ON THE DECOMPOSABILITY OF A SYZYGY OF THE RESIDUE FIELD

Ryo Takahashi

1. Introduction

Throughout the present paper, we assume that all rings are commutative noetherian local rings and all modules are finitely generated modules.

Dutta [10] proved the following theorem in his research into the homological conjectures:

Theorem 1.1 (Dutta)**.** *Let $(R, \mathfrak{m}, k)$ be a local ring. Suppose that the nth syzygy module of k has a non-zero direct summand of finite projective dimension for some $n \geq 0$. Then R is regular.*

Since G-dimension is similar to projective dimension, this theorem naturally leads us to the following question:

Question 1.2. Let $(R, \mathfrak{m}, k)$ be a local ring. Suppose that the nth syzygy module of k has a non-zero direct summand of finite G-dimension for some $n \geq 0$. Then is R Gorenstein?

It is obviously seen from the indecomposability of k that this question is true if $n = 0$. Hence this question is worth considering just in the case where $n \geq 1$.

We are able to answer in this paper that the above question is true if $n \leq 2$. Furthermore, we can even determine the structure of a ring satisfying the assumption of the above question for $n = 1, 2$.

2. Main results

For a local ring R, we denote by $\mathrm{mod}\, R$ the category of finitely generated R-modules. First of all, we recall the definition of G-dimension.

Definition 2.1. (1) We denote by $\mathcal{G}(R)$ the full subcategory of $\mathrm{mod}\, R$ consisting of all R-modules M satisfying the following three conditions:

(i) M is reflexive,
(ii) $\mathrm{Ext}_R^i(M, R) = 0$ for every $i > 0$,
(iii) $\mathrm{Ext}_R^i(M^*, R) = 0$ for every $i > 0$.

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(2) Let M be an R-module. If n is a non-negative integer such that there is an exact sequence

$$0 \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$$

of R-modules with $G_i \in \mathcal{G}(R)$ for every i, then we say that M has *G-dimension at most n*, and write $\text{G-dim}_R M \leq n$. If such an integer n does not exist, then we say that M has *infinite G-dimension*, and write $\text{G-dim}_R M = \infty$.

For properties of G-dimension, we refer to [3] or [9].

Proposition 2.2. *Let $(R, \mathfrak{m}, k)$ be a local ring. Suppose that there is a direct sum decomposition $\mathfrak{m} = I \oplus J$ where I, J are non-zero ideals of R. Let M be a non-free indecomposable module in $\mathcal{G}(R)$. Then there exist elements $x, y \in \mathfrak{m}$ such that*

(1) *$I = (x)$ and $J = (y)$,*
(2) *$(0 : x) = (y)$ and $(0 : y) = (x)$,*
(3) *M is isomorphic to either (x) or (y).*

Hence the minimal free resolution of k is as follows:

$$\cdots \xrightarrow{\left(\begin{smallmatrix} y & 0 \\ 0 & x \end{smallmatrix}\right)} R^2 \xrightarrow{\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)} R^2 \xrightarrow{\left(\begin{smallmatrix} y & 0 \\ 0 & x \end{smallmatrix}\right)} R^2 \xrightarrow{(x\ y)} R \longrightarrow k \longrightarrow 0.$$

Proof. The modules M^* and ΩM are also non-free indecomposable modules in $\mathcal{G}(R)$. There are isomorphisms

$$\begin{aligned} M^* &\cong \mathrm{Hom}_R(M, \mathfrak{m}) \\ &= \mathrm{Hom}_R(M, I \oplus J) \\ &\cong \mathrm{Hom}_R(M, I) \oplus \mathrm{Hom}_R(M, J). \end{aligned}$$

The indecomposability of M^* implies that either $\mathrm{Hom}_R(M, I) = 0$ or $\mathrm{Hom}_R(M, J) = 0$. We may assume that

$$\mathrm{Hom}_R(M, J) = 0. \tag{2.2.1}$$

There is an exact sequence

$$0 \to \Omega M \to R^n \to M \to 0. \tag{2.2.2}$$

Dualizing this by J, we obtain another exact sequence

$$\mathrm{Hom}_R(M, J) \to J^n \to \mathrm{Hom}_R(\Omega M, J).$$

We have $\mathrm{Hom}_R(\Omega M, J) \neq 0$ by (2.2.1). Applying the above argument to the module ΩM yields

$$\mathrm{Hom}_R(\Omega M, I) = 0. \tag{2.2.3}$$

Also, dualizing (2.2.2) by I, we get an exact sequence

$$0 \to \mathrm{Hom}_R(M, I) \to I^n \to \mathrm{Hom}_R(\Omega M, I),$$

and hence $M^* \cong \mathrm{Hom}_R(M, I) \cong I^n$. The indecomposability of M^* implies that $n = 1$ (i.e. M is cyclic), and $M^* \cong I$.

We also have

$$\begin{aligned} M &\cong M^{**} \\ &\cong \mathrm{Hom}_R(M^*, \mathfrak{m}) \\ &\cong \mathrm{Hom}_R(M^*, I) \oplus \mathrm{Hom}_R(M^*, J). \end{aligned}$$

Note that $\mathrm{Hom}_R(M^*, I)$ is isomorphic to $\mathrm{Hom}_R(I, I)$, which contains the identity map of I. Hence $\mathrm{Hom}_R(M^*, I) \neq 0$ and therefore

$$\mathrm{Hom}_R(M^*, J) = 0.$$

Applying the above argument to the module M^*, we see that M^* is also cyclic and $M \cong M^{**} \cong I$. Thus, we have shown that $M \cong M^* \cong I$ and these modules are cyclic. Noting (2.2.3) and applying the above argument to the module ΩM, we see that $\Omega M \cong (\Omega M)^* \cong J$ and these modules are cyclic.

Now, writing $I = (x)$ and $J = (y)$, we can prove $(x) = (0 : y)$ and $(0 : x) = (y)$. Thus we obtain the minimal free resolutions of (x) and (y):

$$\begin{cases} \cdots \xrightarrow{y} R \xrightarrow{x} R \xrightarrow{y} R \to (x) \to 0, \\ \cdots \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{x} R \to (y) \to 0. \end{cases}$$

Taking the direct sum of these exact sequence, we get

$$\cdots \xrightarrow{\left(\begin{smallmatrix} y & 0 \\ 0 & x \end{smallmatrix}\right)} R^2 \xrightarrow{\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)} R^2 \xrightarrow{\left(\begin{smallmatrix} y & 0 \\ 0 & x \end{smallmatrix}\right)} R^2 \longrightarrow \mathfrak{m} \longrightarrow 0.$$

Joining this to the natural exact sequence $0 \to \mathfrak{m} \to R \to k \to 0$ constructs the minimal free resolution of k in the assertion. $\square$

We denote by $\operatorname{edim} R$ the embedding dimension of a local ring R. When a homomorphic image of a regular local ring is given, we can choose a minimal presentation of the ring in the following sense:

Proposition 2.3. *Let R be a homomorphic image of a regular local ring. Then there exist a regular local ring $(S, \mathfrak{n})$ and an ideal I of S contained in $\mathfrak{n}^2$ such that $R \cong S/I$.*

Here we introduce a famous result due to Tate [17, Theorem 6]. See also [5, Remarks 8.1.1(3)].

Lemma 2.4 (Tate)**.** *Let $(S, \mathfrak{n}, k)$ be a regular local ring, I an ideal of S contained in $\mathfrak{n}^2$, and $R = S/I$ a residue class ring. Suppose that the complexity of k over R is at most one. (In other words, the set of all the Betti numbers of the R-module k is bounded.) Then I is a principal ideal.*

We denote by $\beta_i^R(M)$ the ith Betti number of a module M over a local ring R. Handling the above results, we can determine the structure of a local ring with decomposable maximal ideal having a non-free module of G-dimension zero, as follows:

Theorem 2.5. *Let $(S, \mathfrak{n}, k)$ be a regular local ring, I an ideal of S contained in $\mathfrak{n}^2$, and $R = S/I$ a residue class ring. Suppose that there exists a non-free R-module in $\mathcal{G}(R)$. Then the following conditions are equivalent:*

(1) *The maximal ideal of R is decomposable;*
(2) *$\dim S = 2$ and $I = (xy)$ for some regular system of parameter x, y of S.*

Proof. Let $\mathfrak{m} = \mathfrak{n}/I$ be the maximal ideal of R.

(2) $\Rightarrow$ (1): It is easy to see that $\mathfrak{m} = xR \oplus yR$ and that xR, yR are non-zero.

(1) $\Rightarrow$ (2): First of all, note from the condition (1) that R is not an integral domain, hence is not a regular local ring.

Proposition 2.2 says that $\mathfrak{m} = xR \oplus yR$ for some $x, y \in \mathfrak{n}$, and that $\beta_i^R(k) = 2$ for every $i \geq 2$. It follows from Lemma 2.4 that I is a principal ideal. Hence R is a hypersurface.

We write $I = (f)$ for some $f \in \mathfrak{n}^2$. Since $\mathfrak{m}$ is decomposable, the local ring R is not artinian. (Over an artinian Gorenstein local ring, the intersection of non-zero ideals is also non-zero; cf. [8, Exercise 3.2.15].) Hence we have $0 < \dim R < \operatorname{edim} R = 2$, which says that $\dim R = 1$ and $\dim S = 2$.

Note that $\mathfrak{n} = (x, y, f)$. Because $\operatorname{edim} S = \dim S = 2$, one of the elements x, y, f belongs to the ideal generated by the other two elements. Noting that the images of elements x, y in $\mathfrak{m}$ form a minimal system of generators of $\mathfrak{m}$, we see that $f \in (x, y)$, and hence x, y is a regular system of parameters of S.

On the other hand, noting $xR \cap yR = 0$, we get $xy \in I = (f)$. Write $xy = cf$ for some $c \in S$. Since the associated graded ring $\operatorname{gr}_{\mathfrak{n}}(S)$ is a polynomial ring over k in two variables $\overline{x}, \overline{y} \in \mathfrak{n}/\mathfrak{n}^2$, we especially have $\overline{xy} \neq 0$ in $\mathfrak{n}^2/\mathfrak{n}^3$, namely, $xy \notin \mathfrak{n}^3$. It follows that $c \notin \mathfrak{n}$ because $f \in \mathfrak{n}^2$. Therefore the element c is a unit of S, and thus $I = (xy)$. □

Using Theorem 2.5 and Cohen's structure theorem, we obtain the following corollary.

Corollary 2.6. *Let $(R, \mathfrak{m})$ be a complete local ring. The following conditions are equivalent:*

(1) *There is a non-free module in $\mathcal{G}(R)$, and $\mathfrak{m}$ is decomposable;*
(2) *R is Gorenstein, and $\mathfrak{m}$ is decomposable;*
(3) *There are a complete regular local ring S of dimension two and a regular system of parameters x, y of S such that $R \cong S/(xy)$.*

The finiteness of G-dimension is independent of completion. Thus, Corollary 2.6 not only gives birth to a generalization of [15, Proposition 2.3] but also guarantees that Question 1.2 is true if $n = 1$.

As far as here, we have observed a local ring whose maximal ideal is decomposable. From here to the end of this paper, we will observe a local ring such that the second syzygy module of the residue class field is decomposable. We begin with the following theorem, which implies that Question 1.2 is true if $n = 2$.

Theorem 2.7. *Let $(R, \mathfrak{m}, k)$ be a local ring. Suppose that $\mathfrak{m}$ is indecomposable and that $\Omega_R^2 k$ has a non-zero proper direct summand of finite G-dimension. Then R is a Gorenstein ring of dimension two.*

Proof. Replacing R with its $\mathfrak{m}$-adic completion, we may assume that R is a complete local ring. In particular, note that R is Henselian.

We have $\Omega_R^2 k = M \oplus N$ for some non-zero R-modules M and N with $\text{G-dim}_R M < \infty$. There is an exact sequence

$$0 \longrightarrow M \oplus N \xrightarrow{(f,g)} R^e \longrightarrow \mathfrak{m} \longrightarrow 0$$

of R-modules, where $e = \operatorname{edim} R$. Setting $A = \operatorname{Coker} f$ and $B = \operatorname{Coker} g$, we get exact sequences

$$\begin{cases} 0 \to M \xrightarrow{f} R^e \xrightarrow{\alpha} A \to 0, \\ 0 \to N \xrightarrow{g} R^e \xrightarrow{\beta} B \to 0. \end{cases} \tag{2.7.1}$$

It is easily observed that there are exact sequences

$$0 \longrightarrow R^e \xrightarrow{\binom{\alpha}{\beta}} A \oplus B \longrightarrow \mathfrak{m} \longrightarrow 0 \tag{2.7.2}$$

and

$$\text{(2.7.3)} \qquad \begin{cases} 0 \longrightarrow M \xrightarrow{\beta f} B \longrightarrow \mathfrak{m} \longrightarrow 0, \\ 0 \longrightarrow N \xrightarrow{\alpha g} A \longrightarrow \mathfrak{m} \longrightarrow 0. \end{cases}$$

Using (2.7.1), (2.7.2) and (2.7.3), we can prove that $\mathrm{Ext}_R^2(k, R) \neq 0$. (Hence $\operatorname{depth} R \leq 2$.)

Fix a non-free indecomposable module $X \in \mathcal{G}(R)$. Applying the functor $\mathrm{Hom}_R(X, -)$ to (2.7.2) gives an exact sequence

$$0 \to (X^*)^e \to \mathrm{Hom}_R(X, A) \oplus \mathrm{Hom}_R(X, B) \to \mathrm{Hom}_R(X, \mathfrak{m}) \to 0$$

and an isomorphism

$$\text{(2.7.4)} \qquad \mathrm{Ext}_R^1(X, A) \oplus \mathrm{Ext}_R^1(X, B) \cong \mathrm{Ext}_R^1(X, \mathfrak{m}).$$

We have $(X^*)^e \in \mathcal{G}(R)$ and $\mathrm{Hom}_R(X, \mathfrak{m}) \in \mathcal{G}(R)$, hence

$$\mathrm{Hom}_R(X, A) \in \mathcal{G}(R).$$

Take the first syzygy module of X; we have an exact sequence

$$0 \to \Omega X \to R^n \to X \to 0.$$

Dualizing this sequence by A, we obtain an exact sequence

$$0 \to \mathrm{Hom}_R(X, A) \to A^n \to \mathrm{Hom}_R(\Omega X, A) \to \mathrm{Ext}_R^1(X, A) \to 0.$$

Divide this into two short exact sequences

$$\text{(2.7.5)} \qquad \begin{cases} 0 \to \mathrm{Hom}_R(X, A) \to A^n \to C \to 0, \\ 0 \to C \to \mathrm{Hom}_R(\Omega X, A) \to \mathrm{Ext}_R^1(X, A) \to 0 \end{cases}$$

of R-modules. Since ΩX is also a non-free indecomposable module in $\mathcal{G}(R)$, applying the above argument to ΩX instead of X shows that the module $\mathrm{Hom}_R(\Omega X, A)$ also belongs to $\mathcal{G}(R)$. We have $\text{G-dim}_R(A^n) < \infty$ by the first sequence in (2.7.1). Hence it follows from (2.7.5) that $\text{G-dim}_R C < \infty$, and

$$\text{(2.7.6)} \qquad \text{G-dim}_R(\mathrm{Ext}_R^1(X, A)) < \infty.$$

On the other hand, applying the functor $\mathrm{Hom}_R(X, -)$ to the natural exact sequence $0 \to \mathfrak{m} \to R \to k \to 0$, we get an exact sequence

$$0 \to \mathrm{Hom}_R(X, \mathfrak{m}) \to X^* \to \mathrm{Hom}_R(X, k) \to \mathrm{Ext}_R^1(X, \mathfrak{m}) \to 0.$$

There is an isomorphism $\mathrm{Hom}_R(X, k) \cong \mathrm{Ext}_R^1(X, \mathfrak{m})$, hence $\mathrm{Ext}_R^1(X, \mathfrak{m})$ is a k-vector space. Since $\mathrm{Ext}_R^1(X, A)$ is contained in $\mathrm{Ext}_R^1(X, \mathfrak{m})$ by (2.7.4),

$$\text{(2.7.7)} \qquad \mathrm{Ext}_R^1(X, A) \text{ is a } k\text{-vector space.}$$

Using (2.7.6) and (2.7.7), we can prove that the local ring R is Gorenstein.

Since the only number i such that $\mathrm{Ext}_R^i(k, R) \neq 0$ is the Krull dimension of R if R is Gorenstein, it follows from the above two claims that R is a Gorenstein local ring of dimension two, which completes the proof of the theorem. $\square$

The above theorem interests us in the observation of a Gorenstein local ring of dimension two such that the second syzygy module of the residue class field is decomposable. We introduce here a related result due to Yoshino and Kawamoto.

A homomorphic image of a convergent power series ring over a field k is called an *analytic* ring over k. Any complete local ring containing a field is an analytic ring over its coefficient field, and it is known that any analytic local ring is Henselian; see [14, Chapter VII]. Yoshino and Kawamoto observed the decomposability of the fundamental module of an analytic normal domain.

Theorem 2.8 (Yoshino-Kawamoto)**.** *Let R be an analytic normal local domain of dimension two. Suppose that the residue class field of R is algebraically closed and has characteristic zero. Then the following conditions are equivalent:*

(1) *The fundamental module of R is decomposable;*
(2) *R is an invariant subring of a regular local ring by a cyclic group. (In other words, R is a cyclic quotient singularity.)*

For the details of this theorem, see [21, Theorem (2.1)] or [19, Theorem (11.12)].

With the notation of the above theorem, suppose in addition that R is a complete Gorenstein ring such that $\Omega_R^2 k$ is decomposable. Then R satisfies the condition (1) in the above theorem. Hence the proof of the above theorem shows that R is of finite Cohen-Macaulay representation type (i.e. there exist only finitely many non-isomorphic maximal Cohen-Macaulay R-modules); see [21] or [19]. Therefore it follows from a theorem of Herzog [12] that R is a hypersurface. Thus the local ring R is a rational double point of type (A_n) for some $n \geq 1$ by [21, Proposition (4.1)], namely, R is isomorphic to

$$k[[X,Y,Z]]/(XY - Z^{n+1}).$$

From a more general viewpoint, we can give a characterization as follows:

Theorem 2.9. *Let $(S, \mathfrak{n}, k)$ be a regular local ring, I an ideal of S contained in $\mathfrak{n}^2$, and $R = S/I$ a residue class ring. Suppose that R is a Henselian Gorenstein ring of dimension two. Then the following conditions are equivalent:*

(1) *$\Omega_R^2 k$ is decomposable;*
(2) *$\dim S = 3$ and $I = (xy - zf)$ for some regular system of parameters x, y, z of S and $f \in \mathfrak{n}$.*

It is necessary to prepare three elementary lemmas to prove this theorem. The first one is both well-known and easy to check, and we omit the proof.

Lemma 2.10. *Let $(S, \mathfrak{n}, k)$ be a regular local ring of dimension three and $R = S/(f)$ a hypersurface with $f \in \mathfrak{n}^2$. Then $f = xf_x + yf_y + zf_z$ for some $f_x, f_y, f_z \in \mathfrak{n}$, and the minimal free resolution of k over R is as follows:*

$$\cdots \xrightarrow{C} R^4 \xrightarrow{D} R^4 \xrightarrow{C} R^4 \xrightarrow{D} R^4 \xrightarrow{C} R^4 \xrightarrow{B} R^3 \xrightarrow{A} R \longrightarrow k \longrightarrow 0,$$

where

$$A = \begin{pmatrix} x & y & z \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -z & y & f_x \\ z & 0 & -x & f_y \\ -y & x & 0 & f_z \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & -f_z & f_y & x \\ f_z & 0 & -f_x & y \\ -f_y & f_x & 0 & z \\ -x & -y & -z & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & -z & y & f_x \\ z & 0 & -x & f_y \\ -y & x & 0 & f_z \\ -f_x & -f_y & -f_z & 0 \end{pmatrix}.$$

Lemma 2.11. *Let $(R, \mathfrak{m}, k)$ be a local ring and $x \in \mathfrak{m} - \mathfrak{m}^2$ an R-regular element. Then we have a split exact sequence*

$$0 \to k \xrightarrow{\theta} \mathfrak{m}/x\mathfrak{m} \xrightarrow{\pi} \mathfrak{m}/xR \to 0,$$

where θ is defined by $\theta(\overline{a}) = \overline{xa}$ for $\overline{a} \in R/\mathfrak{m} = k$ and π is the natural surjection.

Proof. Let $x_1, x_2, \dots, x_n$ be a minimal system of generators of $\mathfrak{m}$ with $x_1 = x$. Define a homomorphism $\varepsilon : \mathfrak{m}/x\mathfrak{m} \to k$ by $\varepsilon(\sum_{i=1}^n \overline{x_i a_i}) = \overline{a_1}$. We easily see that the composite map $\varepsilon\theta$ is the identity map of k, which means that θ is a split-monomorphism. $\square$

Lemma 2.12. *Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring of dimension one. Then the following conditions are equivalent:*

(1) *R is a discrete valuation ring;*
(2) *$\mathfrak{m}^*$ is a cyclic R-module.*

Proof. (1) $\Rightarrow$ (2): This implication is obvious since the maximal ideal $\mathfrak{m}$ is a free R-module of rank one.

(2) $\Rightarrow$ (1): We have $\mathfrak{m}^* \cong R/I$ for some ideal I of R. Dualizing the natural exact sequence $0 \to \mathfrak{m} \to R \to k \to 0$, we obtain an exact sequence

$$\mathrm{Hom}_R(k, R) \to R \to \mathfrak{m}^*.$$

Since $\mathrm{Hom}_R(k, R) = 0$ by the assumption that R is Cohen-Macaulay, there is an injective homomorphism $R \to R/I$. We easily observe that $I = 0$, equivalently, $\mathfrak{m}^* \cong R$. This implies the condition (1). $\square$

Let R be a local ring and I an ideal of R. We recall that the *grade* of I is defined to be the infimum of the integers n such that $\mathrm{Ext}_R^n(R/I, R) \neq 0$, and is denoted by $\operatorname{grade} I$. As is well-known, it coincides with the length of any maximal R-sequence in I. Now let us prove Theorem 2.9.

Proof of Theorem 2.9. (2) $\Rightarrow$ (1): We have $xy - zf = x \cdot 0 + y \cdot x + z \cdot (-f)$. Lemma 2.10 gives a finite free presentation

$$R^4 \xrightarrow{C} R^4 \longrightarrow \Omega_R^2 k \longrightarrow 0$$

of the R-module $\Omega_R^2 k$, where $C = \begin{pmatrix} 0 & f & x & x \\ -f & 0 & 0 & y \\ -x & 0 & 0 & z \\ -x & -y & -z & 0 \end{pmatrix}$. Putting $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, we obtain

$$PCQ = \begin{pmatrix} U & 0 \\ 0 & {}^tU \end{pmatrix}$$

where $U = \left(\begin{smallmatrix} x & f \\ z & y \end{smallmatrix}\right)$. It is easily seen that the matrices P, Q are invertible. Denoting by M (resp. N) the cokernel of the homomorphism defined by the matrix U (resp. ${}^tU$), we get an isomorphism $\Omega_R^2 k \cong M \oplus N$.

(1) $\Rightarrow$ (2): First of all, note that the local ring R is not regular. We denote by $\mathfrak{m}$ the maximal ideal $\mathfrak{n}/I$ of R.

We can choose an element $z \in \mathfrak{n} - \mathfrak{n}^2$ whose image in $\mathfrak{m}$ is an R-regular element and that the module $\mathfrak{m}/zR$ is decomposable. Put $\overline{(-)} = (-) \otimes_S S/(z)$. Note that $\overline{S}$ is also a regular local ring because z is a minimal generator of the maximal ideal $\mathfrak{n}$ of S (see the proof of Proposition 2.3). Since the maximal ideal $\mathfrak{m}\overline{R}$ of $\overline{R}$ is decomposable, we can apply Theorem 2.5 and see that $\dim \overline{S} = 2$ and $I\overline{S} = xy\overline{S}$ for some $x, y \in \mathfrak{n}$ whose images in $\overline{S}$ form a regular system of parameter of $\overline{S}$. Hence $\overline{R} = \overline{S}/xy\overline{S}$ is a hypersurface, in particular a complete intersection, of dimension one. Therefore R is a complete intersection of dimension two by [8, Theorem 2.3.4(a)]. Since S is a regular local ring of dimension three with regular system of parameter x, y, z, the ideal I is generated by an S-sequence by [8, Theorem 2.3.3(c)]. Noting $\operatorname{ht} I = \dim S - \dim R = 1$, we see that I is a principal ideal. Write $I = (l)$ for some $l \in I$. There is an element $f \in S$ such that $l = xy - zf$. Assume that $f \notin \mathfrak{n}$. Then f is a unit of S, and we see that $zR \subseteq xyR$. Hence $\mathfrak{m} = (x, y)R$, and $\operatorname{edim} R = \dim R = 2$. This implies that R is regular, which is a contradiction. It follows that $f \in \mathfrak{n}$. $\square$

Combining Theorem 2.7 with Theorem 2.9 gives birth to the following corollary. Compare it with Corollary 2.6.

Corollary 2.13. *Let $(R, \mathfrak{m}, k)$ be a complete local ring. Suppose that $\mathfrak{m}$ is indecomposable. Then the following conditions are equivalent:*

(1) *$\Omega_R^2 k$ has a non-zero proper direct summand of finite G-dimension;*
(2) *R is Gorenstein, and $\Omega_R^2 k$ is decomposable;*
(3) *There are a complete regular local ring $(S, \mathfrak{n})$ of dimension three, a regular system of parameters x, y, z of S, and $f \in \mathfrak{n}$ such that $R \cong S/(xy - zf)$.*

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Department of Mathematics,
Faculty of Science,
Okayama University,
1-1 Naka 3-chome Tsushima,
Okayama, 700-8530, JAPAN

E-mail address: `takahasi@math.okayama-u.ac.jp`