A construction of Auslander-Gorenstein rings*

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This is a summary of my joint work with T. Shiba.

Recall that a ring A is said to be an Auslander-Gorenstein ring if it is a left and right noetherian ring and for a minimal injective resolution $E^\bullet$ of A_A we have flat dim $E^n \le n$ for all $n \ge 0$ (cf. [1]). It is well known that the group ring of a finite group over a commutative Gorenstein ring is an Auslander-Gorenstein ring. We refer to[1] for other examples of Auslander-Gorenstein rings. Our main aim is to construct another type of Auslander-Gorenstein rings.

Let R be a ring. In this talk, a ring A is said to be a Frobenius extension of R if it contains R as a subring and satisfies the conditions (F1) A_R and $_RA$ are finitely generated projective, (F2) $A_A \cong \mathrm{Hom}_R({}_AA_R, R_R)$ and $_AA \cong \mathrm{Hom}_R({}_RA_A, {}_RR)$, and (F3) the inclusion $R \to A$ is a split monomorphism of R-R-bimodules (cf. [2]). Again, the group ring of a finite group is a typical example of Frobenius extensions. If A is a Frobenius extension of R, then (1) inj dim A_A = inj dim R_R, (2) inj dim $_AA$ = inj dim $_RR$, and (3) A is an Auslander-Gorenstein ring if R is so. Therefore we will provide a way to construct Frobenius extensions of a given ring.

Let R be a ring, $n \ge 2$ an integer and ν a permutation of $I = \{1, 2, ----, n\}$. We will construct a family of Frobenius extensions A of R such that (i) $1_A = \Sigma_{i \in I}\, e_i$ with the e_i orthogonal idempotents, (ii) $e_iA_A \not\cong e_jA_A$ unless $i = j$, (iii) $xe_i = e_ix$ for all $i \in I$ and $x \in R$, (iv) $e_iA_A \cong \mathrm{Hom}_R({}_AAe_{\nu(i)R}, R_R)$ and $_AAe_{\nu(i)} \cong \mathrm{Hom}_R({}_Re_iA_A, {}_RR)$ for all $i \in I$, and (v) there exists $\eta \in \mathrm{Aut}(A)$ with $\eta(e_i) = e_{\nu(i)}$ for all $i \in I$. Furthermore, the rings e_iAe_i are local if R is so. In particular, if R is a quasi-Frobenius local ring, then A is a quasi-Frobenius ring with $\mathrm{soc}(e_iA_A) \cong e_{\nu(i)}A/e_{\nu(i)}J$ for all $i \in I$, where $J = \mathrm{rad}(A)$. In case ν has no fixed point, we can construct a desired Frobenius extension A of R as a skew matrix ring over R, the notion of which was first introduced in [3] (cf. also [4] and [5]). If ν has a fixed point, then we can not construct a desired Frobenius extension of R as a skew matrix ring over R, but we can construct a desired Frobenius extension B of R which contains an ideal V with B/V a skew matrix ring over R, where $V^2 \neq 0$ in general.

Throughout this note, rings are associative rings with identity. For a ring R, we denote by $R^\times$ the set of units and by rad(R) the Jacobson radical. We use the notation X_R (resp., $_RX$) to denote that the module X considered is a right(resp., left) R-module. Also, we use the notation $_SX_R$ to denote that X is an S-R-bimodule. We denote by Mod-R the category of right R-modules.

1. Frobenius extension of rings

In this section, we introduce a notion of Frobenius extension of rings (cf. [2]).

Definition 1.1. A ring A is said to be a Frobenius extension of a ring R if there exists an injective ring homomorphism $\varphi : R \to A$ satisfying the following conditions:

(F1) A_R and $_RA$ are finitely generated projective;

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*The detailed version will be submitted for publication elsewhere.

(F2) $A_A \cong \mathrm{Hom}_R({}_AA_R, R_R)$ and ${}_AA \cong \mathrm{Hom}_R({}_RA_A, {}_RR)$; and
(F3) φ is a split monomorphism of R-R-bimodules.

Remark 1.2. Let A be a Frobenius extension of R. Then for an isomorphism $\phi : A_A \xrightarrow{\sim} \mathrm{Hom}_R({}_AA_R, R_R)$ we have a unique ring homomorphism $\theta : R \to A$ such that $x\phi(1) = \phi(1)\theta(x)$ for all $x \in R$. Similarly, for an isomorphism $\psi : {}_AA \xrightarrow{\sim} \mathrm{Hom}_R({}_RA_A, {}_RR)$ we have a unique ring homomorphism $\eta : R \to A$ such that $\psi(1)x = \eta(x)\psi(1)$ for all $x \in R$.

Definition 1.3 (cf. [1]). A ring R is said to be an Auslander-Gorenstein ring if it is left and right noetherian and for a minimal injective resolution $E^\bullet$ of R_R we have flat dim $E^n \leq n$ for all $n \geq 0$.

Definition 1.4. A ring R is said to be a quasi-Frobenius ring if it is left and right artinian and left and right selfinjective.

Proposition 1.5. *Let A be a Frobenius extension of R. Then the following hold.*
(1) inj dim A_A = inj dim R_R.
(2) inj dim ${}_AA$ = inj dim ${}_RR$.
(3) *A is an Auslander-Gorenstein ring if R is so.*
(4) *A is a quasi-Frobenius ring if R is so.*

Remark 1.6. The converse holds in (3) and (4) of Proposition 1.5. However, we do not need this fact in the present note.

In the following, taking Proposition 1.5 into account, we provide a way to construct Frobenius extensions of an arbitrary ring.

2. Skew matrix rings

In the following, we fix a ring R and a pair of $\sigma \in \mathrm{Aut}(R)$ and $c \in R$ such that

$$\sigma(c) = c \quad \text{and} \quad xc = c\sigma(x) \quad \text{for all } x \in R.$$

In this section, we develop the construction of skew matrix rings given in [3], [4] and [5]. Let $n \geq 2$ be an integer and $I = \{1, 2, ..., n\}$. Let $\omega : I \times I \to \mathbb{Z}$ be a mapping and set

$$\lambda(i, j, k) = \omega(i, j) + \omega(j, k) - \omega(i, k)$$

for $i, j, k \in I$. We assume the following conditions are satisfied:

$$\omega(i, i) = 0 \quad \text{for all } i \in I \quad \text{and} \quad \lambda(i, j, k) \geq 0 \text{ for all } i, j, k \in I.$$

Definition 2.1. Let A be a free right R-module with a basis $\{e_{ij}\}_{i,\, j\, \in\, I}$ and define the multiplication on A subject to the following axioms:

(A1) $xe_{ij} = e_{ij}\sigma^{\omega(i,j)}(x)$ for all $i, j \in I$ and $x \in R$;
(A2) $e_{ij}e_{kl} = 0$ unless $j = k$; and
(A3) $e_{ij}e_{jk} = e_{ik}c^{\lambda(i,j,k)}$ for all $i, j, k \in I$.

Proposition 2.2. *The following hold.*
(1) *A is an associative ring with $1_A = \sum_{i \in I} e_{ii}$, where the e_{ii} are orthogonal idempotents.*
(2) *We have an injective ring homomorphism $\varphi : R \to A$, $x \mapsto \sum_{i \in I} e_{ii}x$ which is a split monomorphism of R-R-bimodules.*
(3) $e_{ii}A_A \cong e_{jj}A_A$ for all $i, j \in I$ *with $\lambda(i, j, i) = 0$. In case $c \notin R^\times$, the converse holds.*

In the following, we consider R as a subring of A via $\varphi : R \to A$. Note that A is a free left R-module with a basis $\{e_{ij}\}_{i,j \in I}$. Also, for any $i \in I$, since $xe_{ii} = e_{ii}x$ for all $x \in R$, Ae_{ii} is an A-R-bimodule and $e_{ii}A$ is an R-A-bimodule.

Remark 2.3. Denote by $\mathrm{M}_n(R)$ the $n \times n$ full matrix ring over R. Then for any $i \in I$ there exists a ring homomorphism of the form

$$\zeta: A \xrightarrow{\sim} \mathrm{M}_n(R),\ \sum_{j,k \in I} e_{jk}x_{jk} \mapsto (\sigma^{-\omega(i,j)}(c^{\lambda(i,j,k)}x_{jk}))_{j,k \in I}$$

which is an isomorphism if either $c \in R^\times$ or $\lambda = 0$. Also, if c is regular, then ζ is injective.

In the following, taking Remark 2.3 into account, we assume $c \notin R^\times$. However, for the sake of convenience, we do not exclude the case where $\lambda = 0$.

Definition 2.4. There exists a basis $\{\alpha_{ij}\}_{i,j \in I}$ for ${}_R\mathrm{Hom}_R(A_R, {}_RR_R)$ such that $a = \sum_{i,j \in I} e_{ij}\alpha_{ij}(a)$ for all $a \in A$. Similarly, there exists a basis $\{\beta_{ij}\}_{i,j \in I}$ for $\mathrm{Hom}_R({}_RA, {}_RR_R)_R$ such that $a = \sum_{i,j \in I} \beta_{ij}(a)e_{ij}$ for all $a \in A$.

Lemma 2.5. *For any $i, k \in I$ the following are equivalent.*
(1) $\lambda(i, j, k) = 0$ *for all* $j \in I$.
(2) *There exist isomorphisms of the form*

$$\phi_{ik} : e_{ii}A_A \xrightarrow{\sim} \mathrm{Hom}_R({}_AAe_{kkR}, R_R),\ a \mapsto \alpha_{ik}a,$$
$$\psi_{ki} : {}_AAe_{ii} \xrightarrow{\sim} \mathrm{Hom}_R({}_Re_{ii}A_A, {}_RR),\ a \mapsto a\beta_{ik}.$$

(3) *Either* $e_{ii}A_A \cong \mathrm{Hom}_R({}_AAe_{kkR}, R_R)$ *or* ${}_AAe_{kk} \cong \mathrm{Hom}_R({}_Re_{ii}A_A, {}_RR)$.

Proposition 2.6. *Assume R is a local ring. Then the following hold.*
(1) *Every e_{ii} is a local idempotent. In particular, A is a semiperfect ring.*
(2) *Assume either* $A_A \cong \mathrm{Hom}_R({}_AA_R, R_R)$ *or* ${}_AA \cong \mathrm{Hom}_R({}_RA_A, {}_RR)$. *Then there exists* $\nu \in \mathrm{Aut}(I)$ *such that* $\lambda(i, j, \nu(i)) = 0$ *for all* $i, j \in I$.

In the next Proposition, we refer to [6] for derived equivalence of rings.

Proposition 2.7. *Assume c is regular. Then for any $i \in I$ the following hold.*

(1) *A is derived equivalent to a generalized triangular matrix ring*

$$\begin{bmatrix} R & \mathrm{Ext}^1_A(Ae_{ii}A, e_{ii}A) \\ 0 & A/Ae_{ii}A \end{bmatrix}$$

(2) *Assume there exists $j \in I \backslash \{i\}$ such that $\lambda(j, k, i) = 0$ for all $k \in I$. Then $\mathrm{Ext}^1_A(A/Ae_{ii}A, e_{ii}A) \cong e_{jj}(A/Ae_{ii}A)$ as R-$A/Ae_{ii}A$-bimodules.*

In the following, we do not fix ω and use the notation A_ω to denote that the multiplication of A is defined by ω.

3. Classification of ω

In the following, with each $\omega : I \times I \to \mathbb{Z}$ we associate $\lambda : I \times I \times I \to \mathbb{Z}$ such that

$$\lambda(i, j, k) = \omega(i, j) + \omega(j, k) - \omega(i, k)$$

for all $i, j, k \in I$.

Lemma 3.1. *For any ω, $\omega' : I \times I \to \mathbb{Z}$ the following are equivalent.*

(1) $\lambda = \lambda'$.

(2) *There exists $\chi : I \to \mathbb{Z}$ such that $\omega'(i, j) = \omega(i, j) - \chi(i) + \chi(j)$ for all $i, j \in I$.*

Definition 3.2. For ω, $\omega' : I \times I \to \mathbb{Z}$, we set $\omega \equiv \omega'$ if the equivalent conditions of Lemma 3.1 are satisfied. Also, for $\omega : I \times I \to \mathbb{Z}$, we write $\omega \geq 0$ if $\omega(i, j) \geq 0$ for all $i, j \in I$.

Lemma 3.3. *Let ω, $\omega' : I \times I \to \mathbb{Z}$ with $\omega \equiv \omega'$ and assume there exists $i_0 \in I$ such that $\omega(i_0, j) = \omega'(i_0, j)$ for all $j \in I$. Then $\omega = \omega'$*

Definition 3.4. We denote by Ω the set of $\omega : I \times I \to \mathbb{Z}$ such that $\omega(i, i) = 0$ for all $i \in I$ and $\lambda(i, j, k) \geq 0$ for all $i, j, k \in I$. Also, for each $\nu \in \mathrm{Aut}(I)$, we set $I(\nu) = \{i \in I \mid \nu(i) = i\}$ and denote by $\Omega(\nu)$ the set of $\omega \in \Omega$ such that

(1) $\lambda(i, j, \nu(i)) = 0$ for all $i \in I \backslash I(\nu)$ and $j \in I$, and

(2) there exists $t_\omega \geq 0$ such that $\lambda(i, j, i) = t_\omega$ for all $i \in I(\nu)$ and $j \in I \backslash \{i\}$.

Lemma 3.5. *For any $\omega \in \Omega$ and $i_0 \in I$ the following hold.*

(1) *If $\lambda(i_0, j, i_0) = 0$ for all $j \in I$, then $\omega \equiv 0$.*

(2) *If $\omega(i_0, j) = 0$ for all $j \in I$, then $\omega \geq 0$.*

(3) *There exists $\omega' \in \Omega$ such that $\omega \equiv \omega'$, $\omega' \geq 0$ and $\omega'(i_0, j) = 0$ for all $j \in I$.*

Definition 3.6. We denote by Ω_b the set of $\omega \in \Omega$ such that $\lambda(i, j, i) > 0$ for all $i, j \in I$

with $i \neq j$. Also, we set $\Omega_b(\nu) = \Omega_b \cap \Omega(\nu)$ for $\nu \in \mathrm{Aut}(I)$.

Proposition 3.7. *For any* $\nu \in \mathrm{Aut}(I)$ *we have* $\Omega_b(\nu) \neq \varnothing$.

4. Automorphisms of A_ω

In this section, we show that for any $\nu \in \mathrm{Aut}(I)$ and $\omega \in \Omega$ there exists $\eta \in \mathrm{Aut}(A_\omega)$ with $\eta(e_{ij}) = e_{\nu(i),\,\nu(j)}$ for all $i, j \in I$.

Lemma 4.1. *For any* $\omega, \omega' \in \Omega$ *the following hold*.

(1) *If there exists* $\nu \in \mathrm{Aut}(I)$ *such that* $\omega' = \omega \circ (\nu \times \nu)$, *then there exists a ring isomorphism of the form*

$$\xi_1 : A_\omega \xrightarrow{\sim} A_{\omega'},\ \textstyle\sum_{i,\,j\in I} e_{ii}x_{ii} \mapsto \sum_{i,\,j\in I} e_{ii}x_{\nu(i),\,\nu(j)}.$$

(2) *If there exists* $\chi : I \to \mathbb{Z}$ *such that* $\omega'(i, j) = \omega(i, j) - \chi(i) + \chi(j)$ *for all* $i, j \in I$, *then there exists a ring isomorphism of the form*

$$\xi_2 : A_\omega \xrightarrow{\sim} A_{\omega'},\ \textstyle\sum_{i,\,j\in I} e_{ii}x_{ij} \mapsto \sum_{i,\,j\in I} e_{ii}\sigma^{\chi(j)}(x_{ij}).$$

Proposition 4.2. *For any* $\nu \in \mathrm{Aut}(I)$ *and* $\omega \in \Omega(\nu)$ *the following hold*.

(1) *Define* $\chi : I \to \mathbb{Z}$ *as follows*:

$$\chi(i) = \begin{cases} \omega(i, \nu(i)) & \text{if } i \notin I(\nu), \\ t_\omega & \text{if } i \in I(\nu). \end{cases}$$

Then $\omega(\nu(i), \nu(j)) = \omega(i, j) - \chi(i) + \chi(j)$ *for all* $i, j \in I$.

(2) *There exist ring automorphisms of the form*

$$\theta : A_\omega \xrightarrow{\sim} A_\omega,\ \textstyle\sum_{i,j\in I} e_{ii}x_{ij} \mapsto \sum_{i,\,j\in I} e_{ii}\sigma^{-\chi(j)}(x_{\nu(i),\,\nu(j)}),$$
$$\eta : A_\omega \xrightarrow{\sim} A_\omega,\ \textstyle\sum_{i,j\in I} e_{ii}x_{ij} \mapsto \sum_{i,\,j\in I} e_{\nu(i),\nu(j)}\sigma^{\chi(j)}(x_{ij})$$

which are mutually inverse.

(3) *Let* $e_0 = \sum_{i\in N(\nu)} e_{i,\,\nu(i)} + \sum_{i\in I(\nu)} e_{ii}\,c^t$, *where* $t = t_\omega$ *if* $I(\nu) \neq \varnothing$. *Then* $\eta(e_0) = e_0$ *and* $ae_0 = e_0\eta(a)$ *for all* $a \in A_\omega$.

5. The case of ν with $I(\nu) = \varnothing$

In this section, we deal with the case of $\nu \in \mathrm{Aut}(I)$ with $I(\nu) = \varnothing$. Let $\omega \in \Omega_b(\nu)$ and $A = A_\omega$. We set $\alpha_0 = \sum_{i\in I} \alpha_{i,\,\nu(i)}$ and $\beta_0 = \sum_{i\in I} \beta_{i,\,\nu(i)}$ (see Definition 2.4). By Proposition 4.2, we have ring automorphisms

$$\theta : A_\omega \xrightarrow{\sim} A_\omega,\ \textstyle\sum_{i,j\in I} e_{ii}x_{ij} \mapsto \sum_{i,j\in I} e_{ii}\sigma^{-\chi(j)}(x_{\nu(i),\nu(j)}),$$
$$\eta : A_\omega \xrightarrow{\sim} A_\omega,\ \textstyle\sum_{i,j\in I} e_{ii}x_{ij} \mapsto \sum_{i,j\in I} e_{\nu(i),\nu(j)}\sigma^{\chi(j)}(x_{ij})$$

which are mutually inverse.

Lemma 5.1. *There exist isomorphisms of the form*

$$\phi: A_A \xrightarrow{\sim} \mathrm{Hom}_R({}_A Ae_{kkR}, R_R),\ a \mapsto \alpha_{ik}a,$$
$$\psi: {}_A A \xrightarrow{\sim} \mathrm{Hom}_R({}_R e_{ii}A_A, {}_R R),\ a \mapsto a\beta_{ik}.$$

Remark 5.2. The following hold.

(1) θ is the unique ring automorphism of A such that $\theta(e_{\nu(i),\nu(j)}) = e_{ii}$ for all $i, j \in I$ and $x\alpha_0 = \alpha_0\theta(x)$ for all $x \in R$.

(2) η is the unique ring automorphism of A such that $\eta(e_{ii}) = e_{\nu(i),\nu(j)}$ for all $i, j \in I$ and $\beta_0 x = \eta(x)\beta_0$ for all $x \in R$.

Theorem 5.3. *The ring A is a Frobenius extensin of R.*

Proposition 5.4. *Assume R is a quasi-Frobenius local ring. Then A is a quasi-Frobenius ring with* $\mathrm{soc}(e_{ii}A_A) \cong e_{\nu(i),\nu(i)}A/e_{\nu(i),\nu(i)}J$ *for all $i \in I$, where* $J = \mathrm{rad}(A)$.

6. Another base ring

In this section, we prepare another base ring S which we need in the next section. We fix an integer $t > 0$.

Definition 6.1. Let S be a free right R-module with a basis $\{e, v\}$ and define the multiplication on S subject to the following axioms:

(S1) $e^2 = e$, $v^2 = -vc^t$ and $ev = v = ve$; and

(S2) $xe = ex$ and $xv = v\sigma^t(x)$ for all $x \in R$.

Lemma 6.2. *The following hold.*

(1) *S is an associative ring with $1_S = e$.*

(2) *We have ring homomorphisms $\varphi : R \to S$, $x \mapsto ex$ and $\pi : S \to R$, $(ex + vy) \mapsto x$ with* $\pi\varphi = \mathrm{id}$.

(3) *S is a local ring if R is so.*

In the following, we consider R as a subring of S via $\varphi : R \to S$. Note that S is a free left R-module with a basis $\{e, v\}$.

Definition 6.3. There exists a basis $\{\alpha, \mu\}$ for ${}_R\mathrm{Hom}_R(S_R, {}_R R_R)$ such that $b = e\alpha(b) + v\mu(b)$ for all $b \in S$. Similarly, there exists a basis $\{\beta, \rho\}$ for $\mathrm{Hom}_R({}_R S, {}_R R_R)_R$ such that $b = \beta(b)e + \rho(b)v$ for all $b \in S$.

Lemma 6.4. *There exist isomorphisms of the form*

$$\phi: S_S \xrightarrow{\sim} \mathrm{Hom}_R({}_SS_R, R_R),\ b \mapsto \mu b,$$
$$\psi: {}_SS \xrightarrow{\sim} \mathrm{Hom}_R({}_RS_S, {}_RR),\ b \mapsto b\rho.$$

Theorem 6.5. *The ring S is a Frobenius extension of R.*

Proposition 6.6. *There exist ring automorphisms of the form*

$$\theta: S \xrightarrow{\sim} S,\ ex + vy \mapsto e\sigma^{-t}(x) + v\sigma^{-t}(y),$$
$$\eta: S \xrightarrow{\sim} S,\ ex + vy \mapsto e\sigma^{t}(x) + v\sigma^{t}(y)$$

which are mutually inverse.

Remark 6.7. The following hold.
(1) θ is the unique ring automorphism of S such that $\theta(v) = v$ and $x\mu = \mu\theta(x)$ for all $x \in R$.
(2) η is the unique ring automorphism of S such that $\eta(v) = v$ and $\rho x = \eta(x)\rho$ for all $x \in R$.

7. The case of ν with $I(\nu) \neq \emptyset$

In this and the next sections, we deal with the case of ν with $I(\nu) \neq \emptyset$. Let $\omega \in \Omega_b(\nu)$ and $t = t_\omega$. We construct a Frobenius extension B of R which contains an ideal V with $B/V \cong A_\omega$, where $V^2 \neq 0$ in general.

Remark 7.1. It may happen that $\omega \in \Omega_b(\tau)$ for some τ with $I(\tau) = \emptyset$. If this is the case, A_ω itself is a Frobenius extension of R.

Definition 7.2. Let B be a free right R-module with a basis $\{e_{ij}\}_{i,j \in I} \cup \{v_i\}_{i \in I(\nu)}$ and define the multiplication on B subject to the following axioms:
(B1) $xe_{ij} = e_{ij}\sigma^{\omega(i,j)}(x)$ for all $i, j \in I$ and $x \in R$;
(B2) $e_{ij}e_{kl} = 0$ unless $j = k$;
(B3) $e_{ij}e_{jk} = e_{ik}c^{\lambda(i,j,k)}$ unless $i = k \in I(\nu)$ and $j \in I\setminus\{i\}$;
(B4) $e_{ij}e_{ji} = v_i + e_{ii}c^t$ for all $i \in I(\nu)$ and $j \in I\setminus\{i\}$;
(B5) $xv_i = v_i\sigma^t(x)$ for all $i \in I(\nu)$ and $x \in R$;
(B6) $v_ie_{jk} = 0 = e_{ij}v_k$ unless $i = j = k$;
(B7) $v_ie_{ii} = v_i = e_{ii}v_i$ for all $i \in I(\nu)$;
(B8) $v_iv_j = 0$ unless $i = j$; and
(B9) $v_iv_i = -v_ic^t$ for all $i \in I(\nu)$.

Proposition 7.3. *The following hold.*
(1) *B is an associative ring with $1_B = \sum_{i \in I} e_{ii}$, where the e_{ii} are orthogonal idempotents.*
(2) *We have an injective ring homomorphism $\varphi: R \to B$, $x \mapsto \sum_{i \in I} e_{ii}x$ which is a split*

monomorphism of R-R-bimodules.

(3) $e_{ii}B_B \cong e_{jj}B_B$ *only when* $i = j$.

(4) $V = \sum_{i \in I(\nu)} v_iR$ *is an ideal of* B *with* $B/V \cong A_\omega$.

In the following, we consider R as a subring of B via $\varphi : R \to B$. Note that B is a free left R-module with a basis $\{e_{ij}\}_{i,j \in I} \cup \{v_i\}_{i \in I(\nu)}$. Also, for any $i \in I$, since $xe_{ii} = e_{ii}x$ for all $x \in R$, Be_{ii} is a B-R-bimodule and $e_{ii}B$ is an R-B-bimodule.

Definition 7.4. There exists a basis $\{\alpha_{ij}\}_{i,j \in I} \cup \{\mu_i\}_{i \in I(\nu)}$ for ${}_R\mathrm{Hom}_R(B_R, {}_RR_R)$ such that $b = \sum_{i,j \in I} e_{ii}\alpha_{ij}(b) + \sum_{i \in I(\nu)} v_i\mu_i(b)$ for all $b \in B$. Similarly, we have a basis $\{\beta_{ij}\}_{i,j \in I} \cup \{\rho_i\}_{i \in I(\nu)}$ for $\mathrm{Hom}_R({}_RB, {}_RR_R)_R$ such that $b = \sum_{i,j \in I} \beta_{ij}(b)e_{ii} + \sum_{i \in I(\nu)} \rho_i(b)v_i$ for all $b \in B$. We set

$$\mu_0 = \sum_{i \in I\setminus I(\nu)} \alpha_{i,\nu(i)} + \sum_{i \in I(\nu)} \mu_i \quad \text{and} \quad \rho_0 = \sum_{i \in I\setminus I(\nu)} \beta_{i,\nu(i)} + \sum_{i \in I(\nu)} \rho_i.$$

Lemma 7.5. *The following hold.*

(1) *For any* $i \in I(\nu)$ *there exist isomorphisms of the form*

$$\phi_i : e_{ii}B_B \xrightarrow{\sim} \mathrm{Hom}_R({}_BBe_{kkR}, R_R),\ b \mapsto \mu_i b,$$
$$\psi_i : {}_BBe_{ii} \xrightarrow{\sim} \mathrm{Hom}_R({}_Re_{ii}B_B, {}_RR),\ b \mapsto b\rho_i.$$

(2) *For any* $i \in I\setminus I(\nu)$ *there exist isomorphisms of the form*

$$\phi_i : e_{ii}B_B \xrightarrow{\sim} \mathrm{Hom}_R({}_BBe_{\nu(i),\nu(i)R}, R_R),\ b \mapsto \alpha_{i,\nu(i)}b,$$
$$\psi_i : {}_BBe_{\nu(i),\nu(i)} \xrightarrow{\sim} \mathrm{Hom}_R({}_Re_{ii}B_B, {}_RR),\ b \mapsto b\beta_{i,\nu(i)}.$$

(3) *There exist isomorphisms of the form*

$$\phi : B_B \xrightarrow{\sim} \mathrm{Hom}_R({}_BB_R, R_R),\ b \mapsto \mu_0 b,$$
$$\psi : {}_BB \xrightarrow{\sim} \mathrm{Hom}_R({}_RB_B, {}_RR),\ b \mapsto b\rho_0.$$

Theorem 7.8. *The ring* B *is a Frobenius extension of* R.

Proposition 7.9. *Assume* R *is a local ring. Then the following hold.*

(1) *Every* e_{ii} *is a local idempotent. In particular,* B *is a semiperfect ring.*

(2) *Assume further that* R *is a quasi-Frobenius local ring. Then* B *is a quasi-Frobenius ring with* $\mathrm{soc}(e_{ii}B_B) \cong e_{\nu(i),\nu(i)}B/e_{\nu(i),\nu(i)}J$ *for all* $i \in I$, *where* $J = \mathrm{rad}(B)$.

In the next section, we do not fix $\omega \in \Omega_b(\nu)$ and use the notation B_ω to denote that the multiplication of B is defined by ω.

8. Automorphisms of B_ω

In this section, we show that for any $\omega \in \Omega_b(\nu)$ there exists $\eta \in \mathrm{Aut}(B_\omega)$ such that $\eta(e_{ij}) = e_{\nu(i),\, \nu(j)}$ for all $i, j \in I$ and $\eta(v_i) = v_i$ for all $i \in I(\nu)$.

Lemma 8.1. *For any $\omega, \omega' \in \Omega_b(\nu)$ the following hold.*

(1) *If there exists $\tau \in \mathrm{Aut}(I)$ with $\tau\nu = \nu\tau$ such that $\omega' = \omega \circ (\tau \times \tau)$, then there exists a ring isomorphism of the form*

$$\xi_1 : B_\omega \xrightarrow{\sim} B_{\omega'},\ b \mapsto \textstyle\sum_{i,\, j\in I} e_{ij}\alpha_{\tau(i),\, \tau(j)}(b) + \sum_{i \in I(\nu)} v_i\mu_i(b).$$

(2) *If there exists $\chi : I \to \mathbb{Z}$ such that $\omega'(i, j) = \omega(i, j) - \chi(i) + \chi(j)$ for all $i, j \in I$, then there exists a ring isomorphism of the form*

$$\xi_2 : B_\omega \xrightarrow{\sim} B_{\omega'},\ b \mapsto \textstyle\sum_{i,\, j\in I} e_{ij}\sigma^{\chi(j)}(\alpha_{ij}(b)) + \sum_{i \in I(\nu)} v_i\sigma^{\chi(i)}(\mu_i(b)).$$

Proposition 8.2. *For any $\omega \in \Omega_b(\nu)$ the following hold.*

(1) *Define $\chi : I \to \mathbb{Z}$ as follows*:

$$\chi(i) = \begin{cases} \omega(i, \nu(i)) & \text{if } i \notin I(\nu), \\ t_\omega & \text{if } i \in I(\nu). \end{cases}$$

Then $\omega(\nu(i), \nu(j)) = \omega(i, j) - \chi(i) + \chi(j)$ for all $i, j \in I$.

(2) *There exist ring automorphisms of the form*

$$\theta : B_\omega \xrightarrow{\sim} B_\omega,\ b \mapsto \textstyle\sum_{i, j\in I} e_{ij}\sigma^{-\chi(j)}(\alpha_{\nu(i),\, \nu(j)}(b)) + \sum_{i \in I(\nu)} v_i\sigma^{-\chi(i)}(\mu_i(b)),$$
$$\eta : B_\omega \xrightarrow{\sim} B_\omega,\ b \mapsto \textstyle\sum_{i, j\in I} e_{\nu(i),\, \nu(j)}\sigma^{\chi(j)}(\alpha_{ij}(b)) + \sum_{i \in I(\nu)} v_i\sigma^{\chi(i)}(\mu_i(b))$$

which are mutually inverse.

(3) *Let $e_0 = \sum_{i \in N(\nu)} e_{i,\, \nu(i)} + \sum_{i\in I(\nu)} e_{ii}\, c^t$ and $v_0 = \sum_{i \in I(\nu)} v_i$, where $t = t_\omega$. Then $\eta(e_0) = e_0$ and $\eta(v_0) = v_0$.* Also, *$be_0 = e_0\eta(b)$ and $bv_0 = v_0\eta(b)$ for all $b \in B_\omega$.*

Remark 8.3. For any $\omega \in \Omega_b(\nu)$ the following hold.

(1) θ is the unique ring automorphism of B_ω such that $\theta(e_{\nu(i),\, \nu(j)}) = e_{ij}$ for all $i, j \in I$ and $x\mu_0 = \mu_0\theta(x)$ for all $x \in R$.

(2) η is the unique ring automorphism of B_ω such that $\eta(e_{ij}) = e_{\nu(i),\, \nu(j)}$ for all $i, j \in I$ and $\rho_0 x = \eta(x)\rho_0$ for all $x \in R$.

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