MAXIMAL ORDERS AND VALUATION RINGS IN NONASSOCIATIVE QUATERNION ALGEBRAS

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ABSTRACT. A *nonassociative quaternion algebra* over a field F is a four-dimensional F-algebra A whose nucleus is a separable quadratic extension field of F. We define the notion of a valuation ring for A, and we also define a value function on A with values from a totally ordered group. We determine the structure of the set on which a value function assumes non-negative values. The main result of this paper states that, given a valuation ring of a quaternion algebra A, there is a value function associated to it if and only if the valuation ring is invariant under proper F-automorphisms of A and is integral over its center. We later restrict our attention to the case when the nucleus is a tamely ramified and defectless extension of F. With this assumption, we determine the precise connection between value functions, valuation rings, and maximal orders in A – the latter in the event F is discretely valued. We give various examples that illustrates the difference between the associative valuation theory and the nonassociative one.

Key Words: Value functions, valuation rings, maximal orders, nonassociative quaternion algebras.

1. INTRODUCTION

A ring will have a multiplicative unit element and, unless the context demands otherwise, will be assumed to be nonassociative. Let A be an algebra over a field F. The *nucleus* $N = N(A)$ of A is the set of elements of A which associate with every pair of elements of A, that is,

$$(ab)c = a(bc)$$

when one of the elements is in N. It is an associative subalgebra of A. The center $Z(A)$ of A is

$$\{z \in N \mid za = az \forall a \in A\}.$$

The algebra A is said to be *simple* in case 0 and A are the only ideals of A. It is called *central simple* if $A \otimes_F L$ is simple for every field extension L of F. It is said to be a *division algebra* if it is not the zero ring and the equations

$$ax = b, \quad ya = b$$

have unique solutions $x, y \in A$ for all $a \neq 0, b \in A$. *We shall always assume that A is a finite dimensional division algebra over F in this paper.* By [7, Theorem 2.1], $Z(A)$ is a field extension of F and A is a central simple algebra over $Z(A)$. Since A is a division algebra, a routine argument shows that N is an associative division algebra. If R is a ring, let

$$U(R) = \{a \in R \mid \ ba = ac = 1 \text{ for some } b, c \in R\}.$$

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Observe that if R is associative, then $U(R)$ is simply the group of multiplicative units of R. If R is an associative ring, then $J(R)$ will denote its Jacobson radical and $\overline{R} = R/J(R)$.

This paper is organized in the following manner. In Section 1 we define the notions of a valuation ring and value functions on arbitrary nonassociative finite dimensional division algebras, and we also state some elementary general results. The rest of the paper, however, is entirely devoted to how the valuation rings and value functions thus defined relate to nonassociative quaternion algebras. In Section 2, we determine the structure of the set on which a value function assumes non-negative values. The main result of this paper is Theorem 14 in Section 3, which states that, given a valuation ring of a quaternion algebra A, there is a value function associated to it if and only if the valuation ring is invariant under proper F-automorphisms of A and is integral over its center. In Section 4, we restrict our attention to the case when the nucleus is a tamely ramified and defectless extension over F. With this assumption, we determine the precise connection between value functions, valuation rings, and maximal orders in A – the latter in the event F is discretely valued. Finally in Section 5, we give various examples that illustrates the difference between the associative valuation theory and the nonassociative one. Also, the examples demonstrate the necessity of certain assumptions made earlier on in the paper. The reader interested in learning more about nonassociative quaternion algebras is referred to articles [3, 10].

If A is associative, recall that a subring B of A is called a (Dubrovin) valuation ring of A if there is an ideal I of B such that:

(a) B/I is simple Artinian,

(b) if $x \in A \setminus B$, then there are $b_1, b_2 \in B$ with $b_1x, xb_2 \in B \setminus I$

(see [1, 5, 9]). Note that since A is finite dimensional over its center, B/I is a PI-ring for any ideal I of B hence, if I is a maximal ideal of B, then B/I must be Artinian. Therefore condition (a) can be replace by the weaker:

(a′) I is a maximal ideal of B.

We therefore make the following definition in the nonassociative setting, leaving out the Artinianness condition.

Definition 1. Let A be a division algebra finite dimensional over its center F. If B is a subring of A and I is a maximal ideal of B such that, if $x \in A \setminus B$, then there are $b_1, b_2 \in B \cap N$ with $b_1x, xb_2 \in B \setminus I$, then we shall call (B, I) a valuation ring pair of A.

If (B, I) a valuation ring pair of A, we shall sometimes simply refer to B as a valuation ring of A if there is no danger of confusion. We set $\overline{B} = B/I$. Observe that, if A were associative, then our definition of a valuation ring above would become that of a Dubrovin valuation ring.

Now let Γ be a totally ordered group, written additively for convenience although it is not assumed to be abelian.

Definition 2. A value function on A with value group Γ is a surjection $w : A \mapsto \Gamma \cup \{\infty\}$ such that for all $a, b \in A$ we have:

(1) $w(a) = \infty$ if and only if $a = 0$,

(2) $w(a + b) \geq \min\{w(a), w(b)\}$,

(3) $w(ab) \geq w(a) + w(b)$,

(4) $w(a^{-1}) = -w(a) \; \forall \, a \in \; U(N)$,
(5) $im(w) = w(U(N)) \cup \{\infty\}$

(cf [5, Definition 2.1]).

The following lemma is now self-evident, or can be proved in the same manner as the statements in [5, Lemma 2.2].

Lemma 3. *Suppose A has a value function w. Then*
(1) $w(ab) = w(a) + w(b), w(ba) = w(b) + w(a) \; \forall \, a \in U(N)$.
(2) $w|_N$ *is a valuation on* N.
(3) *If* $w(a) \neq w(b)$, *then* $w(a \pm b) = \min\{w(a), w(b)\}$.
(4) $B_w = \{a \in A \mid w(a) \geq 0\}$ *is a subring of* A *and* $J_w = \{a \in A \mid w(a) > 0\}$ *is a two-sided ideal of* B_w.

We will denote B_w/J_w by $\overline{B}_w$.

Remark 4. Given a valuation w on an associative division algebra A, it does satisfy [5, Definition 2.1]. In particular, condition (4) of [5, Definition 2.1] is satisfied, that is,

$$im(w) = w(st(w)) \cup \{\infty\}, \text{ where } st(w) = \{s \in U(A) \mid w(s^{-1}) = -w(s)\}.$$

Notice that if w is a valuation then the stabilizer of w, $st(w)$, coincides with $U(N)$ in the associative case. In fact, by [5, Lemma 2.2], a value function in the sense of [5] is a valuation if and only if $st(w) = U(A) \; (= U(N))$.

If A is nonassociative, we split condition (4) of [5, Definition 2.1] into two parts, namely (4) and (5) of Definition 2. As a result, and in view of Lemma 3(2) above, our value functions generalize valuations on associative division algebras, rather than value functions of [5]. Further, the nucleus plays a role synonymous to that played by the stabilizer in the associative case. Consequently, the value group of A, which coincides with $im(st(w))$ in the associative case, is now equal to the value group of N.

Finally, given a value function w on A, just as in the proof of [5, Theorem 2.4], for each $x \in A \setminus B_w \; \exists \; b_1, b_2 \in B \cap N$ such that $b_1 x, x b_2 \in \; B_w \setminus J_w$: choose $t \in U(N)$ such that $w(t) = w(x)$ and set $b_1 = b_2 = t^{-1}$. Hence (B_w, J_w) is a valuation ring pair if and only if J_w is a maximal ideal of B_w.

Proposition 5. *We have the following:*
(1) Γ *is abelian.*
(2) $aB_w a^{-1} = B_w \forall a \in U(N)$.
(3) *One-sided ideals of the form* $aB_w \; (= B_w a), a \in N$, *are actually two-sided, and are totally ordered by inclusion.*

2. Value Functions on Nonassociative Quaternion Algebras

A nonassociative quaternion algebra over F is a four-dimensional F-algebra A, with a unit element, whose nucleus N is a separable quadratic extension field of F. If $x \mapsto \hat{x}$ denotes the F-involution on N, then by [10]

$$A = N \oplus NJ$$

where $Jx = \hat{x}J \; \forall \, x \in N$ and $J^2 = b \in N \setminus F$.

Incidentally, given a four-dimensional F-algebra of the form $A' = N \oplus NJ'$ where $J'x = \hat{x}J' \; \forall \, x \in N$ and $J'^2 = b' \in N$, then one has the following classification: if $b' = 0$, then A' is not simple, since NJ' is a proper ideal; otherwise if $0 \neq b' \in F$, then one gets the usual cyclic F-algebras of degree 2; if $b' \in N \setminus F$, then one obtains the nonassociative quaternion algebras now under discussion - which are always division algebras (see [10]), and $Z(A') = F$.

If A has a value function, then by Lemma 3(2) F is a valued field. On the other hand, suppose (F, V) is a valued field and let A be a finite dimensional associative division algebra with center F. In [8, Theorem] it was shown that the valuation on F extends to a valuation on A if and only if V is indecomposed in each field K such that $F \subseteq K \subseteq A$. In [4, Theorem 2], it was shown that the valuation on F extends to a valuation on A if and only if $A \otimes_F F_h$ is a division algebra, where (F_h, V_h) is the Henselization of (F, V) (see [2] for the definition and properties of Henselization). We now have the following analogous results in this nonassociative setting:

Proposition 6. *Suppose F is a valued field with valuation ring V. Then the following are equivalent:*

(1) *A has a value function w with $F \cap B_w = V$.*
(2) *V is indecomposed in N.*
(3) *$A \otimes_F F_h$ is a division algebra, where (F_h, V_h) is the Henselization of (F, V).*

Example 7. Let $F = \mathbb{Q}$, $N = \mathbb{Q}(i)$, $b = i \in N \setminus F$, and let v be a valuation on N extending the 3-adic valuation on F. If one defines w on A by

$$w(x + yJ) = \min(v(x), v(y)) \; \forall \, x, y \in N,$$

then it is easily seen that w is a value function on A and

$$B_w = \mathbb{Z}[i]_{(3\mathbb{Z}[i])} \oplus \mathbb{Z}[i]_{(3\mathbb{Z}[i])}J \;\text{ and }\; J_w = 3\mathbb{Z}[i]_{(3\mathbb{Z}[i])} \oplus 3\mathbb{Z}[i]_{(3\mathbb{Z}[i])}J.$$

We will see later (Theorem 17(2)) that (B_w, J_w) is actually a valuation ring pair of A. $\square$

For the rest of this section, A will have a value function w defined on it. Then $w|_N$ is a valuation on N, which for now we will also denote by w. Let S be its corresponding valuation ring and let $V = S \cap F$. Then S is the integral closure of V in N, since V is indecomposed in N. We have $S = N \cap B_w$. We will say that w is a *normalized value function* if $w(J) = 0$. Given an arbitrary value function w, we know there is a $t \in U(N)$ such that $w(t) = w(J)$. Since

$$A = N \oplus NJ = N \oplus N(\frac{1}{t}J),$$

upon replacing J by $\frac{1}{t}J$ if necessary, we may and <u>will assume that w is normalized</u> in this section. Since $J^2 = b$, we see that $w(b) \geq 2w(J) = 0$, hence we will always have $b \in S$ in this section.

By [10], there are only two types of F-automorphisms on A: for the first type, the automorphism ϕ is given by

$$\phi(x + yJ) = x + \gamma yJ \;\text{ where }\; \gamma\hat{\gamma} = 1.$$

Such a map fixes N element-wise and is called a *proper automorphism* of A. We will see in Section 3 that valuation rings that are integral and invariant under proper F-automorphisms of A are precisely those arising from value functions on A. On the other hand, we will make no use of automorphisms of the second type, which occur only when $\hat{b} = -b$ and there is a $\gamma \in N$ satisfying $\gamma\hat{\gamma} = -1$. An automorphism of this type is given by $\phi(x + yJ) = \hat{x} + \gamma\hat{y}J$.

Proposition 8. *If ϕ is a proper automorphism of A, then $w(\phi(z)) = w(z)\ \forall\ z \in A$. In particular, $\phi(B_w) = B_w$ and $\phi(J_w) = J_w$.*

Proposition 9. $U(B_w) = \{x + yJ \in B_w \mid x\hat{x} - y\hat{y}b \in U(S)\}$.

Let $T = T_{N/F} : N \mapsto F$ be the usual trace map, i.e., $T(x) = x + \hat{x}$. As an F-linear mapping, it is known that right multiplication by an element $z = x + yJ$ of A has characteristic polynomial

$$c_z(t) = \{t^2 - T(x)t + x\hat{x} - y\hat{y}b\}\{t^2 - T(x)t + x\hat{x} - y\hat{y}\hat{b}\} \in F[t].$$

If we agree to interpret $c_z(x + yJ)$ as

$$\{(x + yJ)^2 - T(x)(x + yJ) + x\hat{x} - y\hat{y}b\}\{(x + yJ)^2 - T(x)(x + yJ) + x\hat{x} - y\hat{y}\hat{b}\},$$

which is an unambiguous expression, then we have $c_z(x + yJ) = 0$. Given a subring R of F, we will say that $z = x + yJ \in A$ is *integral over* R if $c_z(t) \in R[t]$. A subring B of A will be called *integral* if each one of its elements is integral over $B \cap F$. A valuation ring pair (B, I) of A will be called *integral* if B is integral. A subring of A will be called an *order* in A if it contains an F-basis of A. It will be called an *R-order* if it is an integral order containing R and the field of fractions of R is F. If an R-order is maximal among the R-orders of A with respect to inclusion, it will be called a *maximal R-order* (or just a maximal order if the context is clear). Clearly, every R-order is contained in a maximal order. Note that, if an order B containing R is finitely generated over R and $z \in B$ then, as in the associative case, by computing $c_z(t)$ using an F-basis of A contained in B, one readily sees that $c_z(t) \in R[t]$ and hence B is an R-order if F is the field of fractions of R. Conversely, if R is Noetherian and B is an R-order in A, then the proof of [6, Theorem 10.3] shows that B is finitely generated over R: if $\{u_1, u_2, u_3, u_4\} \subseteq B$ is an F-basis for A and $\alpha = \det(T(u_i u_j)) \in F$, then $\alpha \neq 0$ as was pointed out in the paragraph before [3, Proposition 1.4], and B is a submodule of the finitely generated R-module $\alpha^{-1}(Ru_1 + Ru_2 + Ru_3 + Ru_4)$.

Proposition 10. *B_w is a V-order in A.*

We will encounter more V-orders in §4.

Proposition 11. *We have the following:*

(1) *If $w(x + yJ) = \min(w(x), w(y))\ \forall\ x, y \in N$, then $B_w = S \oplus SJ$ and $J_w = J(S) \oplus J(S)J$.*
(2) *If $B_w = S \oplus SJ$, then $w(x+yJ) = \min(w(x), w(y))\ \forall\ x, y \in N$ and $\overline{B}_w = \overline{S} \oplus \overline{S}\ \overline{J}$, where $\overline{J}\overline{s} = \overline{\hat{s}}\ \overline{J}$ and $\overline{J}^2 = \overline{b}$.*
(3) *If $w(b) > 0$, then $w(x + yJ) = \min(w(x), w(y))\ \forall\ x, y \in N$ and (B_w, J_w) is <u>not</u> a valuation ring pair of A.*

Since (B_w, J_w) cannot be a valuation ring pair when $w(b) > 0$, we turn our attention to the case where we may have $w(b) = 0$. To handle the general situation, we will make use of the following notation: by definition of a value function, for each $u \in U(S)$ there is a $\lambda_u \in S \setminus \{0\}$ such that $w(\lambda_u) = w(1 + uJ)$. Let

$$B_u = S \oplus \lambda_u^{-1} S(1 + uJ),$$

a free S-submodule of B_w.

Theorem 12. *With the notation described above, we have the following:*

1. $w(\lambda_{u_1}) \leq w(\lambda_{u_2})$ *if and only if* $B_{u_1} \subseteq B_{u_2}$. *In particular, the set* $\{B_u \mid u \in U(S)\}$ *is linearly ordered by inclusion.*
2. $S \oplus SJ \subseteq B_u \ \forall\, u \in U(S)$, $B_w = \cup_{u \in U(S)} B_u$, *and* $J_w = \cup_{u \in U(S)} [J(S) \oplus \lambda_u^{-1} J(S)(1+uJ)]$.
3. *For each* $u \in U(S)$, B_u *is a subring of* B_w *and* $T(\frac{1}{\lambda_u} S) \subseteq V$.
4. B_w *is finitely generated over* S *if and only if* $B_w = B_u$ *for some* $u \in U(S)$.

3. Valuation Rings in Nonassociative Quaternion Algebras

Let (B, I) be a valuation ring pair of A. In this section, we are going to determine the precise conditions that will guarantee the existence of a value function w on A such that $(B, I) = (B_w, J_w)$.

A subring B of A will be called *invariant* if $\phi(B) = B$ for every proper F-automorphism ϕ of A. A valuation ring pair (B, I) will be called *invariant* if B is invariant.

A valuation ring pair (B, I) of A will be called *normalized* if $J \in B \setminus I$. Without loss of generality, we may assume that (B, I) is normalized: if $J \notin B$, then we know there is a $t \in N$ such that $tJ \in B \setminus I$; in this case, replace J by tJ. If $J \in I$, then $\frac{1}{b} J \notin B$, otherwise we would have $1 = (\frac{1}{b} J)J \in I$. So there is a $t \in N$ such that $\frac{t}{b} J \in B \setminus I$, in which case we replace J by $\frac{t}{b} J$.

Lemma 13. *If* (B, I) *is normalized, integral, and invariant, then*

1. $S = B \cap N$ *is a valuation ring of* N.
2. *If* $u \in U(S)$, *then there is a* $\sigma_u \in S \setminus \{0\}$ *such that* $\dfrac{\sigma_u}{1 - u\hat{u}b}(1 + uJ) \in B \setminus I$.

   Further, for any $t \in N$,
3. $v(t) = v(\sigma_u)$ *if and only if* $\dfrac{t}{1 - u\hat{u}b}(1 + uJ) \in B \setminus I$, *where* v *is a valuation on* N *corresponding to* S.

If (B, I) is normalized, integral, and invariant and if Γ is the value group of the valuation v, we define a map $w : A \mapsto \Gamma \cup \{\infty\}$ by

$$w(x + yJ) = \begin{cases} \infty & \text{if } x + yJ = 0, \\ \min(v(x), v(y)) & \text{if } v(x) \neq v(y), \\ v(x) + v(1 - u\hat{u}b) - v(\sigma_u) & \text{otherwise, where } u = \frac{y}{x}. \end{cases}$$

By Lemma 13, σ_u exists for each $u \in U(S)$ and $v(\sigma_u)$ depends only on u. Hence w is well defined. This w turns out to be a value function corresponding to (B, I) in the following theorem.

In the associative setting, given a Dubrovin valuation ring of a finite-dimensional division algebra, then in [9, Theorem G & Corollary G] we learn that there is a valuation on the division algebra giving rise to the Dubrovin valuation ring if and only if the Dubrovin valuation ring is invariant under inner automorphisms of the division algebra. In [5], certain value functions are defined on central simple algebras. Given a Dubrovin valuation ring of such an algebra, there is one such value function giving rise to the Dubrovin valuation ring if and only if the Dubrovin valuation ring is integral [5, Corollary 2.5]. We have the following analogue of these two results, but here we need both the invariance and the integralness assumptions.

Theorem 14. *Given a valuation ring pair (B, I) of A, there is a value function w such that $(B, I) = (B_w, J_w)$ if and only if (B, I) is integral and invariant.*

Note that the condition is clearly necessary, by Proposition 10 and Proposition 8.

Corollary 15. *Let (B, I) be a valuation ring pair of A that is invariant and integral. Then:*

1. *I is the unique maximal ideal of B such that, if $z \in A \setminus B$, then there are $b_1, b_2 \in B \cap N$ with $b_1 z, z b_2 \in B \setminus I$.*
2. *$\phi(I) = I$ for every proper F-automorphism ϕ.*
3. *$U(B) = \{x + yJ \in B \mid x\hat{x} - y\hat{y}b \in U(S)\}$.*
4. *If in addition (B, I) is normalized, then $B = \cup_{u \in U(S)}[S \oplus \frac{\sigma_u}{1 - u\hat{u}b} S(1 + uJ)]$ and $I = \cup_{u \in U(S)}[J(S) \oplus \frac{\sigma_u}{1 - u\hat{u}b} J(S)(1 + uJ)]$.*

By Remark 4, we immediately have:

Corollary 16. *Given a value function w on A, if J_w is a maximal ideal of B_w, then it is the unique maximal ideal of B_w satisfying the condition that, if $z \in A \setminus B_w$, then there are $b_1, b_2 \in N \cap B_w$ with $b_1 z, z b_2 \in B_w \setminus J_w$.*

4. The case when N/F is Tamely Ramified and Defectless

All undefined terminology used in this section relating to valuations on fields can be found in [2]. Let us once and for all fix some notation for this section. The quaternion algebra A will have a normalized value function w defined on it. Let $S = N \cap B_w$, a valuation ring of N, and let $V = S \cap F$, a valuation ring of F. We also know that V is indecomposed in N, and so S is the integral closure of V in N.

Let v be a valuation on N with valuation ring S. Let e (resp. f) be the ramification index (resp. residue degree) of S over F. In our case, it is well known that

$$ef \leq 2.$$

If we have equality $ef = 2$, then we say N/F is *defectless*. We call N/F *tamely ramified* if the characteristic of $\overline{V}$ does not divide e and $\overline{S}$ is separable over $\overline{V}$. When N/F is tamely ramified and defectless then, in our situation, there are exactly two cases: N/F is an *inertial* extension if $f = 2$; it is *tamely and totally ramified* when $e = 2$. In the latter case, the characteristic of $\overline{V}$ is not 2, of course.

In this section, we will assume that N/F is tamely ramified and defectless. Under this assumption, B_w has a particulary desirable form and we will determine precise conditions for (B_w, J_w) to be a valuation ring pair of A. This section also shows that there are abundant examples of valuation ring pairs of A when N/F is tamely ramified and defectless.

Theorem 17. *Suppose N/F is tamely ramified and defectless. Then:*

(1) $B_w = S \oplus SJ$.
(2) *If N/F is inertial, then (B_w, J_w) is a valuation ring pair of A if and only if $w(b)=0$. When this occurs, J_w is the unique maximal ideal of B_w and $\overline{B}_w$ is a central simple $\overline{V}$-algebra, which is a division algebra unless $\overline{b}$ is a norm from $\overline{S}$ to $\overline{V}$.*
(3) *If N/F is tamely and totally ramified, then (B_w, J_w) is a valuation ring pair of A if and only if $\overline{b}$ is not a square in $\overline{S}$. When this occurs, then J_w is the unique maximal ideal of B_w and $\overline{B}_w$ is a separable quadratic extension field of $\overline{V}$.*

For the rest of this section, we shall assume that V is a DVR, hence N/F is defectless by [2, Corollary 18.7]. Let $J(S) = \pi S$, and let v be the $J(S)$-adic valuation on N.

The set $\{t \in N \mid v(t\hat{t}b - 1) \geq 0\}$ is clearly non-empty. Let $k \in \{0, 1\}$ be the largest integer such that there is a $u \in N$ with $v(u\hat{u}b - 1) \geq 2k$. If we assume N/F is a tamely and totally ramified extension, then by [3, Proposition 2.5], if $k = 0$, then $B = S \oplus S(1 + uJ)$ is the unique maximal V-order containing S, while if $k = 1$, then there are exactly two maximal orders containing S, namely $B_1 = S \oplus \pi^{-1}S(1 + uJ)$ and $B_2 = S \oplus \pi^{-1}S(1 + u\frac{\hat{\pi}}{\pi}J)$.

Corollary 18. *Suppose V is a DVR and N/F is tamely ramified. Then we have*

(1) *If N/F is inertial, then (B_w, J_w) is a valuation ring pair of A if and only if B_w is a maximal order and $w(b) = 0$.*
(2) *Otherwise if N/F is tamely and totally ramified, then:*
    (a) *If $w(b) = 0$, then B_w is the intersection of (at most two) maximal orders.*
    (b) *(B_w, J_w) is a valuation ring pair of A if and only if B_w is a maximal order and $\overline{b}$ is not a square in $\overline{S}$.*

5. Examples

Example 19. *A subring B_1 of A that is invariant but not integral, a subring B_2 of A that is integral but not invariant, and a valuation ring pair (B, I) that is neither integral nor invariant.*

Let $F = \mathbb{Q}, V = \mathbb{Z}_{(5)}, N = \mathbb{Q}(i)$. Then $J(V)$ splits completely in N. Let $W = \mathbb{Z}[i]_{(2+i)}$, one of the two extensions of V to N. Let S be the integral closure of V in N, that is, $S = \mathbb{Z}[i]_{(2+i)} \cap \mathbb{Z}[i]_{(2-i)}$.

Then $B_1 = W$ is an invariant subring, but not integral. (If A was an associative division algebra, then any subring that is invariant under F-automorphisms of A is integral.)

Now let $b = i \in S \setminus F$. Then $B_2 = S \oplus SJ$ is integral, but not invariant under the proper automorphism $\phi(x + yJ) = x + y(\frac{2-i}{2+i})J$.

Therefore, in general, being integral and being invariant are mutually independent phenomena.

Finally, let $b = i \in W \setminus F$ and let $B = W \oplus WJ$. Note that B is not invariant under the proper automorphism $\phi(x+yJ) = x+y(\frac{2-i}{2+i})J$. Let $I = J(W) \oplus J(W)J$. Then (B, I) is a valuation ring pair of A. It is neither integral nor invariant.

Therefore, unlike in the associative setting, valuation rings over a DVR need not be maximal orders.

Example 20. *An invariant maximal order over a DVR that is not a valuation ring.*

Let $F = \mathbb{Q}$, $V = \mathbb{Z}_{(3)}$, and $N = \mathbb{Q}(i)$. Then $S = \mathbb{Z}[i]_{(3\mathbb{Z}[i])}$ is a valuation ring of N lying over V which is inertial over F. Let $b = 3 + 9i \in S \setminus F$ and let $B = S \oplus SJ$. Then B is a maximal order but, for any maximal ideal I of B, (B, I) is not a valuation ring pair of A.

Therefore, unlike in the associative case, maximal orders over a DVR need not be valuation rings.

The condition that $w(b) = 0$ is necessary in part 2(a) of Corollary 18, as the following example shows. Keeping the notation of Section 4, we have:

Example 21. *A B_w which is not an intersection of maximal orders, but V is a DVR and N/F is tamely and totally ramified.*

Suppose V is a DVR and N/F is tamely and totally ramified. Let $J(S) = \pi S$ and let v be the $J(S)$-adic valuation on N. Let $b = (\hat{\pi}\pi)\pi \in S \setminus F$. If $w(x + yJ) = \min(v(x), v(y))\ \forall\ x, y \in N$, it is easily seen that w is a value function on A and $B_w = S \oplus SJ$. But B_w is not the intersection of maximal orders in A.

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