Relative coherent modules and semihereditary modules

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Contents

Introduction

It is well known that coherent rings and semihereditary rings play important roles in ring theory. Recall that R is a left coherent ring (resp. left semihereditary ring) if every finitely generated left ideal of R is finitely presented (resp. projective).

Introduction

Later, for a given positive integer n, the concepts of n-coherent rings and n-semihereditary rings were introduced.

R is called a left n-coherent ring (Shamsuddin, 2001) (resp. left n-semihereditary ring) (Zhu-Tan, 2005) if every n-generated left ideal of R is finitely presented (resp. projective).

n-coherent rings and n-semihereditary rings were also furthermore studied by Zhang-Chen (2007).

In particular, a left 1-coherent ring coincides with a left P-coherent ring, a left 1-semihereditary ring is exactly a left PP ring (Rickart ring).

Introduction

In this talk, we will generalize the concepts of n-coherent rings and n-semihereditary rings to the general setting of modules.

Main results

Definition 1

Let R be a ring. For a fixed positive integer n, a left R-module M is called n-coherent if every n-generated submodule of M is finitely presented.

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Let R be a ring. For a fixed positive integer n, a left R-module M is called n-coherent if every n-generated submodule of M is finitely presented.

Remark 2

(1) ${}_RR$ is an n-coherent left R-module if and only if R is a left n-coherent ring. ${}_RR^m$ is an n-coherent left R-module if and only if R is a left (m, n)-coherent ring (Zhang-Chen-Zhang, 2005).
(2) M is a coherent left R-module if and only if M is n-coherent for any positive integer n. M is a P-coherent left R-module (Mao, 2010) if and only if M is 1-coherent.
(3) It is easy to see that every submodule of an n-coherent left R-module is n-coherent. In particular, any left ideal of a left n-coherent ring R is an n-coherent left R-module.

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Proposition 3

If R is a left coherent ring and n is a positive integer, then the class of n-coherent left modules is closed under direct sums.

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Theorem 4

Given a positive integer n, the following conditions are equivalent for a left R-module M:

1. M^n is an n-coherent left R-module.
2. $M^{n\times n}$ is a P-coherent left $M_n(R)$-module.

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Proposition 5

The following conditions are equivalent for a left R-module M:

1. M is a coherent left R-module.
2. M^n is a P-coherent left $M_n(R)$-module for any positive integer n.
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Proposition 5

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Corollary 6

The following conditions are equivalent for a ring R:

1. R is a left Noetherian ring.
2. Every left R-module is n-coherent for some positive integer n.
3. Every injective left R-module is n-coherent for some positive integer n.

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Definition 7

A left R-module M is called pseudo-coherent if the left annihilator of any finite subset of M in R is a finitely generated left ideal.

M is said to be a left AFG R-module if the left annihilator of any non-empty subset of M in R is a finitely generated left ideal.

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Proposition 8

The following conditions are equivalent for a left R-module M:

1. M is a pseudo-coherent left R-module.
2. $\bigoplus_{i\in\Lambda} M$ is a P-coherent left R-module for any index set Λ.
3. M^n is a P-coherent left R-module for any positive integer n.

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Definition 10

Let R be a ring. For a fixed positive integer n, a left R-module M is called n-semihereditary if every n-generated submodule of M is projective.

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Remark 11

(1) ${}_RR$ is an n-semihereditary left R-module if and only if R is a left n-semihereditary ring.
(2) Clearly, M is a semihereditary left R-module if and only if M is n-semihereditary for any positive integer n. M is a PP left R-module if and only if M is 1-semihereditary.

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Proposition 12

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Theorem 13

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1. M is an n-semihereditary left R-module.
2. $\bigoplus_{i\in\Lambda} M$ is an n-semihereditary left R-module.
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Corollary 15

The following conditions are equivalent for a ring R:

1. R is a left semihereditary ring.
2. Every projective left R-module is n-semihereditary for any positive integer n.
3. $M_n(R)$ is a left PP ring for any positive integer n.

Corollary 15

The following conditions are equivalent for a ring R:

1. R is a left semihereditary ring.
2. Every projective left R-module is n-semihereditary for any positive integer n.
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Proposition 16

The following conditions are equivalent for a ring R:

1. R is a semisimple Artinian ring.
2. Every left R-module is n-semihereditary for some positive integer n.
3. Every injective left R-module is n-semihereditary for some positive integer n.

Proposition 16

The following conditions are equivalent for a ring R:

1. R is a semisimple Artinian ring.
2. Every left R-module is n-semihereditary for some positive integer n.
3. Every injective left R-module is n-semihereditary for some positive integer n.

Definition 17

Let M be a left R-module. For a fixed positive integer n, a right R-module N is called *n-M-flat* if the induced sequence $0 \to N \otimes K \to N \otimes M$ is exact for any n-generated submodule K of M.

A left R-module L is said to be *n-M-injective* if the induced sequence $\mathrm{Hom}(M, L) \to \mathrm{Hom}(K, L) \to 0$ is exact for any n-generated submodule K of M.

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Remark 18

We observe that an n-R^m-flat right R-module is exactly an (m, n)-flat right R-module (Zhang-Chen-Zhang, 2005), and an n-R^m-injective left R-module is exactly an (m, n)-injective left R-module (Chen-Ding-Li-Zhou, 2001).

In particular, an n-R-flat right R-module is exactly an n-flat right R-module (Shamsuddin, 2001). An n-R-injective left R-module is exactly an n-injective left R-module (Shamsuddin, 2001).

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Let $\mathcal{C}$ be a class of R-modules and M an R-module.

A morphism $\phi : C \to M$ is a $\mathcal{C}$-precover of M if $C \in \mathcal{C}$ and the Abelian group homomorphism $\mathrm{Hom}(C', \phi) : \mathrm{Hom}(C', C) \to \mathrm{Hom}(C', M)$ is surjective for every $C' \in \mathcal{C}$.

A $\mathcal{C}$-precover $\phi : C \to M$ is said to be a $\mathcal{C}$-cover of M if every endomorphism $g : C \to C$ such that $\phi g = \phi$ is an isomorphism.

Dually we have the definitions of a $\mathcal{C}$-preenvelope and a $\mathcal{C}$-envelope.

Lemma 19

Let M be a left R-module.

1. The class of n-M-injective left R-modules is closed under direct sums, direct products and direct summands.
2. The class of n-M-flat right R-modules is closed under pure submodules, pure quotients, direct summands, direct limits and direct sums. Consequently, every right R-module has an n-M-flat cover.

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Lemma 20

Let M be an n-coherent left R-module. Then the class of n-M-flat right R-modules is closed under direct products and every right R-module has an n-M-flat preenvelope.

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Let M be an n-coherent left R-module. Then the class of n-M-flat right R-modules is closed under direct products and every right R-module has an n-M-flat preenvelope.

Theorem 21

The following conditions are equivalent for a finitely presented left R-module M:

1. M is an n-coherent left R-module.
2. The class of n-M-flat right R-modules is closed under direct products.
3. Every right R-module has an n-M-flat preenvelope.
4. The class of n-M-injective left R-modules is closed under pure quotients.
5. The class of n-M-injective left R-modules is closed under direct limits.

Theorem 21

The following conditions are equivalent for a finitely presented left R-module M:

1. M is an n-coherent left R-module.
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3. Every right R-module has an n-M-flat preenvelope.
4. The class of n-M-injective left R-modules is closed under pure quotients.
5. The class of n-M-injective left R-modules is closed under direct limits.

Corollary 22

If M is a finitely presented n-coherent left R-module, then every left R-module has an n-M-injective preenvelope and n-M-injective cover.

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Corollary 23

The following conditions are equivalent for a ring R:

1. R is a left (m, n)-coherent ring.
2. Every left R-module has an (m, n)-injective cover.
3. Every right R-module has an (m, n)-flat preenvelope.

Corollary 23

The following conditions are equivalent for a ring R:

1. R is a left (m, n)-coherent ring.
2. Every left R-module has an (m, n)-injective cover.
3. Every right R-module has an (m, n)-flat preenvelope.

Proposition 24

The following conditions are equivalent for a finitely presented n-coherent left R-module M:

1. ${}_RR$ is n-M-injective.
2. Every right R-module has a monic n-M-flat preenvelope.
3. Every left R-module has an epic n-M-injective cover.

Proposition 24

The following conditions are equivalent for a finitely presented n-coherent left R-module M:

1. ${}_RR$ is n-M-injective.
2. Every right R-module has a monic n-M-flat preenvelope.
3. Every left R-module has an epic n-M-injective cover.

Theorem 25

The following conditions are equivalent for a flat n-coherent left R-module M:

1. M is an n-semihereditary left R-module.
2. Every right R-module has an epic n-M-flat preenvelope.
3. Every left R-module has a monic n-M-injective cover.

Theorem 25

The following conditions are equivalent for a flat n-coherent left R-module M:

1. M is an n-semihereditary left R-module.
2. Every right R-module has an epic n-M-flat preenvelope.
3. Every left R-module has a monic n-M-injective cover.

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Thank you!