g-polytopes of Brauer graph algebras

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Aim of this talk

- To introduce the \textit{g-polytopes} of f.d. algebras, firstly studied by [Asashiba-Mizuno-Nakashima (2019)]
  - cones of \textit{g-vectors}
  - simplicial complexes of two-term silting complexes
  - lattice polytopes

- Convexity and symmetry of (the closure of) \textit{g-polytopes} of Brauer graph algebras.

\begin{itemize}
\item \includegraphics[width=0.2\textwidth]{image1.png}
\item \includegraphics[width=0.2\textwidth]{image2.png}
\item \includegraphics[width=0.2\textwidth]{image3.png}
\item \includegraphics[width=0.2\textwidth]{image4.png}
\end{itemize}
Motivation: idea of L. Hille

$Q$: an acyclic quiver with vertices 1, ..., $n$. $k = \overline{k}$: a field.

In [Hille (2006, 2015)], he studied a simplicial complex of tilting modules over $kQ$ as

$$
\bigcup_{M} C(M) \subseteq \mathbb{R}^n,
$$

where

- $M = \bigoplus_{i=1}^{n} M_i$ runs over all f. g. tilting $kQ$-modules,
- $C(M) := \{ \sum_{i=1}^{n} a_i \dim M_i \mid a_i \in \mathbb{R}_{\geq 0} \}$

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In [Hille (2006, 2015)], he studied a simplicial complex of tilting modules over $kQ$ as

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- $C_{\leq 1}(M) := \left\{ \sum_{i=1}^{n} a_i \text{dim} M_i \mid a_i \in \mathbb{R}_{\geq 0}, \sum_{i=1}^{n} a_i \leq 1 \right\} = \text{conv}\{0, \text{dim} M_i \mid 1 \leq i \leq n\}.$
$\bigcup_{M} C_{\leq 1}(M)$

$Q_1 : 1 \rightarrow 2 \quad \bullet = \text{indec. rigid module}$
$U_M C_{\leq 1}(M)$

$Q_1 : 1 \rightarrow 2$

$Q_2 : 1 \rightsquigarrow 2$
\[ \bigcup_{M} C_{\leq 1}(M) \]

\[ Q_1 : 1 \rightarrow 2 \quad Q_2 : 1 \leftrightarrow 2 \quad Q_3 : 1 \leftrightarrow 2 \]
Motivation: L. Hille’s idea

**Theorem [Hille (2015)]**

If $Q$ is of Dynkin type $\mathbb{A}$, then $\bigcup_M C_{\leq 1}(M)$ is convex.

In this case, we have

$$\bigcup_M C_{\leq 1}(M) = \text{conv} \{0, \dim X \mid X: \text{indec. } kQ\text{-module}\}$$

$$= \text{conv}(\{e_i\}_{i=1}^n \cup \{e_i + \cdots + e_j \mid 1 \leq i < j \leq n\} \cup \{0\})$$

and it does not depend on the orientation of $Q$.

For type $\mathbb{D}$ and $\mathbb{E}$, $\bigcup_M C_{\leq 1}(M)$ is non-convex.
Motivation: L. Hille’s idea

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For type $\mathbb{D}$ and $\mathbb{E}$, $\bigcup_M C_{\leq 1}(M)$ is non-convex.
Let $M = \bigoplus_{i=1}^{n} M_i$ be a tilting $kQ$-module and

$$0 \rightarrow M_i \xrightarrow{f} \bigoplus_{\lambda \in \Lambda} X_\lambda \rightarrow M'_i \rightarrow 0 \quad (X_\lambda : indec.)$$

where $f$ is a left minimal $\text{add}(M/M_i)$-apx. of $M$. $N := M/M_i \oplus M'_i$ is called mutation of $M$ if it is tilting.

**Lemma**

In the above, the following hold:

1. $C_{\leq 1}(M)$, $C_{\leq 1}(N)$ intersect only at their boundary.
2. If $\#\Lambda \leq 2$, then $C_{\leq 1}(M) \cup C_{\leq 1}(N)$ is convex.
3. If $Q$ is of type $\mathbb{A}$, then $\#\Lambda \leq 2$ is always satisfied.
## This talk

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\begin{itemize}
  \item $A$: a finite dimensional $k$-algebra
  \item $P(1), \ldots, P(n)$: indecomposable projective $A$-modules
\end{itemize}

Due to [Asashiba-Mizuno-Nakashima (2019)], we define the following subset $\Delta(A)$ of $\mathbb{R}^n$, which we call \textit{g-polytope} of $A$:

$$
\Delta(A) := \bigcup_{T} C_{\leq 1}(T) \subseteq \mathbb{R}^n, \text{ where }
$$

$T = \bigoplus_{i=1}^{n} T_i$ runs over all two-term silting complexes

$C_{\leq 1}(T) := \{ \sum_{i=1}^{n} a_i g^{T_i} \mid a_i \in \mathbb{R}_{\geq 0}, \sum_{i=1}^{n} a_i \leq 1\}$

For a two-term complex $T = (T^{-1} \to T^0) \in K^b(\text{proj}A)$,

$g^T := (m_1 - m'_1, \ldots, m_n - m'_n) \in \mathbb{Z}^n$ : the $g$-vector of $T$,

where $T^0 \cong \bigoplus_{i=1}^{n} P(i)^{m_i}$ and $T^{-1} \cong \bigoplus_{i=1}^{n} P(i)^{m'_i}$. 
Silting mutation

Let $T = \bigoplus_{i=1}^{n} T_i$ be a two-term silting complex for $A$ and

$$T_i \xrightarrow{f} \bigoplus_{\lambda \in \Lambda} X_{\lambda} \longrightarrow T'_i \rightarrow T_i[1], \quad (X_{\lambda} : indec) \quad (*)$$

where $f$ is a minimal left $\text{add}(T/T_i)$-apx. of $T_i$. Then $U := T/T_i \oplus T'_i$ is again a silting complex, and is called a two-term silting mutation of $T$ if it is two-term.

Lemma (analogues of tilting modules)

In the above, the following hold:

1. $C_{\leq 1}(T), C_{\leq 1}(U)$ intersect only at their boundary.
2. If $\# \Lambda \leq 2$, then $C_{\leq 1}(T) \cup C_{\leq 1}(U)$ is convex.
**Definition**

We say that $A$ is *locally convex* if $\#\Lambda \leq 2$ in $(\ast)$ always satisfied for any two-term silting complex $T$ and any two-term silting mutation of $T$.

**Theorem [Asashiba-Mizuno-Nakashima (2019)]**

Assume that

$$\#\{\text{basic two-term silting complexes for } A\}/_{\text{isom}} < \infty.$$ 

Then the following conditions are equivalent:

1. $A$ is locally convex.
2. $\Delta(A)$ is convex.

In this case, $\Delta(A) = \text{conv}\{g^X | X: \text{indecomposable two-term presilt.}\}$. 
**Definition**

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(2) $\Delta(A)$ is convex.

In this case, $\Delta(A) = \text{conv}\{g^X | X : \text{indec. two-term presilt.}\}$
Brauer tree algebras are f.d. symmetric algebras defined by Brauer trees (= trees embedded in a disk).

- containing the trivial extension of path algebras of type $\tilde{A}$
- closed under derived equivalent
- $\# \{ \text{basic two-term silting complexes for } A \} / \text{isom} < \infty$
Theorem [Asashiba-Mizuno-Nakashima (2019)]

Let $A_G$ be a Brauer tree algebra associated to a Brauer tree $G$. Then the following hold:

1. $\Delta(A_G)$ is convex.
2. $\Delta(A_G)$ is symmetric with respect to origin (i.e. $\Delta(A_G) = -\Delta(A_G)$).

Corollary

For Brauer tree algebras, the $g$-polytope provides a derived invariant in the sense that

$$A_G \sim_{\text{der}} A_{G'} \implies \Delta(A_G) \cong_{\text{SL}} \Delta(A_{G'})$$

$$\mathcal{M}_{A_n} := \text{conv}(\{\pm e_i\}_{i=1}^n \cup \{\pm (e_i + \cdots + e_j) | 1 \leq i < j \leq n\})$$
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For Brauer tree algebras, the $g$-polytope provides a derived invariant in the sense that

$$A_G \sim_{\text{der}} A_{G'} \implies \Delta(A_G) \cong_{\text{SL}} \Delta(A_{G'}) \cong_{\text{SL}} M_{A_n}$$

$$M_{A_n} := \text{conv}(\{\pm e_i\}_{i=1}^n \cup \{\pm (e_i + \cdots + e_j) \mid 1 \leq i < j \leq n\})$$
$n=2$

$G_1 : \bullet --- \bullet --- \bullet$

$A_{G_1} \cong$ the trivial extension of $k(1 \to 2)$
$n=3$

$G_2 : \bullet \overrightarrow{\bullet} \overrightarrow{\bullet} \overrightarrow{\bullet} \overrightarrow{\bullet}$ \hspace{2cm} $G_3 : \bullet \overrightarrow{\bullet} \overrightarrow{\bullet} \overrightarrow{\bullet}$

$A_{G_2} \cong \text{Triv}(k(1 \rightarrow 2 \leftarrow 3))$ \hspace{1cm} $A_{G_3} \cong \text{Triv}(k(1 \rightarrow 2 \rightarrow 3))$
Main Result
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Brauer graph algebras are defined from ribbon graphs (undirected graphs embedded in surfaces).

- a generalization of Brauer tree algebras
- symmetric special biserial algebras (hence, tame-representation type)
- infinitely many two-term silting complexes in general
Main Result

Proposition

Let $A_G$ be a Brauer graph algebra associated to a ribbon graph $G$. Then $A_G$ is locally convex.

It does not imply the convexity of $\Delta(A_G)$, but
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Theorem (A)

Let $A_G$ be a Brauer graph algebra associated to a ribbon graph $G$. Then the following hold:

1. $\Delta(A_G)$ is convex.
2. $\Delta(A_G)$ is symmetric with respect to origin.

Corollary

For Brauer graph algebras, the closure of the $g$-polytope is invariant under iterated tilting mutation ($\Leftrightarrow$ flip of ribbon graphs).
Main Result

Theorem (A)

Let $A_G$ be a Brauer graph algebra associated to a ribbon graph $G$. Then the following hold:

1. $\Delta(A_G)$ is convex.
2. $\Delta(A_G)$ is symmetric with respect to origin.

Corollary

For Brauer graph algebras, the closure of the $g$-polytope is invariant under iterated tilting mutation ($\Leftrightarrow$ flip of ribbon graphs).
The closure $\overline{\Delta(A_{G_1})}$ is given by a rectangular area.
The outline of $\Delta(A_{G_2})$ is a tube of a hexagon.
A proof is given by a geometric (combinatorial) approach due to [Adachi-Aihara-Chan (2014)]:

- Determine all lattice points of $\Delta(A_G)$ combinatorially
- Taking the closure is essentially needed
- The density of cones of $g$-vectors plays an important role

An explicit description of the closure of $g$-polytope
References


Thank you for your attention!!