COHEN-MACAULAY MODULES AND HOLONOMIC MODULES ON FILTERED GORENSTEIN RINGS

HIROKI MIYAHARA

Abstract. This paper is a joint work with K.Nishida. We will study about filtered Gorenstein rings, then Cohen-Macaulay modules and holonomic modules are difined and studied.

1. Introduction and Preliminaries

Definition 1. Let Λ be a (not necessarily commutative) ring. A family of additive subgroups $\{ \mathcal{F}_p\Lambda \mid p \in \mathbb{N} \}$ of Λ, where $\mathbb{N}$ is the set of all non-negative integers, is called a *filtration* of Λ, if

(1) $1 \in \mathcal{F}_0\Lambda$
(2) $\mathcal{F}_p\Lambda \subset \mathcal{F}_{p+1}\Lambda$
(3) $(\mathcal{F}_p\Lambda)(\mathcal{F}_q\Lambda) \subset \mathcal{F}_{p+q}\Lambda$
(4) $\Lambda = \bigcup_{p\in\mathbb{N}} \mathcal{F}_p\Lambda$.

A pair $(\Lambda, \mathcal{F})$ is called a *filtered ring* if Λ has a filtration. Let $\sigma_p : \mathcal{F}_p\Lambda \longrightarrow \mathcal{F}_p\Lambda/\mathcal{F}_{p-1}\Lambda$ $(\mathcal{F}_{-1}\Lambda = 0)$ be a natural homomorphism, then $\mathrm{gr}\Lambda := \bigoplus_{p\in\mathbb{N}} \mathcal{F}_p\Lambda/\mathcal{F}_{p-1}\Lambda$ is a graded ring with multiplication

$$\sigma_p(a)\sigma_q(b) = \sigma_{p+q}(ab), \quad a \in \mathcal{F}_p\Lambda,\ b \in \mathcal{F}_q\Lambda$$

Through the paper, we assume that $\mathrm{gr}\,\Lambda$ is a commutative Noetherian ring. Therefore Λ is a left and right Noetherian ring.

Let Λ be a filtered ring and M a (left) Λ-module. A family of additive subgroups $\{ \mathcal{F}_p\Lambda \mid p \in \mathbb{Z} \}$ of M is called a *filtration* of M, if

(1) $\mathcal{F}_pM \subset \mathcal{F}_{p+1}M$
(2) $\mathcal{F}_pM = 0$ for $p \ll 0$
(3) $(\mathcal{F}_p\Lambda)(\mathcal{F}_qM) \subset \mathcal{F}_{p+q}M$
(4) $M = \bigcup_{p\in\mathbb{Z}} \mathcal{F}_pM$.

A pair $(M, \mathcal{F})$ is called a *filtered Λ-module* if M has a filtration. Let $\tau_p : \mathcal{F}_pM \longrightarrow \mathcal{F}_pM/\mathcal{F}_{p-1}M$ be a natural homomorphism, then $\mathrm{gr}M := \bigoplus_{p\in\mathbb{Z}} \mathcal{F}_pM/\mathcal{F}_{p-1}M$ is a graded $\mathrm{gr}\Lambda$-module by

$$\sigma_p(a)\tau_q(x) = \tau_{p+q}(ax), \quad a \in \mathcal{F}_p\Lambda, x \in \mathcal{F}_qM$$

Let Λ be a filtered ring and let $(M, \mathcal{F})$ be a filtered Λ-module. A filtration $\mathcal{F}$ is called *good* if $\mathrm{gr}M$ is a finitely generated $\mathrm{gr}\Lambda$-module. The module M has a good filtration if and only if M is finitely generated.

The detailed version of this paper will be submitted for publication elsewhere.

Definition 2. A Λ-module M is said to have *Gorenstein dimension zero*, denoted by $\text{G-dim}_\Lambda M = 0$, if $M^{**} \cong M$ and $\mathrm{Ext}^k_\Lambda(M, \Lambda) = \mathrm{Ext}^k_{\Lambda^{\mathrm{op}}}(M^*, \Lambda) = 0$, where $M^* = \mathrm{Hom}_\Lambda(M, \Lambda)$ for $k > 0$. Then, G-dim $M = 0$ if and only if $\mathrm{Ext}^k_\Lambda(M, \Lambda) = \mathrm{Ext}^k_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}M, \Lambda) = 0$ for $k > 0$.

For a positive integer k, M is said to have *Gorenstein dimension less than or equal to* k, denoted by G-dim $M \leq k$, if there exists an exact sequence $0 \to G_k \to \cdots \to G_0 \to M \to 0$ with G-dim $G_i = 0$ for $0 \leq i \leq k$. G-dim $M \leq k$ if and only if G-dim $\Omega^k M = 0$. Also, if G-dim $M < \infty$ then G-dim $M = \sup\{k : \mathrm{Ext}^k_\Lambda(M, \Lambda) \neq 0\}$.

2. Homological property on filtered rings

In this section, we will talk about Gorenstein dimension and grade of filtered modules. The following fact and the lemma are important to prove our main results.

Fact. *Let Λ be a filtered ring and M a filtered Λ-module with a good filtation. Then* $\mathrm{gr}\,\mathrm{Ext}^i_\Lambda(M, \Lambda)$ *is a subfactor of* $\mathrm{Ext}^i_{\mathrm{gr}\Lambda}(\mathrm{gr}M, \mathrm{gr}\Lambda)$ *for* $i \geq 0$.

Lemma 3. *Let Λ be a filtered ring and M a filtered Λ-module with a good filtation. Then there exists an epimorphism* $\alpha : \mathrm{Tr}_{\mathrm{gr}\Lambda}(\mathrm{gr}M) \to \mathrm{gr}(\mathrm{Tr}_\Lambda M)$.
Moreover, if $\text{G-dim}\,\mathrm{gr}M = 0$, *then α is an isomorphism.*

Combining the above fact and the lemma, we get the following theorem.

Theorem 4. *Let Λ be a filtered ring M a filtered Λ-module with a good filtration, and let k be a non-negative integer. Then* $\text{G-dim}\,\mathrm{gr}M \leq k$ *implies* $\text{G-dim}M \leq k$.

Corollary 5. *If* $\text{G-dim}\,\mathrm{gr}M < \infty$, *then* G-dim $M = \text{G-dim}\,\mathrm{gr}M$.

On the other hand, we get the relation between grade of modΛ and grade of mod(grΛ)

Theorem 6. *Let Λ be a filtered ring such that* $\mathrm{gr}\Lambda$ *is a commutative Gorenstein ring and M a filtered Λ-module with a good filtration. Then* $\mathrm{grade}_\Lambda M = \mathrm{grade}_{\mathrm{gr}\Lambda}\mathrm{gr}M$ *holds, where* $\mathrm{grade}M = \inf\{\, i \mid \mathrm{Ext}^i_\Lambda(M, \Lambda) \neq 0 \,\}$.

Remark 7. The above theorem is proved under the assumption that grΛ is Gorenstein, but we can have the equality under a more general condition about a module. We shall study it in another paper.

3. Introduction to filtered Gorenstein rings

A commutative graded ring R is called *local ring if R has a unique maximal graded ideal (*maximal ideal). We assume that grΛ is a commutative Gorenstein *local ring (with unique *maximal ideal $\mathcal{M}$) satisfying the following condition (P):

(P) : There exists an element of positive degree in $\mathrm{gr}\Lambda - \mathfrak{p}$ for any graded prime ideal $\mathfrak{p} \neq \mathcal{M}$

Fact. *Let $(R, \mathcal{M})$ be a commutative *local Gorenstein ring with the condition (P), and let A be a finite graded R-module. Then we have the following* :

$$\text{G-dim } A + {}^*\mathrm{depth}A = {}^*\mathrm{depth}R \qquad ({}^*\mathrm{depth}A := \mathrm{depth}(\mathcal{M}, A))$$
$$\mathrm{grade}A + {}^*\mathrm{dim}A = {}^*\mathrm{dim}R \qquad ({}^*\mathrm{dim}A := \mathrm{ht}\ \mathcal{M}/\mathrm{Ann}_R(A)).$$

Proposition 8. *Let Λ be a filtered ring such that* $\mathrm{gr}\Lambda$ *is a commutative* **local Gorenstein ring with the condition (P), and let M be a filtered Λ-module with a good filtration. Then the following holds* :

$$G\text{-}dim_{\Lambda} M + {}^*\mathrm{depth}\,\mathrm{gr}M = n \quad (n := {}^*\mathrm{dim}\,\mathrm{gr}\Lambda)$$
$$\mathrm{grade}_{\Lambda} M + {}^*\mathrm{dim}\,\mathrm{gr}M = n.$$

Corollary 9. *Let Λ be a filtered ring such that* $\mathrm{gr}\Lambda$ *is a commutative* **local Gorenstein ring with the condition (P). Then,* $\mathrm{id}_{\Lambda}\Lambda = \mathrm{id}_{\Lambda^{op}}\Lambda \leq n$. *Therefore, let* $\mathrm{id}\Lambda = d$, *then* ${}^*\mathrm{dim}\,\mathrm{gr}M \geq n - d$ *for all filtered Λ-module M with a good filtration.*

Definition 10. We call a filtered ring Λ a *filtered Gorenstein ring* if $\mathrm{gr}\Lambda$ is a commutative Gorenstein *local ring with the condition (P)

We can naturally get the following .

Definition 11. Let Λ be a filtered Gorenstein ring. We call filtered Λ-module M with a good filtration a *CM Λ-module*, if $\mathrm{gr}M$ is a graded CM $\mathrm{gr}\Lambda$-module.

The following proposition and corollary are well known for the case commutative rings.

Proposition 12. *Let Λ be a filtered Gorenstein ring and M a filtered Λ-module with a good filtration. Then M is CM if and only if* $\mathrm{grade}M = \mathrm{G\text{-}dim}M$

Corollary 13. *Let Λ be a filtered Gorenstein ring, and put*
$\mathcal{C}_k(\Lambda) = \{\, M \in \mathrm{mod}\Lambda \mid M \text{ is CM with } \mathrm{G}\text{-}dim\, M = k \,\}$.
Then, the functor $\mathrm{Ext}^k_{\Lambda}(-, \Lambda)$ *induces a duality between the categories* $\underline{\mathcal{C}}_k(\Lambda)$ *and* $\underline{\mathcal{C}}_k(\Lambda^{\mathrm{op}})$.

Definition 14. Let Λ be a filtered Gorenstein ring. We call filtered Λ-module M with a good filtration a *holonomic*, if ${}^*\mathrm{dim}\,\mathrm{gr}M = n - d$, where $n = {}^*\mathrm{dim}\,\mathrm{gr}\Lambda$, $d = \mathrm{id}\Lambda$.

Finally, we will show the basic properties of holonomic modules on filtered Gorenstein rings.

Proposition 15. *Let Λ be a filtered Gorenstein ring, and let* $d = \mathrm{id}\Lambda$. *Then a Λ-module M is holonomic if and only if* $\mathrm{grade}M = d$. *Therefore, any holonomic module is CM*

Proposition 16. *Let Λ be a filtered Gorenstein ring, M a holonomic Λ-module, and N a submodule of M. Then N and M/N are holonomic.*

Proposition 17. *A holonomic module is artinian. Therefore, it is of finite length.*

References

[1] Li Huishi and F. van Oystaeyen, *Zariskian Filtrations*, K-Monographs in Mathematics, 2, (1996)
[2] H.Miyahara and K.Nishida, *Cohen-macaulay modules and holonomic modules on filtered Noetherian rings*, preprint

Graduate School of Mathematical Sciences
Shinshu University
Matsumoto, Nagano 390-8621 JAPAN
E-mail address: miyahara_shinshu@yahoo.co.jp