On Enochs Conjecture

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Contents

Introduction

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Introduction

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- **Auslander and Smalø**, mainly concerned with the case of finitely generated modules over **finite dimensional algebras**, stressed the functorial viewpoint and coined the terminology of **contravariant and covariant finiteness**.

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- In this setting, all the existing envelopes and covers can be recovered by specializing the class of modules.

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If ϕ satisfies (a) and perhaps not (b), it is called a **$\mathcal{C}$-precover**.

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Every covering class of modules is closed under direct limits.

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Let $\mathcal{A}$ be the class of absolutely pure modules. If $\mathcal{A}$ is covering, then it is closed under direct limits.

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Thank you!

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