Tate-Hochschild cohomology from the singularity category

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The singularity category (or stable derived category) was introduced by Buchweitz [1] in 1986 and rediscovered in a geometric context by Orlov [8] in 2003. It measures the failure of regularity of an algebra or scheme. Following Buchweitz, one defines the Tate-Hochschild cohomology of an algebra as the Yoneda algebra of the identity bimodule in the singularity category of bimodules. In recent work, Zhengfang Wang [9] has shown that Tate-Hochschild cohomology is endowed with the same rich structure as classical Hochschild cohomology: a Gerstenhaber [5] bracket in cohomology and a B-infinity structure [3] at the cochain level. This suggests that Tate-Hochschild cohomology might be isomorphic to the classical Hochschild cohomology of a (differential graded) category, in analogy with a theorem of Lowen-Van den Bergh [7] in the classical case. We show that indeed, at least as a graded algebra, Tate- Hochschild cohomology is the classical Hochschild cohomology of the singularity category with its canonical dg enhancement. In joint work with Zheng Hua [4], we have applied this to prove a weakened version of a conjecture by Donovan-Wemyss [2] on the reconstruction of a (complete, local, compound Du Val) singularity from its contraction algebra, i.e. the algebra representing the non commutative deformations of the exceptional fiber of a resolution.

References

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