An extension theorem of semibricks

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Setting

- Λ: fin. dim. K-alg. / $K = \overline{K}$
- $\operatorname{mod}\Lambda$: the cat. of fin. gen. Λ-mod's.

Def (motivated by Schur's Lem.)

(1) $B \in \operatorname{mod}\Lambda$: brick $:\Leftrightarrow \begin{cases} B \neq 0 \\ \forall f \neq 0 : B \to B \text{ is iso.} \end{cases}$

$\Leftrightarrow \operatorname{End}_\Lambda(B) \simeq K \quad (\because K = \overline{K})$

$\operatorname{brick}\Lambda$: the set of isoclasses of all bricks in $\operatorname{mod}\Lambda$

(2) $\mathcal{S} \subset \operatorname{brick}\Lambda$: semibrick $:\Leftrightarrow \forall B_1 \neq B_2 \in \mathcal{S},\ \operatorname{Hom}_\Lambda(B_1, B_2) = 0$

$\operatorname{sbrick}\Lambda$: the set of all semibricks

Maximality

$\operatorname{sbrick}\Lambda$ is a poset by inclusions $\subset$.

Thus, maximal semibricks are defined.

$(:\Leftrightarrow \mathcal{S} \in \operatorname{sbrick}\Lambda$ s.t. $\nexists \mathcal{S}' \in \operatorname{sbrick}\Lambda,\ \mathcal{S} \subsetneq \mathcal{S}')$

Finiteness

(1) Many alg. have infinite semibricks

$(:\Leftrightarrow \mathcal{S} \in \operatorname{sbrick}\Lambda$ s.t. $\#\mathcal{S} = \infty$ as a set$)$

ex) $\Lambda = K(1 \rightrightarrows 2)$

$$\left\{ \left(K \underset{\lambda}{\overset{1}{\rightrightarrows}} K \right) \middle| \lambda \in K \right\} : \text{ an infin. semibrick}$$

(2) $\{S_1, S_2, \dots, S_n\}$: maximal finite semibricks

$\subset$ the set of all isoclasses of simple Λ-modules $(n = |\Lambda|)$

$\forall \mathcal{S} \in \operatorname{sbrick}\Lambda$: left finite,

$\exists \mathcal{S}' \in$ 〃 : 〃 , $\mathcal{S} \subset \mathcal{S}'$, $\#\mathcal{S}' \leq |\Lambda|$

Question

When is a fin. semibrick maximal?

Representation schemes

$\Lambda = KQ/I$ (Q : fin. quiver, $I \subset KQ$: admissible)

For $d = (d_i)_{i\in Q_0} \in (\mathbb{Z}_{\geq 0})^{Q_0}$,

$$\mathrm{rep}(\Lambda, d) := \left\{ (\varphi_\alpha)_{\alpha\in Q_1} \in \prod_{\alpha\in Q_1} \mathrm{Hom}_K(K^{d_{s(\alpha)}}, K^{d_{t(\alpha)}}) \;\middle|\; \begin{array}{l}\text{satisfying rel.}\\ \text{corres. to } I\end{array} \right\}$$

the representation scheme for (Λ, d)

$\mathrm{Irr}(\Lambda, d) := \{\text{irr. comp. of } \mathrm{rep}(\Lambda, d)\}$

$\mathrm{Irr}(\Lambda) := \bigsqcup_{d\in(\mathbb{Z}_{\geq 0})^{Q_0}} \mathrm{Irr}(\Lambda, d).$

Action of GL(d)

For $d \in (\mathbb{Z}_{\geq 0})^{Q_0}$, $\overset{\text{connected}}{\mathrm{GL}(d)} := \prod_{i\in Q_0} \mathrm{GL}(d_i)$ acts on $\mathrm{rep}(\Lambda, d)$.

For each $M \in \mathrm{rep}(\Lambda, d)$,

$\mathcal{O}_M :=$ (the $\mathrm{GL}(d)$-orbit of M)

$= \{ N \in \mathrm{rep}(Q, d) \mid N \simeq M \text{ as } \Lambda\text{-mod's}\}$

$\mathcal{Z} \in \mathrm{Irr}(\Lambda, d)$, $M \in \mathcal{Z} \Rightarrow \mathcal{O}_M \subset \overline{\mathcal{O}_M} \subset \mathcal{Z}$.

Def

$B \in \mathrm{brick}\,\Lambda$: open brick $:\Leftrightarrow \overline{\mathcal{O}_B} \in \mathrm{Irr}(\Lambda)$

$\Leftrightarrow \exists \mathcal{Z} \in \mathrm{Irr}(\Lambda)$, $\mathcal{O}_B \subset \mathcal{Z}$: open dense

Thm [A]

Let $\mathcal{S} \in \mathrm{sbrick}\,\Lambda$: finite semibrick.

(1) If $\mathcal{S}$ is maximal, then every $B \in \mathcal{S}$ is an open brick.

(2) If $B \in \mathcal{S}$, $\mathcal{Z} \in \mathrm{Irr}(\Lambda)$ satisfy $\overline{\mathcal{O}_B} \subsetneq \mathcal{Z}$, then

$\exists B' \in \mathcal{Z}$, $\mathcal{S} \sqcup \{B'\} \in \mathrm{sbrick}\,\Lambda$

Rem

- The point of the proof is an argument similar to [Geiss-(Labardini-Fragoso)-Schröer] and [Bongartz]

  Semicontinuous maps on module varieties, 2024

  On degenerations and extensions of finite-dimensional modules, 1996

- See [Mousavand-Paquette] for more properties

  - On the bricks (Schur representations) of finite dimensional algebras
  - Hom-orthogonal modules and brick-Brauer-Thrall conjectures ...

  of bricks and irr. comp.