Wide subcategories and lattices of torsion classes

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Torsion pairs

Let $\mathcal{A}$ be an (ess. small) abelian length category.

- Any object $X \in \mathcal{A}$ has a composition series
  \[ 0 = X_0 \subset X_1 \subset \cdots \subset X_n = X \text{ with } X_i/X_{i-1} : \text{ simple}. \]

**Definition [Dickson]**

Let $\mathcal{T}, \mathcal{F} \subset \mathcal{A}$. $(\mathcal{T}, \mathcal{F})$ is called a torsion pair in $\mathcal{A}$ if

- $\mathcal{F} = \mathcal{T}^\perp := \{ X \in \mathcal{A} \mid \text{Hom}_{\mathcal{A}}(\mathcal{T}, X) = 0 \}$,
- $\mathcal{T} = \perp \mathcal{F} := \{ X \in \mathcal{A} \mid \text{Hom}_{\mathcal{A}}(X, \mathcal{F}) = 0 \}$.

Or equivalently,

- $\text{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{F}) = 0$,
- $\forall X \in \mathcal{A}, \exists (0 \to X' \to X \to X'' \to 0) : \text{ exact, } X' \in \mathcal{T}, X'' \in \mathcal{F}.$
Lattice of torsion classes

Definition

\( \mathcal{T} \subset \mathcal{A} \): torsion class \( \iff (\mathcal{T}, \mathcal{T}^\perp) \): torsion pair

\( \iff \mathcal{T} \) is closed under factor obj’s, extensions

- \( \text{tors } \mathcal{A} := \{ \text{all torsion classes in } \mathcal{A} \} \): poset by \( \subset \).
- For any \( \mathcal{X} \subset \mathcal{A} \), there exists
  \( \mathcal{T}(\mathcal{X}) := (\text{the smallest torsion class containing } \mathcal{X}) \).

Proposition

\( \text{tors } \mathcal{A} \) is a complete lattice with meets and joins

\[
\bigwedge_{\mathcal{T} \in S} \mathcal{T} = \bigcap_{\mathcal{T} \in S} \mathcal{T}, \quad \bigvee_{\mathcal{T} \in S} \mathcal{T} = \mathcal{T} \left( \bigcup_{\mathcal{T} \in S} \mathcal{T} \right) \quad (S \subset \text{tors } \mathcal{A}).
\]
Wide intervals

For intervals \([U, T]\) in tors \(\mathcal{A}\) (with \(U \subset T\)), \(U^\perp \cap T\) gives the “difference” of \(U\) and \(T\).

Today, we deal with the following nice intervals.

**Definition [AP]**

An interval \([U, T]\) in tors \(\mathcal{A}\) is called a **wide interval** if \(U^\perp \cap T\) is a wide subcategory of \(\mathcal{A}\).

\(\mathcal{W} \subset \mathcal{A}\) is called a **wide subcategory** if \(\mathcal{W}\) is closed under kernels, cokernels, extensions.

- A wide subcat. \(\mathcal{W}\) is an abelian length category.
- There exists a bij. \(\{\text{wide subcat.}\} \leftrightarrow \{\text{semibricks}\}\).
  - a **semibrick** = a set of pairwise Hom-orthogonal bricks.
  - \(\mathcal{W} \leftrightarrow \{\text{the simple objects of } \mathcal{W}\}\), \(\text{Filt } S \leftrightarrow S\).
**Brick labeling**

Two torsion classes $\mathcal{U} \subset \mathcal{T}$ are said to be **adjacent** if $\mathcal{U} \neq \mathcal{T}$ and $\exists \mathcal{V} \in \text{tors } \mathcal{A}$, $\mathcal{U} \subset \mathcal{V} \subset \mathcal{T}$.

**Definition**

The **Hasse quiver** of tors $\mathcal{A}$ is defined as follows.

- The vertices are the elements of tors $\mathcal{A}$.
- Write an arrow $\mathcal{T} \rightarrow \mathcal{U}$ if $\mathcal{U} \subset \mathcal{T}$ are adjacent.

**Proposition** [Demonet–Iyama–Reading–Reiten–Thomas]

For any arrow $q: \mathcal{T} \rightarrow \mathcal{U}$, $[\mathcal{U}, \mathcal{T}]$ is a wide interval, and $\mathcal{W} := \mathcal{U}^\perp \cap \mathcal{T}$ has a unique simple object $S_q$, so we label $q: \mathcal{T} \rightarrow \mathcal{U}$ by the brick $S_q$. 
\(\tau\)-tilting reduction

Let \(A\) be a fin. dim. alg. over a field \(K\), and \(\mathcal{A} = \text{mod} A\). For \(N \in \text{mod} A\) and \(Q \in \text{proj} A\), \((N, Q)\) is a \(\tau\)-rigid pair if \(\text{Hom}_A(N, \tau N) = 0\) and \(\text{Hom}_A(Q, N) = 0\).

**Theorem [Jasso, DIRRT]**

For a \(\tau\)-rigid pair \((N, Q)\), set

\[ U := \text{Fac} N, \quad T := N^\perp \cap \perp(\tau N) \cap Q^\perp. \]

1. \([U, T]\) is a wide interval (\(W := U^\perp \cap T\) is wide).
2. \([U, T]\) \cong \text{tors} \(W\) as complete lattices, where

\[ V \mapsto U^\perp \cap V, \quad T(U \cup X) \leftrightarrow X. \]

3. The bijections in (2) preserve brick labeling.

\(W \cong \text{mod} C\) for some fin. dim. alg \(C\).
Main result

Theorem 1 [AP]
Let \([U, T]\) be a wide interval in tors \(A\), \(W := U^\perp \cap T\).

(1) \([U, T]\) \cong tors \(W\) as complete lattices, where
\[
V \mapsto U^\perp \cap V =: \Phi(V),
\]
\[
\Phi^{-1}(X) = T(U \cup X) \leftrightarrow X.
\]

(2) The bijection \(\Phi\) preserves brick labeling: the label of \(V_1 \rightarrow V_2\) is the label of \(\Phi(V_1) \rightarrow \Phi(V_2)\).

(3) The following sets coincide:
   (a) The set of the labels of the arrows from \(T\) in \([U, T]\).
   (b) The set of the labels of the arrows to \(U\) in \([U, T]\).
   (c) The set of the simple objects of \(W\).
“Not $\tau$-tilting” example

Let $K = \overline{K}$, $A = K(1 \Rightarrow 2)$ and $\mathcal{A} = \text{mod } A$. We set $U, T \in \text{tors } \mathcal{A}$ by

- $U := \text{add}\{\text{all preinjective modules}\}$,
- $T := \text{add}\{\text{all regular, preinjective modules}\}.$

Then, $[U, T]$ is a wide interval with

$\mathcal{W} = \text{add}\{\text{all regular modules}\}$

$= \text{Filt}\{M_\lambda \mid \lambda \in \mathbb{P}^1(K)\}$

$$\left( M_\lambda := K \xrightarrow{a} K \xleftarrow{b} K \quad (\lambda = (a : b) \in \mathbb{P}^1(K)) \right)$$

$= \bigoplus_{\lambda \in \mathbb{P}^1(K)} \text{Filt } M_\lambda.$
“Not $\tau$-tilting” example

$[U, \mathcal{T}]$ is a wide interval with

$$\mathcal{W} = \bigoplus_{\lambda \in \mathbb{P}^1(K)} \text{Filt } M_\lambda.$$ 

Since $\text{tors}(\text{Filt } M_\lambda) = \{\text{Filt } M_\lambda, \{0\}\}$,

$$[U, \mathcal{T}] \cong \text{tors } \mathcal{W} \cong \prod_{\lambda \in \mathbb{P}^1(K)} \text{tors}(\text{Filt } M_\lambda) \cong 2^{\mathbb{P}^1(K)}.$$

For $X \in 2^{\mathbb{P}^1(K)}$, the associated torsion class in $[U, \mathcal{T}]$ is

$$\mathcal{V}_X := \mathcal{T}(U \cup \{M_\lambda \mid \lambda \in X\}) \in [U, \mathcal{T}].$$

Any arrow in $[U, \mathcal{T}]$ is of the form

$$\mathcal{V}_{X \cup \{\lambda\}} \xrightarrow{\text{label: } M_\lambda} \mathcal{V}_X \quad (X \in 2^{\mathbb{P}^1(K)}, \lambda \in \mathbb{P}^1(K) \setminus X).$$
Characterization of wide intervals (1)

For any interval $[U, T]$ in tors $A$, we set

$$[U, T]^+ := \{T\} \cup \{V \in [U, T] \mid \exists (T \to V): \text{arrow}\},$$

$$[U, T]^− := \{U\} \cup \{V \in [U, T] \mid \exists (V \to U): \text{arrow}\}.$$  

**Theorem 2 [AP]**

For any interval $[U, T]$ in tors $A$, TFAE.

(a) $[U, T]$ is a wide interval.

(b) $[U, T]$ is a join interval, i.e. $T = \bigvee_{V \in [U, T]^−} V$.

(c) $[U, T]$ is a meet interval, i.e. $U = \bigwedge_{V \in [U, T]^+} V$. 

Characterization of wide intervals (2)

**Question**

How many wide intervals $[U, T]$ exist for $T \in \text{tors } A$?

**Theorem 3 [AP]**

Fix $T \in \text{tors } A$ and $L := \{\text{all labels of arrows from } T\}$.  

1. $L$ is a semibrick with $\text{Filt } L = \alpha(T)$, where 
   
   $$\alpha(T) := \{X \in T \mid \forall Y \in T, \forall f : Y \to X, \text{Ker } f \in T\}.$$  

2. There exists a bijection 
   
   $$2^L \to \{U \in \text{tors } A \mid [U, T]: \text{wide interval}\}$$
   
   $$\mathcal{S} \mapsto T \cap \perp \mathcal{S} =: U_S$$

   and $(U_S) \perp \cap T \subset \text{Filt } S \subset \alpha(T)$: Serre.
Widely generated torsion classes

Theorem [Marks–Šťovíček]
If \( \mathcal{W} \) is a wide subcategory of \( \mathcal{A} \), then \( \alpha(T(\mathcal{W})) = \mathcal{W} \).

Corollary [AP] (cf. [Barnard–Carroll–Zhu])
For \( T \in \text{tors } \mathcal{A} \), TFAE.

(a) \( \exists \mathcal{W} \subset \mathcal{A} \): a wide subcat., \( T = T(\mathcal{W}) \) (widely generated torsion classes).

(b) \( T = T(\alpha(T)) \).

(c) \( T = T(\{\text{all labels of arrows from } T\}) \).

(d) \( \forall U \subsetneq T, \exists (T \to U') \): arrow, \( U \subset U' \).
Thank you for your attention.