The 8th CJK Inter. Symposium on Ring Theory

Gorenstein-projective and semi-Gorenstein-projective modules

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Definition

A: an Artin algebra $\qquad$ A-mod: the cat. of fin. gen. left A-modules
A module ${}_AM$ is **semi-Gorenstein-proj.** if $\mathrm{Ext}^i(M, {}_AA) = 0, \forall\, i \geq 1$.
${}^\perp A$: the full subcat. of A-mod of semi-Gorenstein-proj. modules

Definition (Auslander-Bridger, 1969) An A-module M is Gorenstein-projective, if M satisfies the following conditions:

(G1): M is semi-Gorenstein-proj.;
(G2): $M^* := \mathrm{Hom}(M, A)$ is (right) semi-Gorenstein-proj.;
(G3): M is reflexive, i.e., $\phi_M : M \cong M^{**}$.

$\mathcal{GP}(A)$: the full subcat. of A-mod of Gorenstein-proj. modules
Thus $\qquad \mathcal{P}(A) \subseteq \mathcal{GP}(A) \subseteq {}^\perp A$.

• If gl.dim.$(A) < \infty$, then $\mathcal{P}(A) = \mathcal{GP}(A) = {}^\perp A$. (But the converse is not true.)

• A is self-injective if and only if $\mathcal{GP}(A) = {}^\perp A = A$-mod.

• If A is Iwanaga-Gorenstein of dimension n (i.e., id ${}_AA \leq n$, id$A_A \leq n$), then $\mathcal{GP}(A) = {}^\perp A = \Omega^n(A\text{-mod})$.

Main equivalent definitions

Theorem ([AB, 69], [EJ, 95], [Chris, 00], [AM, 02]) The following are equivalent:

(1) $M \in \mathcal{GP}(A)$;
(2) $M \in {}^{\perp}A$, and the transpose $\mathrm{Tr}M \in {}^{\perp}A_A$;
(3) M has a complete proj. resolution, i.e., $\exists$ an exact sequence

$$P^\bullet : \quad \cdots \longrightarrow P^{-1} \xrightarrow{\quad d^{-1} \quad} P^0 \longrightarrow P^1 \longrightarrow \cdots, \qquad P^{-1} \twoheadrightarrow M \hookrightarrow P^0$$

of projective modules with $\mathrm{Hom}(P^\bullet, A)$ again exact, such that $M \cong \mathrm{Im}d^{-1}$.

Terminologies:

module of G-dimension 0 (Auslander-Bridger [AB], Yoshino [Y], $\cdots$);
Gorenstein-projective module (Enochs-Jenda [EJ], Christensen [Chris], $\cdots$)
totally reflexive module (Avramov-Martsinkovsky [AM], Jorgensen-Şega [JŞ], $\cdots$)
Cohen-Macaulay module (Beligiannis [Bel], Iyama-Kato-Miyachi [IKM], $\cdots$)
maximal Cohen-Macaulay module (Buchweitz [Buch], $\cdots$)

Some properties

1. Let $\mathcal{F}$ be an extension-closed subcategory of A-mod. If $\mathcal{F}$ is a Frobenius subcategory (with the canonical exact structure) with $\mathcal{P}(\mathcal{F}) \subseteq \mathcal{P}(A)$, then $\mathcal{F} \subseteq \mathcal{GP}(\mathcal{A})$ if and only if $\mathrm{Ext}^1(\mathcal{F}, \mathcal{P}(A)) = 0$.

In particular, $\mathcal{GP}(A)$ is the largest resolving Frobenius subcategory of A-mod. So, the stable category $\underline{\mathcal{GP}}(A)$ is a triangulated category.

2. (X.W.Chen) Any Frobenius category is equivalent to a full subcat. of $\mathcal{GP}(\mathcal{A})$, where $\mathcal{A}$ is an abelian cat. with enough proj. objects.

3. (M.Auslander; M.Hoshino) A is Iwanaga-Gorenstein if and only if every module has finite Gorenstein-proj. dimension.

4. (M.Auslander-M.Bridger; E.Enochs-O.Jenda) If A is Iwanaga-Gorenstein, then $\mathcal{GP}(A)$ is contravariantly finite in A-mod, thus functorially finite. In general, this is not true (Y.Yoshino; R.Takahashi).

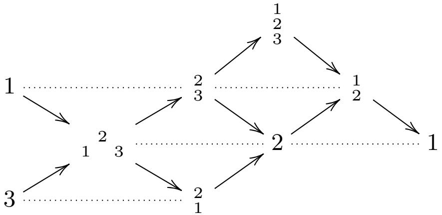
5. If A is Iwanaga-Gorenstein, then $\underline{\mathcal{GP}}(A)$ has Auslander-Reiten triangles, and thus the Serre functor.

6. (R.O.Buchweitz) $\underline{\mathcal{GP}}(A) \overset{\text{can.}}{\hookrightarrow} D^b(A)/K^b(\mathcal{P})$, and if A is Iwanaga-Gorenstein, then $\underline{\mathcal{GP}}(A) \cong D^b(A)/K^b(\mathcal{P})$ canonically. The converse is also true (P.Bergh-D.Jorgensen-S.Oppermann).

Example of a Frobenius subcategory $\nsubseteq \mathcal{GP}(A)$

However, in general, $\mathcal{GP}(A)$ is not the largest Frobenius subcategory of A-mod.

$A:\quad 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \xrightarrow{\ \gamma\ } 3$ with relations $\alpha\beta = 0 = \beta\alpha$. The Auslander-Reiten quiver of A is



Then $\mathcal{GP}(A) = {}^{\perp}A = \mathrm{add}({}_AA \oplus 1 \oplus \begin{smallmatrix}2\\3\end{smallmatrix})$.

Note that $\mathscr{F} := \mathrm{add}(1 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 2 \oplus \begin{smallmatrix}2\\1\end{smallmatrix})$ is a Frobenius subcategory of A-mod with $\mathcal{P}(\mathcal{F}) = \mathcal{I}(\mathcal{F}) = \mathrm{add}(\begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix})$, but $\mathscr{F} \nsubseteq \mathcal{GP}(A)$.

Question 1

$$\mathcal{GP}(A) \subseteq {}^{\perp}A$$

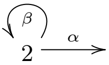
- The first example of an algebra A with ${}^{\perp}A \neq \mathcal{GP}(A)$ is known only in 2006, by D.A.Jorgensen and L.M.Şega [JŞ]. This (commut.) algebra is

$$k[w, x, y, z]/\langle w^2,\ z^2,\ xy,\ wx + qxz,\ wy + yz,\ wx + y^2,\ wy - x^2\rangle.$$

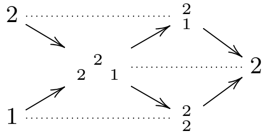
- First noncommut. algebra A such that ${}^{\perp}A \neq \mathcal{GP}(A)$ is obtained by R. Marczinzik in 2017.

- (Y. Yoshino, 2005) To find more classes of algebras such that ${}^{\perp}A = \mathcal{GP}(A)$, which will be called a (left) weakly Iwanaga-Gorenstein algebra.

Example

$A:$ 

1, with relations $\beta^2,\ \alpha\beta$. Then the AR quiver of A is



Then ${}^{\perp}A = \mathcal{P}(A)$. Since

$$\mathcal{P}(A) \subseteq \mathcal{GP}(A) \subseteq {}^{\perp}A$$

it follows that $\mathcal{P}(A) = \mathcal{GP}(A) = {}^{\perp}A$. But A is *not* Iwanaga-Gorenstein.

Descriptions of ${}^{\perp}A = \mathcal{GP}(A)$

Theorem. The following statements are equivalent:

(1) A is (left) weakly Gorenstein (i.e., ${}^{\perp}A = \mathcal{GP}(A)$);

(2) (Z.Y.Huang, 2012) Any semi-Gorenstein-proj. module M is torsionless (i.e., $\phi_M : M \to M^{**}$ is inj.; or equiv., $M \hookrightarrow$ a proj. module);

(3) Any semi-Gorenstein-proj. module is reflexive;

(4) For each $M \in {}^{\perp}A$, the map $\phi_M : M \longrightarrow M^{**}$ is surjective;

(5) For each $M \in {}^{\perp}A$, the module M^* is semi-Gorenstein proj.;

(6) For each $M \in {}^{\perp}A$, $\mathrm{Ext}^1(M^*, A_A) = 0$;

(7) For each $M \in {}^{\perp}A$, $\mathrm{Ext}^1(\mathrm{Tr}\, M, A_A) = 0$;

(8) ${}^{\perp}A$ is a Frobenius category, with the canonical exact structure.

Examples of weakly Gorenstein algebras

Theorem. The following algebras are weakly Iwanaga-Gorenstein (i.e., ${}^{\perp}A = \mathcal{GP}(A)$):

(1) (E.Enochs-O.Jenda) Gorenstein algebras.

(2) (Y.Yoshino, 2005; A. Beligiannis, 2011) $|{}^{\perp}A| < \infty$.

(3) (R.Marczinzik, 2016) Torsionless finite algebras.

(4) $|{}^{\perp}A \cap \mathrm{torless}(A)| < \infty$. In this case, all the Gorenstein-proj. modules are Ω-periodic.

(5) Algebras A such that $A/\mathrm{soc}({}_AA)$ is rep.-finite.

(6) Algebras stably equivalent to hereditary algebras, in particular, all algebras with radical square zero.

(7) Minimal rep.-infinite algebras.

(8) Special biserial algebras without indec. proj-inj. modules.

(9) (T. Aihara, A. Chan, T. Houma, in this conf.) Gendo algebras over a rep.-fin. algebra.

Main technique: the operator $\mho$

For any A-module M, denote by $\mho M$ the cokernel of a minimal left add(A)-approximation of M.

- If M is indec. torsionless and not proj., then $\mho M$ is indec. and not proj., and $\Omega\mho M \cong M$.

- For any A-module M, $\mho M \cong \mathrm{Tr}\Omega\mathrm{Tr}M$. Thus $\mho\mathrm{Tr}M \cong \mathrm{Tr}\Omega M$, but $\mathrm{Tr}\mho M \ncong \Omega\mathrm{Tr}M$ in general. The operator $\mathrm{Tr}\Omega\mathrm{Tr}$ has been studied by Auslander-Reiten in 1996.

- $(\mho, \Omega)$ is an adjoint pair of A-$\underline{\mathrm{mod}}$.

- For $t \geq 0$, there is an exact sequence

$$0 \to \mathrm{Ext}^{t+1}(\mathrm{Tr}M, A_A) \longrightarrow \mho^t M \stackrel{\phi_{\mho^t M}}{\longrightarrow} (\mho^t M)^{**} \longrightarrow \mathrm{Ext}^{t+2}(\mathrm{Tr}M, A_A) \to 0.$$

The Auslander-Bridger exact seq. $0 \to \mathrm{Ext}^1(\mathrm{Tr}M, A_A) \to M \to M^{**} \to \mathrm{Ext}^2(\mathrm{Tr}M, A_A) \to 0$

Question 2

L. Avramov - A. Martsinkovsky, 2002:

"Whether or not the conditions (G1), (G2), (G3) are independent"?

D. A. Jorgensen - L. M. Şega, 2006:

In commutative case: (G1) + (G3) do not imply (G2)
In commutative case: (G2) + (G3) do not imply (G1)

Still open:

- Whether or not (G1) + (G2) imply (G3), in the both cases of commutative or noncommutative?

- In noncommutative case: whether or not (G1) + (G3) imply (G2)?

- In noncommutative case: whether or not (G2) + (G3) imply (G1)?

- whether M^* is reflexive implies that M is reflexive, i.e., $\phi_{M^*} : M^* \cong M^{***} \overset{?}{\Longrightarrow} \phi_M : M \cong M^{**}$.

or, whether M^* is Gorenstein-proj. implies that M is Gorenstein-proj.?

Quantum complete intersections

Let k be a field, $n_1, \cdots, n_t$ be positive integers with $t \geq 2$ and all $n_i \geq 2$, and $\mathbf{q} = (q_{ij})$ a $t \times t$ matrix over k with $q_{ii} = -1$, $q_{ij}q_{ji} = 1$, for all i, j. Let

$$A(\mathbf{q}, n_1, ..., n_t) := k\langle x_1, ..., x_t\rangle / \langle x_i^{n_i}, x_i x_j + q_{ij} x_j x_i,\ 1 \leq i, j \leq t\rangle.$$

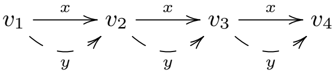
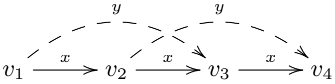
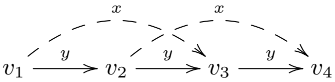
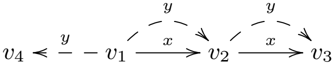
It is called *a quantum complete intersection*, and *a quantum exterior algebra* if all $n_i = 2$. It is local and Frobenius, and it is symmetric if and only if $\prod_{i=1}^{t} (-q_{ij})^{n_i - 1} = 1$ for all j (P.A.Bergh, S.Oppermann).

A quantum complete intersection is either wild or tame, and it is tame if and only if $t = 2$ and $n_1 = n_2 = 2$ (P.A.Bergh - K.Erdmann; C.M.Ringel).

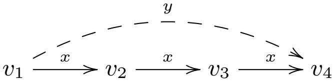
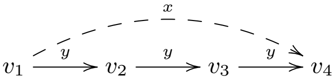
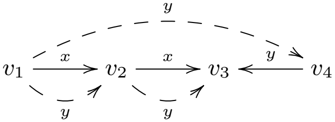
"Hopless": the classification of indec. $A(\mathbf{q}, n_1, ..., n_t)$-modules if $t \geq 3$, or if some $n_i \geq 3$.

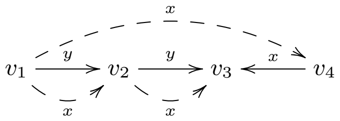
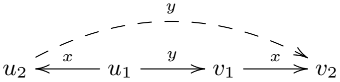
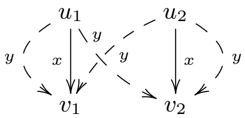
However, it is possible to classify the "low dim." $A(\mathbf{q}, n_1, ..., n_t)$-modules.

Theorem. Let $A(q,m,n) = k\langle x,y\rangle/\langle x^m, y^n, xy+qyx\rangle$ with $q \neq -1, \neq 1$, $m,n \geq 4$. Then all the pairwise non-isomorphic indec. 4-dim. $A(q,m,n)$-modules are as follows (all together 21 classes):

Type	Name	Diagram presentation
$(3,2,1,3,2,1)$	$M_{(3,2,1,3,2,1)}(a)$	 $yv_1 = av_2,\ yv_2 = -q^{-1}av_3,\ yv_3 = q^{-2}av_4$
$(3,2,1,2,0,0)$	$M_{(3,2,1,2,0,0)}(a)$	 $yv_1 = -qav_3,\ yv_2 = av_4$
$(2,0,0,3,2,1)$	$M_{(2,0,0,3,2,1)}(a)$	 $xv_1 = -q^{-1}av_3,\ xv_2 = av_4$
$(2,1,0,2,1,0)$	$M_{(2,1,0,2,1,0)}(a)$	 $yv_1 = -qav_2 + v_4,\ yv_2 = av_3$

Type	Name	Diagram presentation
$(2,1,0,2,1,0)$	$M_{(2,1,0,2,1,0)}(a,0)$	$v_1 \xrightarrow{x} v_2 \xrightarrow{x} v_3 \xleftarrow{y} v_4$ $yv_1 = -qav_2,\ yv_2 = av_3$
$(2,1,0,2,1,0)$	$M_{(2,1,0,2,1,0)}(0,b)$	$v_1 \xrightarrow{x} v_2 \xrightarrow{x} v_3 \xleftarrow{y} v_4$ $yv_1 = bv_4,\ yv_4 = v_3$
$(2,1,0,2,1,0)$	$M_{(2,1,0,2,1,0)}(a,b)$ $b \neq qa^2$	$v_1 \xrightarrow{x} v_2 \xrightarrow{x} v_3 \xleftarrow{y} v_4$ $yv_1 = -qav_2 + bv_4,\ yv_2 = av_3,\ yv_4 = v_3.$
$(2,1,0,1,0,0)$	$M_{(2,1,0,1,0,0)}$	$v_1 \xrightarrow{x} v_2 \xrightarrow{x} v_3 \xleftarrow{y} v_4$
$(2,1,0,1,0,0)$	$M'_{(2,1,0,1,0,0)}$	$v_4 \xleftarrow{y} v_1 \xrightarrow{x} v_2 \xrightarrow{x} v_3$
$(1,0,0,2,1,0)$	$M_{(1,0,0,2,1,0)}$	$v_1 \xrightarrow{y} v_2 \xrightarrow{y} v_3 \xleftarrow{x} v_4$
$(1,0,0,2,1,0)$	$M'_{(1,0,0,2,1,0)}$	$v_4 \xleftarrow{x} v_1 \xrightarrow{y} v_2 \xrightarrow{y} v_3$

Type	Name	Diagram presentation
$(3,2,1,1,0,0)$	$M_{(3,2,1,1,0,0)}(a)$	 $yv_1 = av_4$
$(1,0,0,3,2,1)$	$M_{(1,0,0,3,2,1)}(a)$	 $xv_1 = av_4$
$(3,2,1,0,0,0)$	$M_{(3,2,1,0,0,0)}$ $= k[x]/\langle x^4 \rangle$	$v_1 \xrightarrow{x} v_2 \xrightarrow{x} v_3 \xrightarrow{x} v_4$
$(0,0,0,3,2,1)$	$M_{(0,0,0,3,2,1)}$ $= k[y]/\langle y^4 \rangle$	$v_1 \xrightarrow{y} v_2 \xrightarrow{y} v_3 \xrightarrow{y} v_4$
$(2,1,0,2,0,0)$	$M_{(2,1,0,2,0,0)}(a)$	 $yv_1 = -qa(v_2 - av_4),\ yv_2 = av_3,\ yv_4 = v_3$

Type	Name	Diagram presentation
$(2,0,0,2,1,0)$	$M_{(2,0,0,2,1,0)}(a)$	 $xv_1 = -q^{-1}a(v_2 - av_4),\ xv_2 = av_3,,\ xv_4 = v_3$
$(2,0,0,2,0,0)$	$M_{(2,0,0,2,0,0)}$ $= A(q,2,2)$	 $yu_2 = -q^{-1}v_2$
$(2,0,0,2,0,0)$	$M_{(2,0,0,2,0,0)}(T)$	 $\binom{yu_1}{yu_2} = T\binom{v_1}{v_2}$
$(2,0,0,1,0,0)$	$M_{(2,0,0,1,0,0)}$	$u_1 \xrightarrow{x} v_1 \xleftarrow{y} u_2 \xrightarrow{x} v_2$
$(1,0,0,2,0,0)$	$M_{(1,0,0,2,0,0)}$	$u_1 \xrightarrow{y} v_1 \xleftarrow{x} u_2 \xrightarrow{y} v_2$

Algebra $\Lambda(q)$

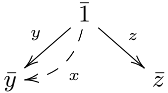
Let $q \in k^*$. Consider the algebra

$$\Lambda = \Lambda(q) := k\langle x, y, z\rangle / \langle x^2,\ y^2,\ z^2,\ yz,\ xy + qyx,\ xz - zx,\ zy - zx\rangle.$$

It has a basis 1, x, y, z, yx, and zx. For $(a, b, c) \in k^3 \setminus \{0\}$, consider 3-dimensional left Λ-module

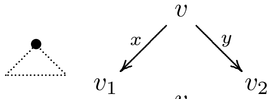
$$M(a, b, c) := {}_\Lambda\Lambda/[\Lambda(ax + by + cz) + \mathrm{soc}\Lambda].$$

For example, $M(1, -q, 0) = {}_\Lambda\Lambda/[\Lambda(x - qy) + \mathrm{soc}\Lambda] = k\bar{1} \oplus \bar{y} \oplus \bar{z}$ with



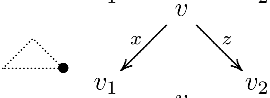
Proposition. Let M be an indecomposable 3-dimensional left Λ-module. Then M is isomorphic to one of the following pairwise non-isomorphic Λ-modules (14 classes):

(1) $M(0,0,1)$



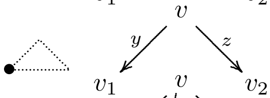
$zM = 0,\ xM \neq yM$

(2) $M(0,1,0)$



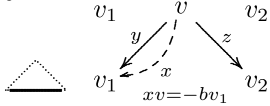
$yM = 0,\ xM \neq zM$

(3) $M(1,0,0)$



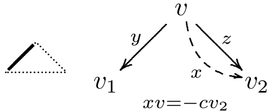
$xM = 0,\ yM \neq zM$

(4) $M(1,b,0)$
$b \in k^*$



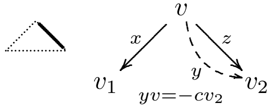
$xM = yM$

(5) $M(1,0,c)$
$c \in k^*$



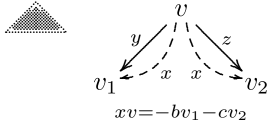
$xM = zM$

(6) $M(0,1,c)$
$c \in k^*$



$yM = zM$

(7) $M(1,b,c)$
$b, c \in k^*$



xM, yM, zM non-zero and pairwise different

(1') $D(M(0,0,1))$:

$zM = 0,\ xM = yM$

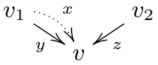
(2') $D(M(0,1,0))$:

$yM = 0,\ xM = zM$

(3') $D(M(1,0,0))$:

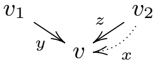
$xM = 0,\ yM = zM$

(4') $D(M(1,b,0))$: $b \in k^*$



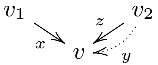
$\mathrm{Ker}\, x = \mathrm{Ker}\, y$

(5') $D(M(1,0,c))$: $c \in k^*$



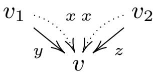
$\mathrm{Ker}\, x = \mathrm{Ker}\, z$

(6') $D(M(0,1,c))$: $c \in k^*$



$\mathrm{Ker}\, y = \mathrm{Ker}\, z$

(7') $D(M(1,b,c))$: $b, c \in k^*$



$\mathrm{Ker}\, x, \mathrm{Ker}\, y, \mathrm{Ker}\, z$ nonzero and pairwise different

Independence Theorem

Theorem Assume that $o(q) = \infty$. Let M be an indecomposable $\Lambda(q)$-module of dimension at most 3. Then

(1) M satisfy (G1) and (G2) such that M is not torsionless if and only if M is isomorphic to $M(1, -q, c)$ with $c \in k$. For example, $M(1, -q, 0)$

(2) M satisfy (G1) and (G3) but not (G2) if and only if M is isomorphic to $M(1, -q^i, c)$ with $i \geq 3, c \in k$. For example, $M(1, -q^3, 0)$

(3) M satisfy (G2) and (G3) but not (G1) if and only if M is isomorphic to $M(1, -q^i, c)$ with $i \leq -1, c \in k$. For example, $M(1, -q^{-1}, 0)$

(4) M is Gorenstein-projective if and only if M is isomorphic to $M(1, b, c)$ with $b \neq -q^i$ for all $i \in \mathbb{Z}$.

(5) Let $L := \Lambda(q)/\Lambda(q)(x - y)$. Then L is a 4-dim. non-reflexive $\Lambda(q)$-module, but $L^* \cong M(1, -1, 0)^*$ is reflexive.

Some open problems

- For commutative algebra A, whether or not (G1) + (G2) imply (G3)?

- If $|\mathcal{GP}(A)| < \infty$, whether or not A is a weakly Iwanaga-Gorenstein algebra, i.e., ${}^{\perp}A = \mathcal{GP}(A)$?

- Even "weaker": if $\mathcal{GP}(A) = \mathcal{P}(A)$, whether or not A is a weakly Iwanaga-Gorenstein algebra?

- Whether A a left weakly Iwanaga-Gorenstein algebra if and only if A a right weakly Iwanaga-Gorenstein algebra?

Thank you very much!

We sincerely thank all the organizers!

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