g-polytopes of Brauer graph algebras

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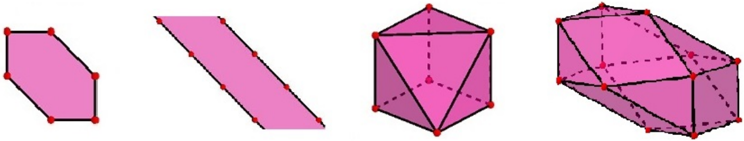
August 27, 2019

Aim of this talk

- To introduce the g-polytopes of f. d. algebras, firstly studied by [Asashiba-Mizuno-Nakashima (2019)]
  - cones of g-vectors
  - simplicial complexes of two-term silting complexes
  - lattice polytopes
- Convexity and symmetry of (the closure of) g-polytopes of Brauer graph algebras.



Motivation: idea of L. Hille

Q: an acyclic quiver with vertices 1,..., n. $k = \overline{k}$: a field.

In [Hille (2006, 2015)], he studied a simplicial complex of tilting modules over kQ as

$$\bigcup_{M} C(M) \subseteq \mathbb{R}^n,$$

where

- $M = \bigoplus_{i=1}^{n} M_i$ runs over all f. g. tilting kQ-modules,
- $C(M) := \{\sum_{i=1}^{n} a_i \underline{\dim} M_i \mid a_i \in \mathbb{R}_{\geq 0}\}$

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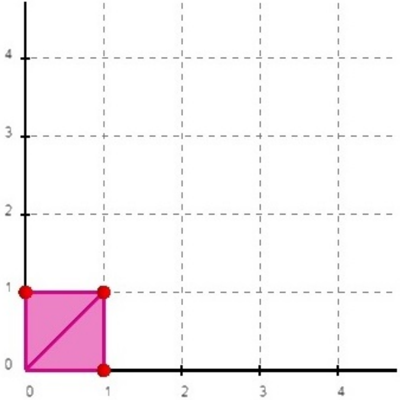
- $C_{\leq 1}(M) := \{\sum_{i=1}^n a_i \underline{\dim} M_i \mid a_i \in \mathbb{R}_{\geq 0}, \sum_{i=1}^n a_i \leq 1\}$
$$= \mathrm{conv}\{0, \underline{\dim} M_i \mid 1 \leq i \leq n\}.$$

$\bigcup_M C_{\leq 1}(M)$

$$Q_1 : 1 \longrightarrow 2 \qquad \bullet = \textit{indec. rigid module}$$



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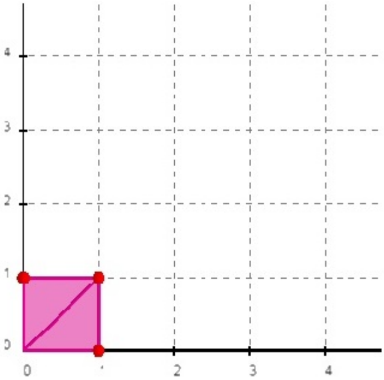
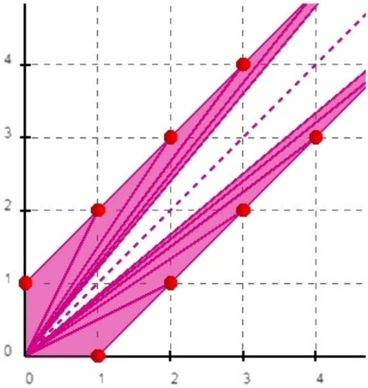


$Q_2 : 1 \rightrightarrows 2$



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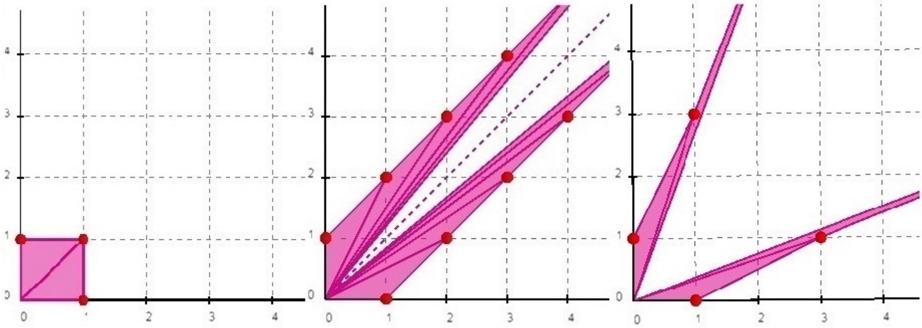
$Q_1 : 1 \longrightarrow 2$ $\qquad$ $Q_2 : 1 \rightrightarrows 2$ $\qquad$ $Q_3 : 1 \overset{\longrightarrow}{\rightrightarrows} 2$



Motivation: L. Hille's idea

Theorem [Hille (2015)]

If Q is of Dynkin type $\mathbb{A}$, then $\bigcup_M C_{\leq 1}(M)$ is convex.

In this case, we have

$$\bigcup_M C_{\leq 1}(M) = \mathrm{conv}\{0, \underline{\dim} X \mid X\text{: indec. } kQ\text{-module}\}$$

$$= \mathrm{conv}(\{\mathbf{e}_i\}_{i=1}^n \cup \{\mathbf{e}_i + \cdots + \mathbf{e}_j \mid 1 \leq i < j \leq n\} \cup \{0\})$$

and it does not depend on the orientation of Q.

For type $\mathbb{D}$ and $\mathbb{E}$, $\bigcup_M C_{\leq 1}(M)$ is non-convex.

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For type $\mathbb{D}$ and $\mathbb{E}$, $\bigcup_M C_{\leq 1}(M)$ is non-convex.

Mutation Let $M = \bigoplus_{i=1}^{n} M_i$ be a tilting kQ-module and

$$0 \longrightarrow M_i \xrightarrow{f} \bigoplus_{\lambda \in \Lambda} X_\lambda \longrightarrow M_i' \longrightarrow 0 \quad (X_\lambda : \textit{indec.})$$

where f is a left minimal $\mathrm{add}(M/M_i)$-apx. of M.
$N := M/M_i \oplus M_i'$ is called *mutation* of M if it is tilting.

Lemma

In the above, the following hold:

1. $C_{\leq 1}(M), C_{\leq 1}(N)$ *intersect only at their boundary.*
2. *If* $\#\Lambda \leq 2$, *then* $C_{\leq 1}(M) \cup C_{\leq 1}(N)$ *is convex.*
3. *If* Q *is of type* $\mathbb{A}$, *then* $\#\Lambda \leq 2$ *is always satisfied.*

This talk

	[H]	this talk
Object	tilting module	2-term silting cpx.
Numerical data	dim. vector	g-vector
Cones	$C(M)$	$C(T)$
Polytope	$\bigcup_{M} C_{\leq 1}(M)$	$\bigcup_{T} C_{\leq 1}(T)$
Intersections	mutation	silting mutation
Locally convexity	the middle term of mutation seq.	defined similarly

- A: a finite dimensional k-algebra
- $P(1), \ldots, P(n)$: indecomposable projective A-modules

Due to [Asashiba-Mizuno-Nakashima (2019)], we define the following subset $\Delta(A)$ of $\mathbb{R}^n$, which we call *g-polytope* of A:

$$\Delta(A) := \bigcup_{T} C_{\leq 1}(T) \ \subseteq \ \mathbb{R}^n, \ \textit{where}$$

- $T = \bigoplus_{i=1}^n T_i$ runs over all two-term silting complexes
- $C_{\leq 1}(T) := \{\sum_{i=1}^n a_i g^{T_i} \mid a_i \in \mathbb{R}_{\geq 0}, \sum_{i=1}^n a_i \leq 1\}$

For a two-term complex $T = (T^{-1} \to T^0) \in \mathrm{K}^{\mathrm{b}}(\mathrm{proj} A)$,

$$g^T := (m_1 - m'_1, \ldots, m_n - m'_n) \in \mathbb{Z}^n : \text{the } g\text{-vector of } T,$$

where $T^0 \cong \bigoplus_{i=1}^n P(i)^{m_i}$ and $T^{-1} \cong \bigoplus_{i=1}^n P(i)^{m'_i}$.

Silting mutation Let $T = \bigoplus_{i=1}^{n} T_i$ be a two-term silting complex for A and

$$T_i \xrightarrow{f} \bigoplus_{\lambda\in\Lambda} X_\lambda \longrightarrow T_i' \to T_i[1], \quad (X_\lambda : \mathit{indec}) \qquad (*)$$

where f is a minimal left $\mathrm{add}(T/T_i)$-apx. of T_i.
Then $U := T/T_i \oplus T_i'$ is again a silting complex, and is called a *two-term silting mutation* of T if it is two-term.

Lemma (analogues of tilting modules)

In the above, the following hold:

1. $C_{\leq 1}(T)$, $C_{\leq 1}(U)$ *intersect only at their boundary.*
2. *If* $\#\Lambda \leq 2$, *then* $C_{\leq 1}(T) \cup C_{\leq 1}(U)$ *is convex.*

Definition

We say that A is *locally convex* if $\#\Lambda \leq 2$ in $(*)$ always satisfied for any two-term silting complex T and any two-term silting mutation of T.

Theorem [Asashiba-Mizuno-Nakashima (2019)]

Assume that

$\#\{$*basic two-term silting complexes for* $A\}/_{isom} < \infty$.

Then the following conditions are equivalent:

(1) *A is locally convex.*

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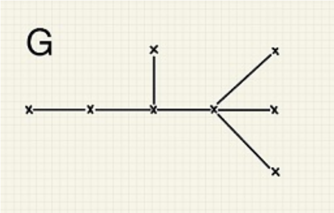
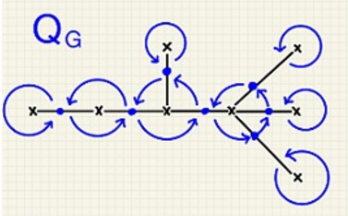
Brauer tree algebras

Brauer tree algebras are f. d. symmetric algebras defined by Brauer trees(= trees embedded in a disk).

- containing the trivial extension of path algebras of type $\mathbb{A}$
- closed under derived equivalent
- $\#\{\text{basic two-term silting complexes for } A\}/_{isom} < \infty$





Theorem [Asashiba-Mizuno-Nakashima (2019)]

Let A_G be a Brauer tree algebra associated to a Brauer tree G. Then the following hold:

1. $\Delta(A_G)$ *is convex.*
2. $\Delta(A_G)$ *is symmetric with respect to origin (i.e. $\Delta(A_G) = -\Delta(A_G)$).*

Corollary

For Brauer tree algebras, the g-polytope provides a derived invariant in the sense that

$$A_G \underset{\text{der}}{\sim} A_{G'} \implies \Delta(A_G) \underset{\text{SL}}{\cong} \Delta(A_{G'})$$

$$\mathcal{M}_{\mathbb{A}_n} := \operatorname{conv}(\{\pm \mathbf{e}_i\}_{i=1}^n \cup \{\pm(\mathbf{e}_i + \cdots + \mathbf{e}_j) \mid 1 \le i < j \le n\})$$

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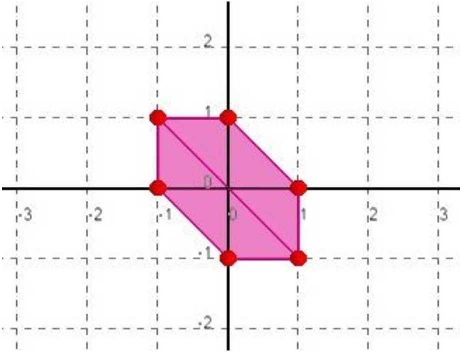
n=2

$$G_1 : \bullet \text{——} \bullet \text{——} \bullet$$

$$A_{G_1} \cong \text{the trivial extension of } k(1 \to 2)$$

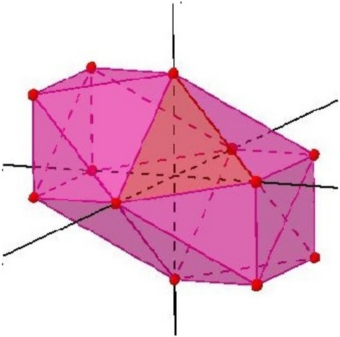


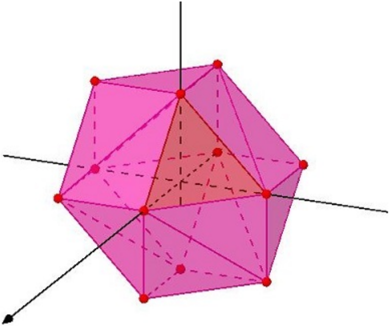
n=3

$G_2 : \bullet - \bullet - \bullet - \bullet$

$$G_3 : \begin{tikzcd} & \bullet \arrow[d, dash] & \\ \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet \end{tikzcd}$$

$A_{G_2} \cong \mathrm{Triv}(k(1 \to 2 \leftarrow 3))$ $\quad$ $A_{G_3} \cong \mathrm{Triv}(k(1 \to 2 \to 3))$





Main Result

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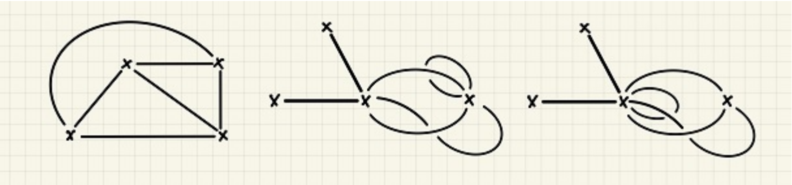
Brauer graph algebras are defined from ribbon graphs(= undirected graphs embedded in surfaces).

- a generalization of Brauer tree algebras
- symmetric special biserial algebras (hence, tame-representation type)
- infinitely many two-term silting complexes in general



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Theorem (A)

Let A_G be a Brauer graph algebra associated to a ribbon graph G. Then the following hold:

1. $\overline{\Delta(A_G)}$ *is convex.*
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Corollary

For Brauer graph algebras, the closure of the g-polytope is invariant under iterated tilting mutation (⇔ flip of ribbon graphs).

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Theorem (A)

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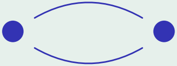
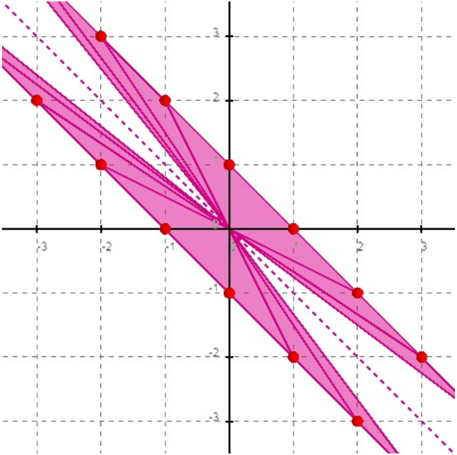
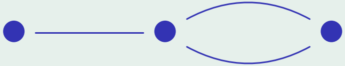
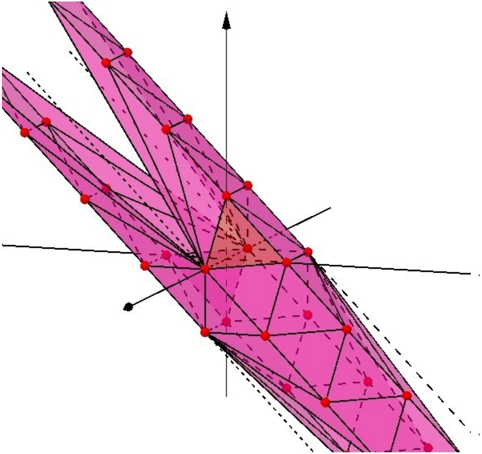
For Brauer graph algebras, the closure of the g-polytope is invariant under iterated tilting mutation ($\Leftrightarrow$ flip of ribbon graphs).

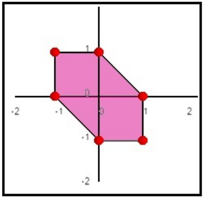
G_1 : 



The closure $\overline{\Delta(A_{G_1})}$ is given by a rectangular area.

G_2 : 





The outline of $\overline{\Delta(A_{G_2})}$ is a tube of a hexagon.

- A proof is given by a geometric (combinatorial) approach due to [Adachi-Aihara-Chan (2014)]:
  - Determine all lattice points of $\overline{\Delta(A_G)}$ combinatorially
  - Taking the closure is essentially needed
  - The density of cones of g-vectors plays an important role
- $\exists$ An explicit description of the closure of g-polytope

References

[1] T. Adachi, T. Aihara, A. Chan, *Classification of two-term tilting complexes over Brauer graph algebras*, 2018

[2] H. Asashiba, Y. Mizuno, K. Nakashima, *Simplicial complexes and tilting theory for Brauer tree algebras*, 2019

[3] L. Hille, *On the volume of a tilting module*, 2006

[4] L. Hille, *Tilting modules over the path algebra of type $\mathbb{A}$, polytopes, and Catalan numbers*, 2015

Thank you for your attention!!