

- $\mathbb{k} = \overline{\mathbb{k}}$ field
- mod. cat. = cat. of f.g. ^{left}mod., Comod. cat. = cat. of f.g. right comod.
- \mathbb{k} -cat. := add. \mathbb{k} -linear category.
meaning $A \cong A\text{-proj. } A : \text{f.d. } \mathbb{k}\text{-alg}$
- Short exact sequence of "functary" \mathbb{k} -cat's [Vasserot-Vergne (6)]

$$\Leftrightarrow \begin{cases} 0 \rightarrow \mathcal{B}_1 \xrightarrow{F} \mathcal{B}_2 \xrightarrow{G} \mathcal{B}_3 \rightarrow 0 \\ \cdot \mathcal{B}_i : \text{functary } \mathbb{k}\text{-cat's} \\ \cdot F \text{ fully faithful} \\ \cdot G \text{ full + dense} \\ \cdot \text{Ker}(G) = \text{Im}(F) := \left\{ f \in \mathcal{B}_1^{\text{Mor}} \mid \begin{array}{l} f \text{ factor through} \\ F(A), \text{ some } A \in \mathcal{A} \end{array} \right\} \end{cases}$$

e.g. $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3) = (\text{add}(Ae), \text{proj } A, \frac{(\text{proj } A)}{[Ae]} \cong \text{proj } A/\langle e \rangle)$.

- Recollement of abelian cat's (\cong ses of ab. cat.)

$$\mathcal{A}_1 \begin{array}{c} \xrightarrow{g} \\ \xleftarrow{i} \\ \xrightarrow{p} \end{array} \mathcal{A}_2 \begin{array}{c} \xleftarrow{l} \\ \xleftrightarrow{e} \\ \xleftarrow{r} \end{array} \mathcal{A}_3$$

- s.t. \mathcal{A}_i : abelian
- $(g, i, p), (l, e, r)$: adjoint triples
 - $\xrightarrow{i}, \xleftarrow{r}$ are fully faithful
 - $\text{Ker}(e) = \text{Im}(i)$

Fact (see, for example, [Psaroudakis - Vitoria])

If $A_e = \text{mod } A$, A : f.d. alg

then recollement is (equivalent to) :

$$\begin{array}{ccc} \text{mod } A_{/\mathbb{K}e} & \begin{matrix} \xrightarrow{\quad A_{/\mathbb{K}e} \otimes A \quad} \\ \xrightarrow{\quad \text{Res} \quad} \\ \xleftarrow{\quad \text{Hom}_A(A_{/\mathbb{K}e}, -) \quad} \end{matrix} & \text{mod } A \\ \left. \begin{array}{c} \xleftarrow{\quad \text{mod } eAe \quad} \\ \xrightarrow{\quad \text{Hom}(Ae, -) \quad} \\ \xleftarrow{\quad eAe \otimes - \quad} \\ \xrightarrow{\quad \text{Hom}_{eAe}(eA, -) \quad} \end{array} \right\} & \text{mod } eAe & \stackrel{=: \text{standard form}}{\longrightarrow} \end{array}$$

where $e = e^2 \in A$ idem. in A

Fact

$$\left\{ \begin{array}{l} \text{Recollements} \\ \text{+ mod. cat.} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{ses of} \\ \text{fin. add. 1k-cat.} \end{array} \right\}$$

$$(\text{mod } A_{/\mathbb{K}e} \xrightleftharpoons{\text{proj}} \text{mod } A \xrightleftharpoons{\text{proj}} \text{mod } eAe) \longleftrightarrow (\text{proj } eAe \xrightarrow{\text{proj } A} \text{proj } A_{/\mathbb{K}e})$$

We will (1) "dualise" : mod \rightsquigarrow comod
 $\text{proj} \rightsquigarrow \text{inj}$ (2) "Upgrade" (\simeq categorify)

Thm [C-Moreira]

\Leftrightarrow finitely many obj + Hom's are finitary cat's

\mathcal{C} : "nice" 2-category

= 2-functors
 $\mathcal{C} \rightarrow \{\text{finitary 1k-cat's}\}$

$$\left\{ \begin{array}{l} \text{recollement of comod. cat.} \\ \text{over } \mathcal{C}-\text{coalgebras} \end{array} \right\} \xrightarrow{1:1} \left\{ \begin{array}{l} \text{s.e.s. of} \\ \text{2-representations of } \mathcal{C} \end{array} \right\}$$

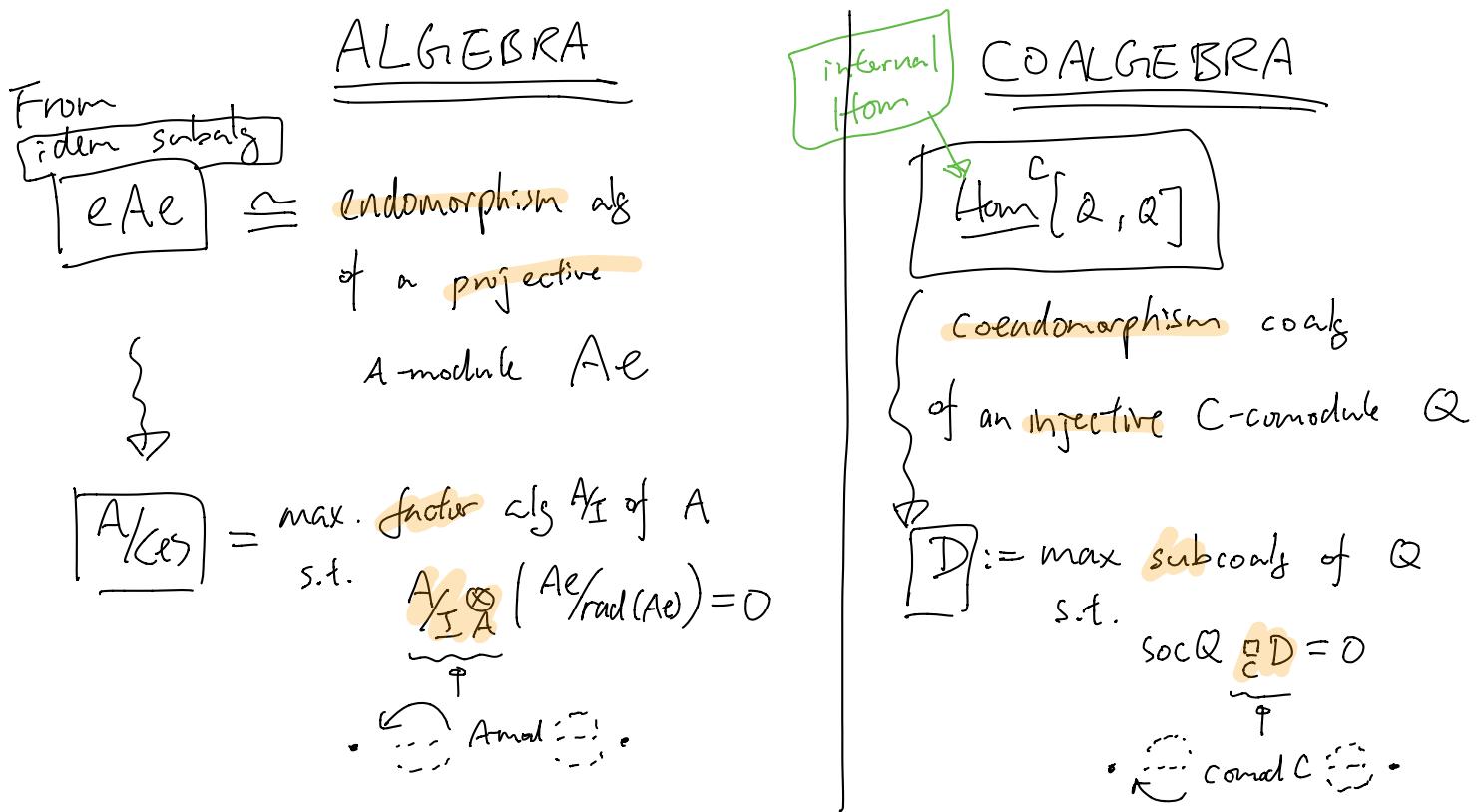
$$[\text{comod}_e(D) \underset{\mathcal{C}}{=} \text{comod}_e(C) \underset{\mathcal{C}}{=} \text{comod}(E)] \hookrightarrow [0 \rightarrow \text{inj } E \rightarrow \text{inj } C \rightarrow \text{inj } D]$$

(\mathcal{C} -coalgebra := coalgebra object in the categorification of the morphism of 1-cats)

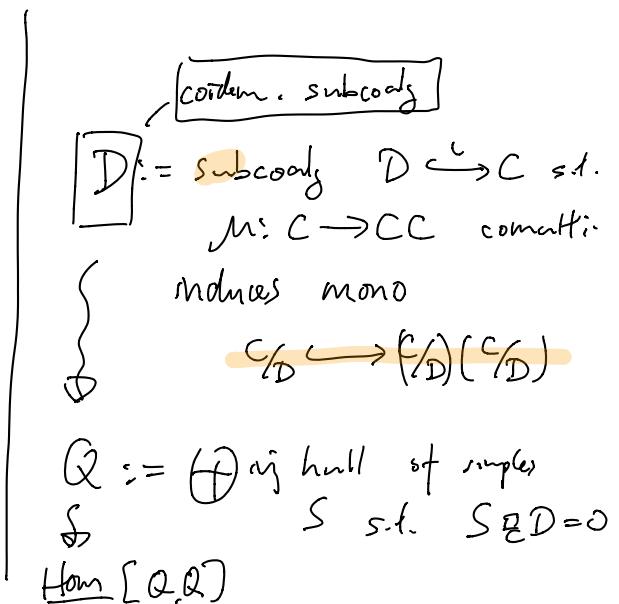
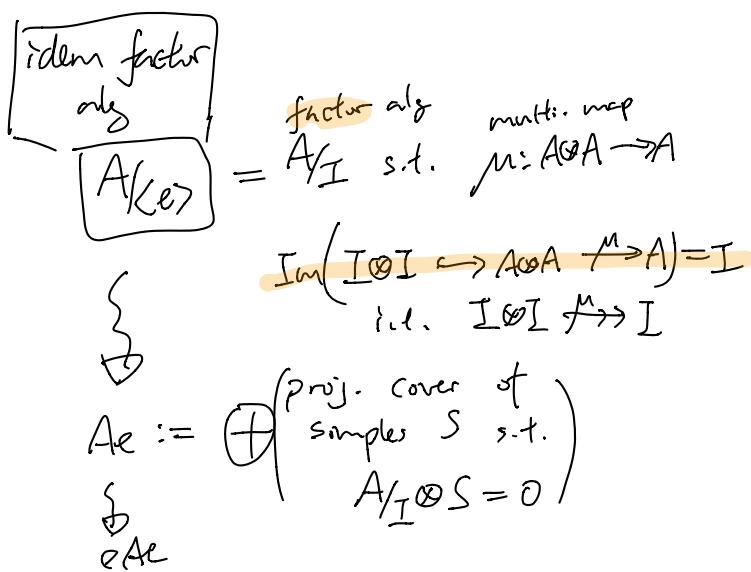
Note: Classical setting is when $\begin{cases} \mathcal{L} = \text{k-mod} \\ \text{horizontal compo.} = -\oplus_{\text{k}} - \\ \text{vertical compo.} = \text{compo. of linear maps} \end{cases}$

Intrinsic description of D & E relative to C

Note: Unlike \mathbb{K} -(co)algebra setting, there is no correspondence between idempotents and injective C -comonads !!.



We can also start from the other side:



Remark

(1) The functors are :

$$\begin{array}{ccc}
 \text{Comod}_{\leq D} & \xrightarrow{\text{Hom}^c[D, -]} & \text{Comod}_{\leq C} \\
 \downarrow - \otimes_D - & \xrightarrow{- \otimes_D C} & \downarrow - \otimes_C - \\
 & & \text{Comod}_{\leq \text{Hom}^c[C, -]}
 \end{array}$$

$$\begin{array}{ccc}
 & \xleftarrow{\text{Hom}^c[C, -]} & \\
 \text{Comod}_{\leq \text{Hom}^c[C, -]} & & \xleftarrow{\text{Hom}^c[C, Q], -} \\
 \downarrow - \otimes_Q - & & \downarrow - \otimes_{\text{Hom}^c[C, Q]} -
 \end{array}$$

(2) What is internal Hom?

\mathcal{C} : "nre" 2-cat, M : 2-representation over \mathcal{C}

$$i : F \rightarrow j$$

internal Hom

\rightsquigarrow \exists bifunctorial isom

$$\text{Hom}_{M(j)}(M, FN) \cong \text{Hom}_{\leq(i,j)}(\underline{\text{Hom}(N, M)}, F)$$



(3) Non-trivial e.g. of "nre" 2-cat?

• single obj case = tensor category (su.e.g., [EhNo])

e.g. cat. of Soergel bimodules (su.e.g., [Elias-Williamson])

• 2-cat. of (finite truncation of)

(upper/lower half of) categorified quantum gp

[Khovanov-Lauda; Rouquier]