Elliptic Algebras

S. Paul Smith

University of Washington
Seattle, WA 98195.
smith@math.washington.edu

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What is repn theory of non-commutative algebras about?

Compare: what is algebraic geometry about?

- solutions to systems of polynomial equations

$$f_1(x_1, \dots, x_n) \;=\; \cdots \;=\; f_r(x_1, \dots, x_n) \;=\; 0$$

  with coefficients in a field $\Bbbk$
- two types of solutions
  - $x_1, \dots, x_n \in \Bbbk$ (1-dimensional solutions/repns) OR $(x_1, \dots, x_n) \in \Bbbk^n$ (points on an algebraic variety)
  - $x_1, \dots, x_n$ are $d \times d$ matrices that commute with each other (d-dimensional solutions/repns) and $f_j(x_1, \dots, x_n) = 0 \quad \forall j$

what is repn theory of non-commutative algebras about?

- solutions to systems of "polyn" equations $f_j(x_1, \dots, x_n) = 0$
  - $x_1, \dots, x_n$ are $d \times d$ matrices such that $f_j(x_1, \dots, x_n) = 0 \quad \forall j$
  - special case: $x_1, \dots, x_n$ are 1×1 matrices (1-dim'l reps)
  - special case: allow ∞-dimensional matrices; i.e., linear operators $x_i : V \to V$ such that $f_j(x_1, \dots, x_n) = 0 \quad \forall j$

Equivalent to a problem in ring theory

very important fact: solutions to a system of "polyn" equations

$$f_1(x_1, \ldots, x_n) = \cdots = f_r(x_1, \ldots, x_n) = 0$$

with coefficients in a field $\Bbbk$ are the same things as

$$\text{left } \frac{\Bbbk\langle x_1, \ldots, x_n \rangle}{(f_1, \ldots, f_r)}\text{-modules}$$

Strategy:

- understand this ring R
- homological properties?
- basis? noetherian? finite dimensional? center?
- domain? prime? graded?
- commutative? finite module over its center?
- von Neumann regular?
- nice subrings? nice quotient rings?
- use this information to study $\mathrm{Mod}(R)$

First: classify/understand "irreducible" solutions
equivalently classify/understand simple modules

typical answers:

- finitely many, combinatorial classification
- infinitely many
  - geometric description, one solution for each point p on an algebraic variety X
  - combinatorial + geometric parameter space
- relate $\mathrm{Mod}(R)$ to other categories, e.g.,
  - modules over other rings
  - representations of Lie algebras, groups, etc.
  - categories of sheaves on algebraic varieties
  - methods: **functors!** Morita theory, quotient categories, tilting, stable categories, derived categories, Fourier-Mukai functors, ...

SECRET WEAPON: algebraic geometry
Kollár: translate your problem into algebraic geometry and I will give it to a graduate student

§0. Origins of elliptic algebras $Q_{n,k}(E,\tau)$

- Elliptic algebras $Q_{n,1}(E,\tau)$ discovered by
  - Sklyanin (1982) $n=4$
  - Artin-Schelter (1986) $n=3$
  - Feigin-Odesskii (1989) $n \geq 3$
  - Artin-Tate-Van den Bergh (1990) $n=3$
  - Connes and Dubois-Violette (2005) $n=4$
- different motivations:
  - physics
  - graded non-commutative analogs of polynomial rings with excellent homological properties
  - generalizing Sklyanin's examples
    elliptic solutions to QYBE with spectral parameter
    holomorphic vector bundles on elliptic curves
  - understanding Artin-Schelter's algebras
  - non-commutative 3-spheres, C^*-algebras

- Fix relatively prime integers $n > k \geq 1$
- lattice $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\eta \subseteq \mathbb{C}$ and $\tau \in \mathbb{C} - \frac{1}{n}\Lambda$
- elliptic curve $E := \mathbb{C}/\Lambda$
- $\Theta_n(\Lambda)$ a space of theta functions with period lattice Λ
- $\Theta_n(\Lambda) =$ irrep of the Heisenberg group of order n^3
- a "good basis" $\theta_0(z), \dots, \theta_{n-1}(z)$ for $\Theta_n(\Lambda)$

Definition: Feigin-Odesskii (1989):

$$Q_{n,k}(E,\tau) \;:=\; \frac{\mathbb{C}\langle x_0, \dots, x_{n-1}\rangle}{(R_{ij}(\tau) \mid i,j \in \mathbb{Z}_n)} \qquad (n^2 \text{ relations})$$

where

$$R_{ij}(\tau) \;:=\; \sum_{r \in \mathbb{Z}_n} \frac{\theta_{j-i+r(k-1)}(0)}{\theta_{j-i-r}(-\tau)\theta_{kr}(\tau)} \, x_{j-r}x_{i+r} \qquad (i,j) \in \mathbb{Z}_n^2$$

Large project: **understand** $Q_{n,k}(E,\tau)$

4 joint papers on the arXiv:

- Alex Chirvasitu (SUNY Buffalo)
- Ryo Kanda (Osaka)
- me

- Feigin-Odesskii (several papers) provide few proofs
- BUT many interesting assertions for τ "close to 0"
- CKS: we prove some of FO's assertions, correct some assertions, but unable to prove or disprove most assertions
- CKS: we prove results for all τ, not just τ close to 0
- many, many open problems
- please join us

Remarks about $Q_{n,k}(E,\tau)$ (fix $n > k \geq 1$)

- graded rings $\deg(x_i) = 1$, homogeneous quadratic relations
- $Q_{n,k}(E,0) \ = \$ polynomial ring $\mathbb{C}[x_0,\dots,x_{n-1}]$ (CKS)
- $\dim Q_{n,k}(E,\tau)_d \ = \ \dim \mathbb{C}[x_0,\dots,x_{n-1}]_d$ for all $d \geq 0$ (CKS)
- $Q_{2,1}(E,\tau) \ = \ \mathbb{C}[x_0,x_1]$ polynomial ring
- $Q_{n,n-1}(E,\tau) \ = \ \mathbb{C}[x_0,\dots,x_{n-1}]$ polynomial ring (CKS)
- $Q_{3,1}(E,\tau) \ = \$ 3-dimensional regular algebra
  discovered by Artin-Schelter 1986 and
- studied by Artin-Tate-Van den Bergh 1989-1991
- $Q_{4,1}(E,\tau)$ discovered/defined/studied by Sklyanin 1982-1983
- studied by Smith-Stafford 1992, Levasseur-Smith 1993
- the $Q_{n,k}(E,\tau)$'s are the most generic deformations of polynomial ring on n variables

$Q_{3,1}(E,\tau)$ discovered by Artin-Schelter (1986)

- Artin-Schelter classified non-commutative analogues of the polyomial ring on 3 variables
- with "good homological properties"
- given (E,τ), $\exists$ $(a,b,c) \in \mathbb{P}^2(\mathbb{C})$ such that
- $Q_{3,1}(E,\tau) \cong \mathbb{C}\langle x,y,z\rangle$ modulo relations

$$\begin{aligned} ax^2 + byz + czy &= 0 \\ ay^2 + bzx + cxz &= 0 \\ az^2 + bxy + cyx &= 0 \end{aligned}$$

- $(a,b,c) = (0,1,-1) \rightsquigarrow$ polynomial ring $\mathbb{C}[x,y,z]$
- $\nexists$ PBW basis except for very special (a,b,c)
- methods to understand $Q_{3,1}(E,\tau)$: algebraic geometry
- elliptic curve: $(a^3+b^3+c^3)xyz - abc(x^3+y^3+z^3) = 0$
- and an automorphism of E: $(x,y,z) \mapsto (acy^2 - b^2xz, abx^2 - c^2yz, bcz^2 - a^2xy)$

Tate and Van den Bergh's results on $Q_{n,1}(E,\tau)$

For all τ, $Q_{n,1}(E,\tau)$

- same Hilbert series as the polynomial ring
- for fixed n and E, the $Q_{n,1}(E,\tau)$'s form a flat family of deformations of the polynomial ring parametrized by E
- right and left noetherian, a domain,
- finite module over its center if and only if τ has finite order
- "excellent" homological properties: regular, gl.dim$= n$, Gorenstein, Cohen-Macaulay, ...
- Koszul algebra
- Koszul dual is a deformation of the exterior algebra $\Lambda(\mathbb{C}^n)$
- behaves like the polynomial ring on n variables

we expect all $Q_{n,k}(E,\tau)$'s have these properties

§2. Why study $Q_{n,k}(E,\tau)$? It's related to interesting things

- quantum Yang-Baxter equation with spectral parameter: for all $u, v \in \mathbb{C}$,

$$R(u)_{12}R(u+v)_{23}R(v)_{12} \;=\; R(v)_{23}R(u+v)_{12}R(u)_{23}$$

where $R(u) : \mathbb{C}^n \otimes \mathbb{C}^n \longrightarrow \mathbb{C}^n \otimes \mathbb{C}^n$ and

$$R_{12}(u)\big(v_1 \otimes v_2 \otimes v_3\big) = R(u)\big(v_1 \otimes v_2\big) \otimes v_3 \qquad \text{etc.}$$

- negative continued fraction

$$\frac{n}{k} \;=\; n_1 - \cfrac{1}{n_2 - \cfrac{1}{\ddots - \frac{1}{n_g}}} \;=\; [n_1, \dots, n_g]$$

unique g and unique $n_1, \dots, n_g$ all ≥ 2

- a distinguished invertible sheaf $\mathcal{L}_{n/k}$ on $E^g = E \times \cdots \times E$, where $g =$ the length of the continued fraction

- the Fourier-Mukai transform

$$\Phi := \mathbf{R}\mathrm{pr}_{1*}(\mathcal{L}_{n/k} \otimes^{\mathbf{L}} \mathrm{pr}_g^*(\,\cdot\,))$$

$$\begin{tikzcd} & E \times \cdots \times E \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_g"] & \\ E & & E \end{tikzcd}$$

is an auto-equivalence of $\mathrm{D}^b(\mathrm{coh}(E))$

- Φ provides a bijection: $\mathcal{E}(1,0) \xrightarrow{\Phi} \mathcal{E}(k,n)$ where

$$\mathcal{E}(r,d) = \left\{ \begin{array}{c} \text{isoclasses of indecomposable bundles} \\ \text{of rank } r \text{ and degree } d \text{ on } E \end{array} \right\}$$

- Feigin-Odesskii's definition (brilliant!):

$$\mathcal{L}_{n/k} := \left(\mathcal{L}^{\otimes n_1} \boxtimes \cdots \boxtimes \mathcal{L}^{\otimes n_g}\right) \otimes \left(\bigotimes_{i=1}^{g-1} \mathrm{pr}_{i,i+1}^* \mathcal{P} \right)$$

- $\mathcal{L} := \mathcal{O}_E((0))$
- $\mathcal{P} :=$ the Poincaré bundle $(\mathcal{L}^{-1} \boxtimes \mathcal{L}^{-1})(\Delta)$ on $E \times E$
- $\mathrm{pr}_{i,i+1} : E^g \to E^2$ is the projection $(z_1, \ldots, z_g) \mapsto (z_i, z_{i+1})$

- **Definition:** The characteristic variety of $Q_{n,k}(E,\tau)$, denoted $X_{n/k}$, is the image of the morphism

$$|\mathcal{L}_{n/k}| : E^g \to \mathbb{P}^{n-1} = \mathbb{P}(H^0(E^g, \mathcal{L}_{n/k})^*)$$

- Kanda's talk: The characteristic variety of $Q_{n,k}(E,\tau)$
- **Definition:** a distinguished automorphism

$$\sigma : E^g = \mathbb{C}^g/\Lambda^g \to E^g = \mathbb{C}^g/\Lambda^g$$

  defined by a complicated formula ... involves τ and the integers in the continued fraction $[n_1, \ldots, n_g]$
- $\exists!$ automorphism $\sigma : X_{n/k} \to X_{n/k}$ such that

$$\begin{tikzcd} E^g \arrow[r, "\sigma"] \arrow[d, "\text{quotient map}"'] & E^g \arrow[d, "\text{quotient map}"] \\ X_{n/k} \arrow[r, "\sigma"'] & X_{n/k} \end{tikzcd}$$

  commutes
- the pair $(X_{n/k}, \sigma)$ "controls" (much of) the representation theory of $Q_{n,k}(E,\tau)$

Some results of Chirvasitu-Kanda-Smith:

- **Theorem:** $X_{n/k} \cong E^g/\Sigma_{n/k}$
  quotient by the action of a finite group determined by the location of the 2's in the continued fraction $[n_1, \ldots, n_g]$
- **Theorem:** $X_{n/k}$ = fiber bundle:

$$\begin{tikzcd} X_{n/k} \arrow[d, "\text{fibers } \cong \mathbb{P}^{j_1} \times \cdots \times \mathbb{P}^{j_s}"] \\ E^r \end{tikzcd}$$

  where $r, s, j_1, \ldots, j_s$ are determined by $[n_1, \ldots, n_g]$
- **Theorem:** There are homomorphisms

$$Q_{n,k}(E, \tau) \to B(X_{n/k}, \sigma, \mathcal{L}_{n/k}) = B(E^g, \sigma, \mathcal{L}_{n/k})^{\Sigma_{n/k}}$$

  of graded algebras where $B(\cdot, \cdot, \cdot) =$
  Artin-Tate-Van den Bergh + Feigin-Odesskii's
  twisted homogeneous coordinate ring

- **Theorem:** When $X_{n/k} = E^g$, then $B(E^g, \sigma, \mathcal{L}_{n/k})$ is generated by its degree one component and its relations are in degrees ≤ 3
- **Corollary:** When $X_{n/k} = E^g$, the homomorphism
$$Q_{n,k}(E,\tau) \to B(X_{n/k} = E^g, \sigma, \mathcal{L}_{n/k})$$
is surjective and its kernel is generated by elements of deg ≤ 3
- **Theorem:** [Artin-Van den Bergh]
we know everything about $B(E^g, \sigma, \mathcal{L}_{n/k})$
- **Corollary:** [Artin-Van den Bergh, Smith] If $X_{n/k} = E^g$, there are functors $i^* \dashv i_* \dashv i^!$

$$\begin{tikzcd} \mathrm{Qcoh}(E^g) \arrow[r, "i_*"] & \mathrm{QGr}(Q_{n,k}(E,\tau)) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \end{tikzcd}$$

i^* = inverse image functor
i_* = direct image functor
where $i : E^g \to \mathrm{Proj}_{nc}(Q_{n,k}(E,\tau))$ is a "closed immersion" (non-commutative algebraic geometry)

- **Odesskii's identity:** If $\alpha, \beta \in \mathbb{Z}_n$ and $z \in \mathbb{C}^g$, then

$$(*) \qquad \sum_{r \in \mathbb{Z}_n} \frac{\theta_{\beta-\alpha+r(k-1)}(0)}{\theta_{\beta-\alpha-r}(-\tau)\theta_{rk}(\tau)} \, w_{\beta-r}(z)\, w_{\alpha+r}(\sigma(z)) \;=\; 0$$

  where
  - $w_0(z), \ldots, w_{n-1}(z)$ are certain theta functions in g variables
  - $\sigma : \mathbb{C}^g \to \mathbb{C}^g$ lifts the automorphism
  - $\sigma : E^g \to E^g = (\mathbb{C}/\Lambda)^g = \mathbb{C}^g/\Lambda^g$
- (*) $\Rightarrow$ **Proposition:** The relations for $Q_{n,k}(E,\tau)$ vanish on the graph of $\sigma : X_{n/k} \to X_{n/k}$. Graph $\subseteq \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$
- **Corollary:** If $n = 2k+1$, then
  - $\frac{2k+1}{k} = [3, 2, \ldots, 2]$
  - $X_{(2k+1)/k} \cong S^k E \subseteq \mathbb{P}^{2k} = \mathbb{P}(V^*)$ where
  - $V = Q_{2k+1,k}(E,\tau)_1$
  - $\sigma((x_1, \ldots, x_k)) = ((x_1+\tau, \ldots, x_k+\tau))$

  and the defining relations for $Q_{2k+1,k}(E,\tau)$ are

$$\{f \in V \otimes V \mid f(x, \sigma(x)) = 0 \ \forall x \in S^k E\}$$

- $\exists$ a distinguished space $\Theta_{n/k}(\Lambda)$ of theta functions in g variables defined in terms of $[n_1, \ldots, n_g]$
  - $\dim_{\mathbb{C}}\left(\Theta_{n/k}(\Lambda)\right) = n$
  - $\Theta_{n/k}(\Lambda) =$ irreducible representation of the Heisenberg group

$$H_n \;:=\; \begin{pmatrix} 1 & \mathbb{Z}_n & \mathbb{Z}_n \\ 0 & 1 & \mathbb{Z}_n \\ 0 & 0 & 1 \end{pmatrix}$$

  - $\exists$ basis $w_0, \ldots, w_{n-1}$ for $\Theta_{n/k}(\Lambda)$ that transforms in a nice way with respect to the "standard" generators for H_n
- there are several useful interpretations of $Q_{n,k}(E, tau)_1$:
  - an anonymous vector space V with basis $x_0, \ldots, x_{n-1}$
  - $\Theta_n(\Lambda) =$ space of theta functions in one variable
  - $H^0(E, \mathcal{L}_n) =$ global sections of degree-n line bundle on E
  - $\Theta_{n/k}(\Lambda) =$ space of theta functions in g variables
  - $H^0(E^g, \mathcal{L}_{n/k})$ global sections of $\mathcal{L}_{n/k}$
- **Proposition:** [Feigin-Odesskii]
  H_n acts as automorphisms of $Q_{n,k}(E, \tau)$

- **Feigin-Odesskii** claim: $Q_{n,k}(E,\tau)$ quantizes a "natural" Poisson bracket $\{-,-\}$ on $\operatorname{Ext}^1_E(\mathcal{V}_{n,k},\mathcal{O}_E)$ where
  - $\mathcal{V}_{n,k}$ = an indecomposable vector bundle on E
  - with $\operatorname{rank}(\mathcal{V}_{n,k}) = k$ and $\deg(\mathcal{V}_{n,k}) = n$
- **Hua-Polishchuk:** Feigin-Odesskii's claim is true when $k = 1$
- the stratification of $\mathbb{P}(\operatorname{Ext}^1_E(\mathcal{V}_{n,k},\mathcal{O}_E)) \cong \mathbb{P}^{n-1}$ by symplectic leaves is closely related to repn. theory of $Q_{n,k}(E,\tau)$ (???)
- **Theorem:** (CKS) $Q_{n,k}(E,\tau)$ has global dimension n and is Koszul.
- **Corollary:** $\Lambda := Q_{n,k}(E,\tau)^!$ is a deformation of the exterior algebra $\Lambda(\mathbb{C}^n)$ and has a family of indecomposable modules M_x parametrized by $x \in X_{n/k}$ with minimal resolution

$$\cdots \to \Lambda(-2) \to \Lambda(-1) \to \Lambda \to M_x \to 0.$$

Question: Is $Q_{n,k}(E,\tau)^!$ Frobenius? If so, then $Q_{n,k}(E,\tau)$ is Artin-Schelter regular.

§3. Why study $Q_{n,k}(E,\tau)$? Sklyanin's motivation (1982)

Sklyanin used Baxter's "elliptic" solutions to the QYBE to define algebras $S(\alpha,\beta,\gamma)$ for

$$\alpha,\beta,\gamma \in \mathbb{C} - \{0,\pm 1\} \text{ such that } \alpha\beta\gamma+\alpha+\beta+\gamma=0$$

Definition: $S(\alpha,\beta,\gamma) := \mathbb{C}\langle x_0,x_1,x_2,x_3\rangle$ modulo relations

$$\begin{aligned} x_0x_1 - x_1x_0 &= \alpha(x_2x_3+x_3x_2) & \qquad x_0x_1 + x_1x_0 &= x_2x_3 - x_3x_2 \\ x_0x_2 - x_2x_0 &= \beta(x_3x_1+x_1x_3) & \qquad x_0x_2 + x_2x_0 &= x_3x_1 - x_1x_3 \\ x_0x_3 - x_3x_0 &= \gamma(x_1x_2+x_2x_1) & \qquad x_0x_3 + x_3x_0 &= x_1x_2 - x_2x_1 \end{aligned}$$

Theorem (Sklyanin)

$S(\alpha,\beta,\gamma) \cong Q_{4,1}(E,\tau)$ *for some* E *and* τ*.*

- **Smith-Stafford (1992):** ring-theoretic properties of $Q_{4,1}(E,\tau)$: noetherian, Koszul, regular, Gorenstein, CM, ...

First sentence in Sklyanin's 1982 paper:

> *" One of the strongest methods of investigating the <u>exactly solvable models of quantum and statistical physics</u> is the quantum inverse problem method (QIPM). The problem of enumerating the discrete quantum systems that can be solved by the QIPM reduces to the problem of enumerating the operator-valued functions $L(u)$ that satisfy the relation ..."*

i.e., the solutions are obtained from $S(\alpha, \beta, \gamma)$-modules
i.e., find matrix solutions to the blue equations
i.e., understand/classify $Q_{4,1}(E, \tau)$-modules

Sklyanin:

> *" During our investigation it turned out that it is necessary to bring into the picture new algebraic structures, namely, the quadratic algebras of Poisson brackets and the quadratic generalization of the universal enveloping algebra of a Lie algebra. The theory of these mathematical objects is surprisingly reminiscent of the theory of Lie algebras, the difference being that it is more complicated. In our opinion, it deserves the greatest attention of mathematicians."*

1. **we agree**
2. **the $Q_{n,k}(E,\tau)$'s are fundamental mathematical objects**
3. **related to other fundamental mathematical objects**
4. see above
5. and a final example on the next slide

- $S(\alpha, \beta, \gamma)$ determines and is determined by a quartic elliptic curve $E \subseteq \mathbb{P}^3$ and translation automorphism $x \mapsto x + \tau$ of E
- $E \cong \mathbb{C}/\Lambda$ where $\Lambda = \mathbb{Z} + \mathbb{Z}\eta \subseteq \mathbb{C}$ (think of $\tau \in \mathbb{C}$)
- Jacobi's theta functions $\theta_{00}, \theta_{01}, \theta_{10}, \theta_{11}$ with period lattice Λ

$$\begin{cases} \theta_{ab}(z+1) & = & (-1)^a \theta_{ab}(z) \\ \theta_{ab}(z+\eta) & = & e^{-\pi i \eta - 2\pi i z - \pi i b} \theta_{ab}(z) \end{cases}$$

$\{z \in \mathbb{C} \mid \theta_{ab}(z) = 0\} = \frac{1+b}{2} + \frac{1+a}{2}\eta + \Lambda$

- define $\alpha = \alpha_{00}$, $\beta = \alpha_{01}$, $\gamma = \alpha_{10}$

$$\alpha_{ab} \;:=\; (-1)^{a+b} \left(\frac{\theta_{11}(\tau)\theta_{ab}(\tau)}{\theta_{ij}(\tau)\theta_{kl}(\tau)} \right)^2$$

where $\{ab, ij, kl\} = \{00, 01, 10\}$

- Riemann's identity: $\theta_{00}(\tau)^4 + \theta_{11}(\tau)^4 = \theta_{01}(\tau)^4 + \theta_{10}(\tau)^4$
$\Longrightarrow \alpha\beta\gamma + \alpha + \beta + \gamma = 0$

What problems should we study?

A balance between examples and theory.

What is the right balance?

Herman Weyl: introduction to The Classical Groups (1939):

> *"Important though the general concepts and propositions may be with the modern industrious passion for axiomatizing and generalizing has presented us ... nevertheless I am convinced that the special problems in all their complexity constitute the stock and the core of mathematics; and to master their difficulty requires on the whole the harder labor."*

Question: Is $Q_{n,k}(E,\tau)^!$ Frobenius? If so, then $Q_{n,k}(E,\tau)$ is Artin-Schelter regular.

Theorem: (CKS) Already stated earlier.
Sometimes the homomorphism

$$Q_{n,k}(E,\tau) \to B(X_{n/k},\sigma,\mathcal{L}_{n/k})$$

is surjective, e.g., when $X_{n/k} = E^g$, $X_{n/k} = S^g E$, and ???
In those cases there is an ideal I in $Q_{n,k}(E,\tau)$ such that

$$\mathrm{QGr}\left(\frac{Q_{n,k}(E,\tau)}{I}\right) \equiv \mathrm{Qcoh}(X_{n/k}).$$

This equivalence follows from:

Theorem: (Artin-Van den Bergh)
$\mathrm{QGr}(B(X,\sigma,\mathcal{L})) \equiv \mathrm{Qcoh}(X)$ in "good situations."

§4. Twisted homogeneous coordinate rings

- let X be a scheme (e.g., an algebraic variety), $\sigma : X \to X$ an automorphism, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module
- define $s := (\mathcal{L} \otimes_{\mathcal{O}_X} -) \circ \sigma^* : \mathrm{Qcoh}(X) \to \mathrm{Qcoh}(X)$
- the graded ring

$$B(X, \sigma, \mathcal{L}) \ := \ \bigoplus_{n=0}^{\infty} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, s^n \mathcal{O}_X)$$

  is called a twisted homogeneous coordinate ring
- **compare** to the pre-projective algebra (Minamoto's talk)

$$\Pi(Q) \ = \ \bigoplus_{n \geq 0} \mathrm{Hom}_{\Gamma}(\Gamma, (\tau^{-})^n \Gamma)$$

  where $\tau^{-} \ =$ inverse of AR-translation

$$B(X, \mathrm{id}, \mathcal{L}) \ = \ \bigoplus_{n=0}^{\infty} H^0(X, \mathcal{L}^{\otimes n}).$$

Important questions in algebraic geometry:
When is

$$\bigoplus_{n=0}^{\infty} H^0(X, \mathcal{L}^{\otimes n})$$

generated by its degree-one component $H^0(X, \mathcal{L})$?
What are the degrees of its relations.

The same questions about $B(X, \mathrm{id}, \mathcal{L})$ are very important in non-commutative algebra.

also possible to study projective algebraic geometry without knowing what a sheaf is (but it might be a bad idea)

- $A = \Bbbk \oplus A_1 \oplus A_2 \oplus \cdots$ = connected graded $\Bbbk$-algebra
- $\mathsf{Gr}(A)$ = the category of $\mathbb{Z}$-graded left A-modules and
- $\mathsf{Fdim}(A)$ = the full subcategory of $M \in \mathsf{Gr}(A)$ such that $M = \sum$ of its finite dimensional submodules, and

$$\mathsf{QGr}(A) \;:=\; \frac{\mathsf{Gr}(A)}{\mathsf{Fdim}(A)} \qquad \leftarrow \text{quotient category}$$

- **Theorem.** [Serre, 1955, FAC] Let A = the polynomial ring on n variables.
  1. $\mathsf{QGr}(A) \equiv \mathsf{Qcoh}(\mathbb{P}^{n-1})$
  2. if I is a graded ideal in A, then $\mathsf{QGr}(A/I) \equiv \mathsf{Qcoh}(Z)$ where $Z = \mathrm{Proj}(A/I) \subseteq \mathbb{P}^{n-1}$ is the zero-locus of I.
- **Message:** study $\mathsf{QGr}(A)$ as if it is $\mathsf{Qcoh}(?)$

Three atypical examples

$$\mathrm{QGr}\left(\frac{\mathbb{C}\langle x,y\rangle}{(xy-qyx)}\right) \equiv \mathrm{Qcoh}(\mathbb{P}^1), \qquad q \in \mathbb{C}-\{0\}$$

$$\mathrm{QGr}\left(\frac{\mathbb{C}\langle x,y\rangle}{(x^2y-yx^2, xy^2-y^2x)}\right) \equiv \mathrm{Qcoh}(\mathbb{P}^1 \times \mathbb{P}^1)$$

$$\mathrm{QGr}\left(\frac{\mathbb{C}\langle x,y\rangle}{(x^5-yxy, y^2-xyx)}\right) \equiv \mathrm{Qcoh}(\mathbb{P}^2 \text{ blown up at 3 points})$$

$$\deg(x) = 1 \text{ and } \deg(y) = 2$$

This is not typical! Usually QGr(A) is "like" Qcoh(?)

$$\frac{\mathbb{C}\langle x,y\rangle}{(xy-qyx)} \cong B(\mathbb{P}^1,\sigma,\mathcal{O}_{\mathbb{P}_1}(1)), \qquad \sigma(\alpha,\beta)=(\alpha,q\beta)$$

$$\frac{\mathbb{C}\langle x,y\rangle}{(x^2y-yx^2,xy^2-y^2x)} \cong B(\mathbb{P}^1\times\mathbb{P}^1,\sigma,\mathcal{O}_{\mathbb{P}_1\times\mathbb{P}^1}(1,0)),\ \sigma(u,v)=(v,u)$$

$$\frac{\mathbb{C}\langle x,y\rangle}{(x^5-yxy,y^2-xyx)} \cong B(X,\sigma,\mathcal{L}), \quad \sigma^6=1$$

Philosophy:

- think of these rings as non-commutative homogeneous coordinate rings of these algebraic varieties
- the equivalence of categories on the previous slide tell us everything about the graded representation of these algebras
- this is the "right" way to understand these rings
- **Secret weapon:** algebraic geometry

THE END