $F(\tilde{u}) = u$. Since A is semisimple, sub-Hopf algebras H and $H^{*\mathrm{cop}} \cong B$ are also semisimple [9, Corollary 2.5]. Thus H is semisimple and cosemisimple. So, we have $\tilde{u}^{(\dim H)^3} = 1$ by [4, Theorem 2.5 & Theorem 4.3], and whence $u^{(\dim H)^3} = 1$. Since $\dim A$ is divided by $\dim H$ [14, Proposition 2], we have $u^{(\dim A)^3} = 1$. □

For a field K, let Z_K denote the integral closure the prime ring of K, that is, if the characteristic of K is 0, then Z_K is the ring of algebraic integers in K, and if the characteristic of K is $p > 0$, then Z_K is the algebraic closure of the prime field $\mathbb{F}_p$ in K.

From the above lemma, we have:

Proposition 6. *Let H be a semisimple and cosemisimple Hopf algebra of finite dimension over a field K. Then, for any absolutely simple left H-module M, the coefficients of the polynomial $P_{H,M}(x)$ are in Z_K. Therefore, $P_H^{(d)}(x) \in Z_K[x]$ for any positive integer d.* □

Next, we examine relationship between polynomial invariants and Galois extensions of fields. Let $K/\boldsymbol{k}$ be a field extension, and H a Hopf algebra over K. By a *$\boldsymbol{k}$-form* of H we mean a Hopf algebra A over $\boldsymbol{k}$ such that $H \cong A^K = A \otimes K$ as K-Hopf algebras[1, p.181].

Theorem 7. *Let $K/\boldsymbol{k}$ be a finite Galois extension of fields, and H a semisimple and cosemisimple Hopf algebra of finite dimension over K. If H possesses a $\boldsymbol{k}$-form, then $P_H^{(d)}(x) \in (\boldsymbol{k} \cap Z_K)[x]$ for each positive integer d.* □

We have two corollaries as applications of the above theorem.

Corollary 8. *Let K be a finite Galois extension field of $\mathbb{Q}$, and H a semisimple Hopf algebra of finite dimension over K. If H possesses a $\mathbb{Q}$-form, then $P_H^{(d)}(x) \in \mathbb{Z}[x]$ for a positive integer d, where $\mathbb{Z}$ is the rational integral ring.*

Proof. By [7] a semisimple Hopf algebra over a field of characteristic 0 of finite dimension is cosemisimple. So, the semisimple Hopf algebra H is also cosemisimple. Since $\mathbb{Q} \cap Z_K = \mathbb{Z}$, applying Theorem 7 to H, we have $P_H^{(d)}(x) \in \mathbb{Z}[x]$. □

Corollary 9. *Let Γ be a finite group, and K a finite Galois extension field of $\mathbb{Q}$. Then, $P_{K[\Gamma]}^{(d)}(x) \in \mathbb{Z}[x]$ for a positive integer d.* □

Next, we discuss on stability of polynomial invariants under extension of fields.

Let A be a Hopf algebra over a field $\boldsymbol{k}$, and L a commutative algebra over $\boldsymbol{k}$. Then, $A^L = A \otimes L$ becomes a Hopf algebra over L. Furthermore, if $R = \sum_i \alpha_i \otimes \beta_i$ is a universal R-matrix of A, then

$$R^L = \sum_i (\alpha_i \otimes 1_K) \otimes_L (\beta_i \otimes 1_K) \quad \in \quad A^L \otimes_L A^L$$

is a universal R-matrix of A^L.

Let $\underline{\mathrm{alg}}_{\boldsymbol{k}}$ denote the $\boldsymbol{k}$-additive category whose objects are commutative algebras over $\boldsymbol{k}$ and morphisms are algebra maps between them. Let A and B be two Hopf algebras over $\boldsymbol{k}$. For a commutative algebra $L \in \underline{\mathrm{alg}}_{\boldsymbol{k}}$, we set

$$\mathrm{Hopf}_L(A \otimes L, B \otimes L) := \{ \text{ the } L\text{-Hopf algebra maps } A \otimes L \longrightarrow B \otimes L \ \},$$

and for an algebra map $f : L_1 \longrightarrow L_2$ between commutative algebras $L_1, L_2 \in \underline{\mathrm{alg}}_{\boldsymbol{k}}$ and $\varphi \in \mathrm{Hopf}_{L_1}(A \otimes L_1, B \otimes L_1)$ we define a map $f_*\varphi \in \mathrm{Hopf}_{L_2}(A \otimes L_2, B \otimes L_2)$ by the composition:

$$\begin{aligned} A \otimes L_2 \xrightarrow{\mathrm{id}\otimes\eta} A \otimes (L_1 \otimes L_2) \cong (A \otimes L_1) \otimes L_2 \xrightarrow{\varphi\otimes\mathrm{id}} (B \otimes L_1) \otimes L_2 \\ \xrightarrow{\mathrm{id}_B\otimes f} (B \otimes L_2) \otimes L_2 \cong B \otimes (L_2 \otimes L_2) \xrightarrow{\mathrm{id}\otimes\mu_{L_2}} B \otimes L_2, \end{aligned}$$

where μ_{L_2} is the multiplication of L_2, and $\eta : L_2 \longrightarrow L_1 \otimes L_2$ is the $\boldsymbol{k}$-algebra map defined by $\eta(y) = 1_{L_1} \otimes y$ $(y \in L_2)$. This $\boldsymbol{k}$-linear map $f_*\varphi$ is directly defined by

$$(f_*\varphi)(a \otimes y) = \sum_i b_i \otimes f(x_i)y, \qquad \Big(\varphi(a \otimes 1_{L_1}) = \sum_i b_i \otimes x_i\Big)$$

for all $a \in A,\ y \in L_2$. Let $\underline{\text{Set}}$ denote the category whose objects are sets and morphisms are maps. Then we have a covariant functor $\mathbf{Hopf}(A,B) : \underline{\text{alg}}_{\boldsymbol{k}} \longrightarrow \underline{\text{Set}}$ such as

$$\begin{aligned} \text{for an object}: &\quad L \longmapsto \mathrm{Hopf}_L(A \otimes L, B \otimes L), \\ \text{for a morphism}: &\quad f \longmapsto \Big(\mathbf{Hopf}(A,B)(f) : \varphi \longmapsto f_*\varphi\Big). \end{aligned}$$

If A and B are of finite dimension over $\boldsymbol{k}$, then the functor $\mathbf{Hopf}(A,B)$ can be represented by some finitely generated commutative algebra $Z \in \underline{\text{alg}}_{\boldsymbol{k}}$ [20, p.4–5 & p.58]. Furthermore, if A is semisimple, and B is cosemisimple, then the representing object Z is separable and of finite dimension[5, Corollary 1.3]. This fact leads to the following theorem.

Theorem 10. *Let A be a cosemisimple Hopf algebra over a field $\boldsymbol{k}$ of finite dimension. Then, there is a separable finite extension field L of $\boldsymbol{k}$ such that*

(i) *there are only finitely many universal R matrices of A^L, and*
(ii) *for any field extension E/L, the map $\underline{\text{Braid}}(A^L) \longrightarrow \underline{\text{Braid}}(A^E),\ R \longmapsto R^E$ is bijective.* □

Corollary 11. *Let A be a semisimple and cosemisimple Hopf algebra over a field $\boldsymbol{k}$ of finite dimension. Then, there is a separable finite extension field L of $\boldsymbol{k}$ such that for any field extension E of L and any positive integer d, $P^{(d)}_{A^E}(x) = P^{(d)}_{A^L}(x)$ in $E[x]$.* □

4. Examples

In this section, we give computational results of polynomial invariants for several Hopf algebras. By comparing polynomial invariants one has new examples of pairs of Hopf algebras such that their representation rings are isomorphic, but they are not monoidally Morita equivalent.

Let $N \geq 1$ be an odd integer and $n \geq 2$, and consider the finite group

$$G_{Nn} = \langle h,\ t,\ w \mid t^2 = h^{2N} = 1,\ w^n = h^N,\ tw = w^{-1}t,\ ht = th,\ hw = wh \rangle.$$

The group G_{Nn} is non-commutative, and the order of it is $4Nn$. We remark that if $N = 1$, then $G_{Nn} \cong D_{4n}$, the dihedral group of order $4n$. Let $\boldsymbol{k}$ be a field of $\mathrm{ch}(\boldsymbol{k}) \nmid 2Nn$ which contains a primitive $4Nn$-th root of unity. The group algebra $\boldsymbol{k}[G_{Nn}]$ has a Hopf algebra structure in a usual way. At the same time, one can define another Hopf algebra structure on $\boldsymbol{k}[G_{Nn}]$ as follows.

$$\begin{aligned} &\Delta(h) = h \otimes h, \quad \Delta(t) = h^N wt \otimes e_1 t + t \otimes e_0 t, \quad \Delta(w) = w \otimes e_0 w + w^{-1} \otimes e_1 w, \\ &\varepsilon(h) = 1, \quad ,\varepsilon(t) = 1, \quad \varepsilon(w) = 1, \\ &S(h) = h^{-1}, \quad S(t) = (-e_1 w + e_0)t, \quad S(w) = e_1 w^{-1} + e_1 w, \end{aligned}$$

where $e_0 = \frac{1+h^N}{2}$, $e_1 = \frac{1-h^N}{2}$. We denote this Hopf algebra by A_{Nn}. If we set

$$\lambda = \begin{cases} -1 & (n \text{ is even}), \\ +1 & (n \text{ is odd}), \end{cases}$$

then the Hopf algebra A_{Nn} is isomorphic to the Hopf algebra $A_{Nn}^{+\lambda}$ which is introduced by Satoshi Suzuki[16]. Properties of the Hopf algebras A_{1n} are studied in detail in [2], and A_{12} especially coincides with the Kac-Paljutkin algebra K_8 [6, 11], which is the unique non-commutative and non-cocommutative semisimple Hopf algebra of dimension 8 up to isomorphism.

Let $\omega \in \boldsymbol{k}$ be a primitive $4Nn$-th root of unity. Then, a full set of non-isomorphic (absolutely) simple left $\boldsymbol{k}[G_{Nn}]$-modules is given by

$$\begin{aligned} &\{\ V_{ijk} \mid i,j = 0,1,\ k = 0,2,\cdots,2N-2\ \} \\ \cup &\{\ V_{jk} \mid k = 0,1,\cdots,2N-1,\ j = 1,2,\cdots,n-1,\ j \equiv k \ (\mathrm{mod}\ 2)\ \}, \end{aligned}$$

where the action χ_{ijk} of $\boldsymbol{k}[G_{Nn}]$ on $V_{ijk} = \boldsymbol{k}$ is given by

$$\chi_{ijk}(t) = (-1)^i,\ \chi_{ijk}(w) = (-1)^j,\ \chi_{ijk}(h) = \begin{cases} \omega^{2kn} & (n \text{ is even}), \\ \omega^{2(k+j)n} & (n \text{ is odd}), \end{cases}$$

and the left action ρ_{jk} of $\boldsymbol{k}[G_{Nn}]$ on $V_{jk} = \boldsymbol{k} \oplus \boldsymbol{k}$ is given by

$$\rho_{jk}(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \rho_{jk}(w) = \begin{pmatrix} \omega^{2jN} & 0 \\ 0 & \omega^{-2jN} \end{pmatrix},\ \rho_{jk}(h) = \begin{pmatrix} \omega^{2kn} & 0 \\ 0 & \omega^{2kn} \end{pmatrix}.$$

Since A_{Nn} is isomorphic to the dual Hopf algebra, we can compute $P^{(d)}_{A_{Nn}}(x)$ $(d = 1,2)$ by using the data of the braidings of $A_{Nn}^{+\lambda}$ determined by S. Suzuki[16]. We set

$$\epsilon(n) = \begin{cases} 0 & (n \text{ is even}), \\ 1 & (n \text{ is odd}). \end{cases}$$

Proposition 12. *(1) In case of $n \geq 3$,*

$$P^{(1)}_{A_{Nn}}(x) = \begin{cases} \displaystyle\prod_{s=0}^{N-1}\prod_{i=0}^{N-1}(x-\omega^{-8nis^2})^{4n}(x-\omega^{-8ins^2}(-1)^{\frac{n}{2}})^{4n} & \textit{if } n \textit{ is even}, \\ \displaystyle\prod_{s=0}^{N-1}\prod_{i=0}^{N-1}(x-\omega^{-8nis^2})^{4n}(x^2-\omega^{-4in(2s+1)^2})^{2n} & \textit{if } n \textit{ is odd}, \end{cases}$$

$$\begin{aligned} P^{(2)}_{A_{Nn}}(x) = &\prod_{s=0}^{N-1}\prod_{t=1}^{\frac{n-\epsilon(n)}{2}}\prod_{i=0}^{N-1}\prod_{j=0}^{n-1}(x^2-\omega^{-4in(2s+1)^2-2N(2t-1)^2(2j+1-\epsilon(n))}) \\ &\times\prod_{s=0}^{N-1}\prod_{t=1}^{\frac{n-2+\epsilon(n)}{2}}\prod_{i=0}^{N-1}\prod_{j=0}^{n-1}(x-\omega^{-8ins^2-4Nt^2(2j+1-\epsilon(n))})^2. \end{aligned}$$

(2) In case of $n=2$,

$$P^{(1)}_{A_{N2}}(x)=\prod_{s=0}^{N-1}\prod_{i=0}^{N-1}(x-\omega^{-16is^2})^{16}(x+\omega^{-8is^2})^{8}(x+\omega^{-16is^2})^{8},$$

$$P^{(2)}_{A_{N2}}(x)=\prod_{s=0}^{N-1}\prod_{i=0}^{N-1}(x^4+\omega^{-16i(2s+1)^2})(x^2-\omega^{-8i(2s+1)^2})^{2}. \quad \square$$

On the other hand, we can determine the universal R-matrices of the group Hopf algebra $\boldsymbol{k}[G_{Nn}]$ by using the method developed in [19], and compute the polynomial invariants $P^{(d)}_{\boldsymbol{k}[G_{Nn}]}(x)$ $(d=1,2)$.

Proposition 13. *(1) In case of $n\geq 3$,*

$$P^{(1)}_{\boldsymbol{k}[G_{Nn}]}(x)=\begin{cases}\displaystyle\prod_{s=0}^{N-1}\prod_{i=0}^{N-1}(x-\omega^{-8nis^2})^{8n} & \textit{if } n \textit{ is even,}\\ \displaystyle\prod_{s=0}^{N-1}\prod_{i=0}^{N-1}(x-\omega^{-8nis^2})^{4n}(x^2-\omega^{-4in(2s+1)^2})^{2n} & \textit{if } n \textit{ is odd,}\end{cases}$$

$$P^{(2)}_{\boldsymbol{k}[G_{Nn}]}(x)=\prod_{s=0}^{N-1}\prod_{t=1}^{\frac{n-\epsilon(n)}{2}}\prod_{i=0}^{N-1}\prod_{j=0}^{n-1}(x^2-\omega^{-4in(2s+1)^2-4Nj(2t-1)^2})$$
$$\times\prod_{s=0}^{N-1}\prod_{t=1}^{\frac{n-2+\epsilon(n)}{2}}\prod_{i=0}^{N-1}\prod_{j=0}^{n-1}(x-\omega^{-8ins^2-8Njt^2})^{2}.$$

(2) In case of $n=2$,

$$P^{(1)}_{\boldsymbol{k}[G_{Nn}]}(x)=\prod_{s=0}^{N-1}\prod_{i=0}^{N-1}(x-\omega^{-16is^2})^{32},$$

$$P^{(2)}_{\boldsymbol{k}[G_{Nn}]}(x)=\prod_{s=0}^{N-1}\prod_{i=0}^{N-1}(x^4-\omega^{-16i(2s+1)^2})(x^2-\omega^{-8i(2s+1)^2})^{2}. \quad \square$$

By comparing polynomial invariants of two Hopf algebras A_{Nn} and $\boldsymbol{k}[G_{Nn}]$, we see immediately that if n is odd, then $P^{(d)}_{A_{Nn}}(x)=P^{(d)}_{\boldsymbol{k}[G_{Nn}]}(x)$ for $d=1,2$. So, our polynomial invariants do not detect the representation categories of A_{Nn} and $\boldsymbol{k}[G_{Nn}]$ for an odd integer n. However, for an even integer n our polynomial invariants are useful.

Theorem 14. *Let $N\geq 1$ be an odd integer and $n\geq 2$, and let $\boldsymbol{k}$ be a field of $ch(\boldsymbol{k})\nmid 2Nn$ which contains a primitive $4Nn$-th root of unity. If n is even, then two Hopf algebras A_{Nn} and $\boldsymbol{k}[G_{Nn}]$ are not monoidally Morita equivalent.* $\square$

Example 15. For a non-negative integer h, Φ_h denotes the h-th cyclotomic polynomial. Then, by using Maple12 software, we see that the polynomial invariants of Hopf algebras $\boldsymbol{k}[G_{Nn}]$ and A_{Nn} for $N=1,3,5$ and $n=2,3,4$ are given as in the following list.

Hopf algebra A	$P_A^{(1)}(x)$	$P_A^{(2)}(x)$
$\boldsymbol{k}[G_{12}]$	Φ_1^{32}	$\Phi_4\Phi_2^3\Phi_1^3$
A_{12}	$\Phi_2^{16}\Phi_1^{16}$	$\Phi_8\Phi_2^2\Phi_1^2$
$\boldsymbol{k}[G_{32}]$	$\Phi_3^{64}\Phi_1^{160}$	$\Phi_{12}^2\Phi_4^5\Phi_6^6\Phi_3^6\Phi_2^{15}\Phi_1^{15}$
A_{32}	$\Phi_6^{32}\Phi_3^{32}\Phi_2^{80}\Phi_1^{80}$	$\Phi_{24}^2\Phi_8^5\Phi_6^4\Phi_3^4\Phi_2^{10}\Phi_1^{10}$
$\boldsymbol{k}[G_{52}]$	$\Phi_5^{128}\Phi_1^{288}$	$\Phi_{20}^4\Phi_{10}^{12}\Phi_5^{12}\Phi_4^9\Phi_2^{27}\Phi_1^{27}$
A_{52}	$\Phi_{10}^{64}\Phi_5^{64}\Phi_2^{144}\Phi_1^{144}$	$\Phi_{40}^4\Phi_{10}^8\Phi_5^8\Phi_8^9\Phi_2^{18}\Phi_1^{18}$
$\boldsymbol{k}[G_{13}]$, A_{13}	$\Phi_2^6\Phi_1^{18}$	$\Phi_6\Phi_3^3\Phi_2\Phi_1^3$
$\boldsymbol{k}[G_{33}]$, A_{33}	$\Phi_6^{12}\Phi_3^{36}\Phi_2^{30}\Phi_1^{90}$	$\Phi_6^9\Phi_3^{27}\Phi_2^9\Phi_1^{27}$
$\boldsymbol{k}[G_{53}]$, A_{53}	$\Phi_{10}^{24}\Phi_5^{72}\Phi_2^{54}\Phi_1^{162}$	$\Phi_{30}^4\Phi_{15}^{12}\Phi_{10}^4\Phi_5^{12}\Phi_6^9\Phi_3^{27}\Phi_2^9\Phi_1^{27}$
$\boldsymbol{k}[G_{14}]$	Φ_1^{32}	$\Phi_8^2\Phi_4^2\Phi_2^6\Phi_1^6$
A_{14}		$\Phi_{16}^2\Phi_4^4$
$\boldsymbol{k}[G_{34}]$	$\Phi_3^{64}\Phi_1^{160}$	$\Phi_{24}^4\Phi_{12}^4\Phi_8^{10}\Phi_6^{12}\Phi_3^{12}\Phi_4^{10}\Phi_2^{30}\Phi_1^{30}$
A_{34}		$\Phi_{48}^4\Phi_{16}^{10}\Phi_{12}^8\Phi_4^{20}$
$\boldsymbol{k}[G_{54}]$	$\Phi_5^{128}\Phi_1^{288}$	$\Phi_{40}^8\Phi_{20}^8\Phi_{10}^{24}\Phi_5^{24}\Phi_8^{18}\Phi_4^{18}\Phi_2^{54}\Phi_1^{54}$
A_{54}		$\Phi_{80}^8\Phi_{20}^{16}\Phi_{16}^{18}\Phi_4^{36}$

Remark 16. For an odd integer $N \geq 1$ and an integer $n \geq 2$, the representation rings of two Hopf algebras A_{Nn} and $\boldsymbol{k}[G_{Nn}]$ are isomorphic as rings with $*$-structure. In the case of $N = 1$, this result is obtained by Masuoka[12]. By the above theorem, hence, for an even integer n, A_{Nn} and $\boldsymbol{k}[G_{Nn}]$ give an example of a pair of Hopf algebras such that their representation rings are isomorphic, but their representation categories are not. Such an example was first found by Tambara and Yamagami[18]. They showed that 8-dimensional non-commutative semisimple Hopf algebras $\mathbb{C}[D_8], \mathbb{C}[Q_8], K_8$ over $\mathbb{C}$ are not mutually monoidally Morita equivalent. From the viewpoint of extension of Hopf algebras Masuoka[12] showed that their result holds in the case where the base field are algebraically closed, and its characteristic does not divide 2. By using our polynomial invariants we can also prove the Tambara and Yamagami's result mentioned above. The polynomial invariants of 8-dimensional non-commutative semisimple Hopf algebras $\boldsymbol{k}[D_8], \boldsymbol{k}[Q_8], K_8$ are given by

$$P^{(1)}_{\boldsymbol{k}[D_8]}(x) = P^{(1)}_{\boldsymbol{k}[Q_8]}(x) = (x-1)^{32}, \qquad P^{(1)}_{K_8}(x) = (x-1)^{16}(x+1)^{16},$$

$$P^{(2)}_{\boldsymbol{k}[D_8]}(x) = x^8 - 2x^6 + 2x^2 - 1,$$
$$P^{(2)}_{\boldsymbol{k}[Q_8]}(x) = x^8 + 2x^6 - 2x^2 - 1,$$
$$P^{(2)}_{K_8}(x) = x^8 - 2x^6 + 2x^4 - 2x^2 + 1.$$

Since polynomials $P^{(2)}_{\boldsymbol{k}[D_8]}(x),\ P^{(2)}_{\boldsymbol{k}[Q_8]}(x),\ P^{(2)}_{K_8}(x)$ are all different, we conclude that by Theorem 4 the Hopf algebras $\boldsymbol{k}[D_8], \boldsymbol{k}[Q_8], K_8$ are not mutually monoidally Morita equivalent.

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Department of Mathematics, Faculty of Engineering Science
Kansai University
3-3-35 Yamate-cho, Suita-shi, Osaka 564-8680, Japan
E-mail address: `wakui@ipcku.kasnai-u.ac.jp`