- $\Bbbk = \overline{\Bbbk}$ field
- mod. cat. = cat. of f.g. left mod., Comod. cat. = cat. of f.cog. right comod.
- $\Bbbk$-cat. := add. $\Bbbk$-linear category.

- Short exact sequence of "finitary" $\Bbbk$-cat's [Vasserot–Varagnolo]

  meaning $\mathcal{A} \simeq A\text{-proj}$, A : f.d. $\Bbbk$-alg

  $:\Leftrightarrow$
  $$0 \to \mathcal{B}_1 \xrightarrow{F} \mathcal{B}_2 \xrightarrow{G} \mathcal{B}_3 \to 0$$
  - $\mathcal{B}_i$: finitary $\Bbbk$-cat's
  - F fully faithful
  - G full & dense
  - $\mathrm{Ker}(G) = \mathrm{Im}(F) := \left\{ f \in \mathrm{Mor}\,\mathcal{B} \;\middle|\; \begin{array}{l} f \text{ factor through} \\ F(A), \text{ some } A \in \mathcal{A} \end{array} \right\}$

  e.g. $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3) = \left(\mathrm{add}(Ae),\ \mathrm{proj}\,A,\ (\mathrm{proj}\,A)/[Ae] \simeq \mathrm{proj}\,A/\langle e\rangle\right)$.

- Recollement of abelian cat's ($\approx$ ses of ab. cat.)

$$\begin{tikzcd} \mathcal{A}_1 \arrow[r, "i"] & \mathcal{A}_2 \arrow[l, bend right, "q"'] \arrow[l, bend left, "p"] \arrow[r, "e"] & \mathcal{A}_3 \arrow[l, bend right, "\ell"'] \arrow[l, bend left, "r"] \end{tikzcd}$$

  s.t.
  - $\mathcal{A}_i$: abelian
  - (q, i, p), (ℓ, e, r) : adjoint triples
  - $\xrightarrow{i}$, $\xleftarrow{\ell}$, $\xleftarrow{r}$ are fully faithful
  - $\mathrm{Ker}(e) = \mathrm{Im}(i)$

Fact (see, for example, [Psaroudakis - Vitoria])

If $\mathcal{A}_2 = \operatorname{mod} A$, A: f.d. alg

then recollement is (equivalent to):

$$\left[\begin{tikzcd} \operatorname{mod} A/\langle e\rangle \arrow[r, "\mathrm{Res}"] & \operatorname{mod} A \arrow[l, bend right, "A/\langle e\rangle \otimes_A -"'] \arrow[l, bend left, "\operatorname{Hom}_A(A/\langle e\rangle, -)"] \arrow[r, "\operatorname{Hom}_A(Ae,-)", "\wr"'] & \operatorname{mod} eAe \arrow[l, bend right, "Ae \otimes_{eAe} -"'] \arrow[l, bend left, "\operatorname{Hom}_{eAe}(eA,-)"] \end{tikzcd}\right] =: \text{standard form}$$

where $e = e^2 \in A$ idem. in A

Fact $\left\{\begin{array}{c}\text{Recollements} \\ \text{of mod. cat.}\end{array}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{array}{c}\text{ses of} \\ \text{fin. add. } \Bbbk\text{-cat.}\end{array}\right\}$

$$\left(\operatorname{mod} A/\langle e\rangle \underset{\leftarrow}{\overset{q}{\leftrightarrows}} \operatorname{mod} A \underset{\leftarrow}{\overset{l}{\leftrightarrows}} \operatorname{mod} eAe\right) \longleftrightarrow \left(\operatorname{proj} eAe \xrightarrow{l} \operatorname{proj} A \xrightarrow{q} \operatorname{proj} A/\langle e\rangle\right)$$

We will ① "dualise": mod ⇝ comod, proj ⇝ inj ② "Upgrade" (≈ categorify)

Thm [C-Miemietz]

$\mathcal{C}$: "nice" 2-category — finitely many obj + Hom's are finitary cat's

$\left\{\begin{array}{l}\text{recollement of comod. cat.} \\ \text{over } \mathcal{C}\text{-coalgebras}\end{array}\right\} \overset{1:1}{\longleftrightarrow} \left\{\text{s.e.s. of 2-representations of } \mathcal{C}\right\}$

$= \text{2-functors } \mathcal{C} \to \left\{\begin{array}{c}\text{finitary} \\ \Bbbk\text{-cat's}\end{array}\right\}$

$$\left[\operatorname{comod}_{\mathcal{C}}(D) \equiv \operatorname{comod}_{\mathcal{C}}(C) \equiv \operatorname{comod}_{\mathcal{C}}(E)\right] \longleftrightarrow \left[0 \to \operatorname{inj} E \to \operatorname{inj} C \to \operatorname{inj} D\right]$$

($\mathcal{C}$-coalgebra := coalgebra object in the abelianisation of the morphism (1-cat's) of $\mathcal{C}$)

Note: Classical setting is when $\begin{cases} \mathcal{C} = \Bbbk\text{-mod} \\ \text{horizontal compo.} = - \otimes_{\Bbbk} - \\ \text{vertical compo.} = \text{compo. of linear maps} \end{cases}$

Intrinsic description of D & E relative to C

Note: Unlike $\Bbbk$-(co)algebra setting, there is no correspondence between idempotents and injective C-comodules. !!

ALGEBRA

From idem subalg

$\boxed{eAe} \cong$ endomorphism alg of a projective A-module Ae

$\rightsquigarrow$

$\boxed{A/\langle e \rangle}$ = max. factor alg A/I of A s.t. $\underbrace{A/I \otimes_A}_{\varphi} \left(Ae/\mathrm{rad}(Ae) \right) = 0$

Amod

COALGEBRA

internal Hom

$\boxed{\mathrm{Hom}^C[Q, Q]}$

coendomorphism coalg of an injective C-comodule Q

$\rightsquigarrow$

$\boxed{D} :=$ max subcoalg of Q s.t. $\mathrm{soc}\,Q \underbrace{\square_C}_{\varphi} D = 0$

Comod C

We can also start from the other side:

idem factor alg

$\boxed{A/\langle e \rangle}$ = factor alg A/I s.t. multi. map $\mu: A \otimes A \to A$

$\mathrm{Im}\left(I \otimes I \hookrightarrow A \otimes A \xrightarrow{\mu} A\right) = I$

i.e. $I \otimes I \overset{\mu}{\twoheadrightarrow} I$

$\rightsquigarrow$

$Ae := \bigoplus \begin{pmatrix} \text{proj. cover of simples } S \text{ s.t.} \\ A/I \otimes S = 0 \end{pmatrix}$

$\rightsquigarrow$

eAe

coidem. subcoalg

$\boxed{D} :=$ subcoalg $D \overset{\iota}{\hookrightarrow} C$ s.t. $\mu: C \to CC$ comulti. induces mono

$C/D \hookrightarrow (C/D)(C/D)$

$\rightsquigarrow$

$Q := \bigoplus$ inj hull of simples S s.t. $S \square D = 0$

$\rightsquigarrow$

$\mathrm{Hom}[Q, Q]$

Remark

(1) The functors are :

$$\begin{tikzcd}[column sep=huge]
\mathrm{Comod}_{\underline{\mathcal{C}}}\, D \arrow[r, "- \square_D D_C"] & \mathrm{Comod}_{\underline{\mathcal{C}}}\, C \arrow[l, bend right, "\underline{\mathrm{Hom}}^{C}[D,-]"'] \arrow[l, bend left, "- \square_C D"] \arrow[r, "\underline{\mathrm{Hom}}^{C}[Q,-]"] & \mathrm{Comod}_{\underline{\mathcal{C}}}\, \underline{\mathrm{Hom}}[Q,Q] \arrow[l, bend right, "\underline{\mathrm{Hom}}^{E}\left[\underline{\mathrm{Hom}}^{C}[Q,C],\, -\right]"'] \arrow[l, bend left, "- \square_{\underline{\mathrm{Hom}}^{C}[Q,Q]} Q"]
\end{tikzcd}$$

(2) <u>What is internal Hom ?</u>

$\mathcal{C}$: "nice" 2-cat , $\mathbb{M}$: 2-representation over $\mathcal{C}$

$\Downarrow$

$i \xrightarrow{F} j$

internal Hom

$\rightsquigarrow$ $\exists$ bifunctorial isom

$$\mathrm{Hom}_{\mathbb{M}(j)}(M, FN) \cong \mathrm{Hom}_{\underline{\mathcal{C}}(i,j)}(\underline{\mathrm{Hom}}[N, M], F)$$

(3) Non-trivial e.g. of "nice" 2-cat ?

- single obj case $=$ $\overbrace{\text{tensor category}}$ $\leftarrow$ see, e.g., [EGNO]
  - e.g. cat. of Soergel bimodules (see, e.g., [Elias-Williamson])

- 2-cat. of (finite truncation of)
  (upper/lower half of) categorified quantum gp
  [Khovanov-Lauda ; Rouquier]