Hochschild cohomology of Beilinson algebras of graded down-up algebras

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Let k be an algebraically closed field of $\operatorname{char} k = 0$. A graded k-algebra $A(\alpha, \beta) := k\langle x, y\rangle/(x^2y - \beta yx^2 - \alpha xyx,\ xy^2 - \beta y^2x - \alpha yxy), \deg x = m, \deg y = n \in \mathbb{N}^+$ with parameters $\alpha, \beta \in k$ is called a *graded down-up algebra*. It is known that a graded down-up algebra $A = (\alpha, \beta)$ is a noetherian AS-regular algebra of dimension 3 if and only if $\beta \neq 0$ ([4]). By the special case of [5, Theorem 4.14], if $A = A(\alpha, \beta)$ is a graded down-up algebra with $\beta \neq 0$, then the Beilinson algebra ∇A of A is extremely Fano of global dimension 2, and there exists an equivalence of triangulated categories $\mathsf{D}^{\mathsf{b}}(\mathsf{tails}\, A) \cong \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, \nabla A)$, where $\mathsf{tails}\, A$ is the noncommutative projective scheme of A in the sense of [1].

The aim of our talk is to investigate the Hochschild cohomology groups $\mathrm{HH}^i(\nabla A)$ of ∇A of a graded down-up algebra $A = A(\alpha, \beta)$ with $\beta \neq 0$. If $\deg x = \deg y = 1$, then a description of the Hochschild cohomology group $\mathrm{HH}^i(\nabla A)$ of ∇A has been obtained using a geometric technique ([2, Table 2]). In this talk, for $\deg x = 1, \deg y = n \geq 2$, we give the dimension formula of $\mathrm{HH}^i(\nabla A)$ for each $i \geq 0$. In this case, the Beilinson algebra ∇A of A is given by the following quiver Q with relations $f_i = 0$ $(1 \leq i \leq n)$, $g = 0$:

$$Q := \begin{tikzcd}
1 \arrow[r, "x_1"] \arrow[rrrr, bend right, "y_1"'] & 2 \arrow[r, "x_2"] \arrow[rrrr, bend right, "y_2"'] & \cdots \arrow[r, "x_{n-1}"] & n \arrow[r, "x_n"] \arrow[rrr, bend right, "y_n"'] & n+1 \arrow[r, "x_{n+1}"] \arrow[rrr, bend right, "y_{n+1}"'] & n+2 \arrow[r, "x_{n+2}"] \arrow[rrr, bend right, "y_{n+2}"'] & \cdots \arrow[r, "x_{2n}"] & 2n+1 \arrow[r, "x_{2n+1}"] & 2n+2
\end{tikzcd},$$

$$f_i := x_i x_{i+1} y_{i+2} - \beta y_i x_{i+n} x_{i+n+1} - \alpha x_i y_{i+1} x_{i+n+1},$$
$$g := x_1 y_2 y_{n+2} - \beta y_1 y_{n+1} x_{2n+1} - \alpha y_1 x_{n+1} y_{n+2}.$$

In particular, it turns out from our dimension formula that the group structure of $\mathrm{HH}^i(\nabla A)$ depends on the values of $\alpha^2 + 4\beta$ and $\delta_n := \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & 1 \\ \beta & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ([3, Theorem 1.4]). Using the fact that Hochschild cohomology is invariant under derived equivalence, our result implies the following: Let $A = A(\alpha, \beta)$ and $A' = A(\alpha', \beta')$ be graded down-up algebras with $\deg x = 1, \deg y = n \geq 1$, where $\beta \neq 0, \beta' \neq 0$. If $\delta_n := \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & 1 \\ \beta & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$ and $\delta'_n := \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha' & 1 \\ \beta' & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq 0$, then $\mathsf{D}^{\mathsf{b}}(\mathsf{tails}\, A) \ncong \mathsf{D}^{\mathsf{b}}(\mathsf{tails}\, A')$ ([3, Corollary 1.5]).

References

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2010 *Mathematics Subject Classification*. 16E40, 16S38, 16E05, 18E30.