### LONG EXACT SEQUENCES COMING FROM TRIANGLES

### AMNON NEEMAN

ABSTRACT. Suppose we are given a homological functor, from a triangulated to an abelian category. It takes triangles to long exact sequences. It turns out that not every long exact sequence can occur; there are restrictions.

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### 0. INTRODUCTION

Suppose  $\mathcal{A}$  is a sufficiently nice abelian category, so that it has a derived category  $\mathbf{D}(\mathcal{A})$ . This will happen, for example, if  $\mathcal{A}$  has enough projectives, or if it has enough injectives; for details see Hartshorne [2] or Verdier [3, 4]. Given a distinguished triangle in  $\mathbf{D}(\mathcal{A})$ 

 $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X ,$ 

we can form the long exact sequence in cohomology. We deduce in  $\mathcal{A}$  a long exact sequence

$$\cdots \longrightarrow H^{-1}(Z) \longrightarrow H^0(X) \longrightarrow H^0(Y) \longrightarrow H^0(Z) \longrightarrow H^1(X) \longrightarrow \cdots$$

We can wonder what long exact sequences can be obtained this way.

It is clear that any sequence of length four is obtainable. If we have an exact sequence in  $\mathcal A$ 

 $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow 0$ 

then it is very easy to deal with it; consider B and C as objects of  $\mathcal{A} \subset \mathbf{D}(\mathcal{A})$ , and complete the morphism  $B \longrightarrow C$  into a triangle in  $\mathbf{D}(\mathcal{A})$ . The reader can easily check that the long exact sequence, obtained from the functor H applied to this triangle, is nothing other than

 $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow 0 \ .$ 

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The remarkable fact, which I do not fully understand, is what comes next. It turns out that not all sequences of length five

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow E \longrightarrow 0$$

are the long exact sequences of triangles. Any exact sequence of length five defines a class in  $\operatorname{Ext}^3_{\mathcal{A}}(E, A)$ , or equivalently a morphism  $E \longrightarrow \Sigma^3 A$  in  $\mathbf{D}(\mathcal{A})$ . It turns out that the sequence will be the long exact sequence of a triangle if and only if this class in  $\operatorname{Ext}^3_{\mathcal{A}}(E, A)$  vanishes. In this article I will only prove the necessity, but the sufficiency is easy enough.

More is true. Given any distinguished triangle  $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$  in the derived category  $\mathbf{D}(\mathcal{A})$ , we can look at its long exact sequence in cohomology. It can be chopped into bits of length five, for example

$$0 \longrightarrow K \longrightarrow H^0(X) \xrightarrow{H^0(u)} H^0(Y) \xrightarrow{H^0(v)} H^0(Z) \longrightarrow Q \longrightarrow 0 ,$$

where K is the kernel of  $H^0(u) : H^0(X) \longrightarrow H^0(Y)$ , while Q is cokernel of  $H^0(v) : H^0(Y) \longrightarrow H^0(Z)$ . We will prove that, for every such length-five bit, the corresponding element in  $\operatorname{Ext}^3_{\mathcal{A}}(Q, K)$  vanishes. I know that this vanishing is necessary, but have no idea whether it suffices. In other words, I do not know whether it characterizes the long exact sequences coming from triangles in  $\mathbf{D}(\mathcal{A})$ .

In the proof we will be slightly more general. We will start with an arbitrary triangulated category  $\mathfrak{T}$ , possessing a *t*-structure; the reader is referred to Beilinson, Bernstein and Deligne [1] for the definitions and elementary properties of *t*-structures. We will let  $\mathcal{A}$  be the heart of the *t*-structure. We will assume that  $\mathfrak{T}$  is nice enough so that the inclusion  $\mathcal{A} \longrightarrow \mathfrak{T}$  factors through a triangulated functor  $F : \mathbf{D}^b(\mathcal{A}) \longrightarrow \mathfrak{T}$ ; this is a very weak hypothesis, usually satisfied. We recall that, for any pair of objects  $\mathcal{A}, \mathcal{B} \in \mathcal{A}$ , we have

$$\operatorname{Hom}_{\mathbf{D}^{b}(\mathcal{A})}(A,B) = \operatorname{Hom}_{\mathfrak{T}}(A,B), \qquad \operatorname{Hom}_{\mathbf{D}^{b}(\mathcal{A})}(A,\Sigma B) = \operatorname{Hom}_{\mathfrak{T}}(A,\Sigma B).$$

The reason for the first equality is that  $\mathcal{A}$  embeds fully faithfully in both  $\mathbf{D}^{b}(\mathcal{A})$  and  $\mathcal{T}$ , and the second equality is because both groups classify extensions  $0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0$ in  $\mathcal{A}$ . But it is perfectly possible for a non-zero morphism  $\alpha : \mathcal{A} \longrightarrow \Sigma^{n} B$ , in the category  $\mathbf{D}^{b}(\mathcal{A})$ , to map to zero in  $\mathcal{T}$ ; all we learn, from the discussion above, is that this can only happen if  $n \geq 2$ .

What we will prove, in the generality of triangulated categories  $\mathcal{T}$  with *t*-structures, is the following. Given any triangle  $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$  in the category  $\mathcal{T}$ , we can still look at its long exact sequence in cohomology. It can still be chopped into bits of length five, for example

$$0 \longrightarrow K \longrightarrow H^0(X) \xrightarrow{H^0(u)} H^0(Y) \xrightarrow{H^0(v)} H^0(Z) \longrightarrow Q \longrightarrow 0 .$$

Every such length-five bit corresponds to an element in  $\operatorname{Ext}^3_{\mathcal{A}}(Q, K)$ , that is to a morphism  $\alpha : Q \longrightarrow \Sigma^3 K$  in  $\mathbf{D}^b(\mathcal{A})$ . We will prove that the functor  $F : \mathbf{D}^b(\mathcal{A}) \longrightarrow \mathfrak{T}$  must take  $\alpha$  to zero.

## 1. The proof

Before all else we need to fix our conventions for this section.

**Notation 1.1.** Let  $\mathcal{T}$  be a triangulated category with a *t*-structure. Let  $\mathcal{A}$  be the heart of this *t*-structure. Assume that the category  $\mathcal{T}$  is "natural" enough so that the embedding of  $\mathcal{A}$  into  $\mathcal{T}$  extends to a triangulated functor  $F : \mathbf{D}^b(\mathcal{A}) \longrightarrow \mathcal{T}$ . We fix these assumptions throughout the section.

Let us also fix the notation that  $H: \mathcal{T} \longrightarrow \mathcal{A}$  will be the homological functor sending an object  $X \in \mathcal{T}$  to the truncation  $H(X) = (X^{\leq 0})^{\geq 0}$ . We will let  $H^n(X) = H(\Sigma^n X)$ .

With these conventions, we are ready to state and prove our main observation:

**Lemma 1.2.** Let  $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$  be a triangle in  $\mathfrak{T}$ , and suppose that

- (1) X and Y lie in  $\mathfrak{T}^{\leq 1} \cap \mathfrak{T}^{\geq 0}$ .
- (2) Z lies in  $\mathcal{A} = \mathfrak{T}^{\leq 0} \cap \mathfrak{T}^{\geq 0}$ .

This implies that the functor H sends the triangle to the long exact sequence

$$(*) \qquad 0 \longrightarrow H^0(X) \longrightarrow H^0(Y) \longrightarrow H^0(Z) \longrightarrow H^1(X) \longrightarrow H^1(Y) \longrightarrow 0$$

with all the other terms vanishing. In the abelian category  $\mathcal{A}$ , this 5-term exact sequence defines a class in  $\operatorname{Ext}^{3}_{\mathcal{A}}(H^{1}(Y), H^{0}(X))$ . This class can also be viewed as a morphism  $\alpha : H^{1}(Y) \longrightarrow \Sigma^{3} H^{0}(X)$ , in the derived category  $\mathbf{D}^{b}(\mathcal{A})$ .

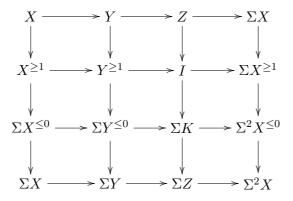
We assert that, under the functor  $F: \mathbf{D}^b(\mathcal{A}) \longrightarrow \mathfrak{T}$ , the image of  $\alpha$  vanishes.

*Proof.* Consider the commutative square

$$\begin{array}{c} X \longrightarrow Y \\ \downarrow & \downarrow \\ X^{\geq 1} \longrightarrow Y^{\geq 1} \end{array}$$

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We may complete to a  $3 \times 3$  diagram, were the rows and columns are triangles



and the proof is by studying this  $3 \times 3$  diagram. In the second row, we have that  $\Sigma(X^{\geq 1}) = H^1(X)$  and  $\Sigma(Y^{\geq 1}) = H^1(Y)$  are both in  $\mathcal{A} \subset \mathcal{T}$ , and that the morphism  $\Sigma(X^{\geq 1}) \longrightarrow \Sigma(Y^{\geq 1})$  is surjective; it is the morphism  $H^1(X) \longrightarrow H^1(Y)$  in the long exact sequence (\*) of the lemma. The second row reduces to the short exact sequence in  $\mathcal{A}$ 

$$0 \longrightarrow I \longrightarrow H^1(X) \longrightarrow H^1(Y) \longrightarrow 0 ,$$

and the map  $Y^{\geq 1} \longrightarrow I$  is the image, under the functor  $F : \mathbf{D}^{b}(\mathcal{A}) \longrightarrow \mathfrak{T}$ , of the morphism in  $\mathbf{D}^{b}(\mathcal{A})$  defining the extension  $0 \longrightarrow I \longrightarrow H^{1}(X) \longrightarrow H^{1}(Y) \longrightarrow 0$ .

So much for the second row. Now look at the commutative diagram

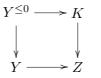
If we apply to it the functor H, we discover the diagram

$$\begin{array}{c} H^0(Z) \longrightarrow H^1(X) \\ & \downarrow \\ & \parallel \\ H^0(I) \longrightarrow H^1(X) \longrightarrow H^1(Y) \end{array}$$

Both Z and I lie in the heart  $\mathcal{A}$ , and the diagram above identifies for us the map  $Z \longrightarrow I$ as the factorization of the morphism from  $Z = H^0(Z)$  to  $H^1(X)$  through the kernel of  $H^1(X) \longrightarrow H^1(Y)$ , which is the image of  $Z \longrightarrow H^1(X)$ . Now the column

 $K \longrightarrow Z \longrightarrow I \longrightarrow \Sigma K$ 

is a triangle, which reduces to the short exact sequence  $0 \longrightarrow K \longrightarrow Z \longrightarrow I \longrightarrow 0$ in  $\mathcal{A} \subset \mathcal{T}$ . We also learn that the map  $I \longrightarrow \Sigma K$  is the image, under the functor  $F: \mathbf{D}^{b}(\mathcal{A}) \longrightarrow \mathcal{T}$ , of the morphism in  $\mathbf{D}^{b}(\mathcal{A})$  corresponding to the extension. Next consider the commutative square



If we apply the functor H we learn that the map  $Y^{\leq 0} \longrightarrow K$ , which is a map between objects in  $\mathcal{A}$ , is just the factorization through K of the morphism  $H^0(Y) \longrightarrow H^0(Z) = Z$ . The triangle

$$X^{\leq 0} \longrightarrow Y^{\leq 0} \longrightarrow K \longrightarrow \Sigma X^{\leq 0}$$

is therefore nothing fancy; it is simply the exact sequence  $0 \longrightarrow H^0(X) \longrightarrow H^0(Y) \longrightarrow K \longrightarrow 0$  in  $\mathcal{A}$ . Moreover, the map  $K \longrightarrow \Sigma X^{\leq 0}$  is just exactly the image, under the functor  $F: \mathbf{D}^b(\mathcal{A}) \longrightarrow \mathfrak{T}$ , of the morphism in  $\mathbf{D}^b(\mathcal{A})$  corresponding to the extension.

What we have learned so far is that three of the six triangles, in our  $3 \times 3$  diagram, amount to short exact sequences in A

$$0 \longrightarrow H^{0}(X) \longrightarrow H^{0}(Y) \longrightarrow K \longrightarrow 0$$
  
$$0 \longrightarrow K \longrightarrow Z \longrightarrow I \longrightarrow 0$$
  
$$0 \longrightarrow I \longrightarrow H^{1}(X) \longrightarrow H^{1}(Y) \longrightarrow 0$$

Moreover, the differentials of these triangles are the classes of the three extensions, and are also part of our  $3 \times 3$  diagram. The composite of these three differentials is the map

$$Y^{\geq 1} \longrightarrow I$$

$$\downarrow$$

$$\Sigma K \longrightarrow \Sigma^2 X^{\leq 0}$$

which the reader will find in our diagram. The commutativity of

coupled with the vanishing of  $Y^{\geq 1} \longrightarrow I \longrightarrow \Sigma X^{\geq 1}$ , tells us that this composite vanishes. In the category  $\mathcal{T}$  the three extensions compose to zero.

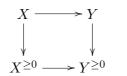
**Proposition 1.3.** Let the conventions be as in Notation 1.1. Suppose  $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$  is a triangle in  $\mathfrak{T}$ . Complete  $H^0(X) \longrightarrow H^0(Y) \longrightarrow H^0(Z)$  to an exact sequence

$$0 \longrightarrow K \longrightarrow H^0(X) \xrightarrow{H^0(u)} H^0(Y) \xrightarrow{H^0(v)} H^0(Z) \longrightarrow Q \longrightarrow 0 ,$$

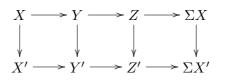
where K must be the kernel of  $H^0(u) : H^0(X) \longrightarrow H^0(Y)$ , while Q is forced to be the cokernel of  $H^0(v) : H^0(Y) \longrightarrow H^0(Z)$ . The sequence defines an element in  $\operatorname{Ext}^3_{\mathcal{A}}(Q, K)$ , or equivalently a morphism  $\alpha : Q \longrightarrow \Sigma^3 K$  in  $\mathbf{D}^b(\mathcal{A})$ .

We assert that the functor  $F: \mathbf{D}^b(\mathcal{A}) \longrightarrow \mathfrak{T}$  takes  $\alpha$  to zero.

*Proof.* Consider the commutative square



It may be extended to a morphism of triangles, which we will write



That is  $X' = X^{\geq 0}$  and  $Y' = Y^{\geq 0}$ . We have

(1) X' and Y' belong to  $\mathcal{T}^{\geq 0}$ , while Z' belongs to  $\mathcal{T}^{\geq -1}$ .

(2) The three maps

$$H^0(X) \longrightarrow H^0(X'), \qquad H^0(Y) \longrightarrow H^0(Y'), \qquad H^0(Z) \longrightarrow H^0(Z')$$

are all isomorphisms. For  $X' = X^{\geq 0}$  and  $Y' = Y^{\geq 0}$  this is obvious, by the definition of the functor H in terms of truncations. For Z' consider the commutative diagram with exact rows

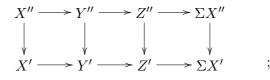
$$\begin{array}{cccc} H^0(X) & \longrightarrow & H^0(Y) & \longrightarrow & H^0(Z) & \longrightarrow & H^1(X) & \longrightarrow & H^1(Y) \\ \rho & & \sigma & & \tau & & \Sigma \rho & & \Sigma \sigma & \\ H^0(X') & \longrightarrow & H^0(Y') & \longrightarrow & H^0(Z') & \longrightarrow & H^1(X') & \longrightarrow & H^1(Y') \end{array}$$

We know that  $\rho$ ,  $\sigma$ ,  $\Sigma \rho$  and  $\Sigma \sigma$  are isomorphisms. The 5-lemma permits us to conclude that so is  $\tau$ .

Now apply the dual construction; consider the commutative square

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We can extend to a morphism of triangles



as before, this means  $Y'' = (Y')^{\leq 0}$  and  $Z'' = (Z')^{\leq 0}$ . We leave it as an exercise to the reader to check that

- (1) X'' belongs to  $\mathfrak{T}^{\leq 1} \cap \mathfrak{T}^{\geq 0}$ , and Y'' belong to  $\mathcal{A} = \mathfrak{T}^{\leq 0} \cap \mathfrak{T}^{\geq 0}$ , while Z'' belongs to  $\mathfrak{T}^{\leq 0} \cap \mathfrak{T}^{\geq -1}$ .
- (2) The three maps

$$H^0(X'') \longrightarrow H^0(X'), \qquad H^0(Y'') \longrightarrow H^0(Y'), \qquad H^0(Z'') \longrightarrow H^0(Z')$$

are all isomorphisms.

The proposition now follows from Lemma 1.2, applied to the triangle  $\Sigma^{-1}Z'' \longrightarrow X'' \longrightarrow Y'' \longrightarrow Z''$ .

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CENTRE FOR MATHEMATICS AND ITS APPLICATIONS, MATHEMATICAL SCIENCES INSTITUTE, JOHN DEDMAN BUILDING, THE AUSTRALIAN NATIONAL UNIVERSITY, CANBERRA, ACT 0200, AUSTRALIA *E-mail address*: Amnon.Neeman@anu.edu.au