ON THE $\mathbb{Z}D_\infty$ CATEGORY [1]

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Abstract In this paper we give a direct proof of the properties of the $\mathbb{Z}D_\infty$ category which was introduced in the classification of noetherian, hereditary categories with Serre duality by Idun Reiten and the author.

1. Introduction

Below k is a field. All categories will be k-linear. An abelian or triangulated category $\mathcal{A}$ is Ext-finite if for all objects $A, B \in \mathcal{A}$ one has that $\oplus_i \operatorname{Ext}^i(A,B)$ is finite dimensional. If $\mathcal{A}$ is triangulated and Ext-finite then we say that $\mathcal{A}$ satisfies Serre duality [1] if there exists an auto-equivalence F of $\mathcal{A}$ together with isomorphisms

$$\operatorname{Hom}_{\mathcal{A}}(A,B) \to \operatorname{Hom}_{\mathcal{A}}(B,FA)^*$$

natural in A, B (where $(-)^*$ is the k-dual). If $\mathcal{A}$ is abelian and Ext-finite then we say that $\mathcal{A}$ satisfies Serre duality if this is the case for $D^b(\mathcal{A})$. The following result can be extracted from [5, Ch. 1].

Theorem 1.1. *Assume that $\mathcal{C}$ is an* Ext*-finite hereditary category without injectives or projectives. Then the following are equivalent*

(1) *$\mathcal{C}$ has almost split sequences.*
(2) *$\mathcal{C}$ satisfies Serre duality.*
(3) *There is an auto-equivalence $V : \mathcal{C} \to \mathcal{C}$ together with natural isomorphisms*

$$\operatorname{Hom}_{\mathcal{C}}(A,B) \to \operatorname{Ext}^1_{\mathcal{C}}(B,VA)^* \tag{1.1}$$

Furthermore the functor V coincides with the Auslander-Reiten translate τ when evaluated on objects.

In the classification of noetherian Ext-finite hereditary categories with Serre duality in [5] we considered a category $\mathcal{C}$ defined by the following pullback diagram

$$\begin{tikzcd} \operatorname{mod}(k) \oplus \operatorname{mod}(k) \arrow[r] & \operatorname{mod}(k) \\ \mathcal{C} \arrow[u] \arrow[r] & \operatorname{gr}(k[x]) \arrow[u] \end{tikzcd} \tag{1.2}$$

where the horizontal map sends (V_1, V_2) to $V_1 \oplus V_2$ and the vertical map is localizing at x followed by restricting to degree zero. It was shown by a rather indirect argument that $\mathcal{C}$ is a noetherian, Ext-finite, hereditary abelian category without injectives or projectives

[1] The paper is in a final form and no version of it will be submitted for publication elsewhere.

which satisfies Serre duality. It was also shown that the AR-quiver of $\mathcal{C}$ has two components, one equal to $\mathbb{Z}A_\infty$ (a "wing") and the other equal to $\mathbb{Z}D_\infty$. For this reason $\mathcal{C}$ was called the "$\mathbb{Z}D_\infty$-category".

The aim of this paper is to give a direct proof of the above facts. In addition we will also establish a link with one-dimensional A_n singularities.

2. Elementary properties

It is easy to see that $\mathcal{C}$, as defined in the introduction is a noetherian abelian category. It will be convenient to consider the locally noetherian Grothendieck category $\tilde{\mathcal{C}}$ associated to $\mathcal{C}$. It follows for example by [3, Prop. 2.14] that $D^b(\mathcal{C})$ and $D^b_{\mathcal{C}}(\tilde{\mathcal{C}})$ are equivalent. Hence the Ext-groups between objects in $\mathcal{C}$ may be computed in $\tilde{\mathcal{C}}$.

The objects of $\tilde{\mathcal{C}}$ are quadruples (M, V_0, V_1, ϕ) where M is a graded $k[x]$-module, V_0, V_1 are k-vector spaces and ϕ is an isomorphism of $k[x]$ modules $M_x \to (V_0 \oplus V_1) \otimes_k k[x, x^{-1}]$. Objects in $\mathcal{C}$ are given by the quadruples (M, V_0, V_1, ϕ) in which M is finitely generated.

Sending (M, V_0, V_1, ϕ) to M defines an faithful exact functor $\tilde{\mathcal{C}} \to \mathrm{Gr}(k[x])$ which we call the restriction functor and which we denote by $(-)_{k[x]}$.

We write $M(n) = (M(n), V_0, V_1, \phi)$ where we have identified $M(n)_x$ with M_x through multiplication with x^n. Furthermore we define $\sigma(M) = (M, V_1, V_0, \phi)$.

We define $\tilde{\mathcal{T}} \subset \tilde{\mathcal{C}}$ and $\tilde{\mathcal{F}} \subset \tilde{\mathcal{C}}$ respectively as the inverse images of the x-torsion and x-torsion free modules in $\mathrm{Gr}(k[x])$. $\mathcal{T}$ and $\mathcal{F}$ are defined similarly, but starting from $\mathcal{C}$.

By $\tilde{\mathcal{C}}_x$ we denote the full subcategory of $\tilde{\mathcal{C}}$ with objects the quadruples (M, V_0, V_1, ϕ) in which x acts invertibly on M.

We denote by $(-)_x$ the functor $\tilde{\mathcal{C}} \to \tilde{\mathcal{C}}_x$ which sends (M, V_0, V_1, ϕ) to (M_x, V_0, V_1, ϕ). Clearly if $M \in \tilde{\mathcal{C}}$ and $N \in \tilde{\mathcal{C}}_x$ then the canonical maps

$$\mathrm{Hom}_{\tilde{\mathcal{C}}}(M, N) \to \mathrm{Hom}_{\tilde{\mathcal{C}}_x}(M, N) \to \mathrm{Hom}_{\tilde{\mathcal{C}}_x}(M_x, N) \tag{2.1}$$

are isomorphisms. We list a few other obvious facts.

(O1) $(\tilde{\mathcal{T}}, \tilde{\mathcal{F}})$ forms a torsion pair in $\tilde{\mathcal{C}}$. That is $\mathrm{Hom}(\tilde{\mathcal{T}}, \tilde{\mathcal{F}}) = 0$ and for any $M \in \tilde{\mathcal{C}}$ there exists an exact sequence (necessarily unique)

$$0 \to T \to M \to F \to 0$$

with $T \in \tilde{\mathcal{T}}$ and $F \in \tilde{\mathcal{F}}$.

(O2) If $T \in \tilde{\mathcal{T}}$ and $M \in \tilde{\mathcal{C}}$ then

$$\mathrm{Hom}_{\tilde{\mathcal{C}}}(T, M) = \mathrm{Hom}_{k[x]}(T, M)$$
$$\mathrm{Hom}_{\tilde{\mathcal{C}}}(M, T) = \mathrm{Hom}_{k[x]}(M, T)$$

(O3) The restriction functor defines an equivalence between $\tilde{\mathcal{T}}$ and $\mathrm{Tors}(k[x])$ where $\mathrm{Tors}(k[x])$ denotes the x-torsion modules is $\mathrm{Gr}(k[x])$.

(O4) The functor $\tilde{\mathcal{C}}_x \to \mathrm{Mod}(k) \oplus \mathrm{Mod}(k)$ which sends (M, V_0, V_1, ϕ) to $V_0 \oplus V_1$ is an equivalence of categories.

Combining (O4) with (2.1) yields in particular

(O5) If $N \in \tilde{\mathcal{C}}_x$ then $\mathrm{Hom}_{\mathcal{C}}(-, N)$ is exact. Hence the objects in $\tilde{\mathcal{C}}_x$ are injective in $\tilde{\mathcal{C}}$.

We now describe the indecomposable injectives in $\tilde{\mathcal{C}}$. For $n \in \mathbb{Z}$ let E_n be the graded injective $k[x]$-module given by $k[x, x^{-1}]/x^{n+1}k[x]$. Since $E_n \in \mathrm{Tors}(k[x])$ there exists by (O3) a corresponding object in $\tilde{\mathcal{T}}$ which we denote by the same symbol. From (O2) it follows that $\mathrm{Hom}(-, E_n)$ is exact and hence E_n is injective in $\tilde{\mathcal{C}}$.

To construct other injectives we note that by (O5) we know that the objects in $\tilde{\mathcal{C}}_x$ are injective in $\tilde{\mathcal{C}}$. Since by (O4) $\tilde{\mathcal{C}}_x$ is equivalent to $\mathrm{Mod}(k) \oplus \mathrm{Mod}(k)$ there must be two corresponding indecomposable injectives in $\tilde{\mathcal{C}}$. They are given by

$$E^0 = (k[x,x^{-1}], k, 0, \mathrm{id}_{k[x,x^{-1}]})$$
$$E^1 = (k[x,x^{-1}], 0, k, \mathrm{id}_{k[x,x^{-1}]})$$

Proposition 2.1.
(1) *$(E_n)_n, E^0, E^1$ forms a complete list of indecomposable injectives in $\tilde{\mathcal{C}}$.*
(2) *Every object in $\tilde{\mathcal{C}}$ has injective dimension one (and hence $\tilde{\mathcal{C}}$ and $\mathcal{C}$ are hereditary* [5, Prop. A.3]*).*
(3) *$\mathcal{C}$ is* Ext*-finite.*

Proof. Since the listed injectives are clearly indecomposable (1) follows if we can show that any indecomposable object can be embedded in a direct sum of them [4].

We prove (1) and (2) together by showing that every object $M \in \tilde{\mathcal{C}}$ has a resolution of length at most two whose terms consist of direct sums of the injectives given in (1). By (O1) it is clearly sufficient to prove this claim separately in the cases $M \in \tilde{\mathcal{T}}$ and $M \in \tilde{\mathcal{F}}$.

Assume first that $M \in \tilde{\mathcal{T}}$. Then M has an injective resolution

$$0 \to M \to I_0 \to I_1 \to 0 \tag{2.2}$$

in $\mathrm{Tors}(k[x])$. By (O3) this resolution corresponds to one in $\tilde{\mathcal{C}}$. Furthermore by the structure of the injectives in $\mathrm{Gr}(k[x])$ the I_i are direct sums of the E_n in $\mathrm{Gr}(k[x])$. Again by (O3) the same is true in $\tilde{\mathcal{C}}$.

Now assume that $M \in \tilde{\mathcal{F}}$. Consider the short exact sequence

$$0 \to M \to M_x \to M_x/M \to 0 \tag{2.3}$$

M_x lies in $\tilde{\mathcal{C}}_x$ and hence by (O4) is a direct sum of copies of E^0 and E^1. M_x/M is x-divisible and lies in $\tilde{\mathcal{T}}$ and so by (O3) M_x/M is a direct sum of copies of E_n. Whence (2.3) is the kind of resolution we were looking for.

To prove (3) we note that if E, F are indecomposable injectives as in (1) then $\dim \mathrm{Hom}_{\tilde{\mathcal{C}}}(E,F) \leq 1$. Thus it suffices to show that every $M \in \mathcal{C}$ has an injective resolution consisting in every degree of a finite number of indecomposable injectives. This follows easily from the construction. $\square$

Proposition 2.2. *If $F \in \tilde{\mathcal{F}}$ and $T \in \tilde{\mathcal{T}}$ then $\mathrm{Ext}^1_{\tilde{\mathcal{C}}}(F,T) = 0$. In particular every object in $\mathcal{C}$ is of the form $F \oplus T$ with $F \in \mathcal{F}$ and $T \in \mathcal{T}$.*

Proof. It follows from (O2) that $\mathrm{Hom}(F,-)$ is exact on $\mathcal{T}$. Since by the proof of the previous proposition T has a $\tilde{\mathcal{C}}$ injective resolution inside $\tilde{\mathcal{T}}$, we are done. $\square$

Now we describe the Ext-groups between objects in $\tilde{\mathcal{F}}$.

Lemma 2.3. *Assume that $F = (F, V_0, V_1, \phi)$, $F' = (F', V_0', V_1', \phi')$ are objects in $\tilde{\mathcal{F}}$. Then there exists an exact sequence of the form*

$$0 \to \mathrm{Hom}_{\tilde{\mathcal{C}}}(F,F') \to \mathrm{Hom}_{k[x]}(F,F') \to \mathrm{Hom}_k(V_0,V_1') \oplus \mathrm{Hom}_k(V_1,V_0') \to \mathrm{Ext}^1_{\tilde{\mathcal{C}}}(F,F') \to 0 \tag{2.4}$$

Proof. We start with the short exact sequence

$$0 \to F' \to F'_x \to F'_x/F' \to 0$$

which according to the proof of lemma 2.1 is an injective resolution of F', both in $\tilde{\mathcal{C}}$ and in $\mathrm{Gr}(k[x])$.

Applying $\mathrm{Hom}_{\mathcal{C}}(F,-)$, $\mathrm{Hom}_{\mathrm{Gr}(k[x])}(F,-)$ and comparing yields a commutative diagram with exact rows and columns.

$$\begin{tikzcd}
 & & 0 \arrow[d] & & & \\
0 \arrow[r] & \mathrm{Hom}_{\tilde{\mathcal{C}}}(F,F') \arrow[r] \arrow[d] & \mathrm{Hom}_k(V_0,V_0') \oplus \mathrm{Hom}_k(V_1,V_1') \arrow[r] \arrow[d] & \mathrm{Hom}_{\tilde{\mathcal{C}}}(F,F'_x/F') \arrow[r] \arrow[d, equal] & \mathrm{Ext}^1_{\tilde{\mathcal{C}}}(F,F') \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Hom}_{\mathrm{Gr}(k[x])}(F,F') \arrow[r] & \mathrm{Hom}_k(V_0 \oplus V_1, V_0' \oplus V_1') \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{Gr}(k[x])}(F,F'_x/F') \arrow[r] & 0 & \\
 & & \mathrm{Hom}(V_0,V_1') \oplus \mathrm{Hom}(V_1,V_0') \arrow[d] & & & \\
 & & 0 & & &
\end{tikzcd}$$

(2.4) now follows from the previous diagram through an easy diagram chase. ☐

Proposition 2.4. *$\mathcal{C}$ has neither injectives nor projectives.*

Proof. Since $\mathcal{T}$ is equivalent to the x-torsion modules in $\mathrm{gr}(k[x])$, it is easy to see that $\mathcal{T}$ does not contain any injective or projectives.

If $0 \neq (F, V_0, V_1, \phi)$ in $\mathcal{F}$ then by considering the faithful restriction functor to $\mathrm{gr}(k[x])$ we see that $\mathrm{Hom}_{\mathcal{C}}(F, \sigma F(-n)) = 0$ for $n \gg 0$. On the other hand V_0 or $V_1 \neq 0$. It follows from the previous lemma that $\mathrm{Ext}^1_{\mathcal{C}}(F, \sigma F(-n)) \neq 0$ for $n \gg 0$. Hence F is not projective. A similar argument shows that F is not injective. ☐

Remark 2.5. The reason why we called this section "Elementary properties" is that the stated results hold in greater generality. For example, suitably adapted versions would be valid for the pullback of

$$\begin{tikzcd}
\mathrm{mod}(k)^{\oplus m} \arrow[r] & \mathrm{mod}(k) \\
 & \mathrm{gr}(k[x]) \arrow[u]
\end{tikzcd}$$

for any m. By contrast, the results in the next section require $m = 2$.

3. Serre duality

§ Our next aim is to prove that $\mathcal{C}$ satisfies Serre duality. First we construct a Serre functor on $\mathcal{F}$. Put $VM = \sigma(M)(-1)$.

The first step in proving Serre duality is constructing a "trace map" $\eta_M : \mathrm{Ext}^1_{\mathcal{C}}(M, VM) \to k$ for $M \in \mathcal{F}$ which should corresponds to the identity map in $\mathrm{Hom}_{\mathcal{C}}(M, M)$ under the isomorphism (1.1).

We now use (2.4) to construct the trace map η_F for $F = (F, V_0, V_1, \phi) \in \mathcal{F}$. In this case $VF = (F(-1), V_1, V_0, \phi)$ and we have an exact sequence

$$\mathrm{Hom}_{k[x]}(F, VF) \to \mathrm{Hom}_k(V_0, V_0) \oplus \mathrm{Hom}_k(V_1, V_1) \to \mathrm{Ext}^1_{\mathcal{C}}(F, VF) \to 0$$

Lemma 3.1. *The composition*

$$\mathrm{Hom}_{k[x]}(F, VF) \to \mathrm{Hom}_k(V_0, V_0) \oplus \mathrm{Hom}_k(V_1, V_1) \xrightarrow{\mathrm{Tr}_{V_0} + \mathrm{Tr}_{V_1}} k \tag{3.1}$$

is the zero map.

Proof. To see this note that $\mathrm{Hom}_{k[x]}(F, VF) = \mathrm{Hom}(F, F(-1))$ and furthermore that (3.1) can be extended to a commutative diagram.

$$\begin{tikzcd}
\mathrm{Hom}_{k[x]}(F, F(-1)) \arrow[r] & \mathrm{Hom}_k(V_0 \oplus V_1, V_0 \oplus V_1) \arrow[r] \arrow[d, "\mathrm{Tr}"] & \mathrm{Hom}_k(V_0, V_0) \oplus \mathrm{Hom}_k(V_1, V_1) \arrow[dl, "\mathrm{Tr}_{V_0} + \mathrm{Tr}_{V_1}"] \\
& k &
\end{tikzcd}$$

By choosing a basis for F as graded $k[x]$-module one easily sees that every element of $\mathrm{Hom}(F, F(-1)) \subset \mathrm{Hom}(F, F)$ is nilpotent. Since nilpotent elements have zero trace it follows that the composition

$$\mathrm{Hom}_{k[x]}(F, F(-1)) \to \mathrm{Hom}_k(V_0 \oplus V_1, V_0 \oplus V_1) \xrightarrow{\mathrm{Tr}} k$$

is zero. This proves what we want. □

From lemma 3.1 together with (2.4) there exists a unique map $\eta_F : \mathrm{Ext}^1_{\mathcal{C}}(F, VF) \to k$ which makes the following diagram commutative.

$$\begin{tikzcd}
\mathrm{Hom}_{k[x]}(F, VF) \arrow[r] & \mathrm{Hom}(V_0, V_0) \oplus \mathrm{Hom}(V_1, V_1) \arrow[r] \arrow[dr, "\mathrm{Tr}_{V_0} + \mathrm{Tr}_{V_1}"] & \mathrm{Ext}^1_{\mathcal{C}}(F, VF) \arrow[r] \arrow[d, "\eta_F"] & 0 \\
& & k &
\end{tikzcd}$$

To continue it will be convenient to use the Yoneda multiplication on $\mathrm{Ext}^*_{\mathcal{C}}(-,-)$. In order to have compatibility with the notation for compositions of maps we will write the Yoneda multiplication as a pairing

$$\mathrm{Ext}^*_{\mathcal{C}}(B, C) \times \mathrm{Ext}^*_{\mathcal{C}}(A, B) \to \mathrm{Ext}^*_{\mathcal{C}}(A, C)$$

We extend η_F to a map $\mathrm{Ext}^*(F, VF) \to k$ by letting it act trivially on $\mathrm{Hom}(F, VF)$.

Lemma 3.2. *Let $F, G \in \mathcal{F}$ and assume that $f \in \mathrm{Ext}^*_{\mathcal{C}}(F, G)$ and $g \in \mathrm{Ext}^*_{\mathcal{C}}(G, VF)$. Then we have $\eta_F(gf) = \eta_G(V(f)g)$.*

Proof. We may assume that f and g are homogeneous. Furthermore the cases where f,g are both of degree 0 or of degree 1 are trivial. Hence we may assume that $(\deg f, \deg g) = (0, 1)$ or $(\deg f, \deg g) = (1, 0)$.

Let us consider the first possibility. We check that $\eta_F(-f) = \eta_G(V(f)-)$ as maps $\mathrm{Ext}^1(G, VF) \to k$. This amounts to the commutativity of

$$\begin{tikzcd}
\mathrm{Ext}^1(G, VF) \arrow[r] \arrow[d] & \mathrm{Ext}^1(G, VG) \arrow[d, "\eta_G"] \\
\mathrm{Ext}^1(F, VF) \arrow[r, "\eta_F"] & k
\end{tikzcd} \tag{3.2}$$

Assume that $F = (F, V_0, V_1, \phi)$, $G = (G, W_0, W_1, \theta)$. Then f induces maps $f_0 : V_0 \to W_0$ and $f_1 : V_1 \to W_1$. Elementary linear algebra yields that we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}(W_0, V_0) \oplus \mathrm{Hom}(W_1, V_1) & \longrightarrow & \mathrm{Hom}(W_0, W_0) \oplus \mathrm{Hom}(W_1, W_1) \\
\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Tr}_{W_0} + \mathrm{Tr}_{W_1}} \\
\mathrm{Hom}(V_0, V_0) \oplus \mathrm{Hom}(V_1, V_1) & \xrightarrow{\mathrm{Tr}_{V_0} + \mathrm{Tr}_{V_1}} & k
\end{array}$$

This diagram, together with the definition of η yields the commutativity of (3.2).

Now we consider the possibility $(\deg f, \deg g) = (1, 0)$. Since we trivially have

$$\eta_{VX} \circ V = \eta_X \tag{3.3}$$

it is sufficient to prove that $\eta_{VF}(V(g)V(f)) = \eta_G(V(f)g)$. Replacing (Vf, g) by (g, f) this reduces to the previous case. $\square$

We are now in a position to prove Serre duality for objects in $\mathcal{F}$. We will show that the pairing

$$\mathrm{Hom}_{\mathcal{C}}(F, G) \times \mathrm{Ext}^1_{\mathcal{C}}(G, VF) \to \mathrm{Ext}^1(F, VF) \xrightarrow{\eta_F} k : (f, g) \mapsto \eta_F(gf) \tag{3.4}$$

is non-degenerate. By lemma 3.2 the non-degeneracy of (3.4) for all F, G is equivalent to the non-degeneracy of the pairing

$$\mathrm{Ext}^1_{\mathcal{C}}(F, G) \times \mathrm{Hom}_{\mathcal{C}}(G, VF) \to \mathrm{Ext}^1(F, VF) \xrightarrow{\eta_F} k : (f, g) \to \eta_F(gf) \tag{3.5}$$

for all F, G. It follows also easily from lemma 3.2 that (3.4) and (3.5) are natural in F and G.

Lemma 3.3. *If we have and exact sequence*

$$0 \to F_1 \to F \to F_2 \to 0 \tag{3.6}$$

in $\mathcal{F}$ and if we have non-degeneracy of (3.4) *and* (3.5) *for two out of the three pairs $(F_1, G), (F, G), (F_2, G)$ then we also have it for the third one. A similar statement holds for an exact sequence*

$$0 \to G_1 \to G \to G_2 \to 0 \tag{3.7}$$

Proof. Assume that we have an exact sequence of the form (3.6). We claim that the following diagram with exact rows is commutative.

$$\begin{array}{ccccccccccccc}
0 & \to & \mathrm{Hom}(F_2, G) & \to & \mathrm{Hom}(F, G) & \to & \mathrm{Hom}(F_1, G) & \to & \mathrm{Ext}^1(F_2, G) & \to & \mathrm{Ext}^1(F, G) & \to & \mathrm{Ext}^1(F_1, G) & \to 0 \\
 & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta_1} & \\
0 & \to & \mathrm{Ext}^1(G, VF_2)^* & \to & \mathrm{Ext}^1(G, VF)^* & \to & \mathrm{Ext}^1(G, VF_1)^* & \to & \mathrm{Hom}(G, VF_2)^* & \to & \mathrm{Hom}(G, VF)^* & \to & \mathrm{Hom}(G, VF_1)^* & \to 0
\end{array}$$

Here the maps labeled by α are obtained from (3.4) whereas those labeled by β are obtained from (3.5).

The commutativity of this diagram follows easily from lemma 3.2 together with the observation that all horizontal arrows are obtained by Yoneda multiplying with elements of suitable Ext-groups. For example the connecting maps are obtained from multiplying with the element of $\mathrm{Ext}^1(F_2, F_1)$ representing the exact sequence (3.6).

If we now have non-degeneracy for two out of the three pairs (F_1, G), (F, G), (F_2, G) then we also have it for the third pair because of the five-lemma.

The case where we have an exact sequence as in (3.7) is treated similarly. $\square$

To continue we define a some canonical objects in $\mathcal{F}$. Let $a \in \mathbb{N}$. Then we write

$$F^0_{0a} = (x^{-a}k[x], k, 0, \mathrm{id}_{k[x,x^{-1}]})$$
$$F^1_{0a} = (x^{-a}k[x], 0, k, \mathrm{id}_{k[x,x^{-1}]})$$

(the reason for this notation will become clear in Section §5).

Lemma 3.4. *Every object F in $\mathcal{F}$ has a finite filtration $0 = F_0 \subset \cdots \subset F_n = F$ such that the corresponding subquotients are among the F^i_{0a}.*

Proof. By the structure of $\mathcal{C}_x$ there must be a surjective map $\phi : F_x \to E^i$ where $i = 0$ or $i = 1$. Hence $\operatorname{im}\phi$ is a non-trivial quotient. Since it is easy to see that the subobjects of E^i in $\mathcal{C}$ are of the form F^i_{0a} we are done. □

Using (2.4) we can compute the Hom and Ext-groups between the F^i_{0a}. The results are given in the next lemma.

Lemma 3.5. *One has*

$$\mathrm{Hom}(F^i_{0a}, F^j_{0b}) = \begin{cases} k & \text{if } i = j \text{ and } a \le b \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{Ext}^1(F^i_{0a}, F^j_{0b}) = \begin{cases} k & \text{if } i = 1 - j \text{ and } a > b \\ 0 & \text{otherwise} \end{cases}$$

Proof. The claim for Hom is trivial, so we concentrate on Ext.

We use (2.4). This immediately yields that $\mathrm{Ext}^1(F^i_{0a}, F^j_{0b}) = 0$ if $j \neq 1 - i$. If $j = 1 - i$ then we have the following exact sequence.

$$\mathrm{Hom}_{k[x]}(x^{-a}k[x], x^{-b}k[x]) \to k \to \mathrm{Ext}^1_{\mathcal{C}}(F^i_{0a}, F^j_{0b}) \to 0 \tag{3.8}$$

This yields that $\mathrm{Ext}^1_{\mathcal{C}}(F^i_{0a}, F^j_{0b}) = k$ if and only if $\mathrm{Hom}_{k[x]}(x^{-a}k[x], x^{-b}k[x]) = 0$, i.e. if and only if $a > b$. □

We are now in a position to prove the main result of this section.

Theorem 3.6. *$\mathcal{C}$ satisfies Serre duality.*

Proof. We show first that $\mathcal{C}$ satisfies Serre duality for objects F, G in $\mathcal{F}$. We will show the non-degeneracy of (3.4) and (3.5) by induction of $\mathrm{rk}_{k[x]}(F)$, $\mathrm{rk}_{k[x]}(G)$. This reduces us to the case where $F = F^i_{0a}$, $G = F^j_{0b}$. So we need to check the non-degeneracy of

$$\mathrm{Hom}_{\mathcal{C}}(F^i_{0a}, F^j_{0b}) \times \mathrm{Ext}^1_{\mathcal{C}}(F^j_{0b}, F^{i'}_{0,a-1}) \to \mathrm{Ext}^1_{\mathcal{C}}(F^i_{0a}, F^{i'}_{0,a-1}) \to k \tag{3.9}$$

$$\mathrm{Ext}^1_{\mathcal{C}}(F^i_{0a}, F^j_{0b}) \times \mathrm{Hom}_{\mathcal{C}}(F^j_{0b}, F^{i'}_{0,a-1}) \to \mathrm{Ext}^1_{\mathcal{C}}(F^i_{0a}, F^{i'}_{0,a-1}) \to k \tag{3.10}$$

where $i' = 1 - i$. We will concentrate ourselves on (3.10). (3.9) is similar. By (3.5) the only non-trivial case is given by $j = 1 - i$ and $a > b$. In that case all vector spaces involved are equal to k and what we want to prove follows from inspecting (3.8).

Now we show that $\mathcal{C}$ has almost split sequences. By Theorem 1.1 this implies that $\mathcal{C}$ satisfies Serre duality.

By Proposition 2.2 it is clearly sufficient to construct almost split sequences ending in indecomposable objects in $\mathcal{F}$ and $\mathcal{T}$. First let $F \in \mathcal{F}$ be indecomposable. Since $\mathrm{Ext}^1_{\mathcal{C}}(F, VF) \cong \mathrm{Hom}_{\mathcal{C}}(F, F)^*$, $\mathrm{Ext}^1_{\mathcal{C}}(F, VF)$ has a simple socle as (left or right) $\mathrm{Hom}_{\mathcal{C}}(F, F)$

module. Let ζ be a non-zero element this socle. It is well-known, and easy to see that ζ defines the almost split sequence in $\mathcal{F}$ ending in F.

$$0 \to VF \to M \to F \to 0$$

But this is also an almost split sequence in $\mathcal{C}$ since if $T \in \mathcal{T}$ then $\operatorname{Hom}(T, F) = 0$.

Now let $T \in \mathcal{T}$. Then there exists an almost split sequence in $\mathcal{T}$

$$0 \to T'' \to T' \to T \to 0 \tag{3.11}$$

(since $\mathcal{T}$ is equivalent to the category of x-torsion modules over $k[x]$).

We have to prove that any pullback for $C \to T$ of (3.11) with C indecomposable is split. Clearly we only have to consider the case $C \in \mathcal{F}$. But then it follows from Proposition 2.2 that the pullback is split. This finishes the proof. □

4. Relation with one-dimensional graded type A_n-singularities

Let $\mathcal{C}$ be the hereditary category which was described in the previous section. We will now show that $\mathcal{C}$ can be considered as a limit of certain graded simple singularities.

If $m \in \mathbb{N}$ then the graded simple A_{2m-1}-singularity of dimension one is by definition the graded subring R_m of $k[x] \oplus k[x]$ generated by $u = (x, x)$ and $v = (x^m, 0)$. It is easy to see that $R \cong k[u, v]/(u^m v - u^{2m})$ and hence this is equivalent to the classical definition (see for example [2]). We put $\mathcal{C}_m = \operatorname{mod}(R_m)$.

Let us consider $k[x]$ as being diagonally embedded in $k[x] \oplus k[x]$. That is we identify x with (x, x). Clearly we have

$$(R_m)_x = k[x, x^{-1}] \oplus k[x, x^{-1}]$$

Hence if $M \in \operatorname{Mod}(R_m)$ then M_x is canonically a sum of two $k[x, x^{-1}]$-modules which we denote by M_x^0 and M_x^1 respectively. This allows us to define the following functor.

$$U_m : \mathcal{C}_m \to \mathcal{C} : M \mapsto (M, (M_x^0)_0, (M_x^1)_0, \mathrm{id})$$

Clearly U_m is faithful. We have inclusions

$$k[x] \subset \cdots \subset R_{m+1} \subset R_m \subset \cdots R_0 = k[x] \oplus k[x]$$

Dualizing these yield restriction functors

$$\mathcal{C}_0 \to \cdots \to \mathcal{C}_m \to \mathcal{C}_{m+1} \to \cdots \to \operatorname{mod}(k[x])$$

It is clear that these restriction functors are compatible with the functors $(U_m)_m$. Define $\mathcal{C}_\infty$ as the 2-direct limit of the $\mathcal{C}_m$. That is the objects in $\mathcal{C}_\infty$ are the objects in $\coprod_m \mathcal{C}_m$ and we put

$$\operatorname{Hom}_{\mathcal{C}_\infty}(M, N) = \operatorname{inj\,lim} \operatorname{Hom}_{\mathcal{C}_m}(M, N)$$

The functors $(U_m)_m$ define a functor $U_\infty : \mathcal{C}_\infty \to \mathcal{C}$.

Proposition 4.1. *The functor U_∞ defined above is an equivalence.*

Proof. From the definition it is clear that U_∞ is faithful. So we only have to show that it is full and essentially surjective.

We will first show that U_∞ is full. Let $M, N \in \mathcal{C}_m$ and let $f : M \to N$ be a homomorphism in $\mathcal{C}$. So f is in fact a $k[x]$-linear homomorphism $f : M \to N$ such that the localization f_x is $k[x, x^{-1}] \oplus k[x, x^{-1}]$-linear.

Let $y = (x, 0)$. Then $y^n \in R_m$ for $n \geq m$ and R_n as subring of R_m is generated by x and y^n. To prove fullness of U_∞ it is sufficient that f is y^n-linear for $n \gg 0$. Let

T be the torsion submodule of N and consider the $k[x]$ linear map $M \to N$ given by $f^{(n)} = f(y^n -) - y^n f(-)$. Since after localizing at x, f is y linear, it follows that the image of $f^{(n)}$ lies in T. Since T is rightbounded it is clear that $f^{(n)}$ must be zero if $n \gg 0$. This proves what we want.

Now we prove essential surjectivity. First let $F \in \mathcal{F}$. Then we claim that $F \subset F_x$ is stable under multiplication by y^n for $n \gg 0$. First note that yF is a finitely generated $k[x]$-submodule of F_x. Hence $x^n yF \subset F$. Since $x^n y = y^{n+1}$ this proves what we want.

Now let $T \in \mathcal{T}$. Then as graded $k[x]$-module T has right bounded grading and since $k[x]_{<m} = (R_n)_{<m}$ for $n \geq m$ it follows that for $n \gg 0$ we may consider T as a graded R_n-module. This proves what we want. □

5. Representation theory

In section we construct the AR-quiver of $\mathcal{C}$. From the above discussion it follows that the components of the AR-quiver of $\mathcal{C}$ lie either in $\mathcal{T}$ or in $\mathcal{F}$. Since $\mathcal{T}$ is equivalent to the x-torsion modules in $\mathrm{gr}(k[x])$ it has a unique component which is $\mathbb{Z}A_\infty$. So the main difficulty is represented by the component(s) in $\mathcal{F}$.

We now describe the indecomposable torsion free objects in $\mathcal{C}$ as well as the associated Auslander-Reiten quiver (see [5]). Using Proposition 4.1 this could be easily obtained by using a graded version of the results in [2]. However for completeness we give an independent proof here.

For $m > 0$ denote by F_{ma} the unique indecomposable projective R_m-module in with grading starting in degree $-a$ (thus $F_{ma} = F_{m0}(a)$). For $m = 0$ we let F^0_{00}, F^1_{00} be the two indecomposable R_0 modules whose gradings starts exactly at 0. We also put $F^i_{0a} = F^i_{00}(a)$ (as in Section §3).

Finally to simplify the notation we will write F^i_{ma} ($i = \emptyset$, if $m \neq 0$) for $U_m(F^i_{ma})$.

Proposition 5.1. *The indecomposable objects in $\mathcal{F}$ are given by F^i_{ma}. Furthermore the associated Auslander-Reiten quiver is given by Figure 1*

Proof. By Serre duality it follows that $\mathrm{Ext}^1_{\mathcal{C}}(F^i_{ma}, VF^i_{ma})$ is one dimensional. Therefore its unique (up to scalar multiplication) non-zero element represents the almost split sequence ending in F^i_{ma}.

Let us now explicitly construct non-split extensions between F^i_{ma} and VF^i_{ma}. First note

$$VF_{ma} = F_{m,a-1}$$

where for simplicity we have written $F_{0a} = F^0_{0a} \oplus F^1_{1a}$, and

$$VF^i_{0a} = F^{1-i}_{0a-1}$$

To construct the extension associated to $F_{m,a}$ we note that $F_{m-1,a-1}$ and $F_{m+1,a}$ are naturally submodules of $F_{m,a}$ whose sum is $F_{m,a}$ and whose intersection is $F_{m,a-1}$. Hence the exact sequence

$$0 \to F_{m,a-1} \to F_{m-1,a-1} \oplus F_{m+1,a} \to F_{m,a} \to 0 \tag{5.1}$$

yields the sought extension.

To construct the extension associated to F^i_{0a} we note that F_{1a} maps surjectively to F^i_{0a} with kernel F^{1-i}_{0a-1}. Thus in this case the sought extension is

$$0 \to F^{1-i}_{0,a-1} \to F_{1a} \oplus F^i_{0a} \to F_{m,a} \to 0 \tag{5.2}$$

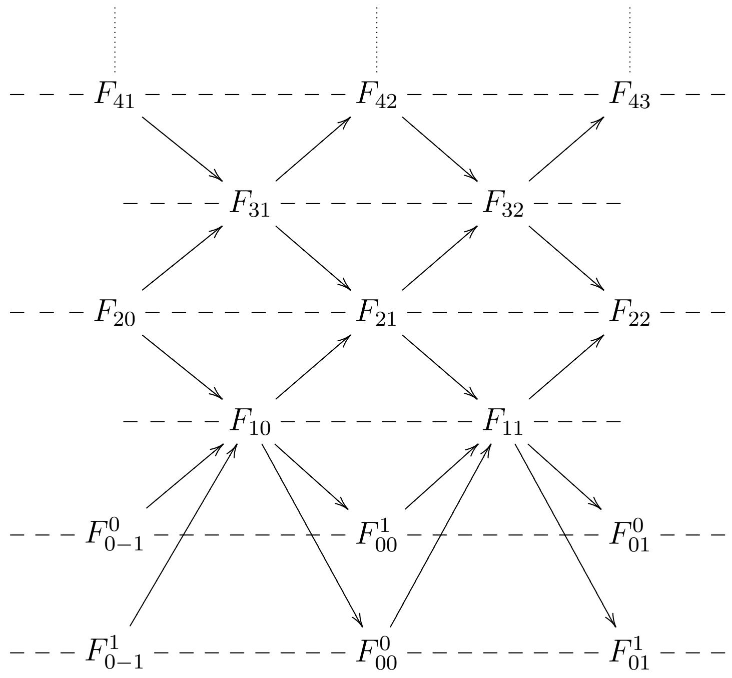

FIGURE 1. The Auslander-Reiten quiver of $\mathcal{F}$

It is now easy to assemble the almost split sequences given by (5.1) and (5.2) into the translation quiver given by Figure 1.

To show that Figure 1 is the entire AR-quiver of $\mathcal{F}$ (and not just a component) we have to show that there are no other indecomposable objects.

So assume that F is an indecomposable object in $\mathcal{F}$, not occurring among the F^i_{ma}. By lemma 3.4 there exist a non-zero map $F^i_{0a} \to F$ for some i, a. Using the defining property of AR-sequences we may use this to construct a non-zero map F^i_{mb} for some i (possibly $\emptyset$) and m, and for b arbitrarily large.

Now note that the only non-trivial torsion free quotients of F^i_{mb} are F^i_{mb} itself and $F^{0,1}_{0b}$ (if $m \neq 0$). Since all these quotients possess a non-trivial element in degree $-b$ it follows that $\mathrm{Hom}_{\mathcal{C}}(F^i_{mb}, F) = 0$ for $b \gg 0$. This finishes the proof. □

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