Noncommutative Auslander Theorem and noncommutative quotient singularities

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Outline

(I) Noncommutative Auslander Theorem

(II) Related to noncommutative resolutions for singularities

(III) Related to noncommutative McKay correspondence

(IV) Noncommutative quadric hypersurfaces

(I) Noncommutative Auslander Theorem

Auslander Theorem

- $\Bbbk$ is an algebraically closed field of characteristic zero.

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- $S = \mathbb{k}[x_1, \ldots, x_n]$ is the polynomial algebra.
- G is a finite small subgroup of $\mathrm{GL}(\mathbb{k}^{\oplus n})$.

  small $=$ G does not contain a pseudo-reflection of $\mathbb{k}^{\oplus n}$.

Auslander Theorem

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where $S * G$ is the skew-group algebra.

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- First appeared at
  M. Auslander, On the purity of the branch locus, Am. J. Math., 1962

  A proof for $n = 2$ at

  Y. Yoshino, Cohen-Macaulay modules over Cohen-Macaulay rings, LMS Lecture Note Series 146, 1990

  A complete proof at

  O. Iyama, R. Takahashi, Tilting and cluster tilting for quotient singularities, Math. Ann.,

  2013

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- If further, $gldim(S) = d$, then S is called an Artin-Schelter regular algebra.

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- **Remark.** Artin-Schelter regular algebras may be viewed as "coordinate rings" for noncommuative projective spaces.

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- Let R be a noetherian graded algebra.

  $\mathsf{gr}\, R$ = the category of graded finitely generated right R-modules
  $\mathsf{tor}\, R$ = finite dimensional graded right R-modules
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- **Mori-Ueyama, T. AMS, 2016**
  R is called an isolated singularity if $\mathsf{qgr}\,R$ has finite global dimension.

Noncommutative Auslander Theorem

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Theorem

The following are equivalent.

- *S^G is an isolated singularity, and there is a natural isomorphism $S * G \cong \mathrm{End}_{S^G}(S)$;*
- *There is an equivalence of abelian categories $\mathrm{qgr}\, S^G \cong \mathrm{qgr}\, S * G$;*

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Question

What will happen when S^G *is not an isolated singularity?*

Noncommutative Auslander Theorem

- Let S be a noetherian graded algebra with finite Gelfand-Kirillov dimension (abbr. GKdim).

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  Let $\int \in H$ be the integral of H such that $\varepsilon(\int) = 1$.

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Definition

The pertinency of the H-action on R is defined to be the number

$$\mathrm{p}(S, H) = \mathrm{GKdim}(S) - \mathrm{GKdim}((S\#H)/I),$$

where I is the ideal of $S\#H$ generated by the element $1\# \int$.

- **Y.-H. Bao, J.-W. He, J.J. Zhang**, Pertinency of Hopf actions and quotient categories of Cohen-Macaulay algebras, J. Noncomm. Geom., 2019

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- **Remark.** $\mathsf{qgr}_0\, S = \mathsf{qgr}\, S$.

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- **Remark.** $\mathsf{qgr}_0\, S = \mathsf{qgr}\, S$.

- Assume $\mathsf{GKdim}(S) = d \geq 2$ and S is a Cohen-Macaulay algebra, that is,

  for every $M \in \mathsf{gr}\, S$, $\mathsf{GKdim}(M) + j(M) = \mathsf{GKdim}(S)$,

  where $j(M) = \min\{i \mid \mathsf{Ext}^i_S(M, S) \neq 0\}$, called the grade of M.

Noncommutative Auslander Theorem

- Let H be a semisimple Hopf algebra which acts on S homogeneously and inner faithfully. Let $S^H = \{a \in S | h \cdot a = \varepsilon(h)a, \forall h \in H\}$ be the fixed subalgebra of S.

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Theorem

The following are equivalent.

- *There is a natural equivalence of abelian categories* $\mathrm{qgr}_{d-2}\, S^H \cong \mathrm{qgr}_{d-2}\, S\#H$;
- *There is a natural isomorphism of graded algebras* $S\#H \cong \mathrm{End}_{S^H}(S)$;
- $\mathrm{p}(S, H) \geq 2$.

Y.-H. Bao, J.-W. He, J.J. Zhang, Pertinency of Hopf actions and quotient categories of Cohen-Macaulay algebras, J. Noncomm. Geom., 2019

- The group actions on the following classes of algebras satisfy the condition $\mathrm{p}(S, H) \geq 2$.

Theorem

(1) *Let $\mathfrak{g}$ be a finite dimensional Lie algebra, and $G \leq \mathrm{Aut}_{Lie}(\mathfrak{g})$ a finite small subgroup. Then $U(\mathfrak{g}) * G \cong End_{U(\mathfrak{g})^G} U(\mathfrak{g})$.*

(2) *Let $S = \Bbbk_{p_{ij}}[x_1, \ldots, x_n]$ be the skew polynomial algebra, and assume $\{p_{ij} | 1 \leq i < j \leq n\}$ are generic. Let G be a finite small group of automorphisms of S. Then $S * G \cong End_{S^G} S$.*

(3) *Let $S = \Bbbk\langle x, y\rangle/(f_1, f_2)$ be the graded down-up algebra, where $f_1 = x^2y - \alpha xyx - \beta yx^2, f_2 = xy^2 - \alpha yxy - \beta y^2x$.*
Let G be any nontrivial finite subgroup of $\mathrm{Aut}_{gr}(S)$.
If $\beta \neq -1$ or $(\alpha, \beta) = (2, -1)$, then $S * G \cong End_{S^G} S$.

Y.-H. Bao, J.-W. He, J.J. Zhang, Noncommutative Auslander Theorem, T. AMS. 2018

J. Gaddis, E. Kirkman, W.F. Moore, R. Won, Auslander's Theorem for permutation actions on noncommutative algebras, P. AMS, 2019

(II) Related to noncommutative resolutions for singularities

Noncommutative crepant resolution (NCCR)

- Let R be a (commutative) Cohen-Macaulay ring, and Λ a module-finite R-algebra.

Definition

(1) Λ is called an R-order if Λ is a maximal Cohen-Macaulay module.

An R-order is non-singular if $\operatorname{gldim} \Lambda_{\mathfrak{p}} = \dim R_{\mathfrak{p}}$ for all $\mathfrak{p} \in SpecR$.

(2) A noncommutative crepant resolution (NCCR) of R is an R-algebra of the form $\Gamma = \operatorname{End}_R(M)$ where M is a reflexive R-module, such that Γ is a non-singular R-order.

M. van den Bergh, Non-commutative crepant resolutions, The legacy of Niels Henrik Abel, 2004

O. Iyama, I. Reiten, Fomin-Zelevinsky mutation and tilting modules over Calabi-Yau algebras, Amer. J. Math., 2008

- Noncommutative Bondal-Orlov conjecture: If R is a normal Gorenstein domain, then all the NCCRs of R are derived equivalent.

NCCR

- Noncommutative Bondal-Orlov conjecture: If R is a normal Gorenstein domain, then all the NCCRs of R are derived equivalent.
- Assume R is a (commutative) d-dimensional Cohen-Macaulay equi-codimensional normal domain with a canonical module.

Theorem

- *If $d = 2$, then all NCCRs of R are Morita equivalent;*
- *If $d = 3$, then all NCCRs of R are derived equivalent.*

O. Iyama, I. Reiten, Fomin-Zelevinsky mutation and tilting modules over Calabi-Yau algebras, Amer. J. Math., 2008

O. Iyama, M. Wemyss, On the noncommutative Bondal-Orlov conjecture, J. Reine Angew.

Math., 2013

NCCR

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O. Iyama, M. Wemyss, On the noncommutative Bondal-Orlov conjecture, J. Reine Angew. Math., 2013

- **Question.** How about the case that R is not a commutative algebra?

Noncommutative quasi-resolution (NQR)

- Let S be noetherian graded Cohen-Macaulay algebra with $\mathrm{GKdim}(S) = d < \infty$.

  Let H be a semisimple Hopf algebra which acts on S homogeneously and inner faithfully.

Theorem

For a positive integer $i \leq \mathrm{p}(S, H)$, we have a natural equivalence of abelian categories

$$\mathrm{qgr}_{d-i}\, S^H \cong \mathrm{qgr}_{d-i}\, S\#H.$$

Y.-H. Bao, J.-W. He, J.J. Zhang, Pertinency of Hopf actions and quotient categories of Cohen-Macaulay algebras, J. Noncomm. Geom., 2019

NQR

- Let A be a noetherian locally finite $\mathbb{N}$-graded algebra with $\mathrm{GKdim}(A) = d \in \mathbb{N}$.

  Assume B be a noetherian locally finite $\mathbb{N}$-graded Auslander regular Cohen-Macaulay algebra with $\mathrm{GKdim}(B) = d$.

Definition

If there are graded modules ${}_BM_A$ and ${}_AN_B$, which are finitely generated on both sides, such that

(1) there is a B-bimodule morphism $f : M \otimes_A N \to B$ such that $\mathrm{GKdim}(\ker f) \leq d - 2$ and $\mathrm{GKdim}(\mathrm{coker} f) \leq d - 2$,

(2) there is an A-bimodule morphism $g : N \otimes_B M \to A$ such that $\mathrm{GKdim}(\ker f) \leq d - 2$ and $\mathrm{GKdim}(\mathrm{coker} f) \leq d - 2$,

then B is called a noncommutative quasi-resolution (NQR) of A.

X.-S. Qin, Y.-H. Wang, J.J. Zhang, Noncommutative quasi-resolutions, J. Algebra, 2019

NQR

Remark

In commutative case, NQR and NCCR are equivalent.

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In commutative case, NQR and NCCR are equivalent.

- The following results generalize Iyama-Wemyss' results.

Theorem

Let A be a noetherian locally finite $\mathbb{N}$-graded algebra.

- *If* $\mathrm{GKdim}(A) = 2$, *then all NQRs of A are Morita equivalent;*
- *If* $\mathrm{GKdim}(A) = 3$, *then all NQRs of A are derived equivalent.*

X.-S. Qin, Y.-H. Wang, J.J. Zhang, Noncommutative quasi-resolutions, J. Algebra, 2019

(III) Related to noncommutative McKay correspondence

Classical McKay correspondence

- Let $G \leq SL(\Bbbk^{\oplus 2})$ be a finite subgroup, which acts on $S = \Bbbk[x, y]$ naturally.

  Auslander Theorem, $S * G \cong \mathrm{End}_{S^G}(S)$.

Theorem

- *There are equivalences of abelian categories*

$$\mathrm{mod} \prod \widetilde{Q}_G \cong \mathrm{mod}\, S * G \cong \mathrm{mod}\, \mathrm{End}_{S^G}(S),$$

  where $\widetilde{Q}_G$ is a quiver whose underlying graph is extended Dynkin of type ADE.

- $D^b(\mathrm{mod} \prod \widetilde{Q}_G) \cong D^b(\widetilde{Spec}(S^G))$, *where $\widetilde{Spec}(S^G)$ is the minimal resolution of the quotient singularity $\mathbb{A}^2/G$.*

M. Kapranov and E. Vasserot, Kleinian singularities, derived categories and Hall algebras, Math. Ann., 2000

Y. Yoshino, Cohen-Macaulay modules over Cohen-Macaulay rings, LMS Lecture Note

Series 146, 1990

Noncommutative McKay correspondence

- Let S be an Artin-Schelter regular algebra of global dimension 2. Let $G \leq HSL(S)$ be a finite subgroup.

Theorem

All the possible choices of (S, G) are as follows.

	S	G
(1)	$\Bbbk[x,y]$	$G \leq SL(\Bbbk^{\oplus 2})$
(2)	$\Bbbk_{-1}[x,y]$	C_n *diagonal action*
(3)	$\Bbbk_{-1}[x,y]$	C_n *non-diagonal action*
(4)	$\Bbbk_{-1}[x,y]$	D_{2n} *($n \geq 3$)*
(5)	$\Bbbk_q[x,y], q^2 \neq 1$	C_n *($n \geq 2$) diagonal action*
(6)	$\Bbbk_J[x,y]$	C_2 *diagonal action*

K. Chan, E. Kirkman, C. Walton, J.J. Zhang, Quantum binary polyhedral groups and their actions on quantum planes, J. Reine Angew. Math., 2016

- Noncommutative Auslander Theorem:

Theorem

Let S be an Artin-Schelter regular algebra of global dimension 2. Let $G \leq HSL(S)$ be a finite subgroup. Then

$$S * G \cong \mathrm{End}_{S^G}(S).$$

K. Chan, E. Kirkman, C. Walton and J.J. Zhang, McKay Correspondence for semisimple Hopf actions on regular graded algebras I, J. Algebra, 2018

Theorem

Let S be an Artin-Schelter regular algebra of global dimension 2. Let $G \leq HSL(S)$ be a finite subgroup. There are bijective correspondences between the isomorphism classes of

- *indecomposable maximal Cohen-Macaulay left S^G-modules, up to degree shift;*
- *indecomposable finitely generated projective left $S * G$-modules;*
- *simple G-modules.*

K. Chan, E. Kirkman, C. Walton and J.J. Zhang, McKay Correspondence for semisimple Hopf actions on regular graded algebras II, J. Noncomm. Geom., 2019

Remark

If $\mathrm{gldim}(S) \geq 2$, **I. Mori** provided an explicit construction of the McKay quiver of the G-action in the case that G is a cyclic group and acts on S diagonally.

I. Mori, McKay-type correspondence for AS-regular algebras, J. LMS, 2013

(IV) Noncommutative quadric hypersurfaces

- Recall a result in the noncommutative McKay correspondence:

Theorem

Let S be an Artin-Schelter regular algebra of global dimension 2, and let $G \leq HSL(S)$ be a finite subgroup.

Then

- *the fixed subalgebra S^G is not regular;*
- *$S^G \cong C/Cw$, where C is an Artin-Schelter regular algebra of global dimension 3, and w is a normal element of C.*

K. Chan, E. Kirkman, C. Walton, J.J. Zhang, Quantum binary polyhedral groups and their actions on quantum planes, J. Reine Angew. Math., 2016

K. Chan, E. Kirkman, C. Walton and J.J. Zhang, McKay Correspondence for semisimple Hopf actions on regular graded algebras I, J. Algebra, 2018

Noncommutative quadric hypersurfaces

- Let S be a Koszul Artin-Schelter regular algebra of global dimension d.
  Let $z \in S_2$ be a central regular element of S.

Noncommutative quadric hypersurfaces

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- The following facts are well-known:

  (1) $A := S/Sz$ is a Koszul algebra;

  (2) A is an Artin-Schelter Gorenstein algebra of injective dimension $d-1$.

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- The following facts are well-known:

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  (2) A is an Artin-Schelter Gorenstein algebra of injective dimension $d-1$.
- $\mathsf{mcm}\, A$ = the category of (finitely generated) maximal Cohen-Macaulay modules over A

  $\underline{\mathsf{mcm}}A$ = the stable category

  $\underline{\mathsf{mcm}}A$ is a triangulated category.

Noncommutative quadric hypersurfaces

- **S.P. Smith, M. van den Bergh**, Noncommutative quadric surfaces, J. Noncomm. Geom., 2013

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Theorem

- *there is an equivalence of triangulated categories*

$$\underline{\mathrm{mcm}} A \cong D^b(\mathrm{mod}\, C(A)).$$

- *If $C(A)$ is semisimple, then A is an isolated singularity (i.e.* qgr A *has finite global dimension).*

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- *If $C(A)$ is semisimple, then A is an isolated singularity (i.e. $\mathrm{qgr}\, A$ has finite global dimension).*

- **Remark.** The finite dimensional algebra $C(A)$ is an important tool to understand the singularities of A.

Clifford deformation of Koszul Frobenius algebra

- Let $E = S^!$ be the quadratic dual of the Koszul Artin-Schelter regular algebra S.

  Then E is a Koszul Frobenius algebra.

  S.P. Smith, Some finite dimensional algebras related to elliptic curves, in: CMS Conf. Proc., 1996

  D.-M. Lu, J.H. Palmieri, Q.-S. Wu, J.J. Zhang, A_∞-algebra for ring theorist, Algebra Colloq., 2004

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- Write $E = T(V)/(R)$, where $R \subseteq V \otimes V$. A linear map $\theta : R \to \Bbbk$ is called a Clifford map if

$$(\theta \otimes 1 - 1 \otimes \theta)(V \otimes R \cap R \otimes V) = 0.$$

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- Let $E(\theta) = T(V)/(r - \theta(r) : r \in R)$.

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- Let $E(\theta) = T(V)/(r - \theta(r) : r \in R)$.
- We call $E(\theta)$ a Clifford deformation of E.

Noncommutative quadric hypersurfaces

Proposition

- *Each central element* $0 \neq z \in S_2$ *corresponding to a Clifford map* θ_z *of* $E = S^!$.
- $E(\theta_z)$ *is a strongly* $\mathbb{Z}_2$*-graded algebra.*
- $C(A) \cong E(\theta_z)_0$.

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Theorem

Let S be a Koszul Artin-Schelter regular algebra, and let $z \in S_2$ be a central regular element.

Then $A = S/Sw$ is an isolated singularity if and only if $C(A) = E(\theta_z)_0$ is a semisimple algebra.

J.-W. He, Y. Ye, Clifford deformations of Koszul Frobenius algebras and noncommutative quadrics, arxiv:1905.04699

I. Mori, K. Ueyama, Noncommutative Knörrer Periodicity Theorem and noncommutative quadric hypersurfaces, arxiv:1905.12266

- **Example.** Let $S = \Bbbk\langle x, y, z\rangle/(f_1, f_2, f_3)$, where

$$f_1 = zx + xz, f_2 = yz + zy, f_3 = x^2 + y^2.$$

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- S is Koszul Artin-Schelter regular algebra of global dimension 3.
- All the possible noncommutative quadric hypersurfaces defined by a central element $w \in S_2$ of S:

w	$E(\theta)_0$	singularities of S/wS
$z^2 + xy + yx + \lambda x^2,\ \ \lambda \neq \pm 2\sqrt{-1}$	$\mathbf{k}^{\oplus 4}$	isolated
$z^2 + xy + yx \pm 2\sqrt{-1}x^2$	$\mathbf{k}[u]/(u^2) \times \mathbf{k}[u]/(u^2)$	nonisolated
z^2	$\mathbf{k}[u, v]/(u^2 - v^2, uv)$	nonisolated
$z^2 + x^2$	$\mathbf{k}^{\oplus 4}$	isolated
$xy + yx + \lambda x^2,\ \ \lambda \neq \pm 2\sqrt{-1}$	$\mathbf{k}[u]/(u^2) \times \mathbf{k}[u]/(u^2)$	nonisolated
$xy + yx \pm 2\sqrt{-1}x^2$	$\mathbf{k}[u, v]/(u^2, v^2)$	nonisolated
x^2	$\mathbf{k}[u]/(u^2) \times \mathbf{k}[u]/(u^2)$	nonisolated

- Another application:

  Clifford deformations provide a new explanation of **Knörrer Periodicity Theorem** for noncommutative quadric hypersurfaces.

Noncommutative quadric hypersurfaces

- Another application:

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- Let S be a Koszul Artin-Schelter regular algebra. Set $A^{\#} = S[v]/(z + v^2)$ and $A^{\#\#} = S[v_1, v_2]/(z + v_1^2 + v_2^2)$.

Theorem

Assume that $\operatorname{gldim} S \geq 2$. *Then*

- *A is a noncommutative isolated singularity if and only if so is $A^{\#}$.*
- *there is an equivalence of triangulated categories* $\underline{\mathrm{mcm}} A \cong \underline{\mathrm{mcm}} A^{\#\#}$.

H. Knörrer, Cohen-Macaulay modules on hypersurface singularities I, Invent. Math., 1987

I. Mori, K. Ueyama, Noncommutative Knörrer's Periodicity Theorem and noncommutative quadric hypersurfaces, arxiv:1905.12266

A. Conner, E. Kirkman, W. F. Moore, C. Walton, Noncommutative Knörrer periodicity and noncommutative Kleinian singularities, arXiv:1809.06524

Thank you for you attention!