The 8th China-Japan-Korea International Symposium on Ring Theory

Nagoya, August 26-31, 2019, August 27, 9-9:50

Singular Hochschild cohomology and the singularity category

Plan: 1. Hochschild cohomology

2. Sing. Hochschild cohomology and the main thm

3. Application: 2 reconstruction thms (with Zheng Hua)

1. Hochschild cohomology by its history

k a field (for simplicity)

A a k-algebra (assoc., with 1, non com.)

$HH^*(A,A) = HH^*(A)$

$\quad$ = Hochschild cohomology (1945: attributed to Eilenberg-MacLane)

$\quad = H^* C(A,A)$

$$C(A,A) = (A \longrightarrow \mathrm{Hom}_k(A,A) \longrightarrow \mathrm{Hom}_k(A\otimes A, A) \longrightarrow \dots \longrightarrow \mathrm{Hom}_k(A^{\otimes p}, A) \to \dots)$$

$$a \longmapsto [a,?],\ f \longmapsto (a\otimes b \mapsto f(a)b - f(ab) + af(b))$$

We see: $HH^0(A) = Z(A)$ a com. alg.

$HH^1(A) = \mathrm{OutDer}(A)$ a Lie alg.

$A^e = A \otimes A^{op}$ enveloping algebra, ${}_A A_A$ = identity bimodule

Cartan-Eilenberg (1956): $HH^*(A) \simeq \mathrm{Ext}^*_{A^e}(A,A)$: algebra : cup product

Gerstenhaber (1963): $HH^*(A)$ is graded com.

modern argument: A = unit in $\mathcal{D}(A^e)$ with $\overset{L}{\underset{A}{\otimes}}$

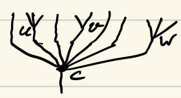
$HH^{*+1}(A)$ is a graded Lie algebra : Gerstenhaber bracket

Getzler-Jones (1994): $(C(A,A), \cup, \text{brace op.})$ is a B_∞-algebra

B : Baues (1981): $C^*_{sg}(X, \mathbb{Z})$ is B_∞, $\forall$ top. space X

brace op. (Kadeishvili 1988):

$$c\{u, v, \dots w\} = \sum \pm$$



Rks 1) The B_∞-str. contains all the info, e.g. $[c,u] = c\{u\} \mp u\{c\}$.

2) The constr. generalizes from k-algebras to k-categories (Mitchell 1972) and to differential graded (= dg) categories

$$\begin{array}{c} HH^*(\mathrm{Inj} A) \\ \parallel \end{array}$$

Thm (Lowen - Van den Bergh 2005): $HH^*(A) \simeq HH^*_{ab}(\mathrm{Mod} A) \simeq HH^*(\mathcal{D}_{dg} A)$

(this lifts to the B_∞-level, K 2003).

Not.: $\mathrm{Mod} A$ = {all (right) A-modules}, $\mathcal{D}A = \mathcal{D}\mathrm{Mod} A$ = unbounded der. cat.

$\mathcal{D}_{dg} A$ = canonical dg enhancement of $\mathcal{D}A$.

Rk: In particular, we get $Z(A) \simeq Z(\mathcal{D}_{dg} A)$.

$Z(\mathcal{D}A)$ is pathological, e.g. $Z(\mathcal{D}^b(k[\varepsilon]/(\varepsilon^2))) \simeq k \ltimes k^{\mathbb{N}}$ (Krause-Ye, 2011)

2. Tate-Hochschild cohomology

A a right Noeth. algebra (for simplicity)

$\mathrm{mod}A = \{$fin. gen. A-modules$\}$

$\mathcal{D}^bA = \mathcal{D}^b(\mathrm{mod}A)$

$\mathrm{per}A = \{X \in \mathcal{D}^b(\mathrm{mod}A) \mid X$ qis to a bded complex of fin. gen. proj. modules$\}$

$sg(A) = \mathcal{D}^bA/\mathrm{per}A$

$=$ stable derived cat. (Buchweitz 1986)

$=$ singularity category (Orlov 2003)

Assume A^e is also Noetherian.

Def.: Sing. Hochs. cohom. $= HH^*_{sg}(A) = \mathrm{Ext}^*_{sg(A^e)}(A,A)$

Rk: $HH^*_{sg}(A)$ is graded com. (although $sg(A^e)$ is <u>not</u> monoidal).

Thm (Zhengfang Wang): a) $HH^*_{sg}(A)$ carries a canonical (but intricate!) Gerstenhaber bracket (2015)

b) There is a can. B_∞-alg. $C_{sg}(A,A)$ computing $HH^*_{sg}(A)$ (2018).

Buchweitz:

I had given the topic of finding a Gerstenhaber bracket on Hochschild-Tate cohomology (what the candidate calls *singular* Hochschild cohomology, an apt term) to one of my Ph.D. students several years ago, and even though that student was quite talented, he got nowhere! Because of my numerous discussions with that student, and my own inability to point him in the right direction I can and do appreciate how important and beautiful Wang's contribution here is.

Main Thm: $HH^*_{sg}(A) \simeq HH^*(sg_{dg}A)$ as graded algebras.

Conj.: This isom. lifts to the B_∞-level.

Thm (Chen-Li-Wang): True for $A = kQ/(Q_1)^2$, where Q is a finite quiver w/o sinks nor sources

Isom. in the main thm: $\mathcal{M} = \mathcal{D}^b_{dg}(\mathrm{mod}A)$, $\mathcal{S} = sg_{dg}(A)$

We have dg functors: $A \xrightarrow{i} \mathcal{M} \xrightarrow{p} \mathcal{S}$, $p \circ i \simeq 0$.

$$\begin{tikzcd}
\mathcal{D}^b(A\otimes A^{op}) \arrow[r, "(1\otimes i)^*"] \arrow[d] & \mathcal{D}(A\otimes \mathcal{M}^{op}) \arrow[r, "(i\otimes 1)^!"] & \mathcal{D}(\mathcal{M}\otimes \mathcal{M}^{op}) \arrow[d, "(p\otimes p)^*"] \\
sg(A^e) \arrow[rr, dashed] & & \mathcal{D}(\mathcal{S}\otimes \mathcal{S}^{op})
\end{tikzcd}$$

$A \longmapsto \mathrm{Id}_{\mathcal{S}}$

induces an isom. in the Yoneda algebras.

3. Applications: 2 reconstruction thms (with Zhēng Huá)

<u>Thm 1</u>: $S = \mathbb{C}[\![x_1, \ldots, x_n]\!] \longrightarrow R = S/(f)$ isol. sing.
Then R is determined by $\dim R$ and $\mathrm{sg}_{dg}(R)$.

Proof: $Z(\mathrm{sg}_{dg} R) = HH^0_{sg}(R) \underset{\substack{\text{matrix fact.}\\ \text{Buchweitz}\\ \text{BACH}}}{\xrightarrow{\sim}}$ Tyurina alg. of $R = S/(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$

$\dim R$ and the Tyurina alg. determine R (Mather-Yau 1982, Greuel-Pham 2017). ✓

R a complete $\overbrace{\text{isol.}}^{\text{local}}$ compound du Val sing. [3-dim., normal, a generic hyperplane section is du Val]
$f: X \longrightarrow \operatorname{Spec} R$ smooth min. model contracting a tree of rat. curves
Λ the associated <u>contraction algebra</u> (Donovan-Wemyss, 2013) [represents the NC defo of the exc. fibre]
$\Lambda \simeq \mathrm{Jac}(Q, W)$ (Van den Bergh)
$\overline{W}$ = image of W in $HH_0(\Lambda)$
<u>Thm 2</u>: The der. eq. class of $(\Lambda, \overline{W})$ determines R.

Rks: 1) Donovan-Wemyss conj. that the der. eq. class of Λ <u>alone</u> determines R.
2) $\mathrm{sg}(R) \simeq \mathcal{C}_{Q,W}$ = gen. cluster category (Amiot 2009)