The Gabriel-Quillen functor for extriangulated categories

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Notation

Categories and functors are always assumed to additive.

- $\bullet~\mathcal{C}$ category.
- $\bullet \ \mathsf{Mod}\, \mathcal{C}$ category of contravariant functors: $\mathcal{C} \to \mathsf{Ab}.$

$Mod\mathcal{C}$	\supseteq	$Lex\mathcal{C}$
UI		\cup I
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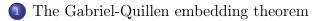
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- 2 Extriangulated category
- 3 The GQ functor for extriangulated categories
- 4 The general heart construction

Theorem

Let $(\mathcal{C}, \mathbb{E})$ be a small exact category.

• There exists the following localization sequence

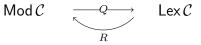


- The composed functor $E_{\mathcal{C}} : \mathcal{C} \hookrightarrow \operatorname{\mathsf{Mod}} \mathcal{C} \xrightarrow{Q} \operatorname{\mathsf{Lex}} \mathcal{C}$ is exact and fully faithful.
- ⁽³⁾ C is extension-closed in Lex C and the exact structure \mathbb{E} is a class of all short sequences which belong to C.

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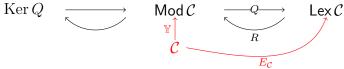


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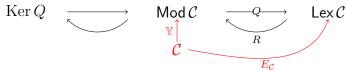


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Definition (Auslander)

Let $\delta: Z \to Y \to X$ be a conflation in \mathcal{C} . Then we have an exact sequence

$$0 \longrightarrow \mathcal{C}(-, Z) \longrightarrow \mathcal{C}(-, Y) \longrightarrow \mathcal{C}(-, X)$$

in $\mathsf{Mod}\,\mathcal{C}$.

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Define the following subcategories in $\mathsf{Mod}\,\mathcal{C}$.

- def C all defects in $\mathsf{Mod} C$.
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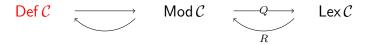
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The definition

The notion of extriangulated category is a simultaneous generalization of triangulated ones and exact ones.

Definition (Nakaoka-Palu'19)

The extriangulated category is a triple $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$, where

- C an additive category;
- \mathbb{E} a biadditive functor $\mathcal{C} \times \mathcal{C}^{op} \to \mathsf{Ab}$;
- \mathfrak{s} an assignment from an element in $\mathbb{E}(X, Z)$ to a sequence $Z \to Y \to X$ in \mathcal{C} ,

with some suitable compatibility.

Some examples

- Extension-closed subcategory in a triangulated category has a natural extriagulated structure.
- Let (U, V) be a torsion pair in a triangulated category. Then U has a natural extriangulated structure.

The GQ type localization sequence

Theorem (O)

Let ${\mathcal C}$ be a small extriangulated category. Then there exists a localization sequence



A finitely presented version

Lemma (Freyd'65)

The following are equivalent for an additive category $\mathcal{C} {:}$

- The category \mathcal{C} admits weak-kernels;
- **2** The full subcategory $\mathsf{mod}\,\mathcal{C}$ is an exact abelian subcategory in $\mathsf{Mod}\,\mathcal{C}$.

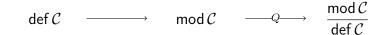
Proposition (O)

Let \mathcal{C} be an extriangulated category with weak-kernels. Then def \mathcal{C} is a Serre subcategory in mod \mathcal{C} .

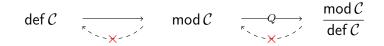
Remark

The quotient functor $Q: \operatorname{\mathsf{mod}} \mathcal{C} \to \frac{\operatorname{\mathsf{mod}} \mathcal{C}}{\operatorname{\mathsf{def}} \mathcal{C}}$ does not necessarily have a right adjoint.

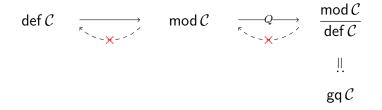
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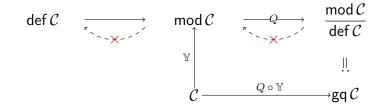
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We put $E_{\mathcal{C}} := Q \circ \mathbb{Y}$ and call it the *GQ functor* for \mathcal{C} .

Theorem (O)

- The GQ functor $E_{\mathcal{C}}: \mathcal{C} \to \operatorname{gq} \mathcal{C}$ is:
 - exact and fully faithful iff C is an exact category;
 - **2** an equivalence iff \mathcal{C} is an abelian category.

Corollary (Auslander's defect formula)

Let \mathcal{C} be an abelian category. Then we have the localization sequence



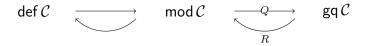
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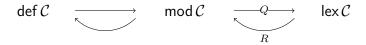
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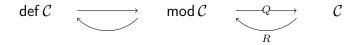
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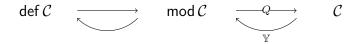
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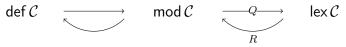


The exact case

Theorem

Let \mathcal{C} be an idempotent complete exact category with weak-kernels. Assume that it has enough projectives.

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- **③** C is extension-closed in lex C and the exact structure \mathbb{E} is a class of all short exact sequences which belong to C.

Torsion class

- \mathcal{T} triangulated category.
- $(\mathcal{U}, \mathcal{V})$ torsion pair.

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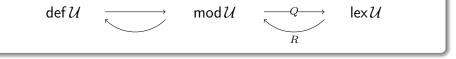


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There exists the following localization sequence.



• $(\mathcal{U}, \mathcal{V})$ - torsion pair in \mathcal{T} .

- Put $\mathcal{W} := \mathcal{U} \cap \mathcal{V}[-1].$
- Put $\mathcal{T}^+ := \mathcal{U}[-1] * \mathcal{W}, \mathcal{T}^- := \mathcal{W} * \mathcal{V} \text{ and } \mathcal{H} := \mathcal{T}^+ \cap \mathcal{T}^-.$
- $\underline{\mathcal{H}} := \mathcal{H}/[\mathcal{W}]$ the Nakaoka heart.

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The Nakaoka heart $\underline{\mathcal{H}}$ is abelian.

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Example

Let $(\mathcal{U}, \mathcal{V})$ be a torsion pair in \mathcal{T} .

- If (U, V[−1]) forms a t-structure, then <u>H</u> is the usual heart of it.
- If \mathcal{U} is a (2-)cluster tilting subcategory, then $\underline{\mathcal{H}}$ is $\mathcal{T}/[\mathcal{C}]$ which has been shown to be abelian by Koenig-Zhu.

Theorem (O) The Nakaoka heart $\underline{\mathcal{H}}$ is equivalent to $\mathsf{lex}\mathcal{U}$.

Thank you for your attention!