

# On Enochs Conjecture

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- **Auslander and Smalø**, mainly concerned with the case of finitely generated modules over **finite dimensional algebras**, stressed the functorial viewpoint and coined the terminology of **contravariant and covariant finiteness**.



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# Preliminaries

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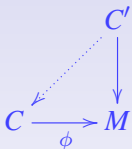
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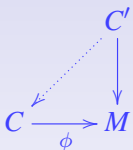


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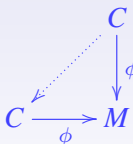
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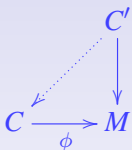


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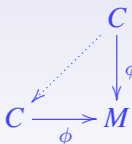
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If  $\phi$  satisfies (a) and perhaps not (b), it is called a  **$\mathcal{C}$ -precover**.



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# Enochs Conjecture

**Enochs Conjecture:**  
*Every covering class of modules is closed under direct limits.*



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


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*Thank you!*

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