FROBENIUS AND QUASI-FROBENIUS PROPERTY FOR mod $\underline{\mathcal{C}}$

Yuji YOSHINO

§1. Introduction

This is a survey report of my recent work [8], and we shall omit every proofs of the result in this monograph. The reader should refer to the original paper [8] [1] for the detail.

In the following R always denotes a commutative Noetherian ring, and $\mathrm{mod}\,R$ is the category of finitely generated R-modules. We are interested in the subcategories $\mathcal{G}$ and $\mathcal{H}$ of $\mathrm{mod}\,R$ that are defined as follows:

Definition 1. $\mathcal{G}$ is defined to be the full subcategory of $\mathrm{mod}\,R$ consisting of all modules $X \in \mathrm{mod}\,R$ that satisfy

$$\mathrm{Ext}^i_R(X, R) = 0 \quad \text{and} \quad \mathrm{Ext}^i_R(\mathrm{Tr}X, R) = 0 \quad \text{for any} \ \ i > 0.$$

We also define $\mathcal{H}$ to be the full subcategory consisting of all modules with the first half of the above conditions, therefore a module $X \in \mathrm{mod}\,R$ is an object in $\mathcal{H}$ if and only if

$$\mathrm{Ext}^i_R(X, R) = 0 \quad \text{for any} \ \ i > 0.$$

Note that $\mathcal{G} \subseteq \mathcal{H}$ and that $\mathcal{G}$ is called the subcategory of modules of G-dimension zero. See [2] for the G-dimension of modules.

Recently, D.Jorgensen and L.M.Sega [5] reported that they constructed an example of an artinian ring R, on which $\mathcal{G} \neq \mathcal{H}$. However, we still expect that the equality $\mathcal{G} = \mathcal{H}$ holds in many cases.

The main purpose of this paper is to characterize functorially these two subcategories and to get the conditions under which a subcategory $\mathcal{C}$ of $\mathcal{H}$ is contained in $\mathcal{G}$.

First we settle the notation which we shall use later. When we say $\mathcal{C}$ is a subcategory of $\mathrm{mod}\,R$, we always mean the following:

- $\mathcal{C}$ is essential in $\mathrm{mod}\,R$, i.e. if $X \cong Y$ in $\mathrm{mod}\,R$ and if $X \in \mathcal{C}$, then $Y \in \mathcal{C}$.
- $\mathcal{C}$ is full in $\mathrm{mod}\,R$, i.e. $\mathrm{Hom}_{\mathcal{C}}(X, Y) = \mathrm{Hom}_R(X, Y)$ for $X, Y \in \mathcal{C}$.
- $\mathcal{C}$ is additive and additively closed in $\mathrm{mod}\,R$, i.e. for any $X, Y \in \mathrm{mod}\,R$, $X \oplus Y \in \mathcal{C}$ if and only if $X \in \mathcal{C}$ and $Y \in \mathcal{C}$.
- $\mathcal{C}$ contains all projective modules in $\mathrm{mod}\,R$.

Of course $\mathcal{G}$ and $\mathcal{H}$ are subcategories in this sense.

Let $\mathcal{C}$ be any subcategory of $\mathrm{mod}\,R$. As in the general notation we denote the associated stable category by $\underline{\mathcal{C}}$. Of course, there is a natural functor $\mathcal{C} \to \underline{\mathcal{C}}$. We should note that the transpose Tr and the syzygy Ω are well-defined functors over tha stable category $\underline{\mathcal{C}}$:

$$\mathrm{Tr} : (\underline{\mathcal{C}})^{op} \to \underline{\mathrm{mod}\,R} \qquad \Omega : \underline{\mathcal{C}} \to \underline{\mathrm{mod}\,R}.$$

[1] The detailed version [8] of this paper has been submitted for publication elsewhere.

We also note just from the definition that Tr gives dualities on $\underline{\mathcal{G}}$ and also on $\underline{\mathrm{mod} R}$.

For an additive category $\mathcal{A}$, a contravariant additive functor from $\mathcal{A}$ to the category (Ab) of abelian groups is referred to as an $\mathcal{A}$-module, and a natural transform between two $\mathcal{A}$-modules is referred to as an $\mathcal{A}$-module morphism. We denote by $\mathrm{Mod}\mathcal{A}$ the category consisting of all $\mathcal{A}$-modules and all $\mathcal{A}$-module morphisms. Note that $\mathrm{Mod}\mathcal{A}$ is obviously an abelian category. An $\mathcal{A}$-module F is called finitely presented if there is an exact sequence

$$\mathrm{Hom}_{\mathcal{A}}(\ \ , X_1) \to \mathrm{Hom}_{\mathcal{A}}(\ \ , X_0) \to F \to 0,$$

for some $X_0, X_1 \in \mathcal{A}$. We denote by $\mathrm{mod}\mathcal{A}$ the full subcategory of $\mathrm{Mod}\mathcal{A}$ consisting of all finitely presented $\mathcal{A}$-modules.

It follows easily from Yoneda's lemma that an $\mathcal{A}$-module is projective in $\mathrm{mod}\mathcal{A}$ if and only if it is isomorphic to $\mathrm{Hom}_{\mathcal{A}}(\ \ , X)$ for some $X \in \mathcal{A}$. Also note that the functor $\mathcal{A}$ to $\mathrm{mod}\mathcal{A}$ which sends X to $\mathrm{Hom}_{\mathcal{A}}(\ \ , X)$ is a full embedding.

Now let $\mathcal{C}$ be a subcategory of $\mathrm{mod} R$ and let $\underline{\mathcal{C}}$ be the associated stable category. Then the category of finitely presented $\mathcal{C}$-modules $\mathrm{mod}\mathcal{C}$ and the category of finitely presented $\underline{\mathcal{C}}$-modules $\mathrm{mod}\underline{\mathcal{C}}$ are defined as in the above course. Note that for any $F \in \mathrm{mod}\mathcal{C}$ (resp. $G \in \mathrm{mod}\underline{\mathcal{C}}$) and for any $X \in \mathcal{C}$ (resp. $\underline{X} \in \underline{\mathcal{C}}$), the abelian group $F(X)$ (resp. $G(\underline{X})$) has naturally an R-module structure, hence F (resp. G) is in fact a contravariant additive functor from $\mathcal{C}$ (resp. $\underline{\mathcal{C}}$) to $\mathrm{mod} R$. As we stated above there is a natural functor $\mathcal{C} \to \underline{\mathcal{C}}$. We can define from this the functor $\iota : \mathrm{mod}\underline{\mathcal{C}} \to \mathrm{mod}\mathcal{C}$ by sending $F \in \mathrm{mod}\underline{\mathcal{C}}$ to the composition functor of $\mathcal{C} \to \underline{\mathcal{C}}$ with F. Then it is well known and is easy to prove that ι gives an equivalence of categories between $\mathrm{mod}\underline{\mathcal{C}}$ and the full subcategory of $\mathrm{mod}\mathcal{C}$ consisting of all finitely presented $\mathcal{C}$-modules F with $F(R) = 0$.

§2. Frobenius property of $\mathrm{mod}\underline{\mathcal{G}}$

Let $\mathcal{C}$ be a subcategory of $\mathrm{mod} R$. We say that $\mathcal{C}$ is closed under kernels of epimorphisms if it satisfies the following condition:

> If $0 \to X \to Y \to Z \to 0$ is an exact sequence in $\mathrm{mod} R$, and if $Y, Z \in \mathcal{C}$, then $X \in \mathcal{C}$.

(In Quillen's terminology, all epimorphisms from $\mathrm{mod} R$ in $\mathcal{C}$ are admissible.)

We say that $\mathcal{C}$ is closed under extension or extension-closed if it satisfies the following condition:

> If $0 \to X \to Y \to Z \to 0$ is an exact sequence in $\mathrm{mod} R$, and if $X, Z \in \mathcal{C}$, then $Y \in \mathcal{C}$.

A subcategory $\mathcal{C}$ is said to be a resolving subcategory if it is extension-closed and closed under kernels of epimorphisms. Also $\mathcal{C}$ is said to be closed under Ω if it satisfies that $\Omega X \in \underline{\mathcal{C}}$ whenever $X \in C$. Similarly to this, the closedness under Tr is defined.

Note that the categories $\mathcal{G}$ and $\mathcal{H}$ are resolving subcategories and that $\mathcal{G}$ is closed under Tr.

We note that, if a subcategory $\mathcal{C}$ of $\mathrm{mod} R$ is closed under kernels of epimorphisms, then it is closed under Ω. And if a subcategory $\mathcal{C}$ of $\mathrm{mod} R$ is extension-closed and closed under Ω, then it is resolving.

The following proposition is shown straightforward from the definitions. Note that the proof of the proposition is completely similar to that of [7, Lemma (4.17)], in which it is proved that $\mathrm{mod}\underline{\mathcal{C}}$ is an abelian category when R is a Cohen-Macaulay local ring and $\mathcal{C}$ is the category of maximal Cohen-Macaulay modules.

Proposition 2. *Let $\mathcal{C}$ be a subcategory of* $\mathrm{mod}\, R$ *which is closed under kernels of epimorphisms. Then* $\mathrm{mod}\underline{\mathcal{C}}$ *is an abelian category with enough projectives.*

A category $\mathcal{A}$ is said to be a Frobenius category if it is an abelian category with enough projectives and with enough injectives, and if the class of projective objects in $\mathcal{A}$ coincides with the clasee of injective objects in $\mathcal{A}$. Likewise, a category $\mathcal{A}$ is said to be a quasi-Frobenius category if it is an abelian category with enough projectives and all projective objects in $\mathcal{A}$ are injective.

The following theorem is the first result I have got and that motivated me to the detail study on the category $\mathrm{mod}\underline{\mathcal{C}}$.

Theorem 3. *Let $\mathcal{C}$ be a subcategory of* $\mathrm{mod}\, R$ *that is closed under kernels of epimorphisms. If $\mathcal{C} \subseteq \mathcal{H}$ then* $\mathrm{mod}\underline{\mathcal{C}}$ *is a quasi-Frobenius category.*

The proof of the theorem is not difficult. It is enough to notice that the injective objects in $\mathrm{mod}\underline{\mathcal{C}}$ are nothing but half-exact functors as a functor on $\mathcal{C}$. See [8, Theorem 3.5].

Theorem 4. *Let $\mathcal{C}$ be a subcategory of* $\mathrm{mod}\, R$. *And suppose the following conditions.*

(1) *$\mathcal{C}$ is a resolving subcategory of* $\mathrm{mod}\, R$.
(2) *$\mathcal{C} \subseteq \mathcal{H}$.*
(3) *The functor $\Omega : \underline{\mathcal{C}} \to \underline{\mathcal{C}}$ yields a surjective map on the set of isomorphism classes of the objects in $\underline{\mathcal{C}}$.*

Then $\mathrm{mod}\underline{\mathcal{C}}$ *is a Frobenius category. In particular,* $\mathrm{mod}\underline{\mathcal{G}}$ *is a Frobenius category.*

From the third assumption in the theorem, the syzygy functor Ω gives an automorphism on the category $\underline{\mathcal{C}}$, hence there exists a cosyzygy functor Ω^{-1}. Using this fact we can easily prove the theorem as in the same course of the proof of the previous theorem.

Now let us consider the following four conditions for a resolving subcategory $\mathcal{C}$ of $\mathrm{mod}\, R$:

(A) $\mathcal{C}$ is a subcategory of $\mathcal{H}$.
(B) $\mathrm{mod}\underline{\mathcal{C}}$ is a quasi-Frobenius category.
(C) $\mathrm{mod}\underline{\mathcal{C}}$ is a Frobenius category.
(D) $\mathcal{C}$ is a subcategory of $\mathcal{G}$.

Then, the following implications hold:

$$(A) \Longrightarrow (B) \Longleftarrow (C) \Longleftarrow (D)$$

The first implication follows from Theorem 3 and the third will follow from 4 (under a suitable condition on syzygy functor). Of course it is obvious that second implication always holds. Our program is that we analyze closely the reverse implications. Actually, in the next section we shall show that

- $(B) \Rightarrow (A)$ holds if R is a henselian local ring.
- $(C) \Rightarrow (D)$ holds under the validity of the Auslander-Reiten conjecture.
- $(B) \Rightarrow (C)$ holds if $\mathcal{C}$ is of finite type (by Nakayama Theorem).

§3. Main theorems

In this section we always assume that R is a henselian local ring with maximal ideal $\mathfrak{m}$ and with the residue class field $k = R/\mathfrak{m}$. In the following, what we shall need from this assumption is the fact that $X \in \mathrm{mod}\, R$ is indecomposable only if $\mathrm{End}_R(X)$ is a (noncommutative) local ring. In fact it is easy to see that $\mathrm{mod}\underline{\mathcal{C}}$ is a Krull-Schmidt category for any subcategory $\mathcal{C} \subseteq \mathrm{mod}\, R$.

We can prove the converse of Theorem 3 under this assumption.

Theorem 5. *Let $\mathcal{C}$ be a resolving subcategory of* $\mathrm{mod}\, R$*, where R is a henselian local ring. Suppose that* $\mathrm{mod}\underline{\mathcal{C}}$ *is a quasi-Frobenius category. Then $C \subseteq \mathcal{H}$.*

In a sense $\mathcal{H}$ is the largest resolving subcategory $\mathcal{C}$ of $\mathrm{mod}\, R$ for which $\mathrm{mod}\underline{\mathcal{C}}$ is a quasi-Frobenius category.

The proof of this theorem is not so easy. Essential part of the proof is to show that if $\mathrm{mod}\underline{\mathcal{C}}$ is quasi-Frobenius, then any object $X \in \mathcal{C}$ satisfies $\mathrm{Ext}^1_R(X, R) = 0$. The reader should refer to the paper [8, Theorem 4.2] for the complete proof.

As to the implication $(C) \Longrightarrow (D)$ in the last paragraph of the previous section, we can show the following result.

Theorem 6. *Let R be a henselian local ring as above. Suppose that*

(1) *$\mathcal{C}$ is a resolving subcategory of* $\mathrm{mod}\, R$.

(2) $\mathrm{mod}\underline{\mathcal{C}}$ *is a Frobenius category.*

(3) *There is no nonprojective module $X \in \mathcal{C}$ with* $\mathrm{Ext}^1_R(\ \ , X)|_{\underline{\mathcal{C}}} = 0$.

Then $\mathcal{C} \subseteq \mathcal{G}$.

Remark 7. We conjecture that $\mathcal{G}$ should be the largest resolving subcategory $\mathcal{C}$ of $\mathrm{mod}\, R$ such that $\mathrm{mod}\underline{\mathcal{C}}$ is a Frobenius category.

Theorem 6 together with Theorem 4 say that this is true modulo Auslander-Reiten conjecture:

(AR) If $\mathrm{Ext}^i_R(X, X \oplus R) = 0$ for any $i > 0$ then X should be projective.

In fact, if the conjecture (AR) is true, then the third assumption of Thereom 6 is automatically satisfied.

The proof of Theorem 6 is not short, and we restrict ourselves to say that the following lemma is essential in its proof.

Lemma 8. *Let R be a henselian local ring and let $\mathcal{C}$ be an extension-closed subcategory of* $\mathrm{mod}\, R$*. For objects $\underline{X}, \underline{Y} \in \underline{\mathcal{C}}$, we assume the following:*

(1) *There is a monomorphism φ in* $\mathrm{Mod}\underline{\mathcal{C}}$*:*

$$\varphi : \underline{\mathrm{Hom}}_R(\ \ , Y)|_{\underline{\mathcal{C}}} \to \mathrm{Ext}^1(\ \ , X)|_{\underline{\mathcal{C}}}$$

(2) *$\underline{X}$ is indecomposable in $\underline{\mathcal{C}}$.*

(3) *$\underline{Y} \not\cong 0$ in $\underline{\mathcal{C}}$.*

Then the module X is isomorphic to a direct summand of ΩY.

Let $\mathcal{A}$ be any additive category. We denote by $\mathrm{Ind}(\mathcal{A})$ the set of nonisomorphic modules which represent all the isomorphism classes of indecomposable objects in $\mathcal{A}$. If $\mathrm{Ind}(\mathcal{A})$ is a finite set, then we say that $\mathcal{A}$ is a category of finite type. The following theorem is a main theorem of the paper [8], which claims that any resolving subcategory of finite type in $\mathcal{H}$ are contained in $\mathcal{G}$. See [8, Theorem 5.5]

Theorem 9. *Let R be a henselian local ring and let $\mathcal{C}$ be a subcategory of* $\mathrm{mod}\, R$ *which satisfies the following conditions.*

(1) $\mathcal{C}$ *is a resolving subcategory of* $\operatorname{mod} R$*.*
(2) $\mathcal{C} \subseteq \mathcal{H}$*.*
(3) $\mathcal{C}$ *is of finite type.*

Then, $\operatorname{mod}\underline{\mathcal{C}}$ *is a Frobenius category and* $\mathcal{C} \subseteq \mathcal{G}$*.*

We should remark about the proof of Theorem 9. Since we assume that $\mathcal{C}$ is of finite type, the category $\operatorname{mod}\underline{\mathcal{C}}$ is isomorphic to the module category of certain artinian algebra A that is called the Auslander algebra of $\underline{\mathcal{C}}$:

$$\operatorname{mod}\underline{\mathcal{C}} \;\cong\; \operatorname{mod} A$$

Note that the ring A is a finite (noncommutative) algebra over a commutative artinian ring. Since we assume that $\mathcal{C} \subseteq \mathcal{H}$, we know that $\operatorname{mod}\underline{\mathcal{C}}$, hence $\operatorname{mod} A$, is a quasi-Frobenius category. (See Theorem 3.) This means that the artinian ring A is left selfinjective. It is known by Nakayam's Theorem (cf. [6] for example) that A is right selfinjective as well, and therefore, using the duality between $\operatorname{mod} A$ and $\operatorname{mod} A^{op}$, we can conclude that $\operatorname{mod} A$, hence $\operatorname{mod}\underline{\mathcal{C}}$, is a Frobenius category.

To prove that $\mathcal{C} \subseteq \mathcal{G}$ in the theorem, we use Theorem 6. Actually, since we have shown that $\operatorname{mod}\underline{\mathcal{C}}$ is a Frobenious category, it is enough to check the following statement:

(*) If $X \in \mathcal{C}$ such that $\underline{X} \not\cong \underline{0}$ in $\underline{\mathcal{C}}$, then we have $\operatorname{Ext}^1_R(\ \ , X)|_{\underline{\mathcal{C}}} \neq 0$.

This can be proved by using the same idea of Nakayama which we can see in the monograph of Yamagata [6].

References

1. M. Auslander, *Representation of algebras I*, Comm. Algebra **1** (1974), 177 – 268: *II*, Comm. Algebra **2**(1974), 269 – 310.
2. M. Auslander and M. Bridger, *Stable module theory*, Mem. Amer. Math. Soc. **94**, 1969.
3. M. Auslander and R. -O. Buchweitz, *The homological theory of maximal Cohen-Macaulay approximations*, Société Mathématique de France, Mémoire No. 38 (1989), 5–37.
4. M. Auslander and I. Reiten, *Representation of algebras III*, Comm. Algebra **3** (1975), 239 – 294: *IV*, Comm. Algebra **5**(1977), 443 – 518.
5. D. Jorgensen and L. M. Sega, *Independence of the total reflexivity conditions for modules*, arXive: math. AC/0410257, (2004).
6. K. Yamagata, *Frobenius algebras.* Handbook of algebra, edited by M. Hazewinkel, North-Holland, Amsterdam, Vol. 1 (1996), 841–887.
7. Y. Yoshino, *Cohen-Macaulay modules over Cohen-Macaulay rings*, London Math. Soc., Lecture Note Series **146**, Cambridge U.P., 1990.
8. Y. Yoshino, *A functorial approach to modules of G-dimension zero*, arXive: math. AC/0408162, (2004).

Math. Department, Faculty of Science
Okayama University
Okayama 700-8530, Japan
E-mail address: yoshino@math.okayama-u.ac.jp