Noncommutative Auslander Theorem and noncommutative quotient singularities

Ji-Wei He
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Aug. 28, 2019
(I) Noncommutative Auslander Theorem

(II) Related to noncommutative resolutions for singularities

(III) Related to noncommutative McKay correspondence

(IV) Noncommutative quadric hypersurfaces
(I) Noncommutative Auslander Theorem
• $k$ is an algebraically closed field of characteristic zero.
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• $S = k[x_1, \ldots, x_n]$ is the polynomial algebra.
• $\mathbb{k}$ is an algebraically closed field of characteristic zero.

• $S = \mathbb{k}[x_1, \ldots, x_n]$ is the polynomial algebra.

• $G$ is a finite **small** subgroup of $\text{GL}(\mathbb{k}^\oplus n)$.

  **small** = $G$ does not contain a pseudo-reflection of $\mathbb{k}^\oplus n$. 
Auslander Theorem

\[ S^G = \{ a \in S | g(a) = a, \ \forall g \in G \} \], the fixed subalgebra of \( S \).
Auslander Theorem

\[ S^G = \{ a \in S \mid g(a) = a, \ \forall g \in G \}, \text{ the fixed subalgebra of } S. \]

Theorem (Auslander Theorem)

There is a natural isomorphism of algebras

\[ S \ast G \cong \text{End}_{S^G}(S), \ s \ast g \mapsto [s' \mapsto sg(s')] \]

where \( S \ast G \) is the skew-group algebra.
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First appeared at


A proof for \( n = 2 \) at


A complete proof at

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if \( \text{injdim}(SS) = \text{injdim}(S_S) = d < \infty \), and

then \( S \) is called an Artin-Schelter Gorenstein algebra.

If further, \( \text{gldim}(S) = d \), then \( S \) is called an Artin-Schelter regular algebra.
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\[ \text{Ext}^i_S(S\mathbb{k}, SS) = \text{Ext}^i_R(\mathbb{k}S, SS) = \begin{cases} 0, & i \neq d; \\ \mathbb{k}, & i = d. \end{cases} \]
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then \( S \) is called an **Artin-Schelter Gorenstein** algebra.

Remark.

Artin-Schelter regular algebras may be viewed as "coordinate rings" for noncommutative projective spaces.


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**Remark.** Artin-Schelter regular algebras may be viewed as “coordinate rings” for noncommutative projective spaces.
Let $\text{GrAut}(S)$ be the group of graded automorphisms of $S$. 
Noncommutative Auslander Theorem

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- Mori-Ueyama, T. AMS, 2016

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$\text{HSL}(S):=\{\sigma \in \text{GrAut}(S)| \text{hdet} \sigma = 1\}$, called the group of homological special linear group of $S$. 
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Let $R$ be a noetherian graded algebra.

$\text{gr } R = \text{the category of graded finitely generated right } R\text{-modules}$
$\text{tor } R = \text{finite dimensional graded right } R\text{-modules}$
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$R$ is called an isolated singularity if $q\text{gr } R$ has finite global dimension.
Noncommutative Auslander Theorem

Let $S$ be an Artin-Schelter regular algebra of global dimension $d \geq 2$, and let $G \leq \text{GrAut}(S)$ be a finite subgroup.

Theorem
The following are equivalent.

1. $S^G$ is an isolated singularity, and there is a natural isomorphism $S^G \cong \text{End}_{S^G}(S)$;
2. There is an equivalence of abelian categories $\text{qgr} S^G \cong \text{qgr} S^G$;


Question
What will happen when $S^G$ is not an isolated singularity?
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The following are equivalent.

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Let $H$ be a semisimple Hopf algebra, which acts on $S$ homogeneously so that $S$ is a graded $H$-module algebra.

Definition
The pertinency of the $H$-action on $R$ is defined to be the number
$p(S, H) = \text{GKdim}(S) - \text{GKdim}(S \# H / I)$,
where $I$ is the ideal of $S \# H$ generated by the element $1 \# \int$.
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$$p(S, H) = \text{GKdim}(S) - \text{GKdim}((S\#H)/I),$$

where $I$ is the ideal of $S\#H$ generated by the element $1\#\int$.

Assume $\text{GKdim}(S) = d \geq 2$.

$\text{gr}_n S$ = the full subcategory of $\text{gr} S$ consisting of graded $S$-modules with $\text{GKdim} \leq n$. 

Remark. $\text{qgr}_0 S = \text{qgr} S$.
Assume $\text{GKdim}(S) = d \geq 2$.

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**Remark.** $q\text{gr}_0 S = q\text{gr} S$.

Assume $\text{GKdim}(S) = d \geq 2$ and $S$ is a Cohen-Macaulay algebra, that is, for every $M \in \text{gr} S$, $\text{GKdim}(M) + j(M) = \text{GKdim}(S)$, where $j(M) = \min\{i | \text{Ext}^i_S(M, S) \neq 0\}$, called the grade of $M$. 
Noncommutative Auslander Theorem

Let $H$ be a semisimple Hopf algebra which acts on $S$ homogeneously and inner faithfully. Let $S^H = \{a \in S | h \cdot a = \varepsilon(h) a, \forall h \in H\}$ be the fixed subalgebra of $S$. 
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**Theorem**

The following are equivalent.

- There is a natural equivalence of abelian categories $\text{qgr}_{d-2} S^H \cong \text{qgr}_{d-2} S\#H$;
- There is a natural isomorphism of graded algebras $S\#H \cong \text{End}_{S^H}(S)$;
- $p(S, H) \geq 2$.

The group actions on the following classes of algebras satisfy the condition $p(S, H) \geq 2$.

**Theorem**

1. Let $\mathfrak{g}$ be a finite dimensional Lie algebra, and $G \leq \text{Aut}_{\text{Lie}}(\mathfrak{g})$ a finite small subgroup. Then $U(\mathfrak{g}) \ast G \cong \text{End}_{U(\mathfrak{g})^G} U(\mathfrak{g})$.

2. Let $S = \mathbb{k}_{p_{ij}}[x_1, \ldots, x_n]$ be the skew polynomial algebra, and assume \{p_{ij} | 1 \leq i < j \leq n\} are generic. Let $G$ be a finite small group of automorphisms of $S$. Then $S \ast G \cong \text{End}_{S^G} S$.

3. Let $S = \mathbb{k}\langle x, y \rangle/(f_1, f_2)$ be the graded down-up algebra, where $f_1 = x^2y - \alpha xyx - \beta yx^2$, $f_2 = xy^2 - \alpha yxy - \beta y^2x$. Let $G$ be any nontrivial finite subgroup of $\text{Aut}_{\text{gr}}(S)$. If $\beta \neq -1$ or $(\alpha, \beta) = (2, -1)$, then $S \ast G \cong \text{End}_{S^G} S$.

Y.-H. Bao, J.-W. He, J.J. Zhang, Noncommutative Auslander Theorem, T. AMS. 2018

J. Gaddis, E. Kirkman, W.F. Moore, R. Won, Auslander’s Theorem for permutation actions on noncommutative algebras, P. AMS, 2019
(II) Related to noncommutative resolutions for singularities
Let \( R \) be a (commutative) Cohen-Macaulay ring, and \( \Lambda \) a module-finite \( R \)-algebra.

**Definition**

(1) \( \Lambda \) is called an \( R \)-order if \( \Lambda \) is a maximal Cohen-Macaulay module.

An \( R \)-order is non-singular if \( \text{gldim} \Lambda_p = \text{dim} R_p \) for all \( p \in \text{Spec} R \).

(2) A noncommutative crepant resolution (NCCR) of \( R \) is an \( R \)-algebra of the form \( \Gamma = \text{End}_R(M) \) where \( M \) is a reflexive \( R \)-module, such that \( \Gamma \) is a non-singular \( R \)-order.

M. van den Bergh, Non-commutative crepant resolutions, The legacy of Niels Henrik Abel, 2004

Noncommutative Bondal-Orlov conjecture: If $R$ is a normal Gorenstein domain, then all the NCCRs of $R$ are derived equivalent.

Assume $R$ is a (commutative) $d$-dimensional Cohen-Macaulay equi-codimensional normal domain with a canonical module.

Theorem
If $d = 2$, then all NCCRs of $R$ are Morita equivalent; if $d = 3$, then all NCCRs of $R$ are derived equivalent.


Question.
How about the case that $R$ is not a commutative algebra?

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Question. How about the case that $R$ is not a commutative algebra?
Let $S$ be noetherian graded Cohen-Macaulay algebra with $\text{GKdim}(S) = d < \infty$.

Let $H$ be a semisimple Hopf algebra which acts on $S$ homogeneously and inner faithfully.

**Theorem**

For a positive integer $i \leq p(S, H)$, we have a natural equivalence of abelian categories

$$\text{qgr}_{d-i} S^H \cong \text{qgr}_{d-i} S\#H.$$
Let $A$ be a noetherian locally finite $\mathbb{N}$-graded algebra with $\text{GKdim}(A) = d \in \mathbb{N}$.

Assume $B$ be a noetherian locally finite $\mathbb{N}$-graded Auslander regular Cohen-Macaulay algebra with $\text{GKdim}(B) = d$.

**Definition**

If there are graded modules $B M_A$ and $A N_B$, which are finitely generated on both sides, such that

1. there is a $B$-bimodule morphism $f : M \otimes_A N \to B$ such that $\text{GKdim}(\ker f) \leq d - 2$ and $\text{GKdim}(\text{coker} f) \leq d - 2$,

2. there is an $A$-bimodule morphism $g : N \otimes_B M \to A$ such that $\text{GKdim}(\ker f) \leq d - 2$ and $\text{GKdim}(\text{coker} f) \leq d - 2$,

then $B$ is called a **noncommutative quasi-resolution (NQR)** of $A$. 

X.-S. Qin, Y.-H. Wang, J.J. Zhang, Noncommutative quasi-resolutions, J. Algebra, 2019
Remark

In commutative case, NQR and NCCR are equivalent.
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The following results generalize Iyama-Wemyss’ results.

Theorem

Let $A$ be a noetherian locally finite $\mathbb{N}$-graded algebra.

- If $\text{GKdim}(A) = 2$, then all NQRs of $A$ are Morita equivalent;
- If $\text{GKdim}(A) = 3$, then all NQRs of $A$ are derived equivalent.

X.-S. Qin, Y.-H. Wang, J.J. Zhang, Noncommutative quasi-resolutions, J. Algebra, 2019
(III) Related to noncommutative McKay correspondence
Let $G \leq SL(\mathbb{K}^{\oplus 2})$ be a finite subgroup, which acts on $S = \mathbb{K}[x, y]$ naturally.

Auslander Theorem, $S \ast G \cong \text{End}_{SG}(S)$.

**Theorem**

There are equivalences of abelian categories

$$\text{mod} \prod \tilde{Q}_G \cong \text{mod} S \ast G \cong \text{mod} \text{End}_{SG}(S),$$

where $\tilde{Q}_G$ is a quiver whose underlying graph is extended Dynkin of type ADE.

$$D^b(\text{mod} \prod \tilde{Q}_G) \cong D^b(\text{Spec}(S^G)),$$

where $\text{Spec}(S^G)$ is the minimal resolution of the quotient singularity $\mathbb{A}^2 / G$.


Let $S$ be an Artin-Schelter regular algebra of global dimension 2. Let $G \leq HSL(S)$ be a finite subgroup.

**Theorem**

*All the possible choices of $(S, G)$ are as follows.*

<table>
<thead>
<tr>
<th></th>
<th>$S$</th>
<th>$G$</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>$\mathbb{k}[x, y]$</td>
<td>$G \leq SL(\mathbb{k}^{\oplus 2})$</td>
</tr>
<tr>
<td>2</td>
<td>$\mathbb{k}_{-1}[x, y]$</td>
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<tr>
<td>3</td>
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<td>$C_n$ non-diagonal action</td>
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<td>4</td>
<td>$\mathbb{k}_{-1}[x, y]$</td>
<td>$D_{2n}$ $(n \geq 3)$</td>
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<tr>
<td>5</td>
<td>$\mathbb{k}_q[x, y], q^2 \neq 1$</td>
<td>$C_n$ $(n \geq 2)$ diagonal action</td>
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Noncommutative Auslander Theorem:

**Theorem**

Let $S$ be an Artin-Schelter regular algebra of global dimension 2. Let $G \leq HSL(S)$ be a finite subgroup. Then

$$S \rtimes G \cong \text{End}_{S^G}(S).$$

K. Chan, E. Kirkman, C. Walton and J.J. Zhang, McKay Correspondence for semisimple Hopf actions on regular graded algebras I, J. Algebra, 2018
Theorem

Let $S$ be an Artin-Schelter regular algebra of global dimension 2. Let $G \leq HSL(S)$ be a finite subgroup. There are bijective correspondences between the isomorphism classes of

- indecomposable maximal Cohen-Macaulay left $S^G$-modules, up to degree shift;
- indecomposable finitely generated projective left $S \rtimes G$-modules;
- simple $G$-modules.

Remark

If gl.dim(S) \geq 2, I. Mori provided an explicit construction of the McKay quiver of the G-action in the case that G is a cyclic group and acts on S diagonally.

I. Mori, McKay-type correspondence for AS-regular algebras, J. LMS, 2013
(IV) Noncommutative quadric hypersurfaces
Recall a result in the noncommutative McKay correspondence:

**Theorem**

Let $S$ be an Artin-Schelter regular algebra of global dimension 2, and let $G \leq \text{HSL}(S)$ be a finite subgroup. Then

- the fixed subalgebra $S^G$ is **not regular**;
- $S^G \cong C/Cw$, where $C$ is an Artin-Schelter regular algebra of global dimension 3, and $w$ is a normal element of $C$.


K. Chan, E. Kirkman, C. Walton and J.J. Zhang, McKay Correspondence for semisimple Hopf actions on regular graded algebras I, J. Algebra, 2018
Let $S$ be a Koszul Artin-Schelter regular algebra of global dimension $d$.

Let $z \in S_2$ be a central regular element of $S$. 
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The following facts are well-known:

1. $A := S/Sz$ is a Koszul algebra;
2. $A$ is an Artin-Schelter Gorenstein algebra of injective dimension $d - 1$. 
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$mcm A = \text{the category of (finitely generated) maximal Cohen-Macaulay modules over } A$

$mcm A = \text{the stable category}$

$mcm A$ is a triangulated category.
Smith-van den Bergh constructed a finite dimensional algebra $C(A)$,
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**Theorem**

- there is an equivalence of triangulated categories
  
  $$\text{mcm}A \cong D^b(\text{mod } C(A)).$$

- If $C(A)$ is semisimple, then $A$ is an **isolated singularity** (i.e. $\text{qgr } A$ has finite global dimension).
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**Theorem**

- there is an equivalence of triangulated categories

$$\text{mcm} A \cong D^b(\text{mod } \mathcal{C}(A)).$$

- If $\mathcal{C}(A)$ is semisimple, then $A$ is an isolated singularity (i.e. $\text{qgr } A$ has finite global dimension).

**Remark.** The finite dimensional algebra $\mathcal{C}(A)$ is an important tool to understand the singularities of $A$. 
Let $E = S^!$ be the quadratic dual of the Koszul Artin-Schelter regular algebra $S$.

Then $E$ is a **Koszul Frobenius** algebra.


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Write $E = T(V)/(R)$, where $R \subseteq V \otimes V$. A linear map $\theta : R \to \mathbb{k}$ is called a **Clifford map** if

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(\theta \otimes 1 - 1 \otimes \theta)(V \otimes R \cap R \otimes V) = 0.
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- Let $E(\theta) = T(V)/(r - \theta(r) : r \in R)$. 

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Let $E(\theta) = T(V)/(r - \theta(r) : r \in R)$.

We call $E(\theta)$ a **Clifford deformation** of $E$. 
Proposition

- Each central element $0 \neq z \in S_2$ corresponding to a Clifford map $\theta_z$ of $E = S^!$.
- $E(\theta_z)$ is a strongly $\mathbb{Z}_2$-graded algebra.
- $C(A) \cong E(\theta_z)_0$. 

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- $E(\theta_z)$ is a strongly $\mathbb{Z}_2$-graded algebra.
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Theorem

Let $S$ be a Koszul Artin-Schelter regular algebra, and let $z \in S_2$ be a central regular element.

Then $A = S/Sw$ is an isolated singularity if and only if $C(A) = E(\theta_z)_0$ is a semisimple algebra.
**Example.** Let $S = \mathbb{k}\langle x, y, z \rangle/(f_1, f_2, f_3)$, where

$$f_1 = zx + xz, \quad f_2 = yz + zy, \quad f_3 = x^2 + y^2.$$
Example. Let $S = \mathbb{k} \langle x, y, z \rangle / (f_1, f_2, f_3)$, where

$$f_1 = zx + xz, f_2 = yz + zy, f_3 = x^2 + y^2.$$ 

$S$ is Koszul Artin-Schelter regular algebra of global dimension 3.
Example. Let $S = \mathbb{k}\langle x, y, z \rangle/(f_1, f_2, f_3)$, where

$$f_1 = zx + xz, f_2 = yz + zy, f_3 = x^2 + y^2.$$ 

$S$ is Koszul Artin-Schelter regular algebra of global dimension 3.

All the possible noncommutative quadric hypersurfaces defined by a central element $w \in S_2$ of $S$:

<table>
<thead>
<tr>
<th>$w$</th>
<th>$E(\theta)_0$</th>
<th>singularities of $S/wS$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z^2 + xy + yx + \lambda x^2, \lambda \neq \pm 2\sqrt{-1}$</td>
<td>$\mathbb{k}^{\oplus 4}$</td>
<td>isolated</td>
</tr>
<tr>
<td>$z^2 + xy + yx \pm 2\sqrt{-1}x^2$</td>
<td>$\mathbb{k}[u]/(u^2) \times \mathbb{k}[u]/(u^2)$</td>
<td>nonisolated</td>
</tr>
<tr>
<td>$z^2$</td>
<td>$\mathbb{k}[u, v]/(u^2 - v^2, uv)$</td>
<td>nonisolated</td>
</tr>
<tr>
<td>$z^2 + x^2$</td>
<td>$\mathbb{k}^{\oplus 4}$</td>
<td>isolated</td>
</tr>
<tr>
<td>$xy + yx + \lambda x^2, \lambda \neq \pm 2\sqrt{-1}$</td>
<td>$\mathbb{k}[u]/(u^2) \times \mathbb{k}[u]/(u^2)$</td>
<td>nonisolated</td>
</tr>
<tr>
<td>$xy + yx \pm 2\sqrt{-1}x^2$</td>
<td>$\mathbb{k}[u, v]/(u^2, v^2)$</td>
<td>nonisolated</td>
</tr>
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<td>$x^2$</td>
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<td>nonisolated</td>
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</tbody>
</table>
Another application:

Clifford deformations provide a new explanation of Knörrer Periodicity Theorem for noncommutative quadric hypersurfaces.
Noncommutative quadric hypersurfaces

- Another application:

  Clifford deformations provide a new explanation of Knörrer Periodicity Theorem for noncommutative quadric hypersurfaces.

- Let $S$ be a Koszul Artin-Schelter regular algebra. Set $A^\# = S[v]/(z + v^2)$ and $A^{##} = S[v_1, v_2]/(z + v_1^2 + v_2^2)$.

**Theorem**

*Assume that $\text{gldim } S \geq 2$. Then*

- *$A$ is a noncommutative isolated singularity if and only if so is $A^\#$.*

- *there is an equivalence of triangulated categories* 
  
  $\text{mcm} A \cong \text{mcm} A^{##}$.

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**References**


I. Mori, K. Ueyama, Noncommutative Knörrer’s Periodicity Theorem and noncommutative quadric hypersurfaces, arxiv:1905.12266

Thank you for your attention!