On the 2-test modules of projectivity and weakly $\mathfrak{m}$-full ideals

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Throughout this talk, let R be a commutative noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. All modules considered in this paper are assumed to be finitely generated. The notion of a strong test module for projectivity has been introduced and studied by Ramras [3]. An R-module M is called a *strong test module for projectivity* if every R-module N with $\mathrm{Ext}_R^1(N, M) = 0$ is projective. The residue field k and the unique maximal ideal $\mathfrak{m}$ are typical examples of a strong test module for projectivity.

Definition 1. Let M be a non-zero module and let n be a positive integer.

(1) M is called *n-test module for projectivity* if every module X with $\mathrm{Ext}_R^{1\sim n}(X, M) = 0$ is projective.

(2) M is called *n-Tor-test module for projectivity* if every module X with $\mathrm{Tor}^R_{1\sim n}(X, M) = 0$ is projective.

The main results in this talk are the following three theorems.

Theorem 2. *If M is an n-Tor-test module for projectivity then $M, \Omega_R M, \Omega_R^2 M, \ldots, \Omega_R^n M$ are n-test modules for projectivity.*

Theorem 3. *If I is weakly $\mathfrak{m}$-full and $\mathrm{Tor}_1^R(M, R/I) = 0$ then a free covering $0 \to N \to F \to M \to 0$ induces a short exact sequence $0 \to N/IN \to F/IF \to M/IM \to 0$ satisfying $\mathrm{depth}_R N/IN > 0$. Moreover, if I is $\mathfrak{m}$-primary then M is projective.*

Theorem 4. *Suppose I is weakly $\mathfrak{m}$-full and $\mathrm{depth}_R R/I = 0$. If $\mathrm{Tor}_n^R(M, R/I) = 0$ and $\mathrm{depth}_R(\mathrm{Tor}^R_{n-1}(M, R/I)) > 0$ then $\mathrm{proj.dim}_R M < n - 1$ for all positive integer n.*

These theorems induce the following corollaries.

Corollary 5. [1] *Let R be a local ring and let I be an $\mathfrak{m}$-primary ideal of R. If I is weakly $\mathfrak{m}$-full then R/I is a 1-Tor-test module for projectivity.*

Corollary 6. *Let R be a local ring and let I be an $\mathfrak{m}$-primary ideal of R. If I is weakly $\mathfrak{m}$-full then R/I and I are strong test modules for projectivity.*

Corollary 7. [2] *Suppose I is weakly $\mathfrak{m}$-full and $\mathrm{depth}_R R/I = 0$, the following statements hold.*

(1) *R/I is a 2-Tor-test module for projectivity.*

(2) *R/I and I are 2-test modules for projectivity.*

References

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