# Happel's functor and homologically well-graded Iwanaga-Gorenstein algebras

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**For simplicity,**

- $K$ **: field**
- $D = \mathrm{Hom}_K(-, K)$
- **algebra = finite dimensional** $K$**-algebra**
- **module = finitely generated right module**
- $\mathsf{mod}\,\Lambda$ **: the category of finitely generated right** $\Lambda$**-modules**

# 1. Motivation

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**Def.**

$A$ **: IG-algebra**

- $M \in \mathsf{mod}\, A$ **is Cohen-Macaulay** $\overset{\text{def}}{\iff}$ $\mathrm{Ext}_A^{>0}(M, A) = 0$
- $\mathsf{CM}(A) := \left\{ M \in \mathsf{mod}\, A \;\middle|\; \mathrm{Ext}_A^{>0}(M, A) = 0 \right\}$

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**Fact.**

**Since** $A$ **is IG,** $\mathsf{CM}(A)$ **is a Frobenius category.**

**The stable category** $\underline{\mathsf{CM}}(A)$ **has a structure of triangulated category.**

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**Def.**

$A = \bigoplus_{i=0}^{\ell} A_i$ **: $\mathbb{Z}$-graded IG-algebra**

- $M \in \mathsf{mod}^{\mathbb{Z}} A$ **is Cohen-Macaulay** $\overset{\text{def}}{\Longleftrightarrow}$ $\mathrm{Ext}_A^{>0}(M, A) = 0$
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**Fact.**

**Since $A$ is IG,** $\mathsf{CM}^{\mathbb{Z}}(A)$ **is a Frobenius category.**

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**Rem.** **If $A$ is self-injective,** $\mathsf{CM}^{\mathbb{Z}}(A) = \mathsf{mod}^{\mathbb{Z}} A$.

**For a $\mathbb{Z}$-graded IG-algebra** $A = \bigoplus_{i=0}^{\ell} A_i$ **,**

$$\exists\ \mathcal{H} : \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\nabla A) \ \longrightarrow\ \underline{\mathsf{CM}}^{\mathbb{Z}}(A).$$

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**<u>Def.</u> (X-W Chen, Mori)**

$A = \bigoplus_{i=0}^{\ell} A_i$ **: $\mathbb{Z}$-graded algebra**

**The algebra $\nabla A$ is called the Beilinson algebra of $A$ :**

$$\nabla A := \begin{pmatrix} A_0 & A_1 & A_2 & \cdots & A_{\ell-2} & A_{\ell-1} \\ & A_0 & A_1 & \cdots & A_{\ell-3} & A_{\ell-2} \\ & & & \cdots & \cdots & \cdots \\ & & & & A_0 & A_1 \\ O & & & & & A_0 \end{pmatrix}$$

**Def.**

$A=\bigoplus_{i=0}^{\ell} A_i$ **:** $\mathbb{Z}$**-graded IG-algebra.**

$\mathcal{H}$ **is defined as follows.**

$$\mathcal{H} : \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, \nabla A) \;\to\; \mathsf{D}^{\mathrm{b}}(\mathsf{mod}^{\mathbb{Z}} A) \;\to\; \mathsf{D}_{\mathsf{sg}}(A) \;\xrightarrow{\simeq}\; \underline{\mathsf{CM}}^{\mathbb{Z}}(A)$$

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**An abelian subcategory**

$$\mathsf{mod}^{[0,\ell-1]} A := \left\{ M \in \mathsf{mod}^{\mathbb{Z}} A \;\middle|\; M_i = 0 \quad \textbf{for} \quad i \notin [0, \ell - 1] \right\}$$

**of** $\mathsf{mod}^{\mathbb{Z}} A$

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**So by Morita theory**

$$\mathsf{mod}\, \nabla A \simeq \mathsf{mod}^{[0,\ell-1]} A \hookrightarrow \mathsf{mod}^{\mathbb{Z}} A.$$

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**<u>Def.</u> (Buchweitz)**

$A = \bigoplus_{i \geq 0} A_i$ **: $\mathbb{Z}$-graded algebra.**

**The following Verdier quotient is called the singular derived category.**

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**Thm.** **(Buchweitz)**

**If** $A$ **is IG, then**

$$\exists\ \underline{\mathsf{CM}}^{\mathbb{Z}}(A) \xrightarrow{\simeq} \mathsf{D}_{\mathsf{sg}}(A)$$

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**Why we study $\mathcal{H}$ ?**

**This functor $\mathcal{H}$ often becomes fully faithful or an equivalence.**

**In the case $A$ is self-injective,**

**it is known a necessary and sufficient condition for**

**$\mathcal{H}$ to be fully faithful or an equivalence.**

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$A$ **is right $\ell$-strictly well-graded**
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**Ex.** $A = A_0 \oplus A_1 \oplus A_2$

$A$ **is right** $2$**-swg** $\Leftrightarrow$

$$\begin{array}{cccc} \text{deg} & & & \\ 0 & A_0 & \xrightarrow{0} & A_0 \\ 1 & & & A_1 \\ 2 & & & A_2 \end{array} \qquad \begin{array}{cccc} \text{deg} & & & \\ -1 & & & A_0 \\ 0 & A_0 & \xrightarrow{0} & A_1 \\ 1 & & & A_2 \end{array} \qquad \begin{array}{cccc} \text{deg} & & & \\ -2 & & & A_0 \\ -1 & & & A_1 \\ 0 & A_0 & \to & A_2 \end{array}$$

$\operatorname{Hom}_A^{\mathbb{Z}}(A_0, A) = 0$ $\qquad$ $\operatorname{Hom}_A^{\mathbb{Z}}(A_0, A(1)) = 0$ $\qquad$ $\operatorname{Hom}_A^{\mathbb{Z}}(A_0, A(2)) = \operatorname{Hom}_A(A_0, A)$

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**Rem.**

- $A = \bigoplus_{i=0}^{\ell} A_i$ **:** $\mathbb{Z}$**-graded self-injective algebra**

$$A \text{ is right } \ell\text{-swg} \;\Leftrightarrow\; A \text{ is left } \ell\text{-swg}$$

- $A = \bigoplus_{i=0}^{\ell} A_i$ **: basic** $\mathbb{Z}$**-graded algebra**

$$A \text{ is swg self-injective} \;\Leftrightarrow\; DA \simeq A(\ell) \text{ in } \mathsf{mod}^{\mathbb{Z}} A.$$

**<u>Thm.</u> (X-W Chen, Happel, Minamoto-Mori Orlov, Y)**

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$\mathcal{H} : \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, \nabla A) \ \to \ \underline{\mathsf{mod}}^{\mathbb{Z}}(A)$

**(1)** $\mathcal{H}$ **is fully faithful** $\ \Leftrightarrow\ $ $A$ **is swg.**

**(2)** $\mathcal{H}$ **is an equivalence** $\ \Leftrightarrow\ $ $\begin{cases} A \textbf{ is swg} \\ \mathrm{gl.dim}\, A_0 < \infty \end{cases}$

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**Original result due to Happel. He had studied the case that** $A = \Lambda \oplus D\Lambda$ **is the trivial extension of an algebra** $\Lambda$ **by** $D\Lambda$**.**

**So we call** $\mathcal{H}$ **Happel's functor.**

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**<u>Aim.</u> Give an IG-analogue of this result.**

# 2. Our results

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**Recall.**

$A = \bigoplus_{i=0}^{\ell} A_i$ **:** $\mathbb{Z}$**-graded algebra**

$A$ **is right strictly well-graded** $\iff$ $\mathrm{Hom}_A^{\mathbb{Z}}(A_0, A(j)) = 0$ **for all** $j \neq \ell$

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$A$ **is right homologically well-graded** $\overset{\text{def}}{\Longleftrightarrow}$ $\mathbb{R}\mathrm{Hom}_A^{\mathbb{Z}}(A_0, A(j)) = 0$ **for all** $j \neq \ell$

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- **If** $A$ **is self-injective, then**

$$A \text{ is right hwg} \Leftrightarrow A \text{ is right swg}$$

**Main Thm.** **(Minamoto-Y)**

$A = \bigoplus_{i=0}^{\ell} A_i$ **:** $\mathbb{Z}$**-graded IG-algebra**

$\mathcal{H} : \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\nabla A) \;\to\; \underline{\mathsf{CM}}^{\mathbb{Z}}(A)$

**(1)** $\mathcal{H}$ **is fully faithful** $\;\Leftrightarrow\;$ $A$ **is right hwg**

**(2)** $\mathcal{H}$ **is an equivalence** $\;\Leftrightarrow\;$ $\begin{cases} A \text{ is right hwg} \\ \mathrm{gl.dim}\, A_0 < \infty \end{cases}$

**Thm.** **(Symmetry of hwg IG-algebras)**

$A = \bigoplus_{i=0}^{\ell} A_i$ **:** $\mathbb{Z}$**-graded algebra**

**TFAE :**

**(1)** $A$ **is right hwg IG.**

**(2)** $A$ **satisfies the following conditions:**

**(i)** $A_\ell$ **is a cotilting bimodule over** $A_0$

**(ii)** $A(\ell) \simeq \mathbb{R}\mathrm{Hom}_{A_0}(A, A_\ell)$ **in** $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}^{\mathbb{Z}} A)$

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**Thm.** **(Miyachi)**

**A cotitling bimodule gives a contravariant equivalences :**

$$\mathbb{R}\mathrm{Hom}_{A_0}(-, A_\ell) : \ \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A_0) \ \simeq \ \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A_0^{\mathrm{op}}) \ : \mathbb{R}\mathrm{Hom}_{A_0^{\mathrm{op}}}(-, A_\ell).$$

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**<u>Ex.</u> (M. Lu)**

$\Lambda$ **: algebra with** $\mathrm{gl.dim}\,\Lambda < \infty$**.**

$A := \Lambda \otimes_K K[x]/(x^{\ell+1})$ **with** $\deg x = 1$

**(1)** $A$ **is an** $\ell$**-hwg IG-algebra.**

**(2)** $\nabla A$ **is isomorphic to**

$$U_\ell(\Lambda) := \begin{pmatrix} \Lambda & \Lambda & \Lambda & \cdots & \Lambda & \Lambda \\ & \Lambda & \Lambda & \cdots & \Lambda & \Lambda \\ & & & \cdots & \cdots & \cdots \\ & & & & \Lambda & \Lambda \\ O & & & & & \Lambda \end{pmatrix}.$$

**(3)** $\mathcal{H}$ **is equivalence :**

$$\mathcal{H} : \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, U_\ell(\Lambda)) \xrightarrow{\simeq} \underline{\mathsf{CM}}^{\mathbb{Z}}(A)$$

**Note.**

**This algebra $A$ has been studied by many researchers (e.g. Ringel-Zhu, Lu...).**

**The equivalence (3) was shown by M. Lu.**

**His strategy is to find a tilting object in $\underline{\mathsf{CM}}^{\mathbb{Z}}(A)$ and apply tilting theory.**

**We have studied hwg IG-algebras from viewpoint of tilting theory.**

**If you are interested, please check our paper arXiv:1811.08036.**

**Thank you for your attention.**