Knörrer's periodicity for skew quadric hypersurfaces

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k : an algebraically closed field of characteristic not 2.

Theorem 1 (Knörrer's periodicity theorem)

$S = k[x_1, \ldots, x_n] \quad \deg x_i \in \mathbb{N}^+$,
$0 \neq f \in S_{2e}$ *(homog. polynomial of even degree* $2e$*).*
Then

$$\underline{\mathrm{CM}}^{\mathbb{Z}}(S/(f)) \cong \underline{\mathrm{CM}}^{\mathbb{Z}}(S[u, v]/(f + u^2 + v^2))$$

where $\deg u = \deg v = e$.

Theorem 2

$S = k[x_1, \ldots, x_n] \;\; \deg x_i = 1, \quad f = x_1^2 + \cdots + x_n^2 \in S_2$.
(1) If n *is odd, then*

$$\underline{\mathrm{CM}}^{\mathbb{Z}}(S/(f)) \cong \underline{\mathrm{CM}}^{\mathbb{Z}}(k[x_1]/(x_1^2)) \cong \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, k).$$

(2) If n *is even, then*

$$\underline{\mathrm{CM}}^{\mathbb{Z}}(S/(f)) \cong \underline{\mathrm{CM}}^{\mathbb{Z}}(k[x_1, x_2]/(x_1^2 + x_2^2)) \cong \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, k^2).$$

In this talk, we study a "skew version" of Theorem 2.

Setting

For $\varepsilon := (\varepsilon_{ij}) \in M_n(k)$ s.t. $\varepsilon_{ii} = 1$ and $\boldsymbol{\varepsilon_{ij} = \varepsilon_{ji} = \pm 1}$, we fix the following notation:

- $S_\varepsilon := k\langle x_1, \ldots, x_n\rangle/(x_i x_j - \varepsilon_{ij} x_j x_i) \qquad \deg x_i = 1$
  ((± 1)-skew polynomial algebra generated in degree 1).
- $f_\varepsilon := x_1^2 + \cdots + x_n^2 \in S_\varepsilon$ (cental element).
- $A_\varepsilon := S_\varepsilon/(f_\varepsilon)$.
- $\mathrm{CM}^{\mathbb{Z}}(A_\varepsilon) := \{M \in \operatorname{mod}^{\mathbb{Z}} A_\varepsilon \mid \operatorname{Ext}^i_{A_\varepsilon}(M, A_\varepsilon) = 0 \ (i > 0)\}$
  (the category of graded MCM modules).
- $\underline{\mathrm{CM}}^{\mathbb{Z}}(A_\varepsilon)$: stable category of $\mathrm{CM}^{\mathbb{Z}}(A_\varepsilon)$ (triang. cat.).

Aim

To study $\underline{\mathrm{CM}}^{\mathbb{Z}}(A_\varepsilon)$.

Example

$$S_\varepsilon = k\langle x_1, x_2, x_3\rangle/(x_1x_2 + x_2x_1,\ x_1x_3 + x_3x_1,\ x_2x_3 + x_3x_2)$$
$$(\varepsilon_{12} = \varepsilon_{13} = \varepsilon_{23} = -1)$$

$f_\varepsilon = x_1^2 + x_2^2 + x_3^2$.

Then we have

$$\begin{aligned} f_\varepsilon =& (x_1 + x_2 + x_3)(x_1 + x_2 + x_3) = (x_1 - x_2 + x_3)(x_1 - x_2 + x_3) \\ =& (x_1 + x_2 - x_3)(x_1 + x_2 - x_3) = (x_1 - x_2 - x_3)(x_1 - x_2 - x_3) \end{aligned}$$

in S_ε (matrix factorizations of f_ε of rank 1).

$$\begin{aligned} M_1 &= A_\varepsilon/(x_1 + x_2 + x_3)A_\varepsilon, & M_2 &= A_\varepsilon/(x_1 - x_2 + x_3)A_\varepsilon \\ M_3 &= A_\varepsilon/(x_1 + x_2 - x_3)A_\varepsilon, & M_4 &= A_\varepsilon/(x_1 - x_2 - x_3)A_\varepsilon \end{aligned}$$

are non-isomorphic MCM modules over $A_\varepsilon(= S_\varepsilon/(f_\varepsilon))$.
In fact,

$$\underline{\mathrm{CM}}^{\mathbb{Z}}(A_\varepsilon) \cong \mathrm{D}^{\mathrm{b}}(\operatorname{mod} k^4).$$

Graphical methods for computation of $\underline{\mathrm{CM}}^{\mathbb{Z}}(A_\varepsilon)$

Definition 3

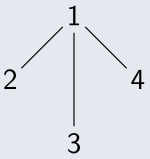
For $\varepsilon := (\varepsilon_{ij}) \in M_n(k)$ s.t. $\varepsilon_{ii} = 1$ and $\varepsilon_{ij} = \varepsilon_{ji} = \pm 1$, we define the graph G_ε by

- (vertices) $V(G_\varepsilon) := \{1, 2, \ldots, n\}$
- (edges) $E(G_\varepsilon) := \{(i,j) \mid \varepsilon_{ij} = \varepsilon_{ji} = 1\}$

Example

$(n = 4)$ $\varepsilon_{12} = \varepsilon_{13} = \varepsilon_{14} = \mathbf{+1}$ $\varepsilon_{23} = \varepsilon_{24} = \varepsilon_{34} = \mathbf{-1}$

Then

$G_\varepsilon =$ 

Two Mutations

Definition 4

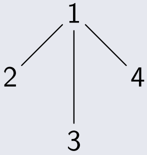
G : a simple graph, $\quad v \in V(G)$.

$\mu_v(G)$: the mutation of G at v $\overset{\text{def}}{\Longleftrightarrow}$

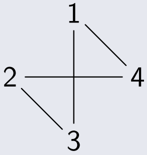
$\mu_v(G)$ is the graph such that $V(\mu_v(G)) := V(G)$ and

- for $u \neq v$, $\quad (v, u) \in E(\mu_v(G)) :\Leftrightarrow (v, u) \notin E(G)$,
- for $u, u' \neq v$, $\quad (u, u') \in E(\mu_v(G)) :\Leftrightarrow (u, u') \in E(G)$.

Example

$G =$ 

$\Longrightarrow$

$\mu_2(G) =$ 

Definition 5

G : a simple graph, $v, w \in V(G)$.

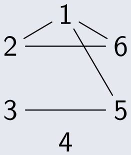
$\mu_{v \leftarrow w}(G)$: the relative mutation of G at v by w $\overset{\text{def}}{\Longleftrightarrow}$

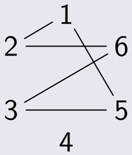
$\mu_{v \leftarrow w}(G)$ is the graph such that $V(\mu_v(G)) := V(G)$ and

- for $u \neq v, w$, $(v, u) \in E(\mu_{v \leftarrow w}(G)) :\Leftrightarrow$

  $(v, u) \in E(G),\ (w, u) \notin E(G)$ or $(v, u) \notin E(G),\ (w, u) \in E(G)$,
- $(v, w) \in E(\mu_{v \leftarrow w}(G)) :\Leftrightarrow (v, w) \in E(G)$,
- for $u, u' \neq v$, $(u, u') \in E(\mu_{v \leftarrow w}(G)) :\Leftrightarrow (u, u') \in E(G)$.

Example

$G =$ 

$\Longrightarrow$ $\mu_{6 \leftarrow 5}(G) =$ 

Theorem 6 (Mutation [MU])

If $G_{\varepsilon'} = \mu_v(G_\varepsilon)$, then

$$\underline{\mathrm{CM}}^{\mathbb{Z}}(A_\varepsilon) \cong \underline{\mathrm{CM}}^{\mathbb{Z}}(A_{\varepsilon'}).$$

Theorem 7 (Relative Mutation [MU])

Assume that G_ε has an isolated vertex u.
If $G_{\varepsilon'} = \mu_{v \leftarrow w}(G_\varepsilon)$ $(v, w \neq u)$, then

$$\underline{\mathrm{CM}}^{\mathbb{Z}}(A_\varepsilon) \cong \underline{\mathrm{CM}}^{\mathbb{Z}}(A_{\varepsilon'}).$$

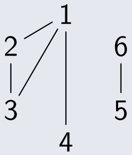
Two Reductions

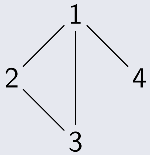
Theorem 8 (Knörrer Reduction [MU])

Assume that G_ε has an isolated segment $[v, w]$.
If $G_{\varepsilon'} = G_\varepsilon \setminus [v, w]$, then

$$\underline{\mathsf{CM}}^{\mathbb{Z}}(A_\varepsilon) \cong \underline{\mathsf{CM}}^{\mathbb{Z}}(A_{\varepsilon'}).$$

Example

$G_\varepsilon =$ 

$\Longrightarrow$ $G_\varepsilon \setminus [5, 6] =$ 

Remark 9

Knörrer reduction is a consequence of noncommutative Knörrer's periodicity theorem presented in Mori's talk.

Theorem 10 (Two Points Reduction [MU])

Assume that G_ε has two distinct isolated vertices v, w.
If $G_{\varepsilon'} = G_\varepsilon \setminus \{v\}$, then

$$\underline{\mathrm{CM}}^{\mathbb{Z}}(A_\varepsilon) \cong \underline{\mathrm{CM}}^{\mathbb{Z}}(A_{\varepsilon'})^{\times 2}.$$

Theorem 11 ([MU])

By using mutation, relative mutation, Knörrer reduction, and two points reduction, we can completely compute $\underline{\mathrm{CM}}^{\mathbb{Z}}(A_\varepsilon)$ up to $n = 6$.

This result suggests that these methods are powerful!
I plan to generalize for any n in future work.

Demonstration

$(n=6)\quad S_\varepsilon = k\langle x_1,\dots,x_6\rangle/(x_ix_j-\varepsilon_{ij}x_jx_i)$ where

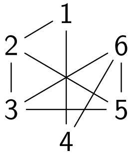
$$\varepsilon_{12}=\varepsilon_{14}=\varepsilon_{23}=\varepsilon_{25}=\varepsilon_{35}=\varepsilon_{36}=\varepsilon_{46}=\varepsilon_{56}=\mathbf{+1}$$

$$\varepsilon_{13}=\varepsilon_{15}=\varepsilon_{16}=\varepsilon_{24}=\varepsilon_{26}=\varepsilon_{34}=\varepsilon_{45}=\mathbf{-1}$$

$f_\varepsilon = x_1^2+\cdots+x_6^2 \in S_\varepsilon$

$A_\varepsilon = S_\varepsilon/(f_\varepsilon)$

Then

$G_\varepsilon =$ 

We can transform G_ε to a disjoint union of two isolated segments and two isolated vertices by applying mutation and relative mutation several times. Hence we have

$$\underline{\mathrm{CM}}^{\mathbb{Z}}(A_\varepsilon)\cong\underline{\mathrm{CM}}^{\mathbb{Z}}(k[x]/(x^2))^{\times 2}\cong \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,k)^{\times 2}\cong \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,k^2).$$

$E_\varepsilon := \bigcap_{\varepsilon_{ij}\varepsilon_{jh}\varepsilon_{hi}=-1} \mathcal{V}(x_i x_j x_h) \subset \mathbb{P}^{n-1}$ (point scheme of S_ε)

Corollary 12 ([MU])

Let ℓ be the number of irreducible components of E_ε that are isomorphic to $\mathbb{P}^1$. Assume that $n \le 6$.

(1) If n is odd, then $\ell \le 10$ and

$$\ell = 0 \iff \underline{\mathrm{CM}}^{\mathbb{Z}}(A_\varepsilon) \cong \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, k),$$

$$0 < \ell \le 3 \iff \underline{\mathrm{CM}}^{\mathbb{Z}}(A_\varepsilon) \cong \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, k^4),$$

$$3 < \ell \le 10 \iff \underline{\mathrm{CM}}^{\mathbb{Z}}(A_\varepsilon) \cong \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, k^{16}).$$

(2) If n is even, then $\ell \le 15$ and

$$0 \le \ell \le 1 \iff \underline{\mathrm{CM}}^{\mathbb{Z}}(A_\varepsilon) \cong \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, k^2),$$

$$1 < \ell \le 6 \iff \underline{\mathrm{CM}}^{\mathbb{Z}}(A_\varepsilon) \cong \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, k^8),$$

$$6 < \ell \le 15 \iff \underline{\mathrm{CM}}^{\mathbb{Z}}(A_\varepsilon) \cong \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, k^{32}).$$

Note that this corollary does not hold in the case $n = 7$.