ELLIPTIC ALGEBRAS

S. Paul Smith

University of Washington, Seattle

Email: smith@math.washington.edu

Ongoing work with Alex Chirvasitu (SUNY, Buffalo) and Ryo Kanda (Osaka).

This talk concerns the elliptic algebras $Q_{n,k}(E,\tau)$ defined by Odesskii and Feigin in 1989. Each $Q_{n,k}(E,\tau)$ is a connected graded $\mathbb{C}$-algebra, usually not commutative, depending on a pair of relatively prime integers $n > k \geq 1$, an elliptic curve $E = \mathbb{C}/\Lambda$, and a translation automorphism $z \mapsto z + \tau$ of E. At first glance, its definition as the free algebra $\mathbb{C}\langle x_0, \dots, x_{n-1}\rangle$ modulo the n^2 relations

$$\sum_{r \in \mathbb{Z}_n} \frac{\theta_{j-i+r(k-1)}(0)}{\theta_{j-i-r}(-\tau)\theta_{kr}(\tau)}\, x_{j-r}x_{i+r} \qquad (i,j) \in \mathbb{Z}_n^2$$

reveals nothing. Here the $\theta_\alpha(z)$, $\alpha \in \mathbb{Z}_n$, are theta functions of order n that are quasi-periodic with respect to the lattice Λ. For a fixed (n,k,E) the $Q_{n,k}(E,\tau)$'s form a flat family of deformations of the polynomial ring $\mathbb{C}[x_0, \dots, x_{n-1}]$. They are Koszul algebras so their Koszul duals form a flat family of finite dimensional algebras that are deformations of the exterior algebra $\wedge(\mathbb{C}^n)$. In a sequence of fascinating papers Feigin and Odesskii proved and claimed that the $Q_{n,k}(E,\tau)$'s have a number of remarkable properties. The ingredients that appear in the study of these algebras indicate the richness of the subject:

- the quantum Yang-Baxter equation with spectral parameter;
- the negative continued fraction expansion for $\frac{n}{k}$;
- a distinguished invertible sheaf $\mathcal{L}_{n/k}$ on $E^g = E \times \cdots \times E$, where g is the length of the continued fraction;
- the Fourier-Mukai transform $\mathbf{R}\mathrm{pr}_{1*}(\mathcal{L}_{n/k} \otimes^{\mathbf{L}} \mathrm{pr}_g^*(\,\cdot\,))$ is an auto-equivalence of the bounded derived category $\mathsf{D}^b(\mathrm{coh}(E))$ that provides a bijection $\mathcal{E}(1,0) \to \mathcal{E}(k,n)$ where $\mathcal{E}(r,d)$ is the set of isomorphism classes of indecomposable bundles of rank r and degree d on E;
- identities for theta functions in one and in g variables;
- the variety $X_{n/k}$ defined as the image of the morphism $|\mathcal{L}_{n/k}| : E^g \to \mathbb{P}^{n-1} = \mathbb{P}(H^0(E^g, \mathcal{L}_{n/k})^*)$, and an automorphism $\sigma : X_{n/k} \to X_{n/k}$ defined in terms of τ and the continued fraction;
- $X_{n/k} \cong E^g/\Sigma_{n/k}$, the quotient modulo the action of a subgroup of the symmetric group Σ_{g+1} defined in terms of the location of the 2's in the continued fraction;
- a homomorphism $Q_{n,k}(E,\tau) \to B(X_{n/k}, \sigma, \mathcal{L}_{n/k}) = B(E^g, \sigma, \mathcal{L}_{n/k})^{\Sigma_{n/k}}$ where $B(\cdot,\cdot,\cdot)$ is a twisted homogeneous coordinate ring à la Artin-Tate-Van den Bergh;
- when $X_{n/k}$ is E^g, an adjoint triple of functors $i^* \dashv i_* \dashv i^!$ where $i_* : \mathrm{Qcoh}(E^g) \to \mathrm{QGr}(Q_{n,k}(E,\tau))$ plays the role of a direct image functor for a morphism $E^g \to \mathrm{Proj}_{nc}(Q_{n,k}(E,\tau))$ in the sense of non-commutative algebraic geometry;
- a similar result when $X_{n/k}$ is the symmetric power S^gE;

The algebras $Q_{n,1}(E,\tau)$ when $n = 3, 4$ are the 3- and 4-dimensional Sklyanin algebras discovered by Artin-Schelter (1986) and Sklyanin (1982) and studied by Artin-Tate-Van den Bergh and Smith-Stafford and Levasseur-Smith. For $n \geq 5$, a lot is known about $Q_{n,1}(E,\tau)$ due to work of Tate-Van den Bergh and Staniszkis.

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