Baer-Kaplansky Classes in Categories

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• [L. Fuchs, Theorem 108.1] L. Fuchs, Infinite Abelian Groups, Pure and Applied Mathematics, 36-II, Academic Press, New York, 1973.

Baer-Kaplansky Theorem: Any two torsion abelian groups having isomorphic endomorphism rings are isomorphic.

Other Classes

An interesting topic of research has been to find other classes of abelian groups, and more generally, of modules, for which a Baer-Kaplansky-type theorem is still true. Such classes have been called Baer-Kaplansky classes by Ivanov and Vámos:

• [G. Ivanov, P. Vámos] G. Ivanov, P. Vámos, A Characterization of FGC rings, Rocky Mountain J. Math. 32 (2002), 1485-1492.

Motivation

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Motivation-Examples

• [D. Keskin Tütüncü, R. Tribak, Example 1.3] D. Keskin Tütüncü, R. Tribak, On Baer-Kaplansky Classes of Modules, Algebra Colloq. 24 (2017), 603-610.

The class of finitely generated abelian groups is Baer-Kaplansky (but the class of torsion-free abelian groups is not).

• [K. Morita, Lemma 7.4] K. Morita, Category-isomorphisms and Endomorphism Rings of Modules, Trans. Amer. Math. Soc. 103 (1962), 451-469.

The class of all modules over a primary artinian uniserial ring is Baer-Kaplansky.

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• [G. Ivanov, Theorem 9] G. Ivanov, Generalizing the Baer-Kaplansky Theorem, J. Pure Appl. Algebra 133 (1998), 107-115.

The class of all modules over a nonsingular artinian uniserial ring is Baer-Kaplansky.
Note that in this paper, Ivanov introduced and proposed in the study of Baer-Kaplansky classes of modules the use of the stronger notion of IP-isomorphism (i.e., indecomposable-preserving isomorphism) instead of isomorphism, together with direct sum decompositions into indecomposables.

Motivation

For some ring R with identity:

- Mod-R: The category of right R-modules.
- R-Mod: The category of left R-modules.
- mod-R: The category of finitely presented right R-modules.
- R-mod: The category of finitely presented left R-modules.

Motivation

It is well known that there is a fully faithful functor

$$H : \mathrm{Mod}\text{-}R \to ((\mathrm{mod}\text{-}R)^{\mathrm{op}}, \mathrm{Ab})$$

defined by $H(M) = \mathrm{Hom}_R(-, M)$, which induces an equivalence between Mod-R and the full subcategory of flat functors in the category $((\mathrm{mod}\text{-}R)^{\mathrm{op}}, \mathrm{Ab})$ of contravariant (additive) functors from mod-R to the category Ab of abelian groups.

Also it is well known that there is a fully faithful functor

$$T : R\text{-}\mathrm{Mod} \to (\mathrm{mod}\text{-}R, \mathrm{Ab})$$

defined by $T(M) = - \otimes_R M$, which induces an equivalence between R-Mod and the full subcategory of FP-injective functors in the category $(\mathrm{mod}\text{-}R, \mathrm{Ab})$ of covariant (additive) functors from mod-R to Ab.

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Motivation

These functors have been used by several authors in order to relate properties of module categories and of the corresponding functor categories.

- M. Auslander, Coherent Functors. In: Proc. Conf. on Categorical Algebra (La Jolla, 1965), pp. 189-231, Springer, New York, 1966.
- L. Gruson, C. U. Jensen, Dimensions Cohomologiques Reliees Aux Foncteurs $\varinjlim^{(i)}$. In: Lecture Notes in Mathematics, 867, pp. 234–294, Springer-Verlag, Berlin, 1981.
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What are our techniques?

We use functor categories techniques in order to relate Baer-Kaplansky classes in (Grothendieck) categories to Baer-Kaplansky classes in finitely accessible additive categories (in particular, the category of torsion-free abelian groups), exactly definable additive categories (in particular, the category of divisible abelian groups) and categories $\sigma[M]$ (in particular, the category of comodules over a coalgebra over a field). Even if our results in these categories are somehow similar to each other, we point out that the above three types of categories are independent in general.

Our Definition

Let $\mathcal{C}$ be a preadditive category and let $\mathcal{M}$ be a class of objects of $\mathcal{C}$. Following Ivanov and Vámos, $\mathcal{M}$ is called a Baer-Kaplansky class if for any two objects M and N of $\mathcal{M}$ such that $\mathrm{End}_{\mathcal{C}}(M) \cong \mathrm{End}_{\mathcal{C}}(N)$ (as rings), one has $M \cong N$.

[S. Crivei, D. Keskin Tütüncü, Proposition 2.1] S. Crivei, D. Keskin Tütüncü, Baer-Kaplansky Classes in Grothendieck Categories and Applications, Mediterr. J. Math. (2019) 16:90 (17 pages)

Let $F : \mathcal{A} \to \mathcal{B}$ be a fully faithful covariant functor between preadditive categories $\mathcal{A}$ and $\mathcal{B}$. Then a class $\mathcal{M}$ of objects of $\mathcal{A}$ is a Baer-Kaplansky class if and only if so is the class $\mathcal{N} = \{F(M) \mid M \in \mathcal{M}\}$.

RESULTS (Baer-Kaplansky Classes in Grothendieck Categories)

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Cotorsion Pair

Let $\mathcal{C}$ be an abelian category and let $\mathcal{M}$ be a class of objects of $\mathcal{C}$. Denote

$$\mathcal{M}^{\perp} = \{C \in \mathcal{C} \mid \mathrm{Ext}^1_{\mathcal{C}}(M, C) = 0 \text{ for every } M \in \mathcal{M}\},$$

$$^{\perp}\mathcal{M} = \{C \in \mathcal{C} \mid \mathrm{Ext}^1_{\mathcal{C}}(C, M) = 0 \text{ for every } M \in \mathcal{M}\}.$$

Recall that a pair $(\mathcal{A}, \mathcal{B})$ of classes of objects of $\mathcal{C}$ is called a cotorsion pair if $\mathcal{A}^{\perp} = \mathcal{B}$ and $^{\perp}\mathcal{B} = \mathcal{A}$.

IP-Isomorphism

Recall that a ring isomorphism $\Phi : \mathrm{End}_{\mathcal{C}}(M) \to \mathrm{End}_{\mathcal{C}}(N)$ is called an IP-isomorphism if for every primitive idempotent $e \in \mathrm{End}_{\mathcal{C}}(M)$, one has $\Phi(e)N \cong eM$ [G. Ivanov].

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Recall that a ring isomorphism $\Phi : \mathrm{End}_{\mathcal{C}}(M) \to \mathrm{End}_{\mathcal{C}}(N)$ is called an *IP*-isomorphism if for every primitive idempotent $e \in \mathrm{End}_{\mathcal{C}}(M)$, one has $\Phi(e)N \cong eM$ [G. Ivanov].

[S. Crivei, D. Keskin Tütüncü, Theorem 2.2, Corollaries 2.3, 2.4 and 2.5]

Let M and N be objects of a Grothendieck category $\mathcal{C}$ such that M has a direct sum decomposition into indecomposable objects and there exists an IP-isomorphism $\Phi : \mathrm{End}_{\mathcal{C}}(M) \to \mathrm{End}_{\mathcal{C}}(N)$. Then:

1. M is (isomorphic to) a pure subobject of N.
2. If $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair in $\mathcal{C}$, $M \in \mathcal{B}$ and $N/M \in \mathcal{A}$, then M and N are isomorphic.

RESULTS (Baer-Kaplansky Classes in Grothendieck Categories)

Corollaries of the previous theorem

- Let $\mathcal{C}$ be a Grothendieck category. Let M and N be objects of $\mathcal{C}$ such that M has a direct sum decomposition into indecomposable objects and there exists an IP-isomorphism $\Phi : \mathrm{End}_{\mathcal{C}}(M) \to \mathrm{End}_{\mathcal{C}}(N)$. If one of the following conditions holds:
  1. M is injective;
  2. N/M is projective;

  then M and N are isomorphic.
- Let $\mathcal{C}$ be a (locally coherent) Grothendieck category. Let M and N be objects of $\mathcal{C}$ such that M is cotorsion and N is flat (N is FP-injective and N/M is FP-projective). Assume that M has a direct sum decomposition into indecomposable objects and there exists an IP-isomorphism $\Phi : \mathrm{End}_{\mathcal{C}}(M) \to \mathrm{End}_{\mathcal{C}}(N)$. Then M and N are isomorphic.

Recall that an object M of any Grothendieck category $\mathcal{C}$ is called:

- flat if every epimorphism $C \to M$ is pure;
- FP-injective if every monomorphism $M \to C$ is pure;
- cotorsion if $\mathrm{Ext}^1_{\mathcal{C}}(F, M) = 0$ for every flat object F of $\mathcal{C}$;
- FP-projective if $\mathrm{Ext}^1_{\mathcal{C}}(M, F) = 0$ for every FP-injective object F of $\mathcal{C}$.

- J. Xu, Flat Covers of Modules, Lecture Notes in Math., 1634, Springer, Berlin, 1996.
- L. Mao, On Covers and Envelopes in some Functor Categories, Comm. Algebra 41 (2013), 1655-1684.

[S. Crivei, D. Keskin Tütüncü, Theorem 2.7]

Let $\mathcal{C}$ be a Grothendieck category. Let $\mathcal{M}$ be a class of objects of $\mathcal{C}$ closed under summands such that every object of $\mathcal{M}$ has a direct sum decomposition into indecomposable summands and the finite embedding property. Then $\mathcal{M}$ is Baer-Kaplansky if and only if the class of indecomposable objects of $\mathcal{M}$ is Baer-Kaplansky.

Let $\mathcal{C}$ be a category and let M be an object of $\mathcal{C}$. Let $M = \bigoplus_{k \in K} M_k$ be a direct sum decomposition into indecomposable summands. A subobject L of M is called finitely embedded in M with respect to the above direct sum decomposition of M if $L \subseteq \bigoplus_{k \in F} M_k$ for some finite $F \subseteq K$. Then M has the finite embedding property if every indecomposable summand of M is finitely embedded in M with respect to the above direct sum decomposition [G. Ivanov].

Corollary of the previous theorem

Let R be a ring with enough idempotents.

- The class of semisimple right R-modules is Baer-Kaplansky if and only if the class of simple right R-modules is Baer-Kaplansky ([D. Keskin Tütüncü, R. Tribak, Proposition 2.12]).
- Assume that R is semiperfect. Then the class of finitely generated projective right R-modules is Baer-Kaplansky if and only if the class of projective local right R-modules is Baer-Kaplansky.
- Assume that R is right noetherian. Then the class of finitely generated injective right R-modules is Baer-Kaplansky if and only if the class of finitely generated indecomposable injective right R-modules is Baer-Kaplansky.

A module M over a ring (with enough idempotents) is called *local* if M has a proper submodule which contains all other proper submodules (e.g., see [R. Wisbauer, Foundations of module and ring theory, Gordon and Breach, Reading, 1991]).

RESULTS (Baer-Kaplansky Classes in Finitely Accessible Categories)

Some Terminology

- W. Crawley-Boevey, Locally finitely presented additive categories, Comm. Algebra **22** (1994), 1641–1674.
- M. Prest, Definable additive categories: purity and model theory, Mem. Amer. Math. Soc., 210, No. 987 (2011).

An additive category $\mathcal{C}$ is called finitely accessible if it has direct limits, the class of finitely presented objects is skeletally small, and every object is a direct limit of finitely presented objects. The category of unitary modules over a ring with enough idempotents, the category of torsion abelian groups and the category of torsion-free abelian groups are typical examples of finitely accessible additive categories.

RESULTS (Baer-Kaplansky Classes in Finitely Accessible Categories)

Some Terminology

Let $\mathcal{C}$ be a finitely accessible additive category. By a sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in the additive category $\mathcal{C}$ we mean a pair of composable morphisms $f : X \to Y$ and $g : Y \to Z$ such that $gf = 0$. The sequence is called pure exact if it induces an exact sequence of abelian groups
$0 \to \mathrm{Hom}_{\mathcal{C}}(P, X) \to \mathrm{Hom}_{\mathcal{C}}(P, Y) \to \mathrm{Hom}_{\mathcal{C}}(P, Z) \to 0$ for every finitely presented object P of $\mathcal{C}$. This implies that f and g form a kernel-cokernel pair, that f is a monomorphism and g an epimorphism. In such a pure exact sequence f is called a pure monomorphism and g a pure epimorphism. An object M of $\mathcal{C}$ is called pure-injective if every pure exact sequence in $\mathcal{C}$ with the first term M splits, and pure-projective if every pure exact sequence in $\mathcal{C}$ with the third term M splits.

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RESULTS (Baer-Kaplansky Classes in Finitely Accessible Categories)

Some Terminology-Facts

- W. Crawley-Boevey, Locally finitely presented additive categories, Comm. Algebra 22 (1994), 1641–1674.
- N. V. Dung, J. L. García, Additive categories of locally finite representation type, J. Algebra 238 (2001), 200–238.
- I. Herzog, Pure-injective envelopes, J. Algebra Appl. 4 (2003), 397–402.

Let $\mathcal{C}$ be a finitely accessible additive category. Then there is a Grothendieck category $\mathcal{A}(\mathcal{C})$ (uniquely determined up to equivalence) and a fully faithful functor $H : \mathcal{C} \to \mathcal{A}(\mathcal{C})$ (naturally isomorphic to the inclusion functor), which induces an equivalence between $\mathcal{C}$ and the full subcategory of flat objects of $\mathcal{A}(\mathcal{C})$. Moreover, a sequence in $\mathcal{C}$ is pure exact if and only if H takes it into an exact sequence in $\mathcal{A}(\mathcal{C})$.

RESULTS (Baer-Kaplansky Classes in Finitely Accessible Categories)

[S. Crivei, D. Keskin Tütüncü, Theorem 3.4]

Let $\mathcal{C}$ be a finitely accessible additive category. Let X and Y be objects of $\mathcal{C}$ such that X has a direct sum decomposition into indecomposable subobjects and there exists an *IP*-isomorphism $\Phi : \mathrm{End}_{\mathcal{C}}(X) \to \mathrm{End}_{\mathcal{C}}(Y)$. If one of the following conditions holds:

1. Y/X is pure-projective;
2. X is pure-injective;

then X and Y are isomorphic.

[S. Crivei, D. Keskin Tütüncü, Corollary 3.5]

Let $\mathcal{C}$ be a f. a. a. category. Let X and Y be objects of $\mathcal{C}$ such that X is Σ-pure-injective. If there exists an *IP*-isomorphism $\Phi : \mathrm{End}_{\mathcal{C}}(X) \to \mathrm{End}_{\mathcal{C}}(Y)$, then X and Y are isomorphic.

RESULTS (Baer-Kaplansky Classes in Finitely Accessible Categories)

A finitely accessible additive category $\mathcal{C}$ is called Krull-Schmidt if every finitely presented object of $\mathcal{C}$ has a finite direct sum decomposition into subobjects with local endomorphism rings

[S. Crivei, D. Keskin Tütüncü, Theorem 3.7]

Let $\mathcal{C}$ be a Krull-Schmidt finitely accessible additive category. The following are equivalent:

1. The class of finitely presented objects of $\mathcal{C}$ is Baer-Kaplansky.
2. The class of pure-projective indecomposable objects of $\mathcal{C}$ is Baer-Kaplansky.
3. The class of finitely presented indecomposable objects of $\mathcal{C}$ is Baer-Kaplansky.

RESULTS (Baer-Kaplansky Classes in Finitely Accessible Categories)

A finitely accessible additive category $\mathcal{C}$ is called pure semisimple if every pure exact sequence in $\mathcal{C}$ splits.

[S. Crivei, D. Keskin Tütüncü, Corollary 3.9]

Let $\mathcal{C}$ be a pure semisimple finitely accessible additive category. Then the class of finitely presented objects of $\mathcal{C}$ is Baer-Kaplansky if and only if the class of (finitely presented) indecomposable objects of $\mathcal{C}$ is Baer-Kaplansky.

RESULTS (Baer-Kaplansky Classes in Exactly Definable Categories)

Some Terminology

- H. Krause, Exactly definable categories, J. Algebra 201 (1998), 456–492.
- M. Prest, Definable additive categories: purity and model theory, Mem. Amer. Math. Soc., 210, No. 987 (2011).

An additive category $\mathcal{C}$ is called exactly definable if it is equivalent to the category $\mathrm{Ex}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$ of exact contravariant additive functors from $\mathcal{A}$ to the category Ab of abelian groups for some skeletally small abelian category $\mathcal{A}$.

RESULTS (Baer-Kaplansky Classes in Exactly Definable Categories)

Some Terminology-Facts

- Every finitely accessible additive category with products is exactly definable ([W. Crawley-Boevey, 3.3, Locally finitely presented additive categories, Comm. Algebra 22 (1994), 1641–1674]).
- More generally, an additive category is exactly definable if and only if it is a definable subcategory (in the sense that it is closed under products, direct limits and pure subobjects) of a finitely accessible additive category with products ([M. Prest, Proposition 11.1, Definable additive categories: purity and model theory, Mem. Amer. Math. Soc., 210, No. 987 (2011)]).

RESULTS (Baer-Kaplansky Classes in Exactly Definable Categories)

Some Terminology-Facts

- [M. Prest, Examples 10.2, 10.3 and 10.5], Definable additive categories: purity and model theory, Mem. Amer. Math. Soc., 210, No. 987 (2011)).
- [M. Prest, Theorem 3.6], Abelian categories and definable additive categories, arXiv:1202.0426, 2012.

- The category of unitary modules over a ring with enough idempotents,
- The category of torsion abelian groups,
- The category of torsion-free abelian groups

are not only finitely accessible categories, but also exactly definable.

Some Terminology-Facts

- A Grothendieck category is exactly definable if and only if it is finitely accessible (3.6).
- In general, exactly definable additive categories need not be finitely accessible. For instance, the category of divisible abelian groups is a definable subcategory of the category of abelian groups, hence it is exactly definable, but not finitely accessible (10.3).

RESULTS (Baer-Kaplansky Classes in Exactly Definable Categories)

Some Terminology

Let $\mathcal{C}$ be an exactly definable additive category. An object M of $\mathcal{C}$ is called pure-injective if for every set I the summation morphism $M^{(I)} \to M$ factors through the canonical morphism $M^{(I)} \to M^I$. A sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in $\mathcal{C}$ is called pure exact if it induces an exact sequence of abelian groups $0 \to \mathrm{Hom}_{\mathcal{C}}(Z, Q) \to \mathrm{Hom}_{\mathcal{C}}(Y, Q) \to \mathrm{Hom}_{\mathcal{C}}(X, Q) \to 0$ for every pure-injective object Q of $\mathcal{C}$. This implies that f and g form a kernel-cokernel pair, that f is a monomorphism and g an epimorphism. In such a pure exact sequence f is called a pure monomorphism and g a pure epimorphism. An object M of $\mathcal{C}$ is called pure-projective if every pure exact sequence in $\mathcal{C}$ with the third term M splits.

RESULTS (Baer-Kaplansky Classes in Exactly Definable Categories)

Some Terminology-Facts

Let $\mathcal{C}$ be an exactly definable additive category. Then there is a locally coherent Grothendieck category $\mathcal{D}(\mathcal{C})$ (uniquely determined up to equivalence) and a fully faithful functor $T : \mathcal{C} \to \mathcal{D}(\mathcal{C})$ (naturally isomorphic to the inclusion functor), which induces an equivalence between $\mathcal{C}$ and the full subcategory of *FP*-injective objects of $\mathcal{D}(\mathcal{C})$. Moreover, a sequence in $\mathcal{C}$ is pure exact if and only if T takes it into an exact sequence in $\mathcal{D}(\mathcal{C})$ (Krause).

RESULTS (Baer-Kaplansky Classes in Exactly Definable Categories)

[S. Crivei, D. Keskin Tütüncü, Theorem 4.5]

Let $\mathcal{C}$ be an exactly definable additive category. Let X and Y be objects of $\mathcal{C}$ such that X has a direct sum decomposition into indecomposable subobjects and there exists an IP-isomorphism $\Phi : \mathrm{End}_{\mathcal{C}}(X) \to \mathrm{End}_{\mathcal{C}}(Y)$. If one of the following conditions hold:

1. X is pure-injective;
2. Y/X is pure-projective;

then X and Y are isomorphic.

[S. Crivei, D. Keskin Tütüncü, Corollary 4.6]

Let $\mathcal{C}$ be an e. d. a. category. Let X and Y be objects of $\mathcal{C}$ such that X is Σ-pure-injective. If there exists an IP-isomorphism $\Phi : \mathrm{End}_{\mathcal{C}}(X) \to \mathrm{End}_{\mathcal{C}}(Y)$, then X and Y are isomorphic.

RESULTS (Baer-Kaplansky Classes in Exactly Definable Categories)

An exactly definable additive category $\mathcal{C}$ is called pure semisimple if every pure exact sequence in $\mathcal{C}$ splits.

[S. Crivei, D. Keskin Tütüncü, Theorem 4.8]

Let $\mathcal{C}$ be a pure semisimple exactly definable additive category. Then the class of finitely presented objects of $\mathcal{C}$ is Baer-Kaplansky if and only if the class of finitely presented indecomposable objects of $\mathcal{C}$ is Baer-Kaplansky.

Some Terminology-Facts

Let R be a ring with identity and M a left R-module. Then the category $\sigma[M]$ is the full subcategory of the category of left R-modules consisting of modules isomorphic to M-subgenerated modules ([R. Wisbauer, Section 15, Foundations of module and ring theory, Gordon and Breach, Reading, 1991]). It is the smallest Grothendieck category containing M. For instance, when $M = R$, $\sigma[M]$ is the category of left R-modules. Also, when C is a coalgebra over a field k, C is a left C^*-module, where $C^* = \mathrm{Hom}_k(C, k)$, and $\sigma[_{C^*}C]$ is the category of right C-comodules ([S. Dăscălescu, C. Năstăsescu, Ş. Raianu, Hopf Algebras. An Introduction, Marcel Dekker, New York, 2001]).

Some Terminology-Facts

In general $\sigma[M]$ need not be finitely accessible ([M. Prest, R. Wisbauer, Example 1.7, M. Prest, R. Wisbauer, Finite presentation and purity in categories $\sigma[M]$, Colloq. Math. 99 (2004), 189–202]).

Some Terminology-Facts

One may define purity, pure-injectivity and pure-projectivity in categories $\sigma[M]$ in a similar way as for a usual module category ([R. Wisbauer, Section 34, Foundations of module and ring theory, Gordon and Breach, Reading, 1991]).
If $(U_i)_{i \in I}$ is a representing set of all finitely presented objects of $\sigma[M]$, then one may construct a certain ring with enough idempotents associated to $\sigma[M]$, called the functor ring of $\sigma[M]$ ([R. Wisbauer, Section 52, Foundations of module and ring theory, Gordon and Breach, Reading, 1991]).

• [R. Wisbauer, 52.2] R. Wisbauer, Foundations of module and ring theory, Gordon and Breach, Reading, 1991.

Let R be a ring with identity, M a left R-module and T the functor ring of $\sigma[M]$. Let $(U_i)_{i\in I}$ be a representing set of all finitely presented objects of $\sigma[M]$ and $U = \bigoplus_{i\in I} U_i$. Assume that U is a generator in $\sigma[M]$. Then there is a fully faithful functor

$$F : \sigma[M] \to T\text{-Mod},$$

which induces an equivalence between $\sigma[M]$ and the full subcategory of flat left T-modules. Moreover, a sequence in $\sigma[M]$ is pure exact if and only if F takes it into a (pure) exact sequence in T-Mod.

[S. Crivei, D. Keskin Tütüncü, Theorem 5.2]

Let R be a ring with identity and let M be a left R-module. Let X and Y be objects of $\sigma[M]$ such that X has a direct sum decomposition into indecomposable subobjects and Y/X is pure-projective. If there exists an IP-isomorphism $\Phi : \mathrm{End}_R(X) \to \mathrm{End}_R(Y)$, then X and Y are isomorphic.

A left R-module M is called pure semisimple if every pure exact sequence in $\sigma[M]$ splits.

[S. Crivei, D. Keskin Tütüncü, Corollary 5.4]

Let R be a ring with identity and let M be a pure semisimple left R-module. Then the class of finitely presented objects of $\sigma[M]$ is Baer-Kaplansky if and only if the class of (finitely presented) indecomposable objects of $\sigma[M]$ is Baer-Kaplansky.

THANK YOU VERY MUCH!