Rudimentary rings: Rings have a faithful indecomposable endoregular module
(joint work with Cosmin Roman and Xiaoxiang Zhang)

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A right primitive ring $R$

$\exists$ a faithful simple right $R$-module $M$

**Schur’s Lemma**

If $M$ is a simple right $R$-module then $\text{End}_R(M)$ is a division ring.

**Lee-Rizvi-Roman**

$M$ is an indecomposable endoregular module if and only if $\text{End}_R(M)$ is a division ring.

From the above 3 observations, we consider:

A right rudimentary ring $R$

$\exists$ a faithful right $R$-module $M$

$\exists \ \text{End}_R(M) :$ a division ring

Equivalently, a ring $R$ which has a faithful indecomposable endoregular right $R$-module $M$. 

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A right primitive ring $R$

\[ \exists \text{ a faithful simple right } R\text{-module } M \]

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- \[ \text{End}_R(M) : \text{a division ring} \]

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Plan:

1. Historical Background for Primitive Rings.

2. Historical Background for Endoregular Modules.

3. Observations.

4. A generalization of primitive rings: Rudimentary rings.

5. Applications of indecomposable endoregular modules.
1. Historical background for primitive rings:

In 1908 (Wedderburn)

A simple artinian ring $R \cong$ an $n \times n$ matrix ring over a division ring for some $n \in \mathbb{N}$.

In 1927 (Wedderburn-Artin)

A semisimple artinian ring $R \iff$ a ring direct sum of a finite number of simple artinian rings.

In 1945 (Jacobson)

A right primitive ring $R \cong$ a dense subring of the endomorphism ring of a left vector space over a division ring.

Definition

A ring $R$ is called right primitive if there exists a faithful simple right $R$-module.

A left primitive ring is defined similarly with left $R$-modules.
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Examples for primitive rings

- any simple ring;
- any full linear ring;
e.g., the endomorphism ring of any left vector space over a division ring;
- Weyl algebras over fields of characteristic zero;
- if $M$ is a simple right $R$-module
then $R/r_R(M)$ is a right primitive ring.

In 1961 (Posner)

A right primitive ring $R \iff$ a right primitive ring $\text{Mat}_n(R)$.

In 1964 Bergman (Also, in 1988 Jategaonkar)

right primitive $\neq$ left primitive.
2. Historical background for endoregular modules:
Recall that a ring $R$ is said to be **von Neumann regular** if for any $r \in R$, there exists $s \in R$ such that $r = rsr$.

### In 1958 (L. Fuchs)

Fuchs raised the question of characterizing **abelian groups** whose endomorphism rings are von Neumann regular.

### In 1967 (K.M. Rangaswamy)

Rangaswamy answered the question for groups.

### In 1971 R. Ware (Also, in 1975 B. Stenström)

Ware and Stenström also answered the question for modules.

### In 1948 (G. Azumaya independently)

The following conditions are equivalent for a right $R$-module $M$:

(a) $\text{End}_R(M)$ is a von Neumann regular ring;
(b) $\ker a < \bigoplus M$ and $\text{Im} a < \bigoplus M$ for all $a \in \text{End}_R(M)$.
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There is another general module theoretical setting of the notion of a von Neumann regular ring:

**Definition**

A module $M$ is called **endoregular** if its endomorphism ring is a von Neumann regular ring.

**Example**

- any semisimple module;
- any nonsingular (even $\mathcal{K}$-nonsingular) continuous (injective) module (thus, $\mathbb{Q}^{(\mathbb{R})}$ is an endoregular $\mathbb{Z}$-module);
- any finitely generated projective module over a von Neumann regular ring.

**Theorem**

The following conditions are equivalent for a module $M$:

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$\exists$ a faithful right $R$-module $M$

$\text{End}_R(M)$: a division ring

Equivalently, a ring $R$ which has a faithful indecomposable endoregular right $R$-module $M$. 
4. Rudimentary rings:
We introduce the generalized notion of primitive rings.

**Definition**

A ring $R$ is called **right rudimentary** if there exists a **faithful** right $R$-module $M$ such that $\text{End}_R(M)$ is a **division ring**.
A left rudimentary ring is defined similarly with left $R$-modules.

**Example**

- any right primitive ring (hence any simple ring)
e.g., the endomorphism ring of any left vector space over a division ring;
- any right weakly primitive ring;
- any right Ore domain;
- any prime right Goldie ring;
- if $M$ is an indecomposable endoregular right $R$-module then $R/r_R(M)$ is a right rudimentary ring.
More examples of right rudimentary rings:

**Theorem**

Let $Q$ be a right rudimentary ring with a faithful right $Q$-module $M$ such that $\text{End}_Q(M)$ is a division ring. If $R$ is a right or left order in $Q$, then $M_R$ is also faithful and $\text{End}_Q(M) = \text{End}_R(M)$. Hence $R$ is right rudimentary.

**Corollary**

(i) Every prime right Goldie ring (and hence every prime PI-ring) is right rudimentary.

(ii) Every right Ore domain is right rudimentary.
It is well known result for commutative primitive rings:

**Proposition**

A commutative ring \( R \) is a (right) primitive ring iff \( R \) is a **field**.

We generalize the above Proposition to rudimentary rings for the commutative case.

**Proposition**

A commutative ring \( R \) is rudimentary iff \( R \) is an **integral domain**.

**Corollary**

The center of a right rudimentary ring is a rudimentary ring.

**Remark**

There is no proper central idempotents in a right rudimentary ring.
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*A commutative ring $R$ is rudimentary iff $R$ is an integral domain.*

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*The center of a right rudimentary ring is a rudimentary ring.*

**Remark**

There is no proper central idempotents in a right rudimentary ring.
Next example shows that the right rudimentary property is not inherited by a corner ring, in general.

**Example**

The ring $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} & 0 \\ 0 & 0 & \mathbb{Z} \end{pmatrix}$ is right rudimentary since $M = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \end{pmatrix}$ is a faithful right $R$-module and $\text{End}_R(M) \cong \mathbb{Q}$.

Let $e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ be the idempotent.

Then $eRe \cong \mathbb{Z} \times \mathbb{Z}$, which is not rudimentary.

In spite of the previous example, the rudimentary property is a Morita invariant as shown next. In fact, in the previous example, $ReR \neq R$.

**Theorem**

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Theorem

The rudimentary property is a Morita invariant.
A number of authors have studied rings $R$ for which the Converse of Schur Lemma holds. Such a ring $R$ is said to be a **CSL ring**. Every strongly regular ring is a CSL ring.

Next, we show:

**Proposition**

*Every right rudimentary right CSL ring is right primitive.*
5. Applications of indecomposable endoregular modules:

In 1949, T. Szele showed that there is no noncommutative division ring as the endomorphism ring of an abelian group.

**In 1949 (T. Szele)**

Let $M$ be an abelian group such that $\text{End}_\mathbb{Z}(M)$ is a division ring. Then $M$ is isomorphic to either $\mathbb{Q}$ or $\mathbb{Z}_p$.

In 1970, Ware and Zelmanowitz extended Szele result that there is no noncommutative division ring as the endomorphism ring of an module over a commutative ring.

**In 1970 (Ware and Zelmanowitz)**

Let $R$ be a commutative ring and let $M$ be a right $R$-module. Then $M$ is an indecomposable endoregular module iff $M$ is $R$-isomorphic to $Q(R/P)$ where $P = r_R(M)$ and $Q(R/P)$ is the maximal ring of quotients of $R/P$. 
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Observation

Let $R_1 = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix}$ and $R_2 = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & 0 & \mathbb{Z} & \mathbb{Z} \\ 0 & 0 & 0 & \mathbb{Z} \end{pmatrix}$ be rings.

Consider a right module $M = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \end{pmatrix}$ over each ring. Then $\text{End}_{R_1}(M) \cong \text{End}_{R_2}(M) \cong \mathbb{Q}$. 
An $n \times n$ partial matrix ring $\text{PM}_n(A)$ over a ring $A$ is a subring of a full $n \times n$ matrix ring over $A$, with elements matrices whose entries are either elements of $A$ or are 0 such that nonzero entries are independent of each other.

That is, $\text{PM}_n(A) = \sum_{(i,j) \in \mathcal{U}} e_{ij}A$ where $e_{ij}$ are matrix units and $\mathcal{U}$ is a subset of the index set $\mathcal{I} \times \mathcal{I}$, $\mathcal{I} = \{1, 2, \ldots, n\}$.

**Theorem**

Let $A$ be a commutative ring and $R = \text{PM}_n(A)$. Let $M = \prod_{i=1}^{n} N_i$ be a direct product of right $A$-modules $N_i$. If $\text{End}_R(M)$ is a division ring, then $\text{End}_R(M) \cong Q(A/P)$ for some prime ideal $P$ of $A$.

**Example**

Let $R_3 = \begin{pmatrix} Z & Z & Z & Z \\ 0 & Z & Z & Z \\ 0 & Z & Z & Z \\ 0 & Z & Z & Z \end{pmatrix}$ and $R_4 = \begin{pmatrix} Z & Z & Z & Z \\ 0 & Z & 0 & 0 \\ 0 & 0 & Z & 0 \\ 0 & 0 & 0 & Z \end{pmatrix}$ be rings.

Consider a right module $M = (Q \ Q \ Q \ Q)$ over each ring. Then $\text{End}_{R_3}(M) \cong \text{End}_{R_4}(M) \cong Q$. 
An $n \times n$ partial matrix ring $PM_n(A)$ over a ring $A$ is a subring of a full $n \times n$ matrix ring over $A$, with elements matrices whose entries are either elements of $A$ or are 0 such that nonzero entries are independent of each other.

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**Theorem**

Let $A$ be a commutative ring and $R = PM_n(A)$. Let $M = \prod_{i=1}^{n} N_i$ be a direct product of right $A$-modules $N_i$. If $\text{End}_R(M)$ is a division ring then $\text{End}_R(M) \cong Q(A/P)$ for some prime ideal $P$ of $A$.

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Let $R_3 = \begin{pmatrix} 0 & Z & Z & Z \\ Z & Z & Z & 0 \\ Z & Z & 0 & Z \\ 0 & 0 & 0 & Z \end{pmatrix}$ and $R_4 = \begin{pmatrix} 0 & Z & Z & Z \\ Z & 0 & 0 & 0 \\ 0 & 0 & Z & 0 \\ 0 & 0 & 0 & Z \end{pmatrix}$ be rings.

Consider a right module $M = (Q \ Q \ Q \ Q)$ over each ring. Then $\text{End}_{R_3}(M) \cong \text{End}_{R_4}(M) \cong Q$. 
**Observation**

Let $R_5 = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} & 0 & 0 \\ 0 & 0 & \mathbb{Z} & 0 \\ 0 & 0 & 0 & \mathbb{Z} \end{pmatrix}$ and $R_6 = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & 0 \\ 0 & \mathbb{Z} & 0 & 0 \\ 0 & 0 & \mathbb{Z} & 0 \\ 0 & 0 & 0 & \mathbb{Z} \end{pmatrix}$ be rings.

Consider a right module $M = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \end{pmatrix}$ over each ring.

Then $\text{End}_{R_5}(M) \cong \mathbb{Q}$ and $\text{End}_{R_6}(M) \cong \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbb{Q} \times \mathbb{Q}$.

**Lemma**

Let $M$ be a faithful right $R$-module, $S = \text{End}_R(M)$ be a field, and $A$ be an integral domain such that $\mathbb{Q}(A) = S$.

If $\dim_S(M) = n$ for some $n \in \mathbb{N}$ and the ring $R = \sum_{(i,j) \in U} e_{ij}A$ where $U \subseteq I \times I$, $I = \{1, 2, \ldots, n\}$, then there exist the smallest cardinal index set $V$ and a matrix ring $R' = \sum_{(i,j) \in V} e_{ij}A$ such that $\text{End}_{R'}(M) \cong S$, $V \subseteq U$, and $|V| = 2n - 1$. 
Observation

Let $R_5 = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} & 0 & 0 \\ 0 & 0 & \mathbb{Z} & 0 \\ 0 & 0 & 0 & \mathbb{Z} \end{pmatrix}$ and $R_6 = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & 0 \\ 0 & \mathbb{Z} & 0 & 0 \\ 0 & 0 & \mathbb{Z} & 0 \\ 0 & 0 & 0 & \mathbb{Z} \end{pmatrix}$ be rings.

Consider a right module $M = (\mathbb{Q} \ 0 \ 0 \ 0)$ over each ring.

Then $\text{End}_{R_5}(M) \cong \mathbb{Q}$ and $\text{End}_{R_6}(M) \cong \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbb{Q} \times \mathbb{Q}$.

Lemma

Let $M$ be a faithful right $R$-module, $S = \text{End}_R(M)$ be a field, and $A$ be an integral domain such that $\mathbb{Q}(A) = S$.

If $\dim_S(M) = n$ for some $n \in \mathbb{N}$ and

the ring $R = \sum_{(i,j) \in U} e_{ij}A$ where $U \subseteq I \times I$, $I = \{1, 2, \ldots, n\}$,

then there exist the smallest cardinal index set $\mathcal{V}$

and a matrix ring $R' = \sum_{(i,j) \in \mathcal{V}} e_{ij}A$

such that $\text{End}_{R'}(M) \cong S$, $\mathcal{V} \subseteq U$, and $|\mathcal{V}| = 2n - 1$. 
Observation

Let \( R_7 = \begin{pmatrix} Z & Q & Q & Q \\ 0 & Z & Z & Z \\ 0 & Z & Z & Z \\ 0 & Z & Z & Z \end{pmatrix} \) and \( R_8 = \begin{pmatrix} Z & Q & Q & Q \\ 0 & Z & 0 & 0 \\ 0 & 0 & Z & 0 \\ 0 & 0 & 0 & Z \end{pmatrix} \) be rings.

Consider a right module \( M = (Q Q Q Q) \) over each ring. Then \( \text{End}_{R_7}(M) \cong \text{End}_{R_8}(M) \cong \mathbb{Q} \).

Lemma

Let \( M \) be a faithful right \( R \)-module and \( S = \text{End}_R(M) \) be a field. Consider \( B = \{ A \mid S = Q(A), A \text{ is a ring} \} \) such that \( B \) is closed under the intersection.

Suppose \( \text{dim}_S(M) = n \) for some \( n \in \mathbb{N} \) and \( R = \sum_{(i,j) \in \mathcal{U}} e_{ij}A_{ij} \) where \( A_{ij} \in B \) and \( \mathcal{U} \subseteq I \times I, I = \{1, 2, \ldots, n\} \),

then for \( A = \bigcap_{(i,j) \in \mathcal{U}} A_{ij} \), a ring \( R' = \sum_{(i,j) \in \mathcal{U}} e_{ij}A \) satisfies \( \text{End}_{R'}(M) = S \).
Observation

Let $R_7 = \left( \begin{array}{cccc} \mathbb{Z} & \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{array} \right)$ and $R_8 = \left( \begin{array}{cccc} \mathbb{Z} & \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Z} & 0 & 0 \\ 0 & 0 & \mathbb{Z} & 0 \\ 0 & 0 & 0 & \mathbb{Z} \end{array} \right)$ be rings.

Consider a right module $M = (\mathbb{Q} \mathbb{Q} \mathbb{Q} \mathbb{Q})$ over each ring. Then $\text{End}_{R_7}(M) \cong \text{End}_{R_8}(M) \cong \mathbb{Q}$.

Lemma

Let $M$ be a faithful right $R$-module and $S = \text{End}_R(M)$ be a field. Consider $\mathcal{B} = \{A \mid S = \mathbb{Q}(A), A \text{ is a ring}\}$ such that $\mathcal{B}$ is closed under the intersection.

Suppose $\text{dim}_S(M) = n$ for some $n \in \mathbb{N}$ and $R = \sum_{(i,j) \in \mathcal{U}} e_{ij}A_{ij}$ where $A_{ij} \in \mathcal{B}$ and $\mathcal{U} \subseteq \mathcal{I} \times \mathcal{I}$, $\mathcal{I} = \{1, 2, \ldots, n\}$, then for $A = \bigcap_{(i,j) \in \mathcal{U}} A_{ij}$, a ring $R' = \sum_{(i,j) \in \mathcal{U}} e_{ij}A$ satisfies $\text{End}_{R'}(M) = S$. 
Theorem

Let $M$ be a faithful right $R$-module and $S = \text{End}_R(M)$ be a field. Consider $\mathcal{B} = \{A \mid S = \text{Q}(A), A$ is a ring$\}$ such that $\mathcal{B}$ is closed under the intersection. If $\dim_S(M) = n$ for some $n \in \mathbb{N}$ and the ring $R = \sum_{(i,j) \in \mathcal{U}} e_{ij} A_{ij}$ where $A_{ij} \in \mathcal{B}$ and $\mathcal{U} \subseteq \mathcal{I} \times \mathcal{I}$, $\mathcal{I} = \{1, 2, \ldots, n\}$, then for $A = \bigcap_{(i,j) \in \mathcal{U}} A_{ij}$, there exist the smallest cardinal index set $\mathcal{V}$ and a matrix ring $R' = \sum_{(i,j) \in \mathcal{V}} e_{ij} A$ such that $\text{End}_{R'}(M) = S$, $\mathcal{V} \subseteq \mathcal{U}$, and $|\mathcal{V}| = 2n - 1$.

Example

Let $R_1 = \left( \begin{array}{cc} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{array} \right)$, $R_2 = \left( \begin{array}{cc} \text{Q} & \text{Q} \\ \text{Q} & \text{Q} \end{array} \right)$,

$R_3 = \left( \begin{array}{cc} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{array} \right)$, $R_4 = \left( \begin{array}{cc} \mathbb{Z} & \text{Q} \\ 0 & \mathbb{Z} \end{array} \right)$, $R_5 = \left( \begin{array}{cc} \mathbb{Z} & \text{Q} \\ 0 & \text{Q} \end{array} \right)$, $R_6 = \left( \begin{array}{cc} \text{Q} & \text{Q} \\ 0 & \mathbb{Z} \end{array} \right)$, $R_7 = \left( \begin{array}{cc} \text{Q} & \text{Q} \\ 0 & \text{Q} \end{array} \right)$,

$R_8 = \left( \begin{array}{cc} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{array} \right)$, $R_9 = \left( \begin{array}{cc} \mathbb{Z} & 0 \\ \text{Q} & \mathbb{Z} \end{array} \right)$, $R_{10} = \left( \begin{array}{cc} \mathbb{Z} & 0 \\ \text{Q} & \text{Q} \end{array} \right)$, $R_{11} = \left( \begin{array}{cc} \text{Q} & 0 \\ \text{Q} & \mathbb{Z} \end{array} \right)$, $R_{12} = \left( \begin{array}{cc} \text{Q} & 0 \\ \text{Q} & \text{Q} \end{array} \right)$,

$R_{13} = \left( \begin{array}{cc} \mathbb{Z} & m\mathbb{Z} \\ n\mathbb{Z} & \mathbb{Z} \end{array} \right)$, $R_{14} = \left( \begin{array}{cc} \mathbb{Z} & n\mathbb{Z} \\ 0 & \mathbb{Z} \end{array} \right)$, and $R_{15} = \left( \begin{array}{cc} \mathbb{Z} & 0 \\ \mathbb{Z} & \mathbb{Z} \end{array} \right)$

be rings where $m, n \in \mathbb{Z}$.

Consider a right module $M = \left( \begin{array}{cc} \text{Q} & \text{Q} \end{array} \right)$ over each ring. Then $\text{End}_{R_i}(M) \cong \text{Q}$ where $1 \leq i \leq 15$. 
The condition “$S = Q(A)$” in the set $B$ in the previous theorem is not superfluous as the next example shows.

**Example**

Let $R = \left( \begin{array}{cc} \mathbb{Z}[i] & \mathbb{Q}[i] \\ 0 & \mathbb{Z} \end{array} \right)$ be the ring and $M = \left( \begin{array}{cc} \mathbb{Q}[i] & \mathbb{Q}[i] \end{array} \right)$ be the right $R$-module. Then $\text{End}_R(M) \cong \mathbb{Q}[i]$. However, $\text{End}_R(M) \cong \mathbb{Q}[i] \neq Q(\mathbb{Z}) = \mathbb{Q}$.

**Example**

(i) Let $R = \left( \begin{array}{cc} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{array} \right)$ be the ring and $M = \left( \begin{array}{cc} \mathbb{Q}[i] & \mathbb{Q}[i] \end{array} \right)$ be the right $R$-module. Then $\text{End}_R(M) \cong \left( \begin{array}{cc} \mathbb{Q} & \mathbb{Q} \\ \mathbb{Q} & \mathbb{Q} \end{array} \right)$.

(ii) Let $R = \text{Mat}_2(\mathbb{Z})$ be the ring and $M = \left( \begin{array}{cc} \mathbb{Q}[i] & \mathbb{Q}[i] \end{array} \right)$ be the right $R$-module. Then $\text{End}_R(M) \cong \left( \begin{array}{cc} \mathbb{Q} & \mathbb{Q} \\ \mathbb{Q} & \mathbb{Q} \end{array} \right)$ and $M \cong \left( \begin{array}{cc} \mathbb{Q} & \mathbb{Q} \\ \mathbb{Q} & \mathbb{Q} \end{array} \right)$.

(iii) Let the ring $R = \left( \begin{array}{ccc} \mathbb{Q} & 0 & \mathbb{Q} \\ 0 & \mathbb{Z} & 0 \\ 0 & 0 & \mathbb{Q} \end{array} \right)$ and the right $R$-module $M = \left( \begin{array}{ccc} \mathbb{Q} & \mathbb{Z}_2 & \mathbb{Q} \\ \mathbb{Z}_2 & \mathbb{Q} & \mathbb{Z}_2 \\ 0 & 0 & \mathbb{Z} \end{array} \right)$. Then $\text{End}_R(M) \cong \left( \begin{array}{cc} \mathbb{Q} & 0 \\ 0 & \mathbb{Z}_2 \end{array} \right)$ is semisimple artinian.
We can build a decreasing chain of right rudimentary rings, whose intersection is no longer right rudimentary. The next example illustrates this statement.

**Example**

Let $R_k = \left( \begin{array}{cc} \mathbb{Z} & 2^k \mathbb{Z} \\ 0 & \mathbb{Z} \end{array} \right)$ for any $k \in \mathbb{N}$.

Then $M = \left( \begin{array}{cc} \mathbb{Q} & \mathbb{Q} \end{array} \right)$ has the property that $\text{End}_{R_k}(M) \cong \mathbb{Q}$, $\forall$ $k \in \mathbb{N}$.

However, $\text{End}_{\bigcap_k R_k}(M) \cong \mathbb{Q} \times \mathbb{Q}$ is not a division ring.

Moreover, $\bigcap_k R_k = \left( \begin{array}{cc} \mathbb{Z} & 0 \\ 0 & \mathbb{Z} \end{array} \right)$ is not a right rudimentary ring.
Thank you.
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Bibliography

L. Fuchs, Abelian groups, Publishing House of the Hungarian Academy of Sciences, Budapest (1958)

N. Jacobson, The radical and semi-simplicity for arbitrary rings, Amer. J. Math., 1945 67, 300–320


G. Lee; S.T. Rizvi; C.S. Roman, Modules whose endomorphism rings are von Neumann regular, Comm. Algebra, 2013 41(11), 4066–4088

G. Lee; C.S. Roman; X. Zhang, Modules whose endomorphism rings are division rings, Comm. Algebra, 2014 42(12), 5205–5223


R. Ware; J. Zelmanowicz, Simple endomorphism rings, Amer. Math. Monthly, 1970 77(9), 987–989