Ladders of abelian recollements

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Layout

1. **Ladders of abelian recollements**

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2. **Grid categories**

Motivation

- **Question:** The existence of an abelian recollement often is easy to establish by Buchweitz ([B, 03]), but without further assumptions or information, it does not provide a strong tool.

$$\begin{tikzcd}
\mathrm{Mod}\text{-}B/BeB \arrow[r, "\mathrm{inc}"] & \mathrm{Mod}\text{-}B \arrow[l, bend right, "B/BeB\otimes_B -"'] \arrow[l, bend left, "\mathrm{Hom}_B(B/BeB,-)"] \arrow[r, "e(-)"] & \mathrm{Mod}\text{-}eBe \arrow[l, bend right, "Be\otimes_{eBe}-"'] \arrow[l, bend left, "\mathrm{Hom}_{eBe}(eB,-)"]
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- **Idea:** Systematically enhance the definition of abelian recollements by additional data called ladders, which are sequences of adjoint functors.

- **Trouble:** In contrast to the triangulated situation, this has to be done in an asymmetric way.

Reasons

- Franjou-Pirashvili [FP, 04]: symmetric ladders only can exist for comma categories, which in the case of module categories means that the ring in the middle of the recollement has to be triangular.

$$
\begin{tikzcd}
 & \vdots & & \vdots & \\
\mathcal{A} \arrow[rr, "i"] \arrow[rr, bend left=40, "q^1"] \arrow[rr, bend right=40, "p^1"'] & & \mathcal{B} \arrow[ll, bend right=20, "q"'] \arrow[ll, bend right=60, "q^2"'] \arrow[ll, bend left=20, "p"] \arrow[ll, bend left=60, "p^2"] \arrow[rr, "e"] \arrow[rr, bend left=40, "l^1"] \arrow[rr, bend right=40, "r^1"'] & & \mathcal{C} \arrow[ll, bend right=20, "l^0"'] \arrow[ll, bend right=60, "l^2"'] \arrow[ll, bend left=20, "r^0"] \arrow[ll, bend left=60, "r^2"] \\
 & \vdots & & \vdots &
\end{tikzcd}
$$

- Feng-Zhang [FZ, 18]: when non-trivial symmetric ladders exist, there are only three cases: upwards extension by one step or downwards extension by one step or ladders that are infinite both upwards and downwards.

- ([FZ, 18]): let R be a ring and $\Lambda = \begin{pmatrix} R & 0 \\ R & R \end{pmatrix}$. Then graphically

we have
$$\begin{tikzcd}
\mathrm{Mod}\text{-}R \arrow[r, "\mathsf{i}"] \arrow[r, bend right=40, "\mathsf{p}^1"'] \arrow[r, bend right=80, "\mathsf{p}^3"'] & \mathrm{Mod}\text{-}\Lambda \arrow[l, bend right=30, "\mathsf{q}"'] \arrow[l, bend left=30, "\mathsf{p}"] \arrow[l, bend left=60, "\mathsf{p}^2"] \arrow[l, bend left=90, "\mathsf{p}^4"] \arrow[r, "\mathsf{e}"] \arrow[r, bend left=40, "\mathsf{l}^1"] & \mathrm{Mod}\text{-}R \arrow[l, bend right=30, "\mathsf{l}^0"'] \arrow[l, bend right=70, "\mathsf{l}^2"'] \arrow[l, bend left=30, "\mathsf{r}^0"] \\
& & \mathrm{Mod}\text{-}\Lambda \arrow[from=1-2, to=1-3, bend right=40, "\mathsf{r}^1"', phantom]
\end{tikzcd}$$

Using symmetric ladders severely restricts the scope and range of the theory by limiting it to comma categories or triangular matrix rings, and in addition by allowing for only three kinds of non-trivial ladders, which is much less flexibility than we need for homological characterisations such as in the main results of this talk.

Recollements of abelian categories

A **recollement**([BBD, 82]) between abelian categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ is a diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\mathsf{i}"] & \mathcal{B} \arrow[l, bend right, "\mathsf{q}"'] \arrow[l, bend left, "\mathsf{p}"] \arrow[r, "\mathsf{e}"] & \mathcal{C} \arrow[l, bend right, "\mathsf{l}"'] \arrow[l, bend left, "\mathsf{r}"] \end{tikzcd}$$

denoted by $(\mathcal{A}, \mathcal{B}, \mathcal{C})$, satisfying the following conditions:

1. (q, i, p) and (l, e, r) are adjoint triples.
2. The functors i, l, and r are fully faithful.
3. $\mathrm{Im}\,\mathsf{i} = \mathrm{Ker}\,\mathsf{e}$.

Ladders of abelian recollements

Definition 1([GKP1]) Let $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ be a recollement of abelian categories. Set $\mathsf{l}^0 := \mathsf{l}$ and $\mathsf{r}^0 := \mathsf{r}$. A **ladder** of $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is a finite or infinite diagram of additive functors

$$\vdots$$

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "\mathsf{i}"] && \mathcal{B} \arrow[ll, bend right=30, "\mathsf{q}"'] \arrow[ll, bend left=30, "\mathsf{p}"] \arrow[rr, "\mathsf{e}"] \arrow[rr, bend left=30, "\mathsf{l}^1"] \arrow[rr, bend right=30, "\mathsf{r}^1"'] && \mathcal{C} \arrow[ll, bend right=50, "\mathsf{l}^2"'] \arrow[ll, bend right=15, "\mathsf{l}^0"'] \arrow[ll, bend left=15, "\mathsf{r}^0"] \arrow[ll, bend left=50, "\mathsf{r}^2"]
\end{tikzcd}$$

$$\vdots$$

such that $(\mathsf{l}^{i+1},\ \mathsf{l}^{i})$ and $(\mathsf{r}^{i},\ \mathsf{r}^{i+1})$ are adjoint pairs for all $i \geq 0$.

Definition 2([GKP1])

The l-**height** of a ladder of $(\mathcal{A},\ \mathcal{B},\ \mathcal{C})$ is some positive integer n, if there is a tuple $(\mathsf{l}^{n-1},\ \ldots, \mathsf{l}^{2},\ \mathsf{l}^{1},\ \mathsf{l}^{0})$ of consecutive left adjoints.

The r-**height** of a ladder of $(\mathcal{A},\ \mathcal{B},\ \mathcal{C})$ is defined similarly.

The **height** of a ladder of $(\mathcal{A},\ \mathcal{B},\ \mathcal{C})$ is the sum of the l-height and the r-height.

The given recollement $(\mathcal{A},\ \mathcal{B},\ \mathcal{C})$ is considered to be a ladder of height one.

[GKP1] N. Gao, S. Koenig, C. Psaroudakis, *Ladders of recollements of abelian Categories*, in preparation, 2019.

One example

Let A be a k-algebra over a commutative ring k, $I_1,\ I_2, \cdots, I_{n-1}$ two-sided ideals of A such that $I_{n-1} \subseteq I_i$ for any $1 \leq i \leq n-2$.

Consider the following matrix rings

$$\Lambda = \begin{pmatrix} A & I_1 & I_2 & \cdots & I_{n-2} & I_{n-1} \\ A & A & I_2 & \cdots & I_{n-2} & I_{n-1} \\ A & A & A & \cdots & I_{n-2} & I_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A & A & A & \cdots & A & I_{n-1} \\ A & A & A & \cdots & A & A \end{pmatrix}$$

and

$$\Gamma = \begin{pmatrix} A/I_{n-1} & I_1/I_{n-1} & I_2/I_{n-1} & \cdots & I_{n-2}/I_{n-1} \\ A/I_{n-1} & A/I_{n-1} & I_2/I_{n-1} & \cdots & I_{n-2}/I_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A/I_{n-1} & A/I_{n-1} & A/I_{n-1} & \cdots & A/I_{n-1} \end{pmatrix}$$

Take the idempotent elements $e = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$ and $f = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$. Then the following recollement of module categories $(\mathrm{Mod}\text{-}\Lambda/\Lambda e\Lambda,\ \mathrm{Mod}\text{-}\Lambda,\ \mathrm{Mod}\text{-}e\Lambda e)$ has r-height at least three

$$
\begin{tikzcd}
\mathrm{Mod}\text{-}\Gamma \arrow[rr] & & \mathrm{Mod}\text{-}\Lambda \arrow[ll, bend right] \arrow[ll, bend left] \arrow[rr, "e"] & & \mathrm{Mod}\text{-}A \arrow[ll, bend right, "\mathsf{l}^0"'] \arrow[ll, bend left, "\mathsf{r}^0"] \arrow[ll, bend left=50, "\mathsf{r}^1"] \arrow[ll, bend left=70, "\mathsf{r}^2"]
\end{tikzcd}
$$

where

$$
\begin{cases}
\mathsf{l}^0 = \Lambda e \otimes_A - \\
\mathsf{e} = e\Lambda \otimes_\Lambda - \\
\mathsf{r}^0 = \Lambda f \otimes_A - \\
\mathsf{r}^1 = \mathrm{Hom}_\Lambda(\Lambda f, -) \cong f\Lambda \otimes_\Lambda - \cong f(-) \\
\mathsf{r}^2 = \mathrm{Hom}_A(f\Lambda, -)
\end{cases}
$$

Characterising the height of a ladder

Theorem 3([GKP1]) Let Λ be an artin algebra, e an idempotent element and $\Gamma := e\Lambda e$. Consider the recollement of $\mathrm{Mod}\text{-}\Lambda$ induced by the idempotent element e as follows:

$$\begin{tikzcd}[column sep=huge] \mathrm{Mod}\text{-}\Lambda/\Lambda e\Lambda \arrow[r] & \mathrm{Mod}\text{-}\Lambda \arrow[l, bend right, "\Lambda/\Lambda e\Lambda\otimes_\Lambda -"'] \arrow[l, bend left, "\mathrm{Hom}_\Lambda(\Lambda/\Lambda e\Lambda, -)"] \arrow[r, "e(-)"] & \mathrm{Mod}\text{-}\Gamma \arrow[l, bend right, "\Lambda e\otimes_\Gamma -"'] \arrow[l, bend left, "\mathrm{Hom}_\Gamma(e\Lambda, -)"] \end{tikzcd}$$

For $n \geq 0$, define a sequence of Λ-Γ (or Γ-Λ)-bimodules by $M_0 := e\Lambda$, $M_1 := \mathrm{Hom}_\Gamma(M_0, \Gamma)$, $M_2 := \mathrm{Hom}_\Lambda(M_1, \Lambda)$, ..., $M_{2n+1} := \mathrm{Hom}_\Gamma(M_{2n}, \Gamma)$, $M_{2n+2} := \mathrm{Hom}_\Lambda(M_{2n+1}, \Lambda)$, ... Then

1. The recollement admits a ladder of r-height exactly $2n+2$ if and only if M_j is projective as a left Γ-module for all even $j \leq 2n$, M_j is projective as a left Λ-module for all odd $j < 2n+1$ and M_{2n+1} is not projective as a left Λ-module.

2. The recollement admits a ladder of r-height exactly $2n+3$ if and only if M_j is projective as a left Γ-module for all even $j \leq 2n$, M_{2n+2} is not projective as a left Γ-module and M_j is projective as a left Λ-module for all odd $j \leq 2n+1$.

Problem 1

Given a recollement of abelian categories enhanced by a left or right ladder of a certain length, is it possible to produce a recollement of triangulated categories, involving derived or singularity categories?

For derived categories and singularity categories, we denote by D and D_{sg}, respectively.

Deriving recollements with ladders

Theorem 4([GKP1]) Let $(\mathcal{A},\ \mathcal{B},\ \mathcal{C})$ be a recollement of abelian categories. Then the following hold.

- Assume that $(\mathcal{A},\ \mathcal{B},\ \mathcal{C})$ has a ladder of l-height three. Then there exists a triangle equivalence

$$\mathsf{D}_{\mathsf{sg}}(\mathcal{B})/\mathrm{Ker}\mathsf{l}^1 \xrightarrow{\simeq} \mathsf{D}_{\mathsf{sg}}(\mathcal{C})$$

and a recollement of triangulated categories

$$\begin{tikzcd} \mathsf{D}(\mathcal{C}) \arrow[r, "\mathsf{l}^0"] & \mathsf{D}(\mathcal{B}) \arrow[l, bend right, "\mathsf{l}^1"'] \arrow[l, bend left, "\mathsf{e}"] \arrow[r] & \mathsf{D}_{\mathcal{A}}(\mathcal{B}) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

which restricts to the bounded derived categories.

- Assume that $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ has a ladder of l-height three and r-height two. Then there exists a recollement of triangulated categories

$$\begin{tikzcd}
\mathsf{D}_{\mathsf{sg}}(\mathcal{C}) \arrow[r, "\mathsf{l}^0"] & \mathsf{D}_{\mathsf{sg}}(\mathcal{B}) \arrow[l, bend right, "\mathsf{l}^1"'] \arrow[l, bend left, "\mathsf{e}"] \arrow[r] & \mathrm{Ker}\,\mathsf{l}^1 \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

- Assume that $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ has a ladder of l-height four. Then there exists a recollement of triangulated categories

$$\begin{tikzcd}
\mathrm{Ker}\,\mathsf{l}^1 \arrow[r] & \mathsf{D}_{\mathsf{sg}}(\mathcal{B}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "\mathsf{l}^1"] & \mathsf{D}_{\mathsf{sg}}(\mathcal{C}) \arrow[l, bend right, "\mathsf{l}^2"'] \arrow[l, bend left, "\mathsf{l}^0"]
\end{tikzcd}$$

Problem 2

Many abelian categories are derived equivalent, for instance by tilting, and thus occur as hearts of t-structures in the same triangulated category.

No tilting procedure is known that is compatible with abelian recollements.

Is it possible to use enhancements by ladders to produce "tilted" (with respect to torsion theories) recollements both on abelian and on derived level?

The problem uses the following concept of Happel-Reiten-Smalø *tilt* of an abelian category ([CFM, 16]):

Let $\mathcal{C}$ be an abelian category with a torsion pair $(\mathcal{T},\ \mathcal{F})$. Set

$$\mathcal{H}_{\mathcal{C}} := \{C^\bullet \in D(\mathcal{C}) | \mathrm{H}^0(C^\bullet) \in \mathcal{T},\ \ \mathrm{H}^{-1}(C^\bullet) \in \mathcal{F},$$

$$\mathrm{H}^i(C^\bullet) = 0,\ \forall i > 0,\ \ \mathrm{H}^i(C^\bullet) = 0,\ \forall i < -1\}.$$

Then $\mathcal{H}_{\mathcal{C}}$ is called Happel-Reiten-Smalø **tilt** (or just tilt for short) of $\mathcal{C}$ by $(\mathcal{T},\ \mathcal{F})$.

Torsion pairs arising from ladders

Theorem 5([GKP1]) Let $(\mathcal{A},\ \mathcal{B},\ \mathcal{C})$ be a recollement of abelian categories which admits a ladder of l-height three. Assume that $(\mathcal{T},\ \mathcal{F})$ is a torsion pair on $\mathcal{B}$ such that $\mathsf{l}^0 \circ \mathsf{l}^1(\mathcal{F}) \subseteq \mathcal{F}$ and $\mathsf{l}^2 \circ \mathsf{l}^1(\mathcal{T}) \subseteq \mathcal{T}$. Then the following hold.

- $(\mathsf{l}^1(\mathcal{T}),\ \mathsf{l}^1(\mathcal{F}))$ is a torsion pair on $\mathcal{C}$.

We denote by $\mathcal{H}_{\mathcal{C}}$ the tilt of $\mathcal{C}$ by $(\mathsf{l}^1(\mathcal{T}),\ \mathsf{l}^1(\mathcal{F}))$.

- There exists a recollement of abelian categories

$$\begin{tikzcd}
\mathrm{Ker}(\mathsf{l}^1_{\mathcal{H}}) \arrow[r] & \mathcal{H}_{\mathcal{B}} \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "\mathsf{l}^1_{\mathcal{H}}"] & \mathcal{H}_{\mathcal{C}} \arrow[l, bend right, "\mathsf{l}^2_{\mathcal{H}}"'] \arrow[l, bend left, "\mathsf{l}^0_{\mathcal{H}}"]
\end{tikzcd}$$

- There exists a recollement of triangulated categories

$$\begin{tikzcd}
\mathrm{Ker}(\mathsf{l}^1_{\mathcal{H}}) \arrow[r] & D(\mathcal{H}_{\mathcal{B}}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "\mathsf{l}^1_{\mathcal{H}}"] & D(\mathcal{H}_{\mathcal{C}}) \arrow[l, bend right, "\mathbb{L}\mathsf{l}^2_{\mathcal{H}}"'] \arrow[l, bend left, "\mathsf{l}^0_{\mathcal{H}}"]
\end{tikzcd}$$

Problem 3

An abelian recollement, for instance of module categories, does not provide connections between homological properties of the categories, or rings, involved nor between objects and their images under the six functors, in general.

Can the enhanced definition by ladders be used to obtain such connections, for instance in terms of Gorenstein homological algebra?

Let $\mathcal{A}$ be an abelian category with enough projective and injective objects.

♦ ([BR, 07]) An object $X \in \mathcal{A}$ is called **Gorenstein projective** if there exists an exact complex of projective objects

$$\mathrm{P}^\bullet\colon \cdots \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

with the property that $\mathrm{Hom}_{\mathcal{A}}(\mathrm{P}^\bullet, \ P)$ are exact for every projective object $P \in \mathcal{A}$, such that $X \cong \mathrm{Coker}(P^{-1} \longrightarrow P^0)$.

♦ **Gorenstein injective object** can be defined dually.

♦ $\mathrm{GProj}\mathcal{A}$ (resp. $\mathrm{GInj}\mathcal{A}$): the full subcategory of Gorenstein projective (resp. injective) objects of $\mathcal{A}$.

♦ $\mathcal{A}$ is called **n-Gorenstein**, if every object has a finite resolution by Gorenstein projective objects of length at most n.

To transfer Gorenstein properties

Theorem 6([GKP1]) Let $(\mathcal{A},\ \mathcal{B},\ \mathcal{C})$ be a recollement of abelian categories.

- Assume that $(\mathcal{A},\ \mathcal{B},\ \mathcal{C})$ has a ladder of l-height three. Then $\mathsf{l}^1\colon \mathcal{B} \longrightarrow \mathcal{C}$ preserves Gorenstein projectives and $\mathsf{e}\colon \mathcal{B} \longrightarrow \mathcal{C}$ preserves Gorenstein injectives. Furthermore, if $\mathcal{B}$ is n-Gorenstein for some non-negative integer n, then $\mathcal{C}$ is n-Gorenstein.

- Assume that $(\mathcal{A},\ \mathcal{B},\ \mathcal{C})$ admits a ladder of l-height four. Then the functor $\mathsf{l}\colon \mathcal{C} \longrightarrow \mathcal{B}$ preserves Gorenstein injective objects. Moreover, $\mathsf{e} \circ \mathsf{l} \cong \mathrm{id}_{\mathrm{GInj}\mathcal{C}}$.

Motivation

- **Question:** ([GL, 87]) Geigle-Lenzing introduced coherent sheaves on Geigle-Lenzing (GL) weighted projective lines $\mathbb{P}^d$, and showed that these categories are equivalent to module categories $\mathrm{mod}\text{-}A$ of a certain order A on $\mathbb{P}^d$, which we call a GL order.

  ([LO, 17]) To study the module category $\mathrm{mod}\text{-}A$ of the GL order A, Lerner-Oppermann introduced the grid category and used it to give a description of $\mathrm{mod}\text{-}A$, and moreover, they studied the grid category using recollements, which we call LO recollement.

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- **Idea:**

  1. On grid categories-potential structure: when is abelian, when is module category and over which ring, what is dual grids.
  2. Can one enhance LO recollements by ladders?

Defining grid category

- $\mathcal{A}$: an abelian category
- $F\colon \mathcal{A} \longrightarrow \mathcal{A}$: an additive functor
- $\eta\colon F \longrightarrow \mathrm{id}_{\mathcal{A}}$: a natural transformation
- $n \geq 2$: a positive integer

([LO, 17]) Lerner-Oppermann introduced the **grid category** $\mathcal{A}[\sqrt[n]{\eta}]$:

♦ The objects (A, f) are sequences of the form:

$$\begin{tikzcd} F(A_n) \arrow[r, "f_1"] \arrow[rrrr, bend right, "\eta_{A_n}"'] & A_1 \arrow[r, "f_2"] & A_2 \arrow[r, "f_3"] & \cdots \arrow[r, "f_n"] & A_n \end{tikzcd}$$

where A_i lies in $\mathcal{A}$, such that $\eta_{A_n} = f_n f_{n-1} \cdots f_1$ and for all $1 \leq i \leq n-1$,

$$\eta_{A_{n-i}} = f_{n-i} \cdots f_1 F(f_n) \cdots F(f_{n-i+1}).$$

♦ Given two objects (A, f) and (A', f') in $\mathcal{A}[\sqrt[n]{\eta}]$, a morphism between them is a commutative diagram:

$$\begin{tikzcd}
F(A_n) \arrow[r, "f_1"] \arrow[d, "F(a_n)"] & A_1 \arrow[r, "f_2"] \arrow[d, "a_1"] & A_2 \arrow[r, "f_3"] \arrow[d, "a_2"] & \cdots \arrow[r, "f_n"] & A_n \arrow[d, "a_n"] \\
F(A'_n) \arrow[r, "f'_1"] & A'_1 \arrow[r, "f'_2"] & A'_2 \arrow[r, "f'_3"] & \cdots \arrow[r, "f'_n"] & A'_n
\end{tikzcd}$$

that is,

$$f'_1 F(a_n) = a_1 f_1$$

and for all $1 \leq i \leq n-1$,

$$f'_{i+1} a_i = a_{i+1} f_{i+1},$$

where $a_i \colon A_i \longrightarrow A'_i$ are morphisms in $\mathcal{A}$ for all $1 \leq i \leq n$.

• $\mathcal{A}_\eta$: the full subcategory of $\mathcal{A}$ consisting of all objects A such that $\eta(A) = 0$

([LO, 17]) The category $\mathcal{A}_\eta[\sqrt[n-1]{0}]$ is defined as follows:

⧫ The objects $(0, A_1, \ldots, A_{n-1})$ are sequences of the form:

$$0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots \longrightarrow A_{n-1}$$

where A_i lies in $\mathcal{A}_\eta$ for all $1 \leq i \leq n-1$.

♦ Given two objects $(0, A_1, \ldots, A_{n-1})$ and $(0, A'_1, \ldots, A'_{n-1})$ in $\mathcal{A}_\eta[\sqrt[n-1]{0}]$, a morphism between them is a commutative diagram:

$$
\begin{tikzcd}
0 \arrow[r, "0"] \arrow[d, "0"] & A_1 \arrow[r, "f_1"] \arrow[d, "a_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "a_2"] & \cdots \arrow[r, "f_{n-2}"] & A_{n-1} \arrow[d, "a_{n-1}"] \\
0 \arrow[r, "0"] & A'_1 \arrow[r, "f'_1"] & A'_2 \arrow[r, "f'_2"] & \cdots \arrow[r, "f'_{n-2}"] & A'_{m-1}
\end{tikzcd}
$$

that is, for all $1 \leq i \leq n-2$,

$$f'_i a_i = a_{i+1} f_i$$

where $a_i \colon A_i \longrightarrow A'_i$ are morphisms in $\mathcal{A}$ for all $1 \leq i \leq n-1$.

Linear grid category

Theorem 7([GKP2]) Let $\mathcal{A}$ be an abelian category. Then $\mathcal{A}[\sqrt[n]{\eta}]$ is an abelian category if and only if $F: \mathcal{A} \longrightarrow \mathcal{A}$ is a right exact functor. In particular, $\mathcal{A}_\eta[\sqrt[n-1]{0}]$ is abelian.

Theorem 8([GKP2]) Let $\mathcal{A}$ be an abelian category with small coproducts. Let $F : \mathcal{A} \longrightarrow \mathcal{A}$ be a right exact functor preserving small coproducts and $\eta : F \longrightarrow \mathrm{id}_{\mathcal{A}}$ a natural transformation. Assume that $\mathcal{A}$ has enough objects M such that η_M is monic. Then for any integer $n \geq 2$, $\mathcal{A}[\sqrt[n]{\eta}]$ is a module category if and only if $\mathcal{A}$ is a module category.

[GKP2] N. Gao, S. Koenig, C. Psaroudakis, *Grid categories*, in preparation, 2019.

Moreover, if $\mathcal{A} \simeq \mathrm{Mod}\text{-}A$ for some ring A, then $\mathcal{A}[\sqrt[n]{\eta}] \simeq \mathrm{Mod}\text{-}\Gamma$, where Γ is the $n \times n$ Morita matrix ring as follows

$$\Gamma_{(\psi_1,\cdots,\psi_n,\varphi_1,\cdots,\varphi_n)} = \begin{pmatrix} A & N & N & \cdots & N & N \\ A & A & N & \cdots & N & N \\ A & A & A & \cdots & N & N \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A & A & A & \cdots & A & N \\ A & A & A & \cdots & A & A \end{pmatrix}$$

in which N is an A-bimodule such that $F \cong N \otimes_A -$ naturally, and the structure map $\psi_i = \eta_A$ and φ_i is the composition

$$A \otimes_A N \xrightarrow{\cong} A \otimes_A N \otimes_A A \xrightarrow{\mathrm{id}_A \otimes \eta_A} A \otimes_A A \xrightarrow{\cong} A.$$

We say that $\mathcal{A}$ has enough objects M, if there exists an object $M \in \mathcal{A}$ and an epimorphism $M \longrightarrow X$, for any object $X \in \mathcal{A}$.

Examples

An exact functor $\mathsf{i}\colon \mathcal{A} \longrightarrow \mathcal{B}$ between abelian categories is called a **homological embedding**([P, 14]), if the map

$$\mathsf{i}^n_{X,Y}\colon \operatorname{Ext}^n_{\mathcal{A}}(X,Y) \longrightarrow \operatorname{Ext}^n_{\mathcal{B}}(\mathsf{i}(X),\mathsf{i}(Y))$$

is an isomorphism of abelian groups for all X, Y in $\mathcal{A}$ and for all $n \geq 0$.

Example 9([GKP2]) Let $(\mathcal{A},\mathcal{B},\mathcal{C})$ be a recollement of abelian categories with enough projective objects and small coproducts as follows:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\mathsf{i}"] & \mathcal{B} \arrow[l, bend right, "\mathsf{q}"'] \arrow[l, bend left, "\mathsf{p}"] \arrow[r, "\mathsf{e}"] & \mathcal{C} \arrow[l, bend right, "\mathsf{l}"'] \arrow[l, bend left, "\mathsf{r}"] \end{tikzcd}$$

Let i be a homological embedding, $F = \mathsf{le}$ and $\eta : \mathsf{le} \longrightarrow \mathrm{id}_{\mathcal{B}}$ the counit of the adjoint pair. Then $\mathcal{B}$, F and η satisfy conditions in Theorem 8.

Example 10([GKP2]) Let A be an algebra and I the two-sided ideal of A. Let

$$\Gamma = \begin{pmatrix} A & I & I & \cdots & I & I \\ A & A & I & \cdots & I & I \\ A & A & A & \cdots & I & I \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A & A & A & \cdots & A & I \\ A & A & A & \cdots & A & A \end{pmatrix}$$

be an $n \times n$ matrix algebra. Let $F = I \otimes_A - \colon \mathrm{Mod}\text{-}A \longrightarrow \mathrm{Mod}\text{-}A$ and $\eta\colon F \longrightarrow \mathrm{id}_{\mathrm{Mod}\text{-}A}$ induced by the natural inclusion $I \hookrightarrow A$. Then there is an equivalence of abelian categories

$$(\mathrm{Mod}\text{-}A)[\sqrt[n]{\eta}] \simeq \mathrm{Mod}\text{-}\Gamma$$

General grid category

Now let A be an algebra, $I \subseteq J$ the two-sided ideals of A.

Consider

- $F_1 = I \otimes_A -\colon \mathrm{Mod}\text{-}A \longrightarrow \mathrm{Mod}\text{-}A$: a right exact functor
- $\eta_1\colon F \longrightarrow \mathrm{id}_{\mathrm{Mod}\text{-}A}$: the natural transformation, induced by the natural inclusion $I \hookrightarrow A$
- $F_2 = J \otimes_A -\colon \mathrm{Mod}\text{-}A \longrightarrow \mathrm{Mod}\text{-}A$: a right exact functor
- $\eta_2\colon F_2 \longrightarrow \mathrm{id}_{\mathrm{Mod}\text{-}A}$: the natural transformation, induced by the natural inclusion $J \hookrightarrow A$

Proposition 11([GKP2]) Let

$$\Gamma = \begin{pmatrix} A & I & I & \cdots & I & I \\ A & A & I & \cdots & I & I \\ A & A & A & \cdots & I & I \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A & A & A & \cdots & A & I \\ A & A & A & \cdots & A & A \end{pmatrix}$$

be an $n_1 \times n_1$ matrix algebra, and

$$\Upsilon = \begin{pmatrix} J & I & I & \cdots & I & I \\ J & J & I & \cdots & I & I \\ J & J & J & \cdots & I & I \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ J & J & J & \cdots & J & I \\ J & J & J & \cdots & J & J \end{pmatrix}$$

the two-sided ideal of Γ. Let

$$\Sigma = \begin{pmatrix} \Gamma & \Upsilon & \Upsilon & \cdots & \Upsilon & \Upsilon \\ \Gamma & \Gamma & \Upsilon & \cdots & \Upsilon & \Upsilon \\ \Gamma & \Gamma & \Gamma & \cdots & \Upsilon & \Upsilon \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \Gamma & \Gamma & \Gamma & \cdots & \Gamma & \Upsilon \\ \Gamma & \Gamma & \Gamma & \cdots & \Gamma & \Gamma \end{pmatrix}$$

be an $n_1 n_2 \times n_1 n_2$ partitioned matrix algebra. Then there are equivalences of abelian categories

$$(\mathrm{Mod}\text{-}A)[\sqrt[n_1]{\eta_1}, \sqrt[n_2]{\eta_2}] \simeq \mathrm{Mod}\text{-}\Sigma$$

Crucial point: $(\mathrm{Mod}\text{-}A)[\sqrt[n_1]{\eta_1}, \sqrt[n_2]{\eta_2}] = (\mathrm{Mod}\text{-}A)[\sqrt[n_1]{\eta_1}][\sqrt[n_2]{\eta_2}]$

Cogrid category

- $\mathcal{C}$: an abelian category
- $S\colon \mathcal{C} \longrightarrow \mathcal{C}$: an additive functor
- $\eta\colon \mathrm{id}_{\mathcal{C}} \longrightarrow S$: a natural transformation
- $n \geq 2$: a positive integer

The **cogrid category** $\mathcal{C}[\sqrt[n]{\eta}]$:

♦ The objects (X, f) are sequences of the form:

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[rrrrr, bend right, "\eta_{X_1}"'] & X_2 \arrow[r, "f_2"] & X_3 \arrow[r, "f_3"] & \cdots \arrow[r] & X_n \arrow[r, "f_n"] & S(X_1) \end{tikzcd}$$

where X_i lies in $\mathcal{C}$, such that

$$\eta_{X_1} = f_n f_{n-1} \cdots f_1$$

and for all $2 \leq i \leq n$,

$$\eta_{X_i} = S(f_{i-1}) \cdots S(f_1) f_n \cdots f_i$$

♦ Given two objects (X, f) and (X', f') in $\mathcal{C}[\sqrt[n]{\eta}]$, a morphism between them is a commutative diagram:

$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d, "a_1"] \arrow[rrrrr, bend right, "\eta_{X_1}"] & X_2 \arrow[r, "f_2"] \arrow[d, "a_2"] & X_3 \arrow[r, "f_3"] \arrow[d, "a_3"] & \cdots \arrow[r] & X_n \arrow[r, "f_n"] \arrow[d, "a_n"] & S(X_1) \arrow[d, "S(a_1)"] \\
X_1' \arrow[r, "f_1'"] \arrow[rrrrr, bend right, "\eta_{X_1'}"] & X_2' \arrow[r, "f_2'"] & X_3' \arrow[r, "f_3'"] & \cdots \arrow[r] & X_n' \arrow[r, "f_n'"] & S(X_1')
\end{tikzcd}$$

that is,

$$f_n' a_n = S(a_1) f_n$$

and for all $1 \leq i \leq n-1$,

$$f_i' a_i = a_{i+1} f_i$$

where $a_i \colon X_i \longrightarrow X_i'$ are morphisms in $\mathcal{C}$ for all $1 \leq i \leq n$.

- $\mathcal{C}_\eta$: the full subcategory of $\mathcal{C}$ consisting of all objects X such that $\eta(X) = 0$

The category $\mathcal{C}_\eta[\sqrt[n-1]{0}]$ is defined as follows:

♦ The objects $(X_1, \dots, X_{n-1}, 0)$ are sequences of the form:

$$X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_{n-1} \longrightarrow 0$$

where X_i lies in $\mathcal{C}_\eta$ for all $1 \leq i \leq n-1$.

♦ Given two objects $(X_1, \ldots, X_{n-1}, 0)$ and $(X'_1, \ldots, X'_{n-1}, 0)$ in $\mathcal{C}_\eta[\sqrt[n-1]{0}]$, a morphism between them is a commutative diagram:

$$
\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d, "a_1"] & X_2 \arrow[r, "f_2"] \arrow[d, "a_2"] & \cdots \arrow[r, "f_{n-2}"] & X_{n-1} \arrow[r] \arrow[d, "a_{n-1}"] & 0 \arrow[d, "0"] \\
X'_1 \arrow[r, "f'_1"] & X'_2 \arrow[r, "f'_2"] & \cdots \arrow[r, "f'_{n-2}"] & X'_{n-1} \arrow[r] & 0
\end{tikzcd}
$$

that is, for all $1 \leq i \leq n-2$,

$$f'_i a_i = a_{i+1} f_i$$

where $a_i \colon X_i \longrightarrow X'_i$ are morphisms in $\mathcal{C}$ for all $1 \leq i \leq n-1$.

Linear cogrid category

Theorem 12([GKP2]) The following hold.

1. Let $\mathcal{C}$ be an abelian category. Then for any integer $n \geq 2$, $\mathcal{C}[\sqrt[n]{\eta}]$ is an abelian category if and only if $S\colon \mathcal{C} \longrightarrow \mathcal{C}$ is a left exact functor. In particular, $\mathcal{C}_\eta[\sqrt[n-1]{0}]$ is abelian.

2. Let $\mathcal{C}$ be an abelian category with small products. Assume that $\mathcal{C}$ has enough objects N such that η_N is an epimorphism. Then $\mathcal{C}[\sqrt[n]{\eta}]$ is a module category if and only if $\mathcal{C}$ is a module category.

Moreover, if $\mathcal{C} \simeq \mathrm{Mod}\text{-}C$ for some ring C, then $\mathcal{C}[\sqrt[n]{\eta}] \simeq \mathrm{Mod}\text{-}\Delta$, where Δ is the $n \times n$ Morita matrix ring

$$\Delta_{(\psi_1,\cdots,\psi_n,\varphi_1,\cdots,\varphi_n)} = \begin{pmatrix} C & C & C & \cdots & C & C \\ M & C & C & \cdots & C & C \\ M & M & C & \cdots & C & C \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ M & M & M & \cdots & C & C \\ M & M & M & \cdots & M & C \end{pmatrix}$$

in which M is a C-bimodule such that $S \cong \mathrm{Hom}_C(M, -)$ naturally, and the structure maps $\varphi_i = \mathrm{adj}(\lambda_C)$ and ψ_i is the composition

$$C \otimes_C M \otimes_C C \xrightarrow{\mathrm{id}_C \otimes \mathrm{adj}(\lambda_C)} C \otimes_C C \xrightarrow{\cong} C$$

where $\mathrm{adj}(\lambda_C)$ is the image of λ_C under the adjoint isomorphism

$$\mathrm{Hom}_C(C,\ \mathrm{Hom}_C(M,\ C)) \cong \mathrm{Hom}_C(M \otimes_C C,\ C)$$

Examples

- H: a finite dimensional k-algebra over a field k
- A: a left H-module algebra
- $A\sharp H$: the smash product algebra
- ${}_A(-) : \mathrm{Mod}\text{-}A\sharp H \longrightarrow \mathrm{Mod}\text{-}A$: the restriction functor
- $A\sharp H \otimes_A - : \mathrm{Mod}\text{-}A \longrightarrow \mathrm{Mod}\text{-}A\sharp H$: the scalar functor induced by the smash product

Example 13([GKP2]) Let $A\sharp H/A$ be a separable extension and $S = {}_A(A\sharp H \otimes_A -) : \mathrm{Mod}\text{-}A \longrightarrow \mathrm{Mod}\text{-}A$. Then

$$(\mathrm{Mod}\text{-}A)[\sqrt[n]{\eta}] \simeq \mathrm{Mod}\text{-}\Delta$$

where

$$\Delta = \begin{pmatrix} A & A & A & \cdots & A & A \\ A\sharp H & A & A & \cdots & A & A \\ A\sharp H & A\sharp H & A & \cdots & A & A \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A\sharp H & A\sharp H & A\sharp H & \cdots & A & A \\ A\sharp H & A\sharp H & A\sharp H & \cdots & A\sharp H & A \end{pmatrix}$$

Example 14([GKP2]) Let $A=\Sigma_{g\in G}A_g$ be a strongly graded k-algebra over a finite group G. Suppose that A/A_e is separable. Let $S={}_A(A\sharp(kG)^*\otimes_A -):\mathrm{Mod}\text{-}A\longrightarrow \mathrm{Mod}\text{-}A$, where $(kG)^*$ is the duality of the group algebra kG. Then there is a categorical equivalence

$$(\mathrm{Mod}\text{-}A)[\sqrt[n]{\eta}]\simeq \mathrm{Mod}\text{-}\Delta$$

where

$$\Delta=\begin{pmatrix} A & A & A & \cdots & A & A\\ A\sharp(kG)^* & A & A & \cdots & A & A\\ A\sharp(kG)^* & A\sharp(kG)^* & A & \cdots & A & A\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ A\sharp(kG)^* & A\sharp(kG)^* & A\sharp(kG)^* & \cdots & A & A\\ A\sharp(kG)^* & A\sharp(kG)^* & A\sharp(kG)^* & \cdots & A\sharp(kG)^* & A \end{pmatrix}$$

Enhancing grid categories by ladders

Theorem 15([GKP2]) Let $F\colon \mathcal{A} \longrightarrow \mathcal{A}$ be a right exact functor and $\mathcal{A}$ have enough objects M such that η_M is a monomorphism. Then the following statements hold.

1. There is a recollement of abelian categories admitting r-height two for any integer $n \geq 2$:

$$\begin{tikzcd}
\mathcal{A}_\eta[\sqrt[n-1]{0}] \arrow[r, "\mathsf{i}_n"] & \mathcal{A}[\sqrt[n]{\eta}] \arrow[l, bend right, "\mathsf{q}_n"'] \arrow[l, bend left, "\mathsf{p}_n"] \arrow[r, "\mathsf{e}_n"] \arrow[r, bend right=70, "\mathsf{r}'_n"] & \mathcal{A} \arrow[l, bend right, "\mathsf{l}_n"'] \arrow[l, bend left, "\mathsf{r}_n"]
\end{tikzcd}$$

2. The functor i_n is a homological embedding.

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Any questions?

Thank you for your attention!