DERIVED EQUIVALENCES FOR TRIANGULAR MATRIX RINGS

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Abstract. We generalize derived equivalences for triangular matrix rings induced by a certain type of classical tilting module introduced by Auslander, Platzeck and Reiten to generalize reflection functors in the representation theory of quivers due to Bernstein, Gelfand and Ponomarev.

1. Notation

For a ring A, we denote by Mod-A the category of right A-modules, by mod-A the full subcategory of Mod-A consisting of finitely presented modules and by $\mathcal{P}_A$ the full subcategory of Mod-A consisting of finitely generated projective modules. We denote by A^{op} the opposite ring of A and consider left A-modules as right A^{op}-modules. Sometimes, we use the notation X_A (resp., ${}_AX$) to stress that the module X considered is a right (resp., left) A-module. We denote by $\mathcal{K}(\text{Mod-}A)$ (resp., $\mathcal{D}(\text{Mod-}A)$) the homotopy (resp., derived) category of cochain complexes over Mod-A and by $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ the full triangulated subcategory of $\mathcal{K}(\text{Mod-}A)$ consisting of bounded complexes over $\mathcal{P}_A$. We consider modules as complexes concentrated in degree zero. For any integer $n \in \mathbb{Z}$ we denote by $(-)[n]$ the n-shift of complexes. Also, we use the notation $\mathrm{Hom}^\bullet(-,-)$ to denote the single complex associated with the double hom complex.

2. Introduction

Let R be a finite dimensional algebra over a field k and M a finitely generated projective right R-module. Set

$$A = \begin{pmatrix} k & M \\ 0 & R \end{pmatrix} \quad \text{and} \quad e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in A.$$

As pointed out by Brenner and Butler (see [4, p.111]), we know from [1] (cf. also [3]) that $\mathrm{Ext}^1_A(A/AeA, A) \oplus Ae \in \text{Mod-}A^{\mathrm{op}}$ is a tilting module of projective dimension at most one (see [6]) with

$$\mathrm{End}_{A^{\mathrm{op}}}(\mathrm{Ext}^1_A(A/AeA, A) \oplus Ae)^{\mathrm{op}} \cong \begin{pmatrix} R & \mathrm{Hom}_R(M, R) \\ 0 & k \end{pmatrix},$$

so that the triangular matrix rings

$$\begin{pmatrix} k & M \\ 0 & R \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} R & \mathrm{Hom}_R(M, R) \\ 0 & k \end{pmatrix}$$

are derived equivalent to each other. Our aim is to extend this type of derived equivalence to the case where M_R has finite projective dimension.

The detailed version of this paper has been submitted for publication elsewhere.

3. General case

Let A be a ring and $e \in A$ an idempotent satisfying the following conditions:

(E1) Ae admits a projective resolution $\varepsilon : P^\bullet \to Ae$ in Mod-eAe with $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_{eAe})$, in particular, $d = \text{proj dim } Ae_{eAe} < \infty$;

(E2) $\mu : Ae \otimes_{eAe} eA \to A, x \otimes y \mapsto xy$ is monic;

(E3) $\varphi : eA \to \mathrm{Hom}_{eAe}(Ae, eAe), x \mapsto (y \mapsto xy)$ is monic;

(E4) if $d > 0$ then φ is an isomorphism and $\mathrm{Ext}^i_{eAe}(Ae, eAe) = 0$ for $1 \leq i < d$; and

(E5) $\mathrm{Tor}^{eAe}_i(Ae, eA) = 0$ for $i \neq 0$.

Set $T_1^\bullet = eA[d+1]$, let $T_2^\bullet$ be the mapping cone of the composite

$$\mu \circ (\varepsilon \otimes_{eAe} eA) : P^\bullet \otimes_{eAe} eA \to Ae \otimes_{eAe} eA \to A$$

and set $T^\bullet = T_1^\bullet \oplus T_2^\bullet$. Then the following hold.

Theorem 1. *The complex $T^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ is a tilting complex with*

$$\mathrm{End}_{\mathrm{K}(\text{Mod-}A)}(T^\bullet) \cong \begin{pmatrix} eAe & \mathrm{Ext}^{d+1}_A(A/AeA, eA) \\ 0 & A/AeA \end{pmatrix}.$$

Remark 2. Assume $\mathrm{Ext}^i_A(A/AeA, A) = 0$ for $i \neq d+1$. Then we have

$$\mathrm{Hom}^\bullet_A(T^\bullet, A)[d+1] \cong \mathrm{Ext}^{d+1}_A(A/AeA, A) \oplus Ae$$

in $\mathcal{D}(\text{Mod-}A^{\mathrm{op}})$. Thus $\mathrm{Ext}^{d+1}_A(A/AeA, A) \oplus Ae \in \text{Mod-}A^{\mathrm{op}}$ is a tilting module with

$$\mathrm{End}_{A^{\mathrm{op}}}(\mathrm{Ext}^{d+1}_A(A/AeA, A) \oplus Ae)^{\mathrm{op}} \cong \begin{pmatrix} eAe & \mathrm{Ext}^{d+1}_A(A/AeA, eA) \\ 0 & A/AeA \end{pmatrix}.$$

4. Main results

Let R and S be rings and M an S-R-bimodule satisfying the following conditions:

(M1) M admits a projective resolution $P^\bullet \to M$ in Mod-R with $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_R)$, in particular, $d = \text{proj dim } M_R < \infty$; and

(M2) $\mathrm{Ext}^i_R(M, R) = 0$ for $i < d$.

Set

$$A = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix} \quad \text{and} \quad e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in A.$$

Then the conditions (E1)–(E5) in the preceding section are satisfied. Also, we have $\mathrm{Ext}^{d+1}_A(A/AeA, eA) \cong \mathrm{Ext}^d_R(M, R)$. Note that $eAe \cong R$ and $A/AeA \cong S$ as rings. Thus by Theorem 1 the following hold.

Theorem 3. *The triangular matrix rings*

$$\begin{pmatrix} S & M \\ 0 & R \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} R & \mathrm{Ext}^d_R(M, R) \\ 0 & S \end{pmatrix}$$

are derived equivalent to each other.

Consider next the case where R is a finite dimensional algebra over a field k and $S = k$. Then by Theorem 3 the following hold.

Proposition 4. *The triangular matrix algebras*

$$\begin{pmatrix} k & M \\ 0 & R \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} k & D\mathrm{Ext}_R^d(M,R) \\ 0 & R \end{pmatrix}$$

are derived equivalent to each other, where $D = \mathrm{Hom}_k(-,k)$.

Remark 5. Since the algebras above are trivial extensions of $\Lambda = k \times R$ by M and $D\mathrm{Ext}_R^d(M,R)$, respectively (see [5]). On the other hand, if inj dim ${}_R R$ = inj dim $R_R < \infty$, then $D\Lambda \in$ Mod-Λ is a tilting module with $\Lambda \cong \mathrm{End}_\Lambda(D\Lambda)$(see e.g. [7, Proposition 1.6]) and $M \otimes_\Lambda^{\mathbf{L}} D\Lambda[-d] \cong M \otimes_R^{\mathbf{L}} DR[-d] \cong \mathrm{Tor}_d^R(M, DR) \cong D\mathrm{Ext}_R^d(M,R)$ in $\mathcal{D}(\text{Mod-}\Lambda)$. Thus, if inj dim ${}_R R$ = inj dim $R_R < \infty$, Proposition 4 is due to [8, Corollary 5.4] (see also [2]).

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