Density of $g$-vector cones from triangulated surfaces

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Today’s talk

1. Notations

2. $\tau$-tilting theory

3. $g$-vectors and $g$-vector cones

4. Triangulated surfaces

5. Result
Notations

- $A$: a finite dimensional algebra over a field $k$.
- $\text{mod } A$: the category of finitely generated right $A$-modules.
- $\tau$: the Auslander-Reiten translation of $\text{mod } A$.
- $|M|$: the number of non-isomorphic indecomposable direct summands of $M \in \text{mod } A$.
- $n = |A|$.
**τ-tilting pairs**

**Definition**

Let $M \in \text{mod } A$ and $P$ a projective $A$-module. We say that $(M, P)$ is

- a **τ-rigid pair** if $\text{Hom}_A(M, \tau M) = 0$ and $\text{Hom}_A(P, M) = 0$.
- a **τ-tilting pair** if it is a τ-rigid pair and $|M| + |P| = n$.
- an **almost τ-tilting pair** if it is a τ-rigid pair and $|M| + |P| = n - 1$.

- $\tau\text{-tilt } A := \{(\text{basic}) \text{ τ-tilting pairs}\}$.
- $\text{irigid } A := \{\text{indecomposable τ-rigid pairs}\}$
  \[= \{\text{indec. direct summands of τ-tilting pairs in τ-tilt } J_T\}\]

**Remark**

\[\{\text{support τ-tilting modules}\} \ni M \leftrightarrow (M, P) \in \tau\text{-tilt } A\]
Theorem (Adachi-Iyama-Reiten, 2014)

Any almost $\tau$-tilting pair is a direct summand of exactly two $\tau$-tilting pairs.

For $U = \bigoplus_{i=1}^{n} U_i \in \tau\text{-}\text{tilt} A$ and $j \in \{1, \ldots, n\}$, there is a unique $U_j' \in \text{irigid} A$ such that $U_j' \neq U_j$ and $U' := \bigoplus_{i \neq j} U_i \oplus U_j' \in \tau\text{-}\text{tilt} A$.

$$U \xrightarrow{\text{mutation \ at \ } j} U'$$

$\tau\text{-}\text{tilt}^\pm A := \left\{ \begin{array}{l} \text{$\tau$-tilting pairs obtained from } (A, 0) \text{ or } (0, A) \\ \text{by a sequence of mutations} \end{array} \right\} \subseteq \tau\text{-}\text{tilt} A.$

$\text{irigid}^\pm A := \left\{ \begin{array}{l} \text{indec. direct summands of $\tau$-tilting pairs} \\ \text{in $\tau\text{-}\text{tilt}^\pm A$} \end{array} \right\} \subseteq \text{irigid} A.$
We only consider finite dimensional algebras defined from triangulated surface. Before we introduce them, we study a useful tool "$g$-vector cones".
$g$-vectors of $\tau$-rigid pairs

Fix $A = \bigoplus_{i=1}^{n} P_i$, where $P_i$ is an indecomposable projective $A$-module.

**Definition**

Let $M \in \text{mod } A$. There is a minimal projective presentation of $M$

$$\bigoplus_{i=1}^{n} P_i^{m_i} \to \bigoplus_{i=1}^{n} P_i^{m_i} \to M \to 0.$$  

The $g$-vector of $M$ is a vector

$$g^M = (m_1 - m'_1, \ldots, m_n - m'_n) \in \mathbb{Z}^n.$$  

For a $\tau$-rigid pair $(M, P)$, its $g$-vector is $g^{(M,P)} := g^M - g^P$.

**Theorem (AIR)**

For $\bigoplus_{i=1}^{n} U_i \in \tau$-tilt $A$, then $g^{U_1}, \ldots, g^{U_n}$ form a basis of $\mathbb{R}^n$. 

Definition

To $U = \bigoplus_{i=1}^{m} U_i \in \tau$-tilt $A$, we associate the $g$-vector cone

$$C(U) := \left\{ \sum_{i=i}^{m} a_i g^{U_i} \mid a_i \geq 0 \right\}.$$ 

Theorem (Reading, 2014)

If $A$ is $\tau$-tilting finite, that is $\# \tau$-tilt $A < \infty$, then we have

$$\bigcup_{U \in \tau$-tilt $A} C(U) = \mathbb{R}^n.$$
Example (Type $A_2$)

For a quiver $Q = (1 \leftarrow 2)$, the AR quiver of the path algebra $kQ$ is

$$
\begin{array}{c}
(0) \\
(1)
\end{array} \rightarrow
\begin{array}{c}
(1) \\
(0)
\end{array}
\rightarrow
\begin{array}{c}
(0) \\
(-1)
\end{array}
$$

and $\tau$-tilt $kQ = \left\{
\begin{array}{c}
\left((1) \oplus (0 \ 1), 0\right), \\
\left((1), (0 \ 1)\right), \\
\left(0, (1) \oplus (0 \ 1)\right)
\end{array}
\right\}$,

where the modules are described by their $g$-vectors. Then their $g$-vector cones span the ambient space $\mathbb{R}^2$ as the right diagram.

In this talk, we consider an analogue of the theorem for finite dimensional algebras defined from triangulated surfaces.
Triangulated surfaces

- $S$: a connected compact oriented Riemann surface
- $M$: a finite set of marked points on $S$ with at least one marked point on each component of the boundary of $S$

Example
An arc of \((S, M)\) is a non-self-crossing curve (considered up to homotopy) with endpoints in \(M\). A triangulation of \((S, M)\) is a maximal set of pairwise non-crossing arcs.

**Example**

![Diagrams of arcs and triangulations]

In general, we use tagged arcs and tagged triangulations instead of arcs and triangulations.
To a tagged triangulation $T$ of $(S, M)$, Labardini-Fragoso associated the finite dimensional **Jacobian algebra** $J_T$.

Roughly speaking: Let $Q_T$ be a quiver whose

- vertices are arcs of $T$,
- arrows are obtained as follows:

$$J_T = kQ_T / \langle \text{paths of length 2 in each triangle} \rangle.$$

**Example**

$$Q_T = (1 \leftarrow 2)$$  
$$Q_T = (1 \leftrightarrow 2)$$
These algebras have the following property.

**Theorem (AIR, Fomin-Shapiro-Thurston, FT, Fu-Keller)**

For a tagged triangulation $T$ of $(S, M)$, there is a bijection

$$\text{irigid}^\pm J_T \leftrightarrow \{ \text{Tagged arcs of } (S, M) \}.$$

It induces a bijection

$$\tau\text{-tilt}^\pm J_T \leftrightarrow \{ \text{Tagged triangulations of } (S, M) \}$$

which commutes with mutations and flips.
Result

**Theorem**

For a tagged triangulation \( T \) of \((S, M)\), we have

\[
\bigcup_{U \in \tau\text{-tilt}^\pm J_T} C'(U) = \mathbb{R}^n.
\]

To prove this theorem, we study geometric tools (laminations, their shear coordinates, Dehn twists, and so on) and apply them to representation theory via the above bijections.
Example

Let $T = \text{Diagram}$. Then $Q_T = (1 \iff 2)$. In this case, there are infinitely many $\tau$-tilting pairs whose $g$-vector cones are as in the right diagram:

where no $g$-vector cones contain the red ray. However, there are infinitely many $g$-vector cones which are getting closer to it. Therefore,

$$\bigcup_{U \in \tau\text{-tilt } kQ_T} C(U) \neq \mathbb{R}^2 \quad \text{and} \quad \bigcup_{U \in \tau\text{-tilt } kQ_T} C(U) = \mathbb{R}^2.$$
Theorem (Demonet-Iyama-Jasso, 2019)

For $U, V \in \tau$-tilt $A$, if $C(U) \cap C(V)$ has dimension $n$, then $U = V$.

Corollary

For a tagged triangulation $T$ of $(S, M)$, we have

$$\tau$$-tilt $J_T = \tau$$-tilt^{\pm} J_T$.

Corollary

For a tagged triangulation $T$ of $(S, M)$, there is a bijections

$$\text{irigid } J_T \longleftrightarrow \{ \text{Tagged arcs of } (S, M) \}$$

$$\tau$$-tilt $J_T \longleftrightarrow \{ \text{Tagged triangulations of } (S, M) \}$.
Reference


