THE STRUCTURE OF SALLY MODULES AND NORMAL HILBERT COEFFICIENTS

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This talk is based on a joint work with S. K. Masuti and M. E. Rossi, and H. L. Truong.

The Sally module of an ideal is an important tool to interplay between Hilbert coefficients and the properties of the associated graded ring. In this talk we give new insights on the structure of the Sally module. We apply these results characterizing the almost minimal value of the first and the second normal Hilbert coefficients in an analytically unramified Cohen-Macaulay local ring.

Let $(R, \mathfrak{m})$ be an analytically unramified Cohen-Macaulay local ring of dimension $d > 0$ with infinite residue field $R/\mathfrak{m}$ and I an $\mathfrak{m}$-primary ideal of R. Let $\overline{I}$ denote the integral closure of I. Consider the so called *normal filtration* $\{\overline{I^n}\}_{n\in\mathbb{Z}}$ and we are interested in the corresponding Hilbert-Samuel polynomial. It is well-known that there are integers $\overline{\mathrm{e}}_i(I)$, called the *normal Hilbert coefficients* of I, such that for $n \gg 0$

$$\ell_R(R/\overline{I^{n+1}}) = \overline{\mathrm{e}}_0(I)\binom{n+d}{d} - \overline{\mathrm{e}}_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d\overline{\mathrm{e}}_d(I).$$

Let us choose a parameter ideal J of R which forms a reduction of I. We set $\overline{\mathrm{r}}_J(I) := \min\{r \geq 0 \mid \overline{I^{n+1}} = J\overline{I^n} \text{ for all } n \geq r\}$.

Suppose that $d \geq 2$. Then by [1] it is known that

$$\overline{\mathrm{e}}_2(I) \geq \overline{\mathrm{e}}_1(I) - \overline{\mathrm{e}}_0(I) + \ell_R(R/\overline{I}) \geq \ell_R(\overline{I^2}/J\overline{I})$$

hold true and if either of the inequalities is an equality, then $\overline{\mathrm{r}}_J(I) \leq 2$, in particular the associated graded ring $\overline{G}(I) = \bigoplus_{n\geq 0} \overline{I^n}/\overline{I^{n+1}}$ is Cohen-Macaulay. Thus the ideals I with $\overline{\mathrm{e}}_1(I) = \overline{\mathrm{e}}_0(I) - \ell_R(R/\overline{I}) + \ell_R(\overline{I^2}/J\overline{I})$ and/or $\overline{\mathrm{e}}_2(I) = \overline{\mathrm{e}}_1(I) - \overline{\mathrm{e}}_0(I) + \ell_R(R/\overline{I})$ enjoy nice properties.

In this talk we present the structure of the Sally module in the case the first or the second normal coefficient is almost minimal, that is the equality $\overline{\mathrm{e}}_1(I) = \overline{\mathrm{e}}_0(I) - \ell_R(R/\overline{I}) + \ell_R(\overline{I^2}/J\overline{I}) + 1$ or $\overline{\mathrm{e}}_2(I) = \overline{\mathrm{e}}_1(I) - \overline{\mathrm{e}}_0(I) + \ell_R(R/\overline{I}) + 1$ holds true. We present in these cases the structure of the Sally module and, in particular, we investigate the depth of associated graded ring $\overline{G}(I)$.

As the title outlines, an important tool in this talk is the Sally module introduced by W. V. Vasconcelos [2]. The aim of this talk was to define a module in between the associated graded ring and the Rees algebra taking care of important information coming from a minimal reduction. Actually, a more detailed information comes from the graded parts of a suitable filtration $\{C^{(i)}\}$ of the Sally module that was introduced by M. Vaz Pinto in [3]. In this talk we prove some important results on $C^{(2)}$ which will be key ingredients for proving the main result. Some of them are stated in a very general setting. Our hope is that these will be successfully applied to give new insights to problems related to the normal Hilbert coefficients, for instance [1].

References

[1] S. Itoh, *Coefficients of normal Hilbert polynomials*, J. Algebra **150** (1992), 101–117.

[2] W. V. Vasconcelos, *Hilbert Functions, Analytic Spread, and Koszul Homology*, Contemporary Mathematics, Vol **159** (1994), 401–422.

[3] M. Vaz Pinto, *Hilbert functions and Sally modules*, J. Algebra **192** (1997), 504–523.

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