General heart construction and the Gabriel-Quillen embedding

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Recently, the notion of extriangulated category was introduced in [5] as a simultaneous generalization of triangulated category and exact category. A typical example of extriangulated categories is the cotorsion class of a cotorsion pair over a triangulated category.

Our first aim is to provide an analog of the following Gabriel-Quillen embedding theorem for extriangulated categories. It shows that any skeletally small exact category $\mathcal{C}$ can be embedded in the category $\mathsf{Lex}\,\mathcal{C}$ of left exact functors from $\mathcal{C}$ to the category Ab of abelian groups. More precisely, the canonical inclusion $R : \mathsf{Lex}\,\mathcal{C} \to \mathsf{Mod}\,\mathcal{C}$ admits a left adjoint Q and hence we have a localization sequence:

$$\begin{tikzcd} \operatorname{Ker} Q \arrow[r] & \mathsf{Mod}\,\mathcal{C} \arrow[r, "Q"] \arrow[l, bend left] & \mathsf{Lex}\,\mathcal{C}. \arrow[l, bend left, "R"] \end{tikzcd}$$

Moreover, the composed functor $E_{\mathcal{C}} : \mathcal{C} \hookrightarrow \mathsf{Mod}\,\mathcal{C} \xrightarrow{Q} \mathsf{Lex}\,\mathcal{C}$, which is called the Gabriel-Quillen embedding functor, is exact and fully faithful. We show a "finitely presented" version of the theorem for some extriangulated categories with weak-kernels, especially, there exists a Gabriel-Quillen type functor $E_{\mathcal{C}} : \mathcal{C} \to \mathsf{lex}\,\mathcal{C}$, where $\mathsf{lex}\,\mathcal{C}$ denotes the category of the finitely presented left exact functors from $\mathcal{C}$ to Ab. Using the functor $E_{\mathcal{C}}$, we provide necessary and sufficient conditions for an extriangulated category $\mathcal{C}$ to be exact and abelian, respectively.

Our main result is an application for a cotorsion pair $(\mathcal{U}, \mathcal{V})$ in a triangulated category $\mathcal{T}$. In [4, 1], it was proved that there exists an abelian category $\underline{\mathcal{H}}$ associated to the cotorsion pair, called the heart. This result was shown for two extremal cases [2, 3], namely, t-structures and 2-cluster tilting subcategories. Since the cotorsion class $\mathcal{U}$ has a natural extriangulated structure, we have the Gabriel-Quillen type functor $E_{\mathcal{U}} : \mathcal{U} \to \mathsf{lex}\,\mathcal{U}$. Our result provides a good understanding for a construction of the heart, in particular, we have an equivalence $\underline{\mathcal{H}} \simeq \mathsf{lex}\,\mathcal{U}$.

References

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