Hochschild cohomology of Beilinson algebras of graded down-up algebras

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Graded down-up algebras

- Throughout let k be an algebraically closed field of $\operatorname{char} k = 0$.

Definition (Benkart-Roby, 1998)

A graded algebra

$$A(\alpha, \beta) := k\langle x, y\rangle / (x^2y - \beta yx^2 - \alpha xyx,\ xy^2 - \beta y^2x - \alpha yxy)$$

$$\deg x = m, \deg y = n \in \mathbb{N}^+$$

with parameters $\alpha, \beta \in k$ is called a *graded down-up algebra*.

- Down-up algebras were originally introduced by Benkart and Roby in the study of the down and up operators on partially ordered sets.
- Since then, various aspects of these algebras have been investigated.

Down-up algebras and 3-dimensional AS-regular algebra

- For example,
  - structures [Benkart-Witherspoon, 2001], [Kirkman-Musson-Passman, 1999], [Zhao, 1999],
  - representations [Carvalho-Musson, 2000],
  - homological invariants [Chouhy-Herscovich-Solotar, 2018],
  - connections with enveloping algebras of Lie algebras [Benkart, 1999], [Benkart-Roby, 1998],
  - invariant theory [Kirkman-Kuzmanovich, 2005], [Kirkman-Kuzmanovich-Zhang, 2015], and so on.

Theorem (Kirkman-Musson-Passman, 1999)

Let $A = A(\alpha, \beta)$ be a graded down-up algebra.
$\Longrightarrow$ [A*: a noetherian* 3*-dimensional AS-regular algebra* $\Longleftrightarrow$ $\beta \neq 0$.]

Remark

A graded down-up algebra has played a key role as a test case for more complicated situations in noncommutative projective geometry.

Beilinson algebras of graded down-up algebras

- $A := A(\alpha, \beta)$: a graded down-up algebra with $\beta \neq 0$, so that A is 3-dimensional AS-regular.
- $\ell := 2(\deg x + \deg y) = 2(m+n)$ (ℓ: the Gorenstein parameter of A).

Definition (Minamoto-Mori, 2011)

The Beilinson algebra of A is defined by

$$\nabla A := \begin{pmatrix} A_0 & A_1 & \cdots & A_{\ell-1} \\ 0 & A_0 & \cdots & A_{\ell-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 \end{pmatrix}$$

with the multiplication $(a_{ij})(b_{ij}) = \left(\sum_{k=0}^{\ell-1} a_{kj} b_{ik}\right)$.

Remark

The Beilinson algebra ∇A of A is finite-dimensional k-algebra.

Example

1. If $\deg x = 1, \deg y = 1$, then ∇A is given by the quiver

$$\begin{tikzcd} 1 \arrow[r, shift left, "x_1"] \arrow[r, shift right, "y_1"'] & 2 \arrow[r, shift left, "x_2"] \arrow[r, shift right, "y_2"'] & 3 \arrow[r, shift left, "x_3"] \arrow[r, shift right, "y_3"'] & \color{blue}{4} \end{tikzcd}$$

   with relations (the Gorenstein parameter of A: $\ell = 2(1+1) = 4$)

   $x_1x_2y_3 - \beta y_1x_2x_3 - \alpha x_1y_2x_3 = 0$, $x_1y_2y_3 - \beta y_1y_2x_3 - \alpha y_1x_2y_3 = 0$.

2. If $\deg x = 1, \deg y = 2$, then ∇A is given by the quiver

$$\begin{tikzcd} 1 \arrow[r, "x_1"] \arrow[rr, bend right, "y_1"'] & 2 \arrow[r, "x_2"] \arrow[rr, bend right, "y_2"'] & 3 \arrow[r, "x_3"] \arrow[rr, bend right, "y_3"'] & 4 \arrow[r, "x_4"] \arrow[rr, bend right, "y_4"'] & 5 \arrow[r, "x_5"] & \color{blue}{6} \end{tikzcd}$$

   with relations (the Gorenstein parameter of A: $\ell = 2(1+2) = 6$)

   $x_1x_2y_3 - \beta y_1x_2x_3 - \alpha x_1y_2x_3 = 0$, $x_2x_3y_4 - \beta y_2x_4x_5 - \alpha x_2y_3x_5 = 0$,

   $x_1y_2y_4 - \beta y_1y_3x_5 - \alpha y_1x_3y_4 = 0$.

Minamoto-Mori's theorem

- tails A: the quotient category of finitely generated graded right A-modules by the Serre subcategory of finite dimensional modules.
  - tails A: the noncommutative projective scheme of A in the sense of [Arthin-Zhang, 1994].
- The following is obtained as a special case of [Theorem 4.14, Minamoto-Mori, 2011].

Theorem

$A = A(\alpha, \beta)$ *is a graded down-up algebra with* $\beta \neq 0 \Longrightarrow$

1. ∇A*: Fano algebra of* $\operatorname{gldim} \nabla A = 2$,
2. *There exists an equivalence of triangulate category* $\mathsf{D}^{\mathsf{b}}(\mathsf{tails}\, A) \cong \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, \nabla A)$.

Aim

Aim

The aim of our talk is to investigate the Hochschild cohomology groups $\mathrm{HH}^i(\nabla A)$ of ∇A of a graded down-up algebra $A = A(\alpha, \beta)$ with $\beta \neq 0$.

Remark

It is known that $\mathrm{HH}^i(\nabla A)$ of ∇A of an AS-regular algebra A is closely related to the Hochschild cohomology of tails A and the infinitesimal deformation theory of tails A.

$\deg x = \deg y = 1$

If $\deg x = \deg y = 1$, then a description of $\mathrm{HH}^i(\nabla A)$ has been obtained using a geometric technique ([Table 2, Belmans, 2017]).

$\deg x = 1, \deg y = n \geq 2$

In this talk, for $\deg x = 1, \deg y = n \geq 2$, we give the dimension formula of $\mathrm{HH}^i(\nabla A)$ for each $i \geq 0$.

$\deg x = \deg y = 1$

The point schemes of down-up algebras are divided into three cases ($\mathbb{P}^1 \times \mathbb{P}^1$, a double curve of bidegree $(1,1)$, or two curves of bidegree $(1,1)$ in general position).

Theorem (Table 2, Belmans, 2017)

Let $A = A(\alpha, \beta)$ be a graded down-up algebra with $\deg x = \deg y = 1$ and $\beta \neq 0$. $\Longrightarrow$

- $\dim_k \mathrm{HH}^0(\nabla A) = 1;$
- $\dim_k \mathrm{HH}^1(\nabla A) = \begin{cases} 6 & \text{if } \alpha = 0, \\ 3 & \text{if } \alpha \neq 0 \text{ and } \alpha^2 + 4\beta = 0, \\ 1 & \text{if } \alpha \neq 0 \text{ and } \alpha^2 + 4\beta \neq 0; \end{cases}$
- $\dim_k \mathrm{HH}^2(\nabla A) = \begin{cases} 9 & \text{if } \alpha = 0, \\ 6 & \text{if } \alpha \neq 0 \text{ and } \alpha^2 + 4\beta = 0, \\ 4 & \text{if } \alpha \neq 0 \text{ and } \alpha^2 + 4\beta \neq 0; \end{cases}$
- $\dim_k \mathrm{HH}^i(\nabla A) = 0$ *for* $i \geq 3$.

$\deg x = 1,\ \deg y \geq 2$

In this case, the Beilinson algebra ∇A is given by the following quiver Q with relations $f_i = 0$ $(1 \leq i \leq n)$, $g = 0$:
(the Gorenstein parameter of A: $\ell = 2(n+1) = 2n+2$)

$$
Q := \begin{tikzcd}
1 \arrow[r, "x_1"] \arrow[rrrr, bend right, "y_1"'] & 2 \arrow[r, "x_2"] \arrow[rrrr, bend right, "y_2"'] & \cdots \arrow[r, "x_{n-1}"] & n \arrow[r, "x_n"] \arrow[rrr, bend right, "y_n"'] & n+1 \arrow[r, "x_{n+1}"] \arrow[rrrr, bend right, "y_{n+1}"'] & n+2 \arrow[r, "x_{n+2}"] \arrow[rrrr, bend right, "y_{n+2}"'] & \cdots \arrow[r, "x_{2n}"] & 2n+1 \arrow[r, "x_{2n+1}"] & 2n+2
\end{tikzcd},
$$

$$f_i := x_i x_{i+1} y_{i+2} - \beta y_i x_{i+n} x_{i+n+1} - \alpha x_i y_{i+1} x_{i+n+1},$$

$$g := x_1 y_2 y_{n+2} - \beta y_1 y_{n+1} x_{2n+1} - \alpha y_1 x_{n+1} y_{n+2}.$$

Main Thoerem 1 ($\deg x = 1$, $\deg y \geq 2$)

Main Theorem 1 (I-U, 2019)

Let $A = A(\alpha, \beta)$ be a graded down-up algebra with $\deg x = 1, \deg y = n \geq 2$, and $\beta \neq 0$. We define

$$\delta_n := \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & 1 \\ \beta & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in k$$

(e.g. $\delta_2 = \alpha^2 + \beta, \delta_3 = \alpha^3 + 2\alpha\beta, \delta_4 = \alpha^4 + 3\alpha^2\beta + \beta^2, \delta_5 = \alpha^5 + 4\alpha^3\beta + 3\alpha\beta^2$).
$\Longrightarrow$

- $\dim_k \mathrm{HH}^0(\nabla A) = 1$;
- $\dim_k \mathrm{HH}^1(\nabla A) =$
$$\begin{cases} 4 & \text{if } n \text{ is odd and } \alpha = 0 \text{ (in this case } \delta_n = 0\text{)}, \\ 3 & \text{if } n \text{ is odd}, \alpha \neq 0, \text{and } \delta_n = 0, \text{or if } n \text{ is even and } \delta_n = 0, \\ 2 & \text{if } \alpha^2 + 4\beta = 0 \text{ (in this case } \delta_n \neq 0\text{)}, \\ 1 & \text{if } \delta_n \neq 0 \text{ and } \alpha^2 + 4\beta \neq 0; \end{cases}$$

- $\dim_k \mathrm{HH}^2(\nabla A) =$
$$\begin{cases} 8 & \text{if } n = 2 \text{ and } \delta_2 = 0, \\ 7 & \text{if } n = 2 \text{ and } \alpha^2 + 4\beta = 0 \text{ (in this case } \delta_2 \neq 0\text{)}, \\ 6 & \text{if } n = 2, \delta_2 \neq 0, \text{and } \alpha^2 + 4\beta \neq 0, \\ n+5 & \text{if } n \text{ is odd and } \alpha = 0 \text{ (in this case } \delta_n = 0\text{)}, \\ n+4 & \text{if } n \text{ is odd}, \alpha \neq 0, \text{and } \delta_n = 0, \text{or if } n \geq 4 \text{ is even and } \delta_n = 0, \\ n+3 & \text{if } n \geq 3 \text{ and } \alpha^2 + 4\beta = 0 \text{ (in this case } \delta_n \neq 0\text{)}, \\ n+2 & \text{if } n \geq 3, \delta_n \neq 0, \text{and } \alpha^2 + 4\beta \neq 0; \end{cases}$$
- $\dim_k \mathrm{HH}^i(\nabla A) = 0$ for $i \geq 3$.

Remark

Since A is not generated in degree 1, the geometric theory of point schemes does not work naively in our case, so our proof of the above theorem is purely algebraic by using Grenn-Snashall's method.

Corollary

- It is known that Hochschild cohomology is invariant under derived equivalence.
- Using Minamoto-Mori's theorem, Main Theorem 1 and Belmans's theorem, we have the following consequence.

Corollary (I-U, 2019)

Let $A = A(\alpha, \beta)$ and $A' = A(\alpha', \beta')$ be graded down-up algebras with $\deg x = 1, \deg y = n \geq 1$, where $\beta \neq 0, \beta' \neq 0$. If

$$\delta_n = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & 1 \\ \beta & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \text{ and } \delta'_n = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha' & 1 \\ \beta' & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq 0,$$

then $\mathsf{D}^{\mathsf{b}}(\operatorname{tails} A) \not\cong \mathsf{D}^{\mathsf{b}}(\operatorname{tails} A')$.

Application to the study of Grothendieck groups

- T: a triangulated category, $K_0(\mathsf{T})$: the Grothendieck group of T.
- If T admits a full strong exceptional sequence of length r, then $K_0(\mathsf{T})$ is $\mathbb{Z}^r$, so $\operatorname{rk} K_0(\mathsf{T}) = r$.
- If T has the Serre functor S in the sense of [Bondal-Kapranov], then S induces an automorphism $\mathfrak{s}$ of $K_0(\mathsf{T})$.

Theorem ((1) (Bondal-Polishchuk, 1994), (2) (Belmans, 2017))

Let $\mathsf{D}^{\mathsf{b}}(\operatorname{coh} X)$ *be the bounded derived category of coherent sheaves on a smooth projective variety* X.

1. *The action of* $(-1)^{\dim X}\mathfrak{s}$ *on* $K_0(\mathsf{D}^{\mathsf{b}}(\operatorname{coh} X))$ *is unipotent.*
2. *If* $\mathsf{D}^{\mathsf{b}}(\operatorname{coh} X)$ *admits a full strong exceptional sequence, then*

$$\chi(\mathrm{HH}^\bullet(X)) = (-1)^{\dim X} \operatorname{rk} K_0(\mathsf{D}^{\mathsf{b}}(\operatorname{coh} X)).$$

where $\chi(\mathrm{HH}^\bullet(X)) := \sum_{i\in\mathbb{Z}}(-1)^i \dim_k \mathrm{HH}^i(X)$.

Analogue for graded down-up algebras

- $A = A(\alpha, \beta)$: a graded down-up algebra with $\deg x = 1, \deg y = n \geq 1$, and $\beta \neq 0$.
- Then $\mathsf{D}^{\mathsf{b}}(\mathsf{tails}\, A)$ has a full strong exceptional sequence of length $2n+2$ by [Minamoto-Mori], so $\operatorname{rk} K_0(\mathsf{D}^{\mathsf{b}}(\mathsf{tails}\, A)) = 2n+2$.
- Moreover $\mathsf{D}^{\mathsf{b}}(\mathsf{tails}\, A)$ has the Serre functor by [de Naeghel-Van den Bergh].
- Note that $\operatorname{gldim}(\mathsf{tails}\, A) = \operatorname{gldim} \nabla A = 2$.

$\deg y = n = 1$

If $n = 1$, then $\mathfrak{s}$ acts unipotently on $K_0(\mathsf{D}^{\mathsf{b}}(\mathsf{tails}\, A))$ and it follows from [Belmans] that

$$\chi(\mathrm{HH}^\bullet(\nabla A)) = 4 = \operatorname{rk} K_0(\mathsf{D}^{\mathsf{b}}(\mathsf{tails}\, A))$$

where $\chi(\mathrm{HH}^\bullet(\nabla A)) := \sum_{i \in \mathbb{Z}} (-1)^i \dim_k \mathrm{HH}^i(\nabla A)$, so an analogue of the above theorem holds.

Main Theorem 2

By Main Theorem 1 and Happel's trace formula, we have the following.

Main Theorem 2 (I-U, 2019)

1. If $n = 2$, then $\mathfrak{s}$ acts unipotently on $K_0(\mathsf{D}^{\mathsf{b}}(\mathsf{tails}\, A))$ and
$$\chi(\mathrm{HH}^{\bullet}(\nabla A)) = 6 = \mathrm{rk}\, K_0(\mathsf{D}^{\mathsf{b}}(\mathsf{tails}\, A)).$$
2. If $n \geq 3$, then $\mathfrak{s}$ does not act unipotently on $K_0(\mathsf{D}^{\mathsf{b}}(\mathsf{tails}\, A))$ and
$$\chi(\mathrm{HH}^{\bullet}(\nabla A)) = n + 2 \neq 2n + 2 = \mathrm{rk}\, K_0(\mathsf{D}^{\mathsf{b}}(\mathsf{tails}\, A)).$$

Remark

In respect of Main Theorem 2,

- when $n = 2$, $\mathsf{D}^{\mathsf{b}}(\mathsf{tails}\, A)$ behaves a bit like a geometric object (a smooth projective surface),
- but when $n \geq 3$, $\mathsf{D}^{\mathsf{b}}(\mathsf{tails}\, A)$ is not equivalent to the derived category of any smooth projective surface.

Thank you for your attention !

If you have an interest in our talk, please see arXiv:1904.00677.