LINEARITY DEFECT OF GRADED MODULES OVER KOSZUL ALGEBRAS

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Abstract. We study the *regularities* and *linearity defects* of graded modules over a Koszul algebra. These invariants are closely related to Koszul duality. We mainly consider a Koszul *commutative* algebra A and its dual $A^!$. We also introduce results on monomial ideals in an exterior algebra $E := \bigwedge \langle y_1, \dots, y_n \rangle$, which is a primary example of a Koszul algebra. The linearity defects of monomial ideals in E have combinatorial interest, and the results in this part belong to joint work with R. Okazaki.

1. Introduction

Let $A = \bigoplus_{i \in \mathbb{N}} A_i$ be a Koszul algebra over a field $K := A_0$, and ${}^*\mathrm{mod}\, A$ the category of finitely generated graded left A-modules. The Koszul duality is a certain derived equivalence between A and its Koszul dual algebra $A^! := \mathrm{Ext}^\bullet_A(K, K)$.

For $M \in {}^*\mathrm{mod}\, A$, set $\beta_{i,j}(M) := \dim_K \underline{\mathrm{Ext}}^i_A(M, K)_{-j}$. If $P_\bullet : \cdots \to P_1 \to P_0 \to M \to 0$ is a minimal graded free resolution of M, then $P_i \cong \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{i,j}(M)}$. We call

$$\mathrm{reg}_A(M) := \sup\{\, j - i \mid i \in \mathbb{N}, j \in \mathbb{Z} \text{ with } \beta_{i,j}(M) \neq 0 \,\}$$

the *(Castelnuovo-Mumford) regularity* of M. When A is a polynomial ring, $\mathrm{reg}_A(M)$ has been studied by many authors from both geometric and computational interest. Even for a general Koszul algebra A, $\mathrm{reg}_A(M)$ is still an interesting invariant closely related to Koszul duality (see Theorem 5 below).

Let $P_\bullet$ be a minimal graded free resolution of $M \in {}^*\mathrm{mod}\, A$. The *linear part* $\mathrm{lin}(P_\bullet)$ of $P_\bullet$ is the chain complex such that $\mathrm{lin}(P_\bullet)_i = P_i$ for all i and its differential maps are given by erasing all the entries of degree ≥ 2 from the matrices representing the differentials of $P_\bullet$. According to Herzog-Iyengar [7], we call

$$\mathrm{ld}_A(M) := \sup\{\, i \mid H_i(\mathrm{lin}(P_\bullet)) \neq 0 \,\}$$

the *linearity defect* of M. This invariant is related to the regularity via Koszul duality (see Theorem 7 below).

In §4, we study the regularities and linearity defects of modules over a Koszul commutative algebra A or its dual $A^!$. Even in this case, it can occur $\mathrm{ld}_A(M) = \infty$ for some $M \in {}^*\mathrm{mod}\, A$, while Avramov-Eisenbud [1] showed that $\mathrm{reg}_A(M) < \infty$ for all $M \in {}^*\mathrm{mod}\, A$. On the other hand, Herzog-Iyengar [7] proved that if A is complete intersection or Golod then $\mathrm{ld}_A(M) < \infty$ for all $M \in {}^*\mathrm{mod}\, A$. Initiated by these results, we will show the following. Since the results in §4 have not been written elsewhere, we will also give precise proofs.

The detailed versions of this paper will be submitted for publication elsewhere.

Theorem A. *For a Koszul commutative algebra A and $N \in {}^*\mathrm{mod}\, A^!$, we have;*
(1) If $\mathrm{reg}_{A^!}(N) < \infty$, then $\mathrm{ld}_{A^!}(N) < \infty$.
(2) If A is a complete intersection, then $\mathrm{reg}_{A^!}(N) < \infty$ for all $N \in {}^*\mathrm{mod}\, A^!$.
(3) If A is Golod and N has a finite presentation, then $\mathrm{reg}_{A^!}(N) < \infty$.

Let $E := \bigwedge \langle y_1, \dots, y_n \rangle$ be the exterior algebra. It is a primary example of a Koszul algebra. Eisenbud et. al [5] showed that $\mathrm{ld}_E(N) < \infty$ for all $N \in {}^*\mathrm{mod}\, E$ (now this is a special case of Theorem A). If $n \geq 2$, then $\sup\{\, \mathrm{ld}_E(N) \mid N \in {}^*\mathrm{mod}\, E \,\} = \infty$. On the other hand, we can prove that there is a uniform bound C such that

$$\mathrm{ld}_E(E/J) < C \qquad \text{for all graded ideals } J \text{ of } E.$$

While we know little about the actual value of C, we can treat $\mathrm{ld}_E(E/J)$ very precisely if $J \subset E$ is a *monomial* ideal. In §5, we collect results in this direction. Most results in this section belong to joint work with R. Okazaki of Osaka university.

For a simplicial complex $\Delta \subset 2^{[n]}$ (here $[n] := \{1, 2, \dots, n\}$), set $J_\Delta := (\, \prod_{i \in F} y_i \mid F \subset [n], F \not\in \Delta \,)$ to be a monomial ideal of E. Note that any monomial ideal of E is of the form J_Δ for some Δ. Recently, J_Δ has become an important tool of Combinatorial Commutative Algebra.

Theorem B. (Okazaki-Y [10]) *With the above notation, we have the following.*
(1) $\mathrm{ld}_E(E/J_\Delta) \leq \max\{1, n-2\}$.
(2) $\mathrm{ld}_E(J_\Delta)$ *only depends on the topology of the geometric realization $|\Delta^\vee|$ of the Alexander dual $\Delta^\vee$ of Δ (and $\mathrm{char}(K)$).*
(3) *If $n \geq 4$, we have $\mathrm{ld}(E/J_\Delta) = n - 2 \iff \Delta$ is an n-gon.*

2. Koszul Algebras and Koszul Duality

Let $A = \bigoplus_{i \in \mathbb{N}} A_i$ be a graded algebra over a field $K := A_0$ with $\dim_K A_i < \infty$ for all $i \in \mathbb{N}$, ${}^*\mathrm{Mod}\, A$ the category of graded left A-modules, and ${}^*\mathrm{mod}\, A$ the full subcategory of ${}^*\mathrm{Mod}\, A$ consisting of finitely generated modules. We say $M = \bigoplus_{i \in \mathbb{Z}} M_i \in {}^*\mathrm{Mod}\, A$ is *quasi-finite*, if $\dim_k M_i < \infty$ for all i and $M_i = 0$ for $i \ll 0$. If $M \in {}^*\mathrm{mod}\, A$, then it is clearly quasi-finite. We denote the full subcategory of ${}^*\mathrm{Mod}\, A$ consisting of quasi-finite modules by $\mathrm{qf}\, A$. Clearly, $\mathrm{qf}\, A$ is an abelian category with enough projectives. For $M \in {}^*\mathrm{Mod}\, A$ and $j \in \mathbb{Z}$, $M(j)$ denotes the shifted module of M with $M(j)_i = M_{i+j}$. For $M, N \in {}^*\mathrm{Mod}\, A$, set $\underline{\mathrm{Hom}}_A(M, N) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{{}^*\mathrm{Mod}\, A}(M, N(i))$ to be a graded k-vector space with $\underline{\mathrm{Hom}}_A(M, N)_i = \mathrm{Hom}_{{}^*\mathrm{Mod}\, A}(M, N(i))$. Similarly, we also define $\underline{\mathrm{Ext}}^i_A(M, N)$.

Set $\mathfrak{m} := \bigoplus_{i > 0} A_i$, and regard $K = A/\mathfrak{m}$ as a graded left A-module. For $M \in \mathrm{qf}\, A$, $i \in \mathbb{N}$ and $j \in \mathbb{Z}$, set

$$\beta_{i,j}(M) := \dim_K \underline{\mathrm{Ext}}^i_A(M, K)_{-j}.$$

Note that $M \in \mathrm{qf}\, A$ has a minimal graded free resolution $P_\bullet : \cdots \to P_1 \to P_0 \to M \to 0$, which is unique up to isomorphism. In this situation, we have $P_i \cong \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{i,j}(M)}$. It is easy to see that $\beta_{i,j}(M) < \infty$ for all i, j. But, if A is not left noetherian, then $\beta_i(M) := \sum_{j \in \mathbb{Z}} \beta_{i\,j}(M)$ can be infinite even for $M \in {}^*\mathrm{mod}\, A$.

We say A is *Koszul*, if $\beta_{i,j}(K) \neq 0$ implies $i = j$, in other words, the left A-module K has a graded free resolution of the form

$$\cdots \longrightarrow A(-i)^{\oplus \beta_i} \longrightarrow \cdots \longrightarrow A(-2)^{\oplus \beta_2} \longrightarrow A(-1)^{\oplus \beta_1} \longrightarrow A \longrightarrow K \to 0.$$

Even if we regard K as a right A-module, we get an equivalent definition.

The polynomial ring $K[x_1, \dots, x_n]$ and the exterior algebra $\bigwedge\langle y_1, \dots, y_n\rangle$ are primary examples of Koszul algebras. Of course, there are many other important Koszul algebras. In the noncommutative case, many Koszul algebras are not noetherian.

Koszul duality is a derived equivalence between a Koszul algebra A and its dual $A^!$. A standard reference of this subject is Beilinson et.al [2]. But, in this paper, we follow the convention of Mori [9].

Recall that Yoneda product makes $A^! := \bigoplus_{i\in\mathbb{N}} \underline{\operatorname{Ext}}^i_A(K,K)$ a graded K-algebra. If A is Koszul, then so is $A^!$ and we have $(A^!)^! \cong A$. The Koszul dual of the polynomial ring $S := K[x_1, \dots, x_n]$ is the exterior algebra $E := \bigwedge\langle y_1, \dots, y_n\rangle$. In this case, since S is regular and noetherian, the Koszul duality is very simple. It states an equivalence $\mathcal{D}^b({}^*\mathrm{mod}\, S) \cong \mathcal{D}^b({}^*\mathrm{mod}\, E)$ which is sometimes called *Bernstein-Gel'fand-Gel'fand correspondence* (*BGG correspondence* for short). In the general case, the description of the Koszul duality is slightly technical.

Let $\mathcal{C}(\mathrm{qf}\, A)$ be the homotopy category of cochain complexes in $\mathrm{qf}\, A$, and $\mathcal{C}^{\uparrow}(\mathrm{qf}\, A)$ its full subcategory consisting of complexes $X^\bullet$ satisfying

$$X^i_j = 0 \quad \text{for } i \gg 0 \text{ or } i+j \ll 0.$$

We denote for $\mathcal{D}^{\uparrow}(\mathrm{qf}\, A)$ the localization of $\mathcal{C}^{\uparrow}(\mathrm{qf}\, A)$ at quasi-isomorphisms.

We denote V^* for the dual space of a K-vector space V. Note that if $M \in {}^*\mathrm{Mod}\, A$ then $M^* := \bigoplus_{i\in\mathbb{Z}}(M_{-i})^*$ is a graded right A-module. And we fix a basis $\{x_\lambda\}$ of A_1 and its dual basis $\{y_\lambda\}$ of $(A_1)^* \,(= (A^!)_1)$. Let $(X^\bullet, \partial) \in \mathcal{C}^{\uparrow}(\mathrm{qf}\, A)$. In this notation, we define the contravariant functor $F_A : \mathcal{C}^{\uparrow}(\mathrm{qf}\, A) \to \mathcal{C}^{\uparrow}(\mathrm{qf}\, A^!)$ as follows.

$$F_A(X^\bullet)^p_q = \bigoplus A^!_{q+j} \otimes_K (X^{j-p}_{-j})^*$$

with the differential $d = d' + d''$ given by

$$d' : A^!_{q+j} \otimes_K (X^{j-p}_{-j})^* \ni a \otimes m \longmapsto (-1)^p \sum y_\lambda a \otimes m x_\lambda \in A^!_{q+j+1} \otimes_K (X^{j-p}_{-j-1})^*$$

and

$$d'' : A^!_{q+j} \otimes_K (X^{j-p}_{-j})^* \ni a \otimes m \longmapsto a \otimes \partial^*(m) \in A^!_{q+j} \otimes_K (X^{j-p-1}_{-j})^*.$$

The contravariant functor $F_{A^!} : \mathcal{C}^{\uparrow}(\mathrm{qf}\, A^!) \to \mathcal{C}^{\uparrow}(\mathrm{qf}\, A)$ is given by the similar way. They induce the contravariant functors $\mathcal{F}_A : \mathcal{D}^{\uparrow}(\mathrm{qf}\, A) \to \mathcal{D}^{\uparrow}(\mathrm{qf}\, A^!)$ and $\mathcal{F}_{A^!} : \mathcal{D}^{\uparrow}(\mathrm{qf}\, A^!) \to \mathcal{D}^{\uparrow}(\mathrm{qf}\, A)$.

Theorem 1. *The contravariant functors $\mathcal{F}_A$ and $\mathcal{F}_{A^!}$ give an equivalence*

$$\mathcal{D}^{\uparrow}(\mathrm{qf}\, A) \cong \mathcal{D}^{\uparrow}(\mathrm{qf}\, A^!)^{\mathsf{op}}.$$

The next result easily follows from Theorem 1 and the fact that $\mathcal{F}_A(K) = A^!$.

Lemma 2 (cf. [9, Lemma 2.8])**.** *For $M \in \mathrm{qf}\, A$, we have*

$$\beta_{i,j}(M) = \dim H^{i-j}(\mathcal{F}_A(M))_j.$$

3. Castelnuovo-Mumford Regularity and Linearity Defect

Throughout this section, $A = \bigoplus_{i \in \mathbb{N}} A_i$ is a Koszul algebra.

Definition 3. For $M \in \operatorname{qf} A$, we call

$$\operatorname{reg}_A(M) := \sup\{\, j - i \mid i \in \mathbb{N}, j \in \mathbb{Z} \text{ with } \beta_{i,j}(M) \neq 0 \,\}$$

the *(Castelnuovo-Mumford) regularity* of M. For convenience, we set the regularity of the 0 module to be $-\infty$.

If $M \notin {}^*\mathrm{mod}\, A$, then $\beta_{0,j}(M) \neq 0$ for arbitrary large j and $\operatorname{reg}_A(M) = \infty$. So $\operatorname{reg}_A(M)$ is essentially an invariant of $M \in {}^*\mathrm{mod}\, A$. But we regard it as an invariant of $M \in \operatorname{qf} A$ for later convenience. The following is clear.

Lemma 4. *(1) For $M \in \operatorname{qf} A$, we have*

$$\operatorname{reg}_A(M) < \infty \implies \beta_i(M) < \infty \text{ for all } i \implies M \text{ has a finite presentation.}$$

(2) Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence in $\operatorname{qf} A$. If two of M, M' and M'' have finite regularity, so does the third.

(3) If $M \in {}^*\mathrm{mod}\, A$ has finite length, then $\operatorname{reg}_A(M) = \sup\{\, i \mid M_i \neq 0\}$.

If A is a polynomial ring $K[x_1, \dots, x_n]$ (more generally, A is *AS regular*), then $\operatorname{reg}_A(M)$ of $M \in {}^*\mathrm{mod}\, A$ can be defined in terms of the local cohomology modules $H^i_{\mathfrak{m}}(M)$, see [6, 8, 15]. If A is commutative, it is known that $\operatorname{reg}_A(M) < \infty$ for all $M \in {}^*\mathrm{mod}\, A$ (see Theorem 8 below). But this need not be true in the non-commutative case. In fact, if A is not left noetherian, then A has a graded left ideal I such that $\beta_1(A/I) = \infty$. In particular, if A is not left noetherian, then $\operatorname{reg}_A(M) = \infty$ for some $M \in {}^*\mathrm{mod}\, A$. The author does not know any example $M \in {}^*\mathrm{mod}\, A$ such that $\beta_i(M) < \infty$ for all i but $\operatorname{reg}_A(M) = \infty$.

The next result directly follows from Lemma 2.

Theorem 5 (Eisenbud et al [5], Mori [9]). *For $M \in \operatorname{qf} A$, we have*

$$\operatorname{reg}_A(M) = -\inf\{\, i \mid H^i(\mathcal{F}_A(M)) \neq 0 \,\}.$$

Let $P_\bullet : \cdots \to P_1 \to P_0 \to M \to 0$ be a minimal graded free resolution of $M \in \operatorname{qf} A$. The *linear part* $\operatorname{lin}(P_\bullet)$ of $P_\bullet$ is the chain complex such that $\operatorname{lin}(P_\bullet)_i = P_i$ for all i and its differential maps are given by erasing all the entries of degree ≥ 2 from the matrices representing the differentials of $P_\bullet$. It is easy to check that $\operatorname{lin}(P_\bullet)$ is actually a complex, but it is not acyclic in general.

Definition 6 (Herzog-Iyengar [7]). Let $M \in \operatorname{qf} A$ and $P_\bullet$ its minimal graded free resolution. We call

$$\operatorname{ld}_A(M) := \sup\{\, i \mid H_i(\operatorname{lin}(P_\bullet)) \neq 0 \,\}$$

the *linearity defect* of M.

We say $M \in {}^*\mathrm{mod}\, A$ has a *linear free resolution* if there is some $l \in \mathbb{Z}$ such that $\beta_{i,j}(M) \neq 0$ implies that $j - i = l$. It is easy to see that

$$\operatorname{reg}_A(M) = \inf\{\, i \mid M_{\geq i} := \bigoplus_{j \geq i} M_j \text{ has a linear free resolution}\}.$$

For $i \in \mathbb{Z}$ and $M \in \operatorname{qf} A$, $M_{\langle i \rangle}$ denotes the submodule of M generated by the degree i component M_i. We say $M \in \operatorname{qf} A$ is *componentwise linear*, if $M_{\langle i \rangle}$ has a linear free resolution for all $i \in \mathbb{Z}$. For example, if M has a linear free resolution, then it is componentwise linear. Note that M can be componentwise linear even if it is not finitely generated. For example, $\bigoplus_{i \in \mathbb{N}} K(-i)$ is componentwise linear. It is easy to see that $\operatorname{ld}_A(M) = \inf\{\, i \mid \Omega_i(M) \text{ is componentwise linear} \,\}$, here $\Omega_i(M)$ is the ith syzygy of M.

Clearly, we have $\operatorname{ld}_A(M) \leq \operatorname{proj.dim}_A(M)$. The inequality is strict quite often. For example, we have $\operatorname{proj.dim}_A(M) = \infty$ and $\operatorname{ld}_A(M) < \infty$ for many M. On the other hand, sometimes $\operatorname{ld}_A(M) = \infty$.

The next result connects the linearity defect with the regularity via Koszul duality.

Theorem 7 (cf. [15, Theorem 4.7])**.** *For $M \in \operatorname{qf} A$, we have*

$$\operatorname{ld}_A(M) = \sup\{\, \operatorname{reg}_A(H^i(\mathcal{F}_A(M))) + i \mid i \in \mathbb{Z}\}.$$

Proof. For a complex $X^\bullet$, $\mathcal{H}(X^\bullet)$ denotes the complex such that $\mathcal{H}(X^\bullet)^i = H^i(X^\bullet)$ for all i and all differentials are 0. Let $P_\bullet$ be a minimal graded free resolution of M. Then $\operatorname{lin}(P_\bullet)$ is isomorphic to $\mathcal{F}_{A^!}(\mathcal{H}(\mathcal{F}_A(M)))$ (this is proved in [15] under the assumption that A is selfinjective, but the assumption is clearly irrelevant). So the assertion follows from Theorem 5. $\square$

4. Koszul Commutative Algebras and their Quadratic Dual

In this section, we study a Koszul commutative algebra A and its dual $A^!$.

Theorem 8 (Avramov-Eisenbud [1])**.** *If A is a Koszul commutative algebra, then we have $\operatorname{reg}_A(M) < \infty$ for all $M \in {}^*\operatorname{mod} A$.*

On the other hand, even if A is Koszul and commutative, $\operatorname{ld}_A(M)$ can be infinite for some $M \in {}^*\operatorname{mod} A$, as pointed out in [7]. In fact, if $\operatorname{ld}_A(M) < \infty$ then the Poincaré series $P_M(t) = \sum_{i \in \mathbb{N}} \beta_i(M) \cdot t^i$ is rational. But there exists a Koszul commutative algebra A such that $P_M(t)$ is not rational for some $M \in {}^*\operatorname{mod} A$. But we have the following.

Theorem 9 (Herzog-Iyengar [7])**.** *Let A be a Koszul commutative algebra.*

1. *If A is complete intersection, then $\operatorname{ld}_A(M) < \infty$ for all $M \in {}^*\operatorname{mod} A$, while $\sup\{\, \operatorname{ld}_A(M) \mid M \in {}^*\operatorname{mod} A \,\} = \infty$ in most cases.*
2. *If A is Golod, then $\operatorname{ld}_A(M) \leq 2 \cdot \dim_K A_1 < \infty$ for all $M \in {}^*\operatorname{mod} A$.*

Now we are interested in $\operatorname{reg}_{A^!}(N)$ and $\operatorname{ld}_{A^!}(N)$ for a Koszul commutative algebra A.

Theorem 10. *If A is a Koszul commutative algebra, we have the following.*

1. *Let $N \in {}^*\operatorname{mod} A^!$. If $\operatorname{reg}_{A^!}(N) < \infty$, then $\operatorname{ld}_{A^!}(N) < \infty$.*
2. *The following conditions are equivalent.*
   (a) *$\operatorname{ld}_A(M) < \infty$ for all $M \in {}^*\operatorname{mod} A$.*
   (b) *If $N \in {}^*\operatorname{mod} A^!$ has a finite presentation, then $\operatorname{reg}_{A^!}(N) < \infty$.*
3. *Let $N \in \operatorname{qf} A^!$. If there is some $c \in \mathbb{N}$ such that $\dim_K N_i \leq c$ for all $i \in \mathbb{Z}$, then $\operatorname{ld}_{A^!}(N) < \infty$.*

Proof. (1) The complex $F_{A^!}(N)$ is always bounded above. Hence if $\mathrm{reg}_{A^!}(N) < \infty$ then $H^i(\mathcal{F}_{A^!}(N)) \neq 0$ for only finitely many i by Theorem 5. Thus the assertion follows from Theorems 7 and 8.

(2) The implication $(a) \Rightarrow (b)$: First assume that $N \in {}^*\mathrm{mod}\, A^!$ has a finite presentation of the form $A^!(-1)^{\oplus \beta_1} \to A^{!\,\oplus \beta_0} \to N \to 0$. Then there is $M \in {}^*\mathrm{mod}\, A$ with $M = \bigoplus_{i=0,1} M_i$ such that $\mathcal{F}_A(M)$ gives this presentation. Since $\mathrm{ld}_A(M) < \infty$, we have $\mathrm{reg}_{A^!}(N) < \infty$ by Theorem 7.

If $N \in {}^*\mathrm{mod}\, A^!$ has a finite presentation, then for a sufficiently large s, $N_{\geq s} := \bigoplus_{i \geq s} N_i$ has a presentation of the form $A^!(-s-1)^{\oplus \beta_1} \to A^!(-s)^{\oplus \beta_0} \to N_{\geq s} \to 0$. (To see this, consider the short exact sequence $0 \to N_{\geq s} \to N \to N/N_{\geq s} \to 0$, and use the fact that $\mathrm{reg}_{A^!}(N/N_{\geq s}) < s$.) We have shown that $\mathrm{reg}_{A^!}(N_{\geq s}) < \infty$. So $\mathrm{reg}_{A^!}(N) < \infty$ by the above short exact sequence.

The implication $(b) \Rightarrow (a)$: Set $s := \min\{\, i \mid M_i \neq 0\,\}$. Then we have a finite presentation $A^! \otimes_K (M_{s+1})^* \to A^! \otimes_K (M_s)^* \to H^{-s}(\mathcal{F}_A(M)) \to 0$. Hence $\mathrm{reg}_{A^!}(H^{-s}(\mathcal{F}_A(M))) < \infty$ by the assumption. Let ∂^i be the differential map of the complex $F_A(M)$. By the exact sequence

$$0 \longrightarrow \mathrm{Im}\,\partial^{-s-1} \longrightarrow A^! \otimes_K (M_s)^* \longrightarrow H^{-s}(\mathcal{F}_A(M)) \longrightarrow 0,$$

we have $\mathrm{reg}_{A^!}(\mathrm{Im}\,\partial^{s-1}) < \infty$. Similarly, by the short exact sequence

$$0 \longrightarrow \mathrm{Ker}\,\partial^{-s-1} \longrightarrow A^! \otimes (M_{s+1})^* \longrightarrow \mathrm{Im}\,\partial^{-s-1} \longrightarrow 0,$$

we have $\mathrm{reg}_{A^!}(\mathrm{Ker}\,\partial^{-s-1}) < \infty$. Consider the short exact sequence

$$0 \longrightarrow \mathrm{Im}\,\partial^{-s-2} \longrightarrow \mathrm{Ker}\,\partial^{-s+1} \longrightarrow H^{-s-1}(\mathcal{F}_A(M)) \longrightarrow 0.$$

Since there is a surjection $A^! \otimes_K (M_{s+2})^* \to \mathrm{Im}\,\partial^{-s-2}$, $\mathrm{Im}\,\partial^{-s-2}$ is finitely generated. Hence $\mathrm{reg}_{A^!}(H^{-s-1}(\mathcal{F}_A(M)))$ has a finite presentation, and its regularity is finite by the assumption. So we also have $\mathrm{reg}_{A^!}(\mathrm{Im}\,\partial^{-s-2}) < \infty$. Repeating this argument, we can show that $\mathrm{reg}_{A^!}(H^i(\mathcal{F}_A(M))) < \infty$ for all i. On the other hand, by Theorem 5 and Theorem 8, $H^i(\mathcal{F}_A(M)) \neq 0$ for only finitely many i. So the assertion follows from Theorem 7.

(3) Let X be the set of all graded submodules of $A^{\oplus c}$ which are generated by elements of degree 1. By Brodmann [3], there is some $C \in \mathbb{N}$ such that $\mathrm{reg}_A(M) < C$ for all $M \in X$. To prove the theorem, it suffices to show that $\mathrm{reg}_A(H^i(\mathcal{F}_{A^!}(N))) + i < C$ for all i. We may assume that $i = 0$. Note that $H^0(\mathcal{F}_{A^!}(N))$ is the cohomology of the sequence

$$A \otimes_K (N_1)^* \xrightarrow{\partial^{-1}} A \otimes_K (N_0)^* \xrightarrow{\partial^0} A \otimes_K (N_{-1})^*.$$

Since $\mathrm{Im}(\partial^0)(-1)$ is a submodule of $A^{\oplus \dim_K N_{-1}} \subset A^{\oplus c}$ generated by elements of degree 1, we have $\mathrm{reg}_A(\mathrm{Im}(\partial^0)) < C$. Consider the short exact sequence

$$0 \longrightarrow \mathrm{Ker}(\partial^0) \longrightarrow A \otimes_K (N_0)^* \longrightarrow \mathrm{Im}(\partial^0) \longrightarrow 0.$$

Since $\mathrm{reg}_A(A \otimes_K (N_0)^*) = 0$, we have $\mathrm{reg}_A(\mathrm{Ker}(\partial^0)) \leq C$. Similarly, we have $\mathrm{reg}_A(\mathrm{Im}(\partial^{-1})) < C$. By the short exact sequence

$$0 \longrightarrow \mathrm{Im}(\partial^{-1}) \longrightarrow \mathrm{Ker}(\partial^0) \longrightarrow H^0(\mathcal{F}_{A^!}(N)) \longrightarrow 0,$$

we are done. $\square$

Corollary 11. *If A is a Koszul complete intersection, then $\mathrm{reg}_{A^!}(N) < \infty$ and $\mathrm{ld}_{A^!}(N) < \infty$ for all $N \in {}^*\mathrm{mod}\, A^!$.*

Proof. If $N \in {}^*\mathrm{mod}\, A^!$ has a finite presentation, we have $\mathrm{reg}_{A^!}(N) < \infty$ and $\mathrm{ld}_{A^!}(N) < \infty$ by Theorem 9 (1) and Theorem 10. On the other hand, it is know that $A^!$ is noetherian. Hence all $N \in {}^*\mathrm{mod}\, A^!$ has a finite presentation. So we are done. □

Proposition 12. *Let A be a Koszul commutative algebra which is Golod. If $N \in {}^*\mathrm{mod}\, A^!$ has a presentation of the form $A^!(-1)^{\oplus \beta_1} \to A^{!\,\oplus \beta_0} \to N \to 0$, then* $\mathrm{reg}_{A^!}(N) \leq 2{\cdot}\dim_K A_1$.

Proof. Follows from Theorem 9 (2) and the argument similar to the proof of Theorem 10 (2). □

In the situation of the above proposition, A is not necessarily noetherian. So it can occur $\mathrm{reg}_{A^!}(N) = \infty$ for some $N \in {}^*\mathrm{mod}\, A^!$ even if A is Golod.

5. Linearity Defects of Face Rings

Let $S := K[x_1, \dots, x_n]$ be the polynomial ring, and $E := \bigwedge \langle y_1, \dots, y_n \rangle$ the exterior algebra. The next result is now a special case of Theorem 10, but it initiated the study on linearity defect.

Theorem 13 (Eisenbud et. al. [5])**.** *We have $\mathrm{ld}_E(M) < \infty$ for all $M \in {}^*\mathrm{mod}\, E$.*

If $n \geq 2$, there is no uniform bound for $\mathrm{ld}_E(M)$, that is, $\sup\{ \mathrm{ld}_E(M) \mid M \in {}^*\mathrm{mod}\, E \} = \infty$. On the other hand, we have

$$\mathrm{ld}_E(M) \leq c^{n!} \cdot 2^{(n-1)!} \quad (\, c := \max\{ \dim_K M_i \mid i \in \mathbb{Z} \}\,)$$

for $M \in {}^*\mathrm{mod}\, E$. This bound follows from Brodmann's bound for the regularity of $M \in {}^*\mathrm{mod}\, S$. We also remark that the above bound seems very far from sharp. For example, the author does not know a graded ideal $J \subset E$ with $\mathrm{ld}_E(E/J) > n-1$. When J is a monomial ideal, we can actually prove that $\mathrm{ld}_E(E/J) \leq n-1$.

Set $[n] := \{1, 2, \dots, n\}$. We say $\Delta \subset 2^{[n]}$ is an *(abstract) simplicial complex*, if $F \in \Delta$ and $G \subset F$ imply $G \in \Delta$. For a simplicial complex $\Delta \subset 2^{[n]}$, we have monomial ideals

$$I_\Delta := (\, \prod_{i \in F} x_i \mid F \subset [n], F \not\in \Delta \,) \quad \text{of } S,$$

and

$$J_\Delta := (\, \prod_{i \in F} y_i \mid F \subset [n], F \not\in \Delta \,) \quad \text{of } E.$$

We call $K[\Delta] := S/I_\Delta$ the *Stanley–Reisner ring* of Δ, and $K\langle\Delta\rangle := E/J_\Delta$ the *exterior face ring* of Δ. Both are very important in Combinatorial Commutative Algebra, see [4, 12]. In this section, we introduce the results on the linearity defects of $K[\Delta]$ and $K\langle\Delta\rangle$. See [10] for detail.

Theorem 14 (Okazaki-Y [10])**.** *For a simplicial complex $\Delta \subset 2^{[n]}$, we have*

$$\mathrm{ld}_E(K\langle\Delta\rangle) = \mathrm{ld}_S(K[\Delta]).$$

There might exists a direct proof of the above result. But, in [10], we use the fact that BGG correspondence $\mathcal{D}^b(^*\mathrm{mod}\, S) \cong \mathcal{D}^b(^*\mathrm{mod}\, S)$ has special meaning for $K[\Delta]$ and $K\langle\Delta\rangle$ (this is the author's previous result, see [13]). From this, we can show that both $\mathrm{ld}_S(K(\Delta))$ and $\mathrm{ld}_E(K\langle\Delta\rangle)$ equal

$$\max\{\, i - \mathrm{depth}_S(\,\mathrm{Ext}_S^{n-i}(I_{\Delta^\vee}, S)\,) \mid 0 \le i \le n\,\}. \tag{5.1}$$

Here

$$\Delta^\vee := \{\, F \subset [n] \mid [n] \setminus F \notin \Delta \,\}$$

is the *Alexander dual* of Δ (it is easy to check that $\Delta^\vee$ is a simplicial complex again). We also remark that the number in (5.1) is closely related to the notion of *sequentially Cohen-Macaulay modules* (c.f. [12, Theorem 2.11]).

Theorem 14 suggests that we may set

$$\mathrm{ld}(\Delta) := \mathrm{ld}_S(K[\Delta]) = \mathrm{ld}_E(K(\langle\Delta\rangle)).$$

A simplicial complex Δ gives the topological space $|\Delta|$ which is called the *geometric realization* of Δ. In other words, Δ is a "triangulation" of $|\Delta|$. It is well-known that many homological/ring theoretical invariants of $K[\Delta]$ only depend on the topological space $|\Delta|$ (and $\mathrm{char}(K)$). But, for $\mathrm{ld}_S(K[\Delta])$, the Alexander dual $\Delta^\vee$ is essential.

Theorem 15 (Okazaki-Y [10]). *If $\Delta \neq 2^T$ for any $T \subset [n]$, $\mathrm{ld}(\Delta)$ is a topological invariant of the geometric realization $|\Delta^\vee|$ of the Alexander dual $\Delta^\vee$.*

The above result follows from the fact that $\mathrm{ld}(\Delta)$ equals the number given in (5.1) and "sheaf method" in the Stanley-Reisner ring theory, which was introduced by the author ([14]).

As a remark, $\mathrm{ld}(\Delta)$ depends on the characteristic $\mathrm{char}(K)$ of K. In fact, when $|\Delta^\vee|$ is homeomorphic to a real projective plane $\mathbb{P}^2\mathbb{R}$, we have

$$\mathrm{ld}(\Delta) = \begin{cases} 3 & \text{if } \mathrm{char}(K) = 2 \\ 1 & \text{otherwise.} \end{cases}$$

The earlier (and slightly weaker) version of the next result was first given in the thesis of T. Römer [11], and treats $\mathrm{ld}_E(K\langle\Delta\rangle)$. Later, it was improved by the author in [15]. The original proofs were slightly complicated. But, now we can give a simple proof which uses Theorem 14 and the fact that if a free module $S(-i)$ appears in the minimal graded free resolution of $K[\Delta]$ then $i \le n$.

Theorem 16 (Herzog-Römer, Y [15]). *For a simplicial complex $\Delta \subset 2^{[n]}$, we have*

$$\mathrm{ld}(\Delta) \le \max\{\, 1,\, n-2\,\}.$$

So it is natural to ask which simplicial complex attains the equality $\mathrm{ld}(\Delta) = n-2$. For an answer, the following holds.

Theorem 17 (Okazaki-Y [10]). *If $n \ge 4$, we have $\mathrm{ld}(\Delta) = n-2 \iff \Delta$ is an n-gon (i.e., $|\Delta|$ is a circle).*

To prove the theorem, we use $\mathrm{ld}(\Delta) = \mathrm{ld}_S(K[\Delta])$. If Δ is an n-gon, then $\beta_{n-1}(K[\Delta]) = 0$, $\beta_{n-2,n}(K[\Delta]) \neq 0$ and $\beta_{n-3,n-1}(K[\Delta]) = 0$. Hence we have $[H_{n-2}(\mathrm{lin}(P_\bullet))]_n \neq 0$, where $P_\bullet$ is the minimal graded free resolution of $K[\Delta]$. The proof of the converse can be reduced to the case when $\dim \Delta = 1$ (i.e., Δ is essentially a simple graph). Regarding Δ as a graph, we say a subgraph C of Δ is a *minimal cycle*, if it is a cycle with no chords. In this terminology, Δ is an n-gon if and only if Δ itself is a minimal cycle. Anyway, the assertion essentially comes from the following fact: If $\dim \Delta = 1$, $H_1(\Delta; K)$ is generated by $H_1(C; K)$ for minimal cycles C of Δ, in other words, we have a surjection

$$\bigoplus_{C:\text{minimal cycle}} H_1(C;K) \longrightarrow H_1(\Delta;K) \longrightarrow 0. \tag{5.2}$$

Example 18. The Alexander dual of the 5-gon is homeomorphic to the Möbius band. So the above theorem states that if $|\Delta^\vee|$ is homeomorphic to the Möbius band then $\mathrm{ld}(\Delta) = 3$, and any triangulation of the Möbius band requires at least 5 points. In this sense, the problem on " a simplicial complex $\Delta \subset 2^{[n]}$ with small $n - \mathrm{ld}(\Delta)$" is weakly related to the classical combinatorial problem on "a triangulation with small number of vertices". For example, if $|\Delta^\vee|$ is homeomorphic to the cylinder or the real projective plane and $\mathrm{char}(K) = 2$, then $\mathrm{ld}(\Delta) = 3$. In both cases, there is a triangulation with 6 vertices (this is the smallest possible number), and then we have $\mathrm{ld}(\Delta) = 3 = n - 3$.

For a simplicial complex $\Delta \subset 2^{[n]}$ and $F \subset [n]$, the restriction $\Delta|_F := \{\, G \in \Delta \mid G \subset F \,\}$ is a simplicial complex again. If $\dim \Delta = 1$ and $\mathrm{ld}(\Delta) \geq 2$, then we have

$$\mathrm{ld}(\Delta) \geq \min\{\, \#F - 2 \mid \ \Delta|_F \text{ is a } \#F\text{-gon} \,\}.$$

But the inequality can be strict.

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