Pure derived categories and weak balanced big Cohen-Macaulay modules

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Setup R: comm. noeth. ring, $d = \dim R < \infty$

$\operatorname{Cot} R = \{ M \in \operatorname{Mod} R \mid \operatorname{Ext}^i_R(F, M) = 0,\ {}^\forall i > 0,\ {}^\forall F \in \operatorname{Flat} R \}$

$\operatorname{FlCot} R = \operatorname{Flat} R \cap \operatorname{Cot} R$

§ 1

$W_i = \{ \mathfrak{p} \in \operatorname{Spec} R \mid \dim R/\mathfrak{p} = i \}$

$\mathbb{W} = \{ W_i \}_{0 \le i \le n}$

$$\mathrm{id}_{\operatorname{Mod} R} \longrightarrow \bar{\lambda}^{W_i} = \prod_{\mathfrak{p} \in W_i} \Lambda^{\mathfrak{p}}(- \otimes_R R_{\mathfrak{p}}), \quad \Lambda^{\mathfrak{p}} = \varprojlim_{t \ge 1} (- \otimes_R R/\mathfrak{p}^t)$$

$X \in C(\operatorname{Mod} R)$

$$\lambda^{\mathbb{W}} X := \operatorname{tot} \left(\begin{tikzcd}[column sep=small] & \vdots & \vdots & & \vdots & \\ 0 \arrow[r] & \prod\limits_{0 \le i \le d} \bar{\lambda}^{W_i} X^{n+1} \arrow[r] \arrow[u] & \prod\limits_{0 \le i < j \le d} \bar{\lambda}^{W_j} \bar{\lambda}^{W_i} X^{n+1} \arrow[r] \arrow[u] & \cdots \arrow[r] & \bar{\lambda}^{W_d} \bar{\lambda}^{W_{d-1}} \cdots \bar{\lambda}^{W_0} X^{n+1} \arrow[r] \arrow[u] & 0 \\ 0 \arrow[r] & \prod\limits_{0 \le i \le d} \bar{\lambda}^{W_i} X^{n} \arrow[r] \arrow[u] & \prod\limits_{0 \le i < j \le d} \bar{\lambda}^{W_j} \bar{\lambda}^{W_i} X^{n} \arrow[r] \arrow[u] & \cdots \arrow[r] & \bar{\lambda}^{W_d} \bar{\lambda}^{W_{d-1}} \cdots \bar{\lambda}^{W_0} X^{n} \arrow[r] \arrow[u] & 0 \\ & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] & \end{tikzcd} \right)$$

$$X \in C(\operatorname{Flat} R) \Rightarrow X \xrightarrow{\text{quasi-isom.}} \lambda^{\mathbb{W}} X \quad \text{(N-Yoshino 2018)}$$

$$\begin{tikzcd} \lambda^w : K(\mathrm{Flat}\,R) \arrow[rr] \arrow[dr] & & K(\mathrm{Flat}\,R) \\ & K(\mathrm{FlCot}\,R) \arrow[ur] & \end{tikzcd}$$

<u>Thm</u>

$$K(\mathrm{Flat}\,R) \underset{\text{inc}}{\overset{\lambda^w}{\rightleftarrows}} K(\mathrm{FlCot}\,R) \qquad \mathrm{Ker}\,\lambda^w = K_{pac}(\mathrm{Flat}\,R)$$

$(\lambda^w, \mathrm{inc})$: adjoint pair

$\left(C(\mathrm{Mod}\,R) \ni X : \text{pure acyclic} \underset{\text{def}}{\Leftrightarrow} X \otimes_R M : \text{acyclic for } {}^\forall M \in \mathrm{Mod}\,R \right)$

<u>Cor</u> $\quad K(\mathrm{Flat}\,R)/K_{pac}(\mathrm{Flat}\,R) \cong K(\mathrm{FlCot}\,R)$

$$\begin{array}{ccc} \cup & & \cup \\ X & \longmapsto & \lambda^w X \end{array}$$

<u>Remark</u> • $\mathrm{FlCot}\,R \ni F \underset{\text{Enochs 1984}}{\Leftrightarrow} F \overset{\exists}{\cong} \prod_{\mathfrak{p} \in \mathrm{Spec}\,R} T_{\mathfrak{p}}$, $T_{\mathfrak{p}} = \Lambda^{\mathfrak{p}}(\bigoplus_{B_{\mathfrak{p}}} R_{\mathfrak{p}})$

• A : ring , $D(\mathrm{Flat}\,A) := K(\mathrm{Flat}\,A)/K_{pac}(\mathrm{Flat}\,A)$ the pure derived category of flat modules (Murfet-Salarian 2011)

$$K(\mathrm{Proj}\,A) \underset{\text{Neeman 2008}}{\cong} D(\mathrm{Flat}\,A) \underset{\substack{\text{Gillespie 2004} \\ \text{Stovicek 2014}}}{\cong} K(\mathrm{FlCot}\,A)$$

Notations

$$K_{tac}(\mathrm{Proj}\,R) = \{ X \in K_{ac}(\mathrm{Proj}\,R) \mid \mathrm{Hom}_R(X, P) : \text{acyclic}, \ {}^{\forall}P \in \mathrm{Proj}\,R \}$$

$$K_{Ftac}(\mathrm{Flat}\,R) = \{ X \in K_{ac}(\mathrm{Flat}\,R) \mid E \otimes_R X : \text{acyclic}, \ {}^{\forall}E \in \mathrm{Inj}\,R \}$$

$$\cup\!\!\!\mid$$

$$K_{Ftac}(\mathrm{FlCot}\,R)$$

$$\| \ (\text{fact})$$

$$K_{tac}(\mathrm{FlCot}\,R) = \{ X \in K_{ac}(\mathrm{FlCot}\,R) \mid \mathrm{Hom}_R(X, F) : \text{acyclic}, \ {}^{\forall}F \in \mathrm{FlCot}\,R \}$$

$\mathbb{G}\mathrm{Proj}\,R \ni M \underset{\text{def}}{\Longleftrightarrow} {}^{\exists}X \in K_{tac}(\mathrm{Proj}\,R)$ s.t. $M = \mathrm{Ker}\, d_X^0$

$\mathbb{G}\mathrm{Flat}\,R \ni M \underset{\text{def}}{\Longleftrightarrow} {}^{\exists}X \in K_{Ftac}(\mathrm{Flat}\,R)$ s.t. $M = \mathrm{Ker}\, d_X^0$

$\mathbb{G}\mathrm{FlCot}\,R := \mathbb{G}\mathrm{Flat}\,R \cap \mathrm{Cot}\,R \ni M \underset{\text{fact}}{\Longleftrightarrow} {}^{\exists}X \in K_{tac}(\mathrm{FlCot}\,R)$ s.t. $M = \mathrm{Ker}\, d_X^0$

<u>Known</u> $\quad \underline{\mathbb{G}\mathrm{Proj}\,R} \cong K_{tac}(\mathrm{Proj}\,R) \xrightarrow[\quad]{{}^{\exists}\ \cong} K_{tac}(\mathrm{FlCot}\,R) \cong \underline{\mathbb{G}\mathrm{FlCot}}\,R$

(modulo projective) (modulo flat cotorsions)

<u>Thm</u> $\quad K_{tac}(\mathrm{Proj}\,R) \xrightarrow[\lambda^{W}]{\cong} K_{tac}(\mathrm{FlCot}\,R)$

Cor If $(R, \mathfrak{m})$ has an isolated singularity, then

$$\underline{\mathrm{GProj}}\,R \xrightarrow[\Lambda^{\mathfrak{m}}]{\cong} \underline{\mathrm{GFlCot}}\,R$$

§ 2.

Def (Holm 2017) $(R, \mathfrak{m})$: local

An R-module is weak balanced big Cohen-Macaulay (wbbCM) if any system of parameters of $\mathfrak{m}$ is a weak regular sequence.

Rem A wbbCM module M with $M/\mathfrak{m}M \neq 0$ has been traditionally called a balanced big CM module.

Henceforth, R will be assumed to be a CM local ring.

Fact (Holm) $\mathrm{Flat}\,R \subseteq \mathrm{wbbCM}\,R$, and the equality holds if and only if R is regular.

$\underline{\text{Def}}$ $\mathrm{wbbCMC}\,R := \mathrm{wbbCM}\,R \cap \mathrm{Cot}\,R$

$$\mathrm{FlCot}\,R \hookrightarrow \mathrm{wbbCMC}\,R \longrightarrow \underline{\mathrm{wbbCMC}}\,R$$

(modulo flat cotorsions)

$\underline{\text{Fact}}$ R is regular $\Leftrightarrow$ $\underline{\mathrm{wbbCMC}}\,R = \{0\}$

$\underline{\text{Rem}}$ When R is Gorenstein, $\mathrm{wbbCM}\,R = \mathrm{GFlat}\,R$ (Holm), and so $\underline{\mathrm{wbbCMC}}\,R = \underline{\mathrm{GFlCot}}\,R$.

Moreover, $K_{tac}(\mathrm{Proj}\,R) = K_{ac}(\mathrm{Proj}\,R)$, and this is compactly generated (Jorgensen 2005), so we can talk about purity in the triangulated category (Krause 2000), where $\underline{\mathrm{GProj}}\,R \simeq K_{ac}(\mathrm{Proj}\,R) \simeq K_{ac}(\mathrm{FlCot}\,R) \simeq \underline{\mathrm{GFlCot}}\,R$.

$\underline{\text{Thm}}$ R : Gorenstein

$M \in \mathrm{GFlCot}\,R$ is pure-injective in $\mathrm{Mod}\,R$ if and only if M is pure-injective in $\underline{\mathrm{GFlCot}}\,R$.

<u>Rem</u> A similar result does not holds for Gorenstein-projectives.

<u>Ex</u> $R = k[\![x,y]\!]/(x^2)$, char $k \neq 2$, $k = \overline{k}$.

The list of indec. wbbCM pure-inj. R-module is :

$\{(x, y^n) \mid n \geq 0\} \cup \{R/(x)\} \cup \underbrace{\{R_{\mathfrak{p}}, R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, \widetilde{R}\}}_{\text{Puninski (2018)}}$ $\quad \widetilde{R}$: the integral closure of R in $R_{\mathfrak{p}}$, $\mathfrak{p} = (x)$

$$K_{ac}(\mathrm{FlCot}\,R) \simeq \underline{\mathrm{GFlCot}}\,R = \underline{\mathrm{wbbCMC}}\,R$$

$$\mathrm{tot}\begin{pmatrix} \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} \\ \downarrow \quad \downarrow \\ \xrightarrow{x} R_{\mathfrak{p}} \xrightarrow{x} R_{\mathfrak{p}} \xrightarrow{x} \end{pmatrix} \in K_{ac}(\mathrm{FlCot}\,R) \longmapsto \widetilde{R} \in \mathrm{GFlCot}\,R$$